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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO MAXWELL'S EQUATIONS IN BOUNDED DOMAINS WITH APPLICATION TO MAGNETOTELLURICS

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We analyze the solution of the time-harmonic Maxwell equations with vanishing electric permittivity in bounded domains and subject to absorbing boundary conditions. The problem arises naturally in magnetotellurics when considering the propagation of electromagnetic waves within the earth's interior. Existence and uniqueness are shown under the assumption that the source functions are square integrable. In this case, the electric and magnetic fields belong to $H(\text{curl};\Omega)$. If, in addition, the divergences of the source functions are square integrable and the coefficients are Lipschitz-continuous, a stronger regularity result is obtained. A decomposition of the space of square integrable vector functions and a new compact imbedding result are exploited.

1. Introduction

The magnetotelluric method is used to infer distribution of the earth's electric conductivity from measurements of natural electric and magnetic fields on the earth's surface (see Refs. 3, 15, 18 and 21). Applications of the magnetotelluric method include petroleum exploration in regions where the seismic reflection method is very expensive or impossible to perform. The aim of this paper is to derive existence and uniqueness results for a mathematical model arising from magnetotellurics.

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Let $E = E(x, \omega)$ and $H = H(x, \omega)$ denote the electric and magnetic field intensities, respectively, at $x \in \mathbb{R}^3$ and frequency ω. Consider the time-harmonic Maxwell equations in the form

$$(i\omega\varepsilon + \sigma)E - \nabla \times H = f \,, \tag{1.1a}$$

$$i\omega\mu H + \nabla \times E = g \,, \tag{1.1b}$$

where the 3×3 matrix functions ε, σ and μ designate the electric permittivity, electric conductivity, and magnetic permeability in the medium, respectively. In (1.1), $-f$ and $-g$ denote the electric and magnetic current densities, and $\rho = \rho(x) = \nabla \cdot (\varepsilon E)(x)$ and $m = m(x) = \nabla \cdot (\mu H)(x)$ are the electric and magnetic charges.

One of the most important features of the magnetotelluric method is that $\omega\varepsilon \ll \sigma$ when ε and σ are scalars; consequently, the term containing displacement currents associated with the electric permittivity ε is usually dropped from (1.1). Another important feature in magnetotelluric modelling comes from limiting the computational domain; one often introduces an artificial boundary so that the size of the domain is reasonable. An absorbing boundary condition, such as the one we impose below, needs to be imposed on the artificial boundary to reduce the effects of reflections generated on this part of the boundary.

Our problem is, therefore, formulated as follows. Let Ω be a bounded, open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary Γ. Then, find E and H such that

$$\sigma E - \nabla \times H = f \quad \text{in } \Omega \,, \tag{1.2a}$$

$$i\omega\mu H + \nabla \times E = g \quad \text{in } \Omega \,, \tag{1.2b}$$

$$\alpha P_\tau E + \nu \times H = 0 \quad \text{on } \Gamma \,, \tag{1.2c}$$

where α is a 3×3 matrix function defined on Γ, ν the unit outward normal on the boundary Γ; P_τ, the projection of the trace operator, is defined in (2.1). If σ and μ are positive functions and

$$\alpha = \frac{1-i}{\sqrt{2\omega}} \sqrt{\frac{\sigma}{\mu}} \,,$$

the boundary condition (1.2c) is a limit of well-known absorbing boundary conditions (see Refs. 2, 8, 12, 14, 23 and 25) for the full Maxwell equations (1.1), and its effect is such that electromagnetic waves arriving normally at the boundary are transmitted transparently; it reduces reflections from the artificial boundary Γ and is a convenient and effective way of controlling computational costs when performing numerical simulations using discretizations of (1.2).

We assume that, for all $\xi \in \mathbb{C}^3$ and $x \in \overline{\Omega}$, the medium parameters satisfy

$$\begin{aligned} 0 < \sigma_{\min}|\xi|^2 \le \sum_{j,k=1}^{3} \sigma_{jk}(x)\xi_j\overline{\xi_k} \le \sigma_{\max}|\xi|^2\,, \\ 0 < \mu_{\min}|\xi|^2 \le \sum_{j,k=1}^{3} \mu_{jk}(x)\xi_j\overline{\xi_k} \le \mu_{\max}|\xi|^2\,, \end{aligned} \tag{1.3}$$

for positive $\sigma_{\min}$, $\sigma_{\max}$, $\mu_{\min}$ and $\mu_{\max}$. Also assume that α is Lipschitz-continuous on Γ and, again for all $\xi \in \mathbb{C}^3$ and $x \in \overline{\Omega}$, that

$$0 < \alpha_{\min}|\xi|^2 \le \left| \sum_{j,k=1}^{3} \alpha_{jk}(x)\xi_j\overline{\xi_k} \right|, \quad \sum_{j,k=1}^{3} \mathrm{Re}(\alpha_{jk}(x))\xi_j\overline{\xi_k} \ge 0 \tag{1.4}$$

for some positive $\alpha_{\min}$. If σ and μ are positive functions and $\alpha = \frac{1-i}{\sqrt{2\omega}}\sqrt{\frac{\sigma}{\mu}}$, $\mathrm{Re}(\alpha) \ge \frac{1}{\sqrt{2\omega}}\sqrt{\sigma_{\min}/\mu_{\max}} > 0$ and $-\mathrm{Im}(\alpha) \ge \frac{1}{\sqrt{2\omega}}\sqrt{\sigma_{\min}/\mu_{\max}} > 0$.

