6.2 Constant Coeff. Homogeneous DE. (1)

Consider

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

Let

$$P(D) = D^n + a_1 D^{n-1} + \cdots + a_n$$

$=$ polynomial Diff Operator.

It has the associated real polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_n$$

(auxiliary polinomial)

$P(r) = 0$ = auxiliary equation

Theorem 6.2.1 $P(D)$, $Q(D)$ are polyn. dif. op.

$\rightarrow \quad P(D)\,Q(D) = Q(D)\,P(D)$.

Factorization of $P(r)$

$$P(r) = (r-r_1)^{m_1} \cdots (r-r_k)^{m_k}$$

$$m_1 + \cdots + m_k = n$$

$$\rightarrow P(D) = (D-r_1)^{m_1} \cdots (D-r_k)^{m_k}$$

Theorem: If $P(D) = P_1(D)\, P_2(D)\, P_3(D)$

and y solves $P_1(D)\, y = 0 \longrightarrow$

$P(D)\, y = 0$.

Proof: $P(D)\, y = P_2(D)\, P_3(D)\, \underbrace{P_1(D)\, y}_{=0}$

$$= P_2(D)\, P_3(D)\, 0 = 0$$

Consequently, any solution of

$(D-r_i)^{m_i}\, y = 0$ will

also be a solution of $P(D)\, y = 0$.

Theorem 6.2.1 The D.E

$$(D-r)^m y = 0 \qquad m > 0$$

r real or complex, has the following m LI solutions

$$e^{rx}, x e^{rx}, \ldots x^{m-1} e^{rx}$$

Proof: Book — → ~~[illegible]~~

Back to

$$P(D)\,y = (D-r_1)^{m_1} \cdots (D-r_k)^{m_k} y = 0$$

1) Each term $(D-r)^m$ with r real contributes with the LI solutions

$$e^{rx}, x e^{rx}, \ldots, x^{m-1} e^{rx}.$$

2) if $r = a \pm ib$, $(D-r)^m$ contributes

$$e^{(a\pm ib)x}, x e^{(a\pm ib)x}, \ldots, x^{m-1} e^{(a\pm ib)x}$$

$$e^{ax} e^{\pm ibx} = e^{ax} (\cos bx \pm i \sin bx)$$

$\rightarrow$ taking real and imaginary parts we have $\underline{2m}$ real L I solutions

$$e^{ax} \cos bx, \; x e^{ax} \cos bx, \; \text{—}, \; x^{m-1} e^{ax} \cos bx$$

$$e^{ax} \sin bx, \; x e^{ax} \sin bx, \; \text{—}, \; x^{m-1} e^{ax} \sin bx$$

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<u>Examples</u>: Find the general solution of

$$y''' + 2y'' + 3y' + 2y = 0$$

$$P(r) = r^3 + 2r^2 + 3r + 2$$

$$= (r+1)(r^2 + r + 2) \rightarrow P(D) = (D+1)(D^2 + D + 2)$$

$$r_1 = -1 \;, \quad r = \frac{-1 \pm i\sqrt{7}}{2}$$

$$\rightarrow \; y_1 = e^{-x}, \quad y_2 = e^{-x/2} \cos\left(\frac{\sqrt{7}}{2} x\right),$$

$$y_3 = e^{-x/2} \sin\left(\frac{\sqrt{7}}{2} x\right)$$ are l.i. solution

General solution: $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

Ex: find the general solution of

$$(D-3)(D^2+1)\,y = 0$$

$P(r) = (r-3)(r^2+1) = 0 \qquad r = 3,\ i,\ -i$

$y_1 = e^{3x}, \qquad y_2 = \cos x, \qquad y_3 \sin x$

are L.I. solutions

$\rightarrow$ G.S.: $\qquad y = c_1 y_1 + c_2 y_2 + c_3 y_3$

Ex: find the G.S. of

$$D^2(D^2+1)^2\,y = 0$$

$r_1 = 0, \quad r_2 = 0, \quad r_3 = i, \quad r_4 = i, \quad r_5 = -i \quad r_6 = -i$

$y_1 = c, \quad y_2 = x, \quad y_3 = \cos x \quad y_4 = x\cos x$

$y_5 = \sin x \qquad y_6 = x \sin x$

$$y = \sum_{i=1}^{6} c_i y_i$$