Under slightly stronger assumptions on ε, μ and σ, the case when $\varepsilon > 0$ has been analyzed in Ref. 25, where a unique continuation principle is used.

If (1.2) is considered in the space–time domain, it has the nature of a parabolic system rather than that of a hyperbolic system that occurs when $\varepsilon > 0$. The major source of difficulty in analyzing (1.2) is its treatment with the boundary condition (1.2c), where tangential components of electric and magnetic fields are coupled.

Problems similar to (1.1) with a Dirichlet boundary condition for the electric field $\nu \times E = \Phi$ on Γ have been studied by several authors; see, e.g. Refs. 1, 16, 17 and 29. Magnetostatic and electrostatic problems with mixed boundary conditions in inhomogeneous, anisotropic media have been analyzed in Refs. 9 and 13, while a boundary condition of the type (1.2c) has been treated for the time-dependent Maxwell equation in Ref. 4.

Several conforming and nonconforming mixed finite element procedures and domain-decomposition iterative procedures for problems related with (1.2) have been proposed and analyzed in Refs. 7, 21 and 22.

The purpose of this paper is to show the existence and uniqueness of solutions of (1.2). The results are stated precisely in the following theorems.

Theorem 1.1. *Let f, $g \in [L^2(\Omega)]^3$ and $\omega \neq 0$. Then, there exist unique electromagnetic fields $\{E, H\} \in [H(\mathrm{curl};\Omega)]^2$ satisfying* (1.2).

Theorem 1.2. *Assume further that σ, μ are Lipschitz-continuous on $\overline{\Omega}$. Let $f, g \in H(\mathrm{div};\Omega)$ and $\omega \neq 0$. Then, there exist unique electromagnetic fields $\{E, H\} \in [H(\mathrm{curl};\Omega)]^2$ satisfying* (1.2). *Moreover, $\{E, H\}$ belongs to $[H^{1/2}(\Omega)]^6$; more precisely, $\{E, H\} \in [H(\mathrm{curl};\Omega)]^2 \cap [H(\mathrm{div};\Omega)]^2$ with boundary values in $[L^2(\Gamma)]^6$.*

The function spaces will be defined at the beginning of the next section.

The paper is organized as follows. In Sec. 2 we introduce some notation and definitions of the function spaces, along with some preliminary results. Then, in Sec. 3 we prove several lemmas. In order to treat the impedance boundary condition (1.2c), a generalized Green's theorem must be used in several places. Problem (1.2) will be reduced to a tractable one by using a decomposition of a square-integrable vector function into a gradient plus a multiple of a divergence-free vector. A new result on compact imbedding will be proved. Combining several lemmas, we prove Theorems 1.1 and 1.2.

2. Notations and Preliminaries

Functions and inner products are taken to be in the complex field. For a positive integer N, denote by $[L^2(\Omega)]^N$ and $[L^2(\Gamma)]^N$ the spaces of square-integrable vector functions on Ω and Γ, respectively, with corresponding inner products and norms $(\cdot\,,\cdot)$, $\langle\cdot\,,\cdot\rangle_\Gamma$ and $\|\cdot\|$, $|\cdot|_\Gamma$. Also, the space of functions square integrable with respect to the weight function w will be denoted by $L^2_w(\Omega)$. That is, $f \in L^2_w(\Omega)$ implies that $\int_\Omega |f|^2 w\,dx < \infty$ and $\|f\|_{L^2_w(\Omega)} = [\int_\Omega |f|^2 w\,dx]^{1/2}$. Let $[H^m(\Omega)]^N$ and $[H^m(\Gamma)]^N$, for any real number m, denote the usual vector Sobolev spaces with norms $\|\cdot\|_m$ and $|\cdot|_{m,\Gamma}$ (see Ref. 11). Let the Hilbert spaces

$$H(\mathrm{curl};\Omega) = \{u \in [L^2(\Omega)]^3;\ \nabla\times u \in [L^2(\Omega)]^3\}\,,$$
$$H(\mathrm{div};\Omega) = \{u \in [L^2(\Omega)]^3;\ \nabla\cdot u \in L^2(\Omega)\}\,,$$

be endowed with the corresponding inner products

$$(u,v)_{H(\mathrm{curl};\Omega)} = (u,v) + (\nabla\times u, \nabla\times v)\,,\quad (u,v)_{H(\mathrm{div};\Omega)} = (u,v) + (\nabla\cdot u, \nabla\cdot v)\,,$$

and norms

$$\|u\|_{H(\mathrm{curl};\Omega)} = \{\|u\|^2 + \|\nabla\times u\|^2\}^{1/2}\,,\quad \|u\|_{H(\mathrm{div};\Omega)} = \{\|u\|^2 + \|\nabla\cdot u\|^2\}^{1/2}\,.$$

Also, set

$$H(\mathrm{div}\,0;\Omega) = \{u \in [L^2(\Omega)]^3;\ \nabla\cdot u = 0 \text{ in } \Omega\}\,.$$

Also, for $w \neq 0$, let

$$H(\mathrm{curl}\,w;\Omega) = \{u \in [L^2(\Omega)]^3;\ \nabla\times(wu) \in [L^2(\Omega)]^3\}\,,$$

with the inner product

$$(u,v)_{H(\mathrm{curl}\,w;\Omega)} = (u,v) + (\nabla\times(wu), \nabla\times(wv))$$

and the norm

$$\|u\|_{H(\mathrm{curl}\,w;\Omega)} = \{\|u\|^2 + \|\nabla\times(wu)\|^2\}^{1/2}\,.$$

Throughout, P_τ the projection of the trace operator into the plane perpendicular to ν. Therefore, if u is a three-dimensional vector field defined on Γ,

$$P_\tau u = u - \nu(\nu \cdot u) = -\nu \times (\nu \times u) \quad \text{on } \Gamma \,. \tag{2.1}$$

The following generalized Green's theorem for vector functions in $H(\text{curl};\Omega)$ was established in Ref. 24:

$$(\nabla \times U, V) - (U, \nabla \times V) = \langle \nu \times U, V \rangle_\Gamma = \langle \nu \times U, P_\tau V \rangle_\Gamma \,, \quad U, V \in H(\text{curl};\Omega) \,. \tag{2.2}$$

For U, $V \in H(\text{curl};\Omega)$ the traces $\nu \times U$ and $\nu \times V$ belong only to $[H^{-1/2}(\Gamma)]^3$. In this case, and in what follows, the boundary integral $\langle \nu \times U, V \rangle_\Gamma$ is understood as $\langle \nu \times U \cdot \overline{V}, 1 \rangle$, the duality pairing between $\nu \times U \cdot \overline{V} \in \text{Lip}(\Gamma)'$ and $1 \in \text{Lip}(\Gamma)$, where $\text{Lip}(\Gamma)'$ is the dual space of the space $\text{Lip}(\Gamma)$ of Lipschitz-continuous functions on Γ. Clearly, the constant function 1 on Γ belongs to $\text{Lip}(\Gamma)$. For details, see Ref. 24.

3. Proofs of Theorems 1.1 and 1.2

Multiply (1.2a) by the conjugate of $\phi \in H(\text{curl};\Omega)$ and integrate over Ω, use (2.2) on the second term, and apply the boundary condition (1.2c). Also, multiply (1.2b) by the conjugate of $\psi \in [L^2(\Omega)]^3$ and integrate over Ω. The weak formulation of (1.2) is then given by

$$(\sigma E, \phi) - (H, \nabla \times \phi) + \langle \alpha P_\tau E, P_\tau \phi \rangle_\Gamma = (f, \phi) \,, \quad \phi \in H(\text{curl};\Omega) \,, \tag{3.1a}$$

$$i\omega(\mu H, \psi) + (\nabla \times E, \psi) = (g, \psi) \,, \quad \psi \in [L^2(\Omega)]^3 \,. \tag{3.1b}$$

The boundary term has meaning since α^* ($= \overline{\alpha}^{\mathrm{T}}$, the complex conjugate of the transpose of α) is Lipschitz-continuous, and therefore, for some $\widetilde{\alpha}^* \in \text{Lip}(\overline{\Omega})$ such that $\widetilde{\alpha}^*|_\Gamma = \alpha^*$, $\widetilde{\alpha}^*\phi \in H(\text{curl};\Omega)$ and $\nu \times [\alpha^*\phi] \in [H^{-1/2}(\Gamma)]^3$; moreover, $\langle \alpha P_\tau E, \phi \rangle_\Gamma = \langle P_\tau E, \alpha^* P_\tau \phi \rangle_\Gamma$.

Let the Maxwell operator be denoted by

$$M = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix} : [H(\text{curl};\Omega)]^2 \subset [L^2(\Omega)]^6 \to [L^2(\Omega)]^6 \,,$$

and consider the unbounded operator L_ω in $[L^2(\Omega)]^6$, for $\omega > 0$, defined by

$$L_\omega = \begin{bmatrix} \sigma & 0 \\ 0 & i\omega\mu \end{bmatrix} - M : \mathcal{D}(L_\omega) \subset [L^2(\Omega)]^6 \to [L^2(\Omega)]^6 \,,$$

with domain

$$\mathcal{D}(L_\omega) = \{\{U, V\} \in [H(\text{curl};\Omega)]^2 : \alpha P_\tau U + \nu \times V = 0 \quad \text{on } \Gamma\} \,,$$

which is dense in $[L^2(\Omega)]^6$.

Then, given $\{f, g\} \in [L^2(\Omega)]^6$, (1.2) is equivalent to finding $\{E, H\} \in \mathcal{D}(L_\omega)$ such that

$$L_\omega\{E, H\} = \{f, g\} \text{ in } \Omega \,.$$

We then have the following result.

Lemma 3.1. $\mathrm{Ker}(L_\omega) = \{0, 0\}$ *for* $\omega \neq 0$.

Proof. Assume $\{E, H\} \in \mathrm{Ker}(L_\omega)$, so that $f = g = 0$. Then choose $\phi = E$ in (3.1a) and $\psi = H$ in the conjugate of (3.1b), and add the resulting equations to obtain

$$(\sigma E, E) - i\omega(\mu H, H) + \langle \alpha P_\tau E, P_\tau E \rangle_\Gamma = 0\,. \tag{3.2}$$

The real part of (3.2) gives

$$(\sigma E, E) + \langle \mathrm{Re}(\alpha) P_\tau E, P_\tau E \rangle_\Gamma = 0\,.$$

Thus, by the assumptions (1.3) and (1.4), $E \equiv 0$ in Ω. This in turn implies that H is identically zero, by (1.2b). This completes the proof. □

Now, let us analyze the range of the operator L_ω. The adjoint operator L_ω^* of L_ω and its domain $\mathcal{D}(L_\omega^*)$ are given by

$$L_\omega^* = \begin{bmatrix} \sigma^{\mathrm{T}} & 0 \\ 0 & -i\omega\mu^{\mathrm{T}} \end{bmatrix} + M : \mathcal{D}(L_\omega^*) \subset [L^2(\Omega)]^6 \to [L^2(\Omega)]^6\,,$$

$$\mathcal{D}(L_\omega^*) = \{\{U, V\} \in [H(\mathrm{curl};\Omega)]^2 : P_\tau[\alpha^* U] - \nu \times V = 0 \quad \text{on } \Gamma\}\,.$$

Consequently, the adjoint problem to (1.2) is given by

$$\begin{aligned} \sigma^{\mathrm{T}} E - \nabla \times H &= f \quad \text{in } \Omega\,, \\ -i\omega\mu^{\mathrm{T}} H + \nabla \times E &= g \quad \text{in } \Omega\,, \\ P_\tau[\alpha^* E] - \nu \times H &= 0 \quad \text{on } \Gamma\,. \end{aligned}$$

The next lemma follows from exactly the same argument as given in the proof of Lemma 3.1.

Lemma 3.2. $\mathrm{Ker}(L_\omega^*) = \{0, 0\}$ *for* $\omega \neq 0$.

Lemma 3.1 implies uniqueness for (1.2), while Lemma 3.2 implies that $\mathcal{R}(L_\omega)$ is dense in $[L^2(\Omega)]^6$. Thus, to obtain the existence for (1.2), by the Banach closed range theorem, we need only to show that $\mathcal{R}(L_\omega)$ is closed. Therefore, let $\{f_j, g_j\}$ be a sequence in $\mathcal{R}(L_\omega)$ such that

$$\{f_j, g_j\} \to \{f, g\} \text{ in } [L^2(\Omega)]^6 \quad \text{as } j \to \infty\,.$$

We wish to show that $\{f, g\} \in \mathcal{R}(L_\omega)$.

For each j, there exists $\{E_j, H_j\} \in \mathcal{D}(L_\omega)$ such that

$$\sigma E_j - \nabla \times H_j = f_j \quad \text{in } \Omega\,, \tag{3.3a}$$

$$i\omega\mu H_j + \nabla \times E_j = g_j \quad \text{in } \Omega\,, \tag{3.3b}$$

$$\alpha P_\tau E_j + \nu \times H_j = 0 \quad \text{on } \Gamma\,. \tag{3.3c}$$

We recall (see Refs. 6, 9, 17, 19 and 25) the following decomposition of vectors:

$$[L^2_\sigma(\Omega)]^3 = \nabla H^1_0(\Omega) \oplus \sigma^{-1} H(\operatorname{div} 0; \Omega)\,, \quad [L^2_\mu(\Omega)]^3 = \nabla H^1_0(\Omega) \oplus \mu^{-1} H(\operatorname{div} 0; \Omega)\,.$$

(For the closedness of $\nabla H^1_0(\Omega)$ in $[L^2(\Omega)]^3$, see Ref. 9.) Therefore, for each j, there exist ϕ_j, $\psi_j \in H^1_0(\Omega)$ and $e_j, h_j \in H(\operatorname{div} 0; \Omega)$ such that

$$E_j = \nabla\phi_j + \sigma^{-1} e_j\,, \quad H_j = \nabla\psi_j + (i\omega\mu)^{-1} h_j\,. \tag{3.4}$$

Since $\nabla \times \nabla = 0$, (3.3) and (3.4) imply that $\sigma^{-1}e_j$ and $\mu^{-1}h_j$ belong to $H(\operatorname{curl}; \Omega)$ with $\nu \times [\sigma^{-1}e_j]$ and $\nu \times [\mu^{-1}h_j]$ in $[H^{-1/2}(\Gamma)]^3$ and that

$$\sigma\nabla\phi_j + e_j - \nabla \times [(i\omega\mu)^{-1}h_j] = f_j\,, \tag{3.5a}$$

$$i\omega\mu\nabla\psi_j + h_j + \nabla \times [\sigma^{-1}e_j] = g_j\,. \tag{3.5b}$$

Also, since $\nu \times \nabla\phi_j = 0$ for $\phi_j \in H^1_0(\Omega)$,

$$\begin{aligned} P_\tau E_j &= -\nu \times (\nu \times E_j) = -\nu \times [\nu \times (\sigma^{-1}e_j + \nabla\phi_j)] \\ &= -\nu \times (\nu \times \sigma^{-1}e_j) = P_\tau[\sigma^{-1}e_j] \quad \text{on } \Gamma\,. \end{aligned}$$

Similarly,

$$\nu \times H_j = \nu \times [(i\omega\mu)^{-1}h_j + \nabla\phi_j] = \nu \times [(i\omega\mu)^{-1}h_j] \quad \text{on } \Gamma\,.$$

By (3.3c),

$$\alpha P_\tau[\sigma^{-1}e_j] + \nu \times [(i\omega\mu)^{-1}h_j] = 0 \quad \text{on } \Gamma\,. \tag{3.6}$$

An application of the divergence operator in (3.5) leads to the two independent elliptic problems

$$\nabla \cdot (\sigma\nabla\phi_j) = \nabla \cdot f_j \quad \text{in } \Omega\,, \tag{3.7a}$$

$$i\omega\nabla \cdot (\mu\nabla\psi_j) = \nabla \cdot g_j \quad \text{in } \Omega\,, \tag{3.7b}$$

$$\phi_j = \psi_j = 0 \qquad \text{on } \Gamma\,. \tag{3.7c}$$

Note that (3.7a) and (3.7b) are understood as in Ref. 10 or Ref. 26. Thus, it follows that

$$\|\nabla\phi_j\| \le \frac{\|f_j\|}{\sigma_{\min}}\,, \quad \|\psi_j\| \le \frac{\|g_j\|}{\mu_{\min}}\,.$$

Since $\nabla\phi_j$ and $\nabla\psi_j$ are Cauchy sequences in $[L^2(\Omega)]^3$, there exist e_0 and h_0 in $[L^2(\Omega)]^3$ such that

$$\nabla\phi_j \to e_0 \quad \text{and} \quad \nabla\psi_j \to h_0 \tag{3.8}$$

in $[L^2(\Omega)]^3$ as $j \to \infty$. From (3.5) and (3.6), it follows that

$$e_j - \nabla \times (i\omega\mu)^{-1}h_j = \widetilde{f}_j \quad \text{in } \Omega\,, \tag{3.9a}$$

$$h_j + \nabla \times (\sigma^{-1}e_j) = \widetilde{g}_j \quad \text{in } \Omega\,, \tag{3.9b}$$

$$\alpha P_\tau[\sigma^{-1}e_j] + \nu \times [(i\omega\mu)^{-1}h_j] = 0 \quad \text{on } \Gamma\,, \tag{3.9c}$$

where $\widetilde{f}_j = f_j - \sigma\nabla\phi_j$ and $\widetilde{g}_j = g_j - i\omega\mu\nabla\psi_j$ are such that $\nabla\cdot\widetilde{f}_j = \nabla\cdot\widetilde{g}_j = 0$ and $\widetilde{f}_j \to f - \sigma e_0$, $\widetilde{g}_j \to g - i\omega\mu h_0$ in $[L^2(\Omega)]^3$ as $j \to \infty$.

Let us introduce the space

$$W = \{\{U, V\} \in [H(\operatorname{curl}\sigma^{-1};\Omega) \times H(\operatorname{curl}\mu^{-1};\Omega)] \cap [H(\operatorname{div}0;\Omega)]^2 : \\ \alpha P_\tau[\sigma^{-1}U] + \nu \times [(i\omega\mu)^{-1}V] = 0 \text{ on } \Gamma\},$$

equipped with the inner product and norm of $H(\operatorname{curl}\sigma^{-1};\Omega) \times H(\operatorname{curl}\mu^{-1};\Omega)$.

Consider the operator $\widetilde{L}_\omega : W \to [H(\operatorname{div}0;\Omega)]^2$ defined by

$$\widetilde{L}_\omega = I - \begin{bmatrix} 0 & \nabla\times(i\omega\mu)^{-1} \\ -\nabla\times\sigma^{-1} & 0 \end{bmatrix}.$$

It follows from (3.9) that $\{e_j, h_j\} \in W$ and $\widetilde{L}_\omega\{e_j, h_j\} = \{\widetilde{f}_j, \widetilde{g}_j\}$. We wish to show that $\{f - \sigma e_0, g - i\omega\mu h_0\} \in R(\widetilde{L}_\omega)$. The following observation provides a starting point.

Lemma 3.3. *For $\{U, V\} \in W$, the tangential components $\nu\times[\sigma^{-1}U]$ and $\nu\times[\mu^{-1}V]$ belong to $[L^2(\Gamma)]^3$.*

Proof. Let $\{U, V\} \in W$. Since $\sigma^{-1}U \in H(\operatorname{curl};\Omega)$, $P_\tau[\sigma^{-1}U]\cdot P_\tau\overline{[\sigma^{-1}U]} = \nu \times [\sigma^{-1}U]\cdot[\nu\times\overline{\sigma^{-1}U}]$ belongs to $\operatorname{Lip}(\Gamma)'$; for the same reason $P_\tau[\mu^{-1}V]\cdot P_\tau\overline{[\mu^{-1}V]}$ also belongs to $\operatorname{Lip}(\Gamma)'$. Then, the boundary condition $\alpha P_\tau[\sigma^{-1}U] + \nu\times[(i\omega\mu)^{-1}V] = 0$ on Γ yields the following estimate:

$$\begin{aligned} &|\langle\omega\alpha P_\tau[\sigma^{-1}U], P_\tau[\sigma^{-1}U]\rangle_\Gamma| \\ &\quad = \omega|\langle\nu\times[(i\omega\mu)^{-1}V], \sigma^{-1}U\rangle_\Gamma| \\ &\quad = |(\nabla\times[\mu^{-1}V], \sigma^{-1}U) - (\mu^{-1}V, \nabla\times[\sigma^{-1}U])| \\ &\quad \le \|\nabla\times[\mu^{-1}V]\|\,\|\sigma^{-1}U\| + \|\mu^{-1}V\|\,\|\nabla\times[\sigma^{-1}U]\| \\ &\quad \le \frac{1}{2}[\|U\|^2_{H(\operatorname{curl}\sigma^{-1};\Omega)} + \|V\|^2_{H(\operatorname{curl}\mu^{-1};\Omega)}] < \infty. \end{aligned} \tag{3.10}$$

On the other hand, by (1.4), we have

$$\omega\alpha_{\min}\langle\nu\times[\sigma^{-1}U], \nu\times[\sigma^{-1}U]\rangle_\Gamma \\ = \omega\alpha_{\min}\langle P_\tau[\sigma^{-1}U], P_\tau[\sigma^{-1}U]\rangle_\Gamma \le |\langle\omega\alpha P_\tau[\sigma^{-1}U], P_\tau[\sigma^{-1}U]\rangle_\Gamma|,$$

which, together with (3.10), shows that $\nu\times[\sigma^{-1}U] \in [L^2(\Gamma)]^3$. Since $P_\tau[\sigma^{-1}U] = -\nu\times[\nu\times(\sigma^{-1}U)]$, the boundary condition $\alpha P_\tau[\sigma^{-1}U] + \nu\times[(i\omega\mu)^{-1}V] = 0$ on Γ implies that $\nu\times[\mu^{-1}V] \in [L^2(\Gamma)]^3$. This completes the proof. □

For $\eta = \sigma$ or μ, let

$$W_\eta = \{V \in H(\operatorname{curl}\eta^{-1};\Omega) \cap H(\operatorname{div};\Omega) : \nu\times[\eta^{-1}V] \in [L^2(\Gamma)]^3\},$$

and note that

$$W_\eta = \{U \in H(\text{curl};\Omega) \cap H(\text{div}\,\eta;\Omega) : \nu \times U \in [L^2(\Gamma)]^3\}\,.$$

Next, we consider the following compactness result.

Lemma 3.4. *Suppose that η is a 3×3 complex-valued matrix function such that*

$$0 < \eta_{\min}|\xi|^2 \le \left|\sum_{j,k=1}^{3} \eta_{jk}(x)\xi_j\overline{\xi}_k\right| \le \eta_{\max}|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^3$ for positive constants $\eta_{\min}$ and $\eta_{\max}$. Then, the imbedding $W_\eta \hookrightarrow [L^2(\Omega)]^3$ is compact.

Proof. Let $\{U_n\} \subset W_\eta$ be such that

$$\|U_n\| + \|\nabla \times U_n\| + \|\nabla \cdot (\eta U_n)\| + |\nu \times U_n|_\Gamma \le 1\,, \quad \forall\; n\,. \tag{3.11}$$

We wish to show that there is a convergent subsequence of $\{U_n\}$.

For every n, there exists $\phi_n \in H_0^1(\Omega)$ such that

$$U_n = \nabla\phi_n + V_n\,, \tag{3.12}$$

where (by the Lax–Milgram lemma) ϕ_n solves the variational problem

$$(\eta\nabla\phi_n, \nabla z) = (\eta U_n, \nabla z)\,, \quad z \in H_0^1(\Omega)\,, \tag{3.13}$$

or, equivalently, the Dirichlet problem

$$-\nabla \cdot (\eta\nabla\phi_n) = -\nabla \cdot (\eta U_n)\,, \qquad x \in \Omega\,, \tag{3.14a}$$

$$\phi_n = 0\,, \qquad x \in \Gamma\,. \tag{3.14b}$$

Then, by (3.11),

$$\eta_{\min}\|\nabla\phi_n\| \le \eta_{\max}\|U_n\| \le \eta_{\max}\,.$$

By the Poincaré inequality, $\|\nabla\phi_n\|_1 \le C_1$ for all n (here, and in what follows, C_j's will denote generic constants). Thus, by the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, it follows that there exists a subsequence of $\{\phi_n\}$, again denoted by $\{\phi_n\}$, which is strongly convergent to some element ϕ_0 in $L^2(\Omega)$. Since $\|\nabla \cdot (\eta U_n)\| \le 1$ for all n due to (3.11), there exists a subsequence of $\{U_n\}$, denoted by $\{U_n\}$, such that $\{\nabla \cdot (\eta U_n)\}$ is weakly convergent in $L^2(\Omega)$. By (3.13) and integration by parts,

$$\begin{aligned}|-(\eta\nabla(\phi_n - \phi_k), \nabla(\phi_n - \phi_k))| &= |-(\eta(U_n - U_k), \nabla(\phi_n - \phi_k))| \\ &= |(\nabla \cdot \eta(U_n - U_k), \phi_n - \phi_k)| \to 0\end{aligned}$$

as n, $k \to \infty$. By the assumption on η and again the Poincaré inequality,

$$\|\phi_n - \phi_k\|_1 \le C_2\|\nabla(\phi_n - \phi_k)\| \to 0$$

as n, $k \to \infty$. Thus, $\{\phi_n\}$ is a Cauchy sequence in $H_0^1(\Omega)$; therefore, there exists $\phi_0 \in H_0^1(\Omega)$ such that $\|\phi_n - \phi_0\|_1 \to 0$ as $n \to \infty$.

Next, observe that V_n satisfies

$$\begin{aligned} \nabla \times V_n &= \nabla \times U_n \,, \quad && \Omega \,, \\ \nabla \cdot (\eta V_n) &= 0 \,, && \Omega \,, \\ \nu \times V_n &= \nu \times U_n \,, && \Gamma \,, \end{aligned}$$

and

$$\|V_n\| + \|\nabla \times V_n\| + |\nu \times V_n|_\Gamma \le C_3 \,, \quad \forall\, n \,.$$

Therefore, there exists a subsequence of $\{V_n\}$, still written as $\{V_n\}$, such that $\{\nabla \times V_n\}$ and $\{\nu \times V_n\}$ converge weakly in $[L^2(\Omega)]^3$ and $[L^2(\Gamma)]^3$, respectively.

By Theorem 3.4 in Ref. 11, there exists $\{\Psi_n\} \subset [H^1(\Omega)]^3$ such that

$$\begin{aligned} \nabla \times \Psi_n &= \eta V_n \,, \quad && \Omega \,, \\ \nabla \cdot (\Psi_n) &= 0 \,, && \Omega \,, \end{aligned}$$

from which one can extract a subsequence, still denoted by $\{\Psi_n\}$, which converges strongly in $[L^2(\Omega)]^3$ and $\{\nu \times \Psi_n\}$ converges strongly in $[L^2(\Gamma)]^3$. Hence,

$$\begin{aligned} (\eta(V_n - V_k), V_n - V_k) &= (\nabla \times (\Psi_n - \Psi_k), V_n - V_k) \\ &= (\Psi_n - \Psi_k, \nabla \times (V_n - V_k)) - \langle \Psi_n - \Psi_k, \nu \times (V_n - V_k) \rangle_\Gamma \,, \end{aligned}$$

which tends to 0 as n and k tend to ∞. Therefore, $\{V_n\}$ is a Cauchy sequence in $[L^2(\Omega)]^3$; hence, it converges to an element $V_0 \in [L^2(\Omega)]^3$. By the decomposition (3.12),

$$\|U_n - (\nabla \phi_0 + V_0)\| \le \|\nabla \phi_n - \nabla \phi_0\| + \|V_n - V_0\| \to 0$$

as $n \to \infty$. Thus, the resulting subsequence $\{U_n\}$ converges to $\nabla \phi_0 + V_0$ strongly in $[L^2(\Omega)]^3$. Consequently, the imbedding $W_\eta \hookrightarrow [L^2(\Omega)]^3$ is compact. This completes the proof. □

Remark 3.1. For homogeneous boundary conditions, analogues of Lemma 3.4 were proved by Leis (see Refs. 16 and 17), Weber (see Ref. 28), and Witsch (see Ref. 29). Alonso and Valli (see Ref. 1) treated more complicated domains and complex, symmetric η for the homogeneous boundary condition case. For nonhomogeneous boundary conditions in more complicated domains, Fernandes and Gilardi (see Ref. 9) and Hazard and Lenoir (see Ref. 13) proved Lemma 3.4 under certain mild assumptions on some function spaces and an elliptic regularity for the Neumann problem, respectively.

For a more restricted η, a better compactness result due to Costabel (see Ref. 5) holds for W_η.

Lemma 3.5. *Suppose* η *is a Lipschitz-continuous, complex-valued matrix function such that*

$$0 < \eta_{\min}|\xi|^2 \le \left|\sum_{j,k=1}^{3} \eta_{jk}(x)\xi_j\overline{\xi}_k\right| \le \eta_{\max}|\xi|^2$$

for all $x \in \Omega$ *for positive constants* $\eta_{\min}$ *and* $\eta_{\max}$*. Then,* $W_\eta \subset [H^{1/2}(\Omega)]^3$.

Proof. Let $U \in W_\eta$. Then, $U \in H(\mathrm{curl};\Omega)$ with $\nu \times U \in [L^2(\Gamma)]^3$ and $\nabla \cdot (\eta U) = (\nabla\eta)U + \eta\nabla \cdot U \in L^2(\Omega)$. Thus, U also belongs to $H(\mathrm{div};\Omega)$. By the result of Costabel (see Ref. 5), it follows that $U \in [H^{1/2}(\Omega)]^3$. This completes the proof. □

Next, recall the following lemma due to Peetre (see Ref. 20) and Tartar (see Ref. 27).

Lemma 3.6. *Let* L_1 *be a bounded linear map from a Banach space* W *into a normed linear space* W_1*. Suppose that there exists a compact linear map* L_2 *from* W *into another normed linear space* W_2 *such that*

$$\|u\|_W \le C\{\|L_1 u\|_{W_1} + \|L_2 u\|_{W_2}\} \quad \textit{for all } u \in W\,.$$

Then, the range of L_1 *is closed and* $\dim\ker(L_1) < \infty$*. Moreover,*

$$\inf_{v\in\ker(L_1)} \|u + v\|_W \le C_1\|L_1 u\|_{W_1} \quad \textit{for all } u \in W\,.$$

We now apply Lemma 3.6 to demonstrate that the range of $\widetilde{L}_\omega$ is closed.

Lemma 3.7. *The range* $R(\widetilde{L}_\omega)$ *is closed in* $[L^2(\Omega)]^6$*. Moreover, there exists a positive constant* C*, independent of* $\{U, V\}$*, such that*

$$\|\{U,V\}\|_W \le C\|\widetilde{L}_\omega\{U,V\}\|\,, \quad \forall\ \{U,V\} \in W\,. \tag{3.15}$$

Proof. Set $W_1 = W_2 = [L^2(\Omega)]^6$. Apply Lemma 3.4 with $L_1 = \widetilde{L}_\omega : W \to W_1$ and $L_2 = id : W \hookrightarrow W_2$. First, observe that, by Lemmas 3.3 and 3.4, the imbedding $L_2 : W \hookrightarrow [L^2(\Omega)]^6$ is compact. Next, note that for $\{U,V\} \in W$,

$$\begin{aligned}
&\|\widetilde{L}_\omega\{U,V\}\|_0^2 \\
&\quad = (\{U - \nabla\times(i\omega\mu)^{-1}V, V + \nabla\times\sigma^{-1}U\}, \{U - \nabla\times(i\omega\mu)^{-1}V, V + \nabla\times\sigma^{-1}U\}) \\
&\quad = (U - \nabla\times(i\omega\mu)^{-1}V, U - \nabla\times(i\omega\mu)^{-1}V) + (V + \nabla\times\sigma^{-1}U, V + \nabla\times\sigma^{-1}U) \\
&\quad = \|U\|_0^2 - (\nabla\times(i\omega\mu)^{-1}V, U) \\
&\qquad - (U, \nabla\times(i\omega\mu)^{-1}V) + (\nabla\times(i\omega\mu)^{-1}V, \nabla\times(i\omega\mu)^{-1}V) \\
&\qquad + \|V\|_0^2 + (\nabla\times\sigma^{-1}U, V) + (V, \nabla\times\sigma^{-1}U) + (\nabla\times\sigma^{-1}U, \nabla\times\sigma^{-1}U) \\
&\quad \ge \|U\|_0^2 + \|\nabla\times\sigma^{-1}U\|_0^2 + \|V\|_0^2 + \omega^{-2}\|\nabla\times(\mu^{-1}V)\|_0^2 \\
&\qquad - \epsilon_1\omega^{-2}\|\nabla\times(\mu^{-1}V)\|_0^2 - \frac{1}{\epsilon_1}\|U\|_0^2 - \epsilon_2\|\nabla\times(\sigma^{-1}U)\|_0^2 - \frac{1}{\epsilon_2}\|V\|_0^2\,,
\end{aligned}$$

for all ϵ_1, $\epsilon_2 > 0$. For sufficiently small $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\|\widetilde{L}_\omega\{U,V\}\|_0^2 \geq C_1\|\{U,V\}\|_W^2 - C_2\|\{U,V\}\|_0^2$$

so that

$$\|\{U,V\}\|_W \leq C_3\|\widetilde{L}_\omega\{U,V\}\| + C_4\|\{U,V\}\| \leq C_5\|\widetilde{L}_\omega\{U,V\}\|\,,$$

for generic positive constants C_j, $j = 1,\ldots,5$, independent of $\{U,V\} \in W$. This proves (3.15). Obviously, we also have

$$\|\widetilde{L}_\omega\{U,V\}\|_0^2 \leq C\|U\|_W\|V\|_W\,,$$

which means that $\widetilde{L}_\omega : W \to [L^2(\Omega)]^6$ is continuous.

By Lemma 3.6, $R(\widetilde{L}_\omega)$ is closed. This completes the proof. □

Let us turn to the proofs of Theorems 1.1 and 1.2. By Lemma 3.7, $\{f-\sigma e_0, g-i\omega\mu h_0\} \in R(\widetilde{L}_\omega)$. Thus, there exists $\{e_1,h_1\} \in W$ such that

$$\widetilde{L}_\omega\{e_1,h_1\} = \{f-\sigma e_0, g - i\omega\mu h_0\}\,,$$

so that

$$e_1 - \nabla\times(i\omega\mu)^{-1}h_1 = f - \sigma e_0\,,\quad h_1 + \nabla\times\sigma^{-1}e_1 = g - i\omega\mu h_0\,.$$

Let

$$E = e_0 + \sigma^{-1}e_1\,,\quad H = h_0 + (i\omega\mu)^{-1}h_1\,.$$

Then, since $\nabla\times h_0 = 0$ and $\nabla\times e_0 = 0$,

$$\begin{aligned}\sigma E - \nabla\times H &= \sigma(e_0+\sigma^{-1}e_1) - \nabla\times[h_0 + (i\omega\mu)^{-1}h_1]\\ &= \sigma e_0 + e_1 - \nabla\times(i\omega\mu)^{-1}h_1 = f\,,\\ \nabla\times E + i\omega\mu H &= \nabla\times(e_0+\sigma^{-1}e_1) + i\omega\mu(h_0 + (i\omega\mu)^{-1}h_1)\\ &= \nabla\times\sigma^{-1}e_1 + i\omega\mu h_0 + h_1 = g\,,\end{aligned}$$

in Ω. The boundary condition (1.2c) is obviously satisfied by $\{E,H\}$. Therefore,

$$L_\omega\{E,H\} = \{f,g\}\,,$$

which implies that $\mathcal{R}(L_\omega)$ is closed. This proves Theorem 1.1.

To prove Theorem 1.2, assume further that $f \in H(\mathrm{div};\Omega)$ and $g \in H(\mathrm{div};\Omega)$ with corresponding assumptions on σ and μ. By Theorem 1.1, there exists a unique pair $\{E,H\} \in [H(\mathrm{curl};\Omega)]^2$ satisfying (1.2). Since $\nabla\cdot f \in L^2(\Omega)$ and $\nabla\cdot g \in L^2(\Omega)$, it follows from (1.2a) and (1.2b) that $\nabla\cdot[\sigma E] \in L^2(\Omega)$ and $\nabla\cdot[\mu H] \in L^2(\Omega)$. By Lemma 3.5, $E \in W_\sigma \subset [H^{1/2}(\Omega)]^3$ and $H \in W_\mu \subset [H^{1/2}(\Omega)]^3$ with the boundary traces of E and H in $[L^2(\Gamma)]^3$. This proves Theorem 1.2.

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