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$$\left(\tau c\, \frac{\theta^{n+1}-2\theta^{n}+\theta^{n-1}}{(\Delta t)^{2}},\ \varphi_k\right) \qquad (1)$$

$$+\left(c\, \frac{(\theta^{n+1}-\theta^{n-1})}{2\Delta t},\ \varphi_k\right)+\left(\gamma\, \nabla\theta^{n},\ \nabla\varphi_k\right)$$

$$+\left\langle \tau c\, V_\theta \left(\frac{\theta^{n+1}-\theta^{n-1}}{2\Delta t}\right),\ \varphi_k\right\rangle$$

$$= -\left(q^{n},\ \varphi_k\right), \quad k=1, \cdots N_x+1$$

$$\frac{1}{(\Delta t)^{2}}\left(\tau c\, \theta^{n+1},\ \varphi_k\right)+\frac{1}{2\Delta t}\left(c\, \theta^{n+1},\ \varphi_k\right)$$

$$+\frac{1}{2\Delta t}\left\langle \tau c\, V_\theta\, \theta^{n+1},\ \varphi_k\right\rangle$$

$$= \frac{1}{(\Delta t)^{2}}\left(\tau c\, [2\theta^{n}-\theta^{n-1}],\ \varphi_k\right) \qquad (2)$$

$$+\frac{1}{2\Delta t}\left(c\, \theta^{n-1},\ \varphi_k\right)-\left(\gamma\, \frac{\partial\theta^{n}}{\partial x},\ \frac{\partial\varphi^{k}}{\partial x}\right)$$

$$+\frac{1}{2\Delta t}\left\langle \tau c\, V_\theta\, \theta^{n-1},\ \varphi_k\right\rangle-\left(q^{n},\ \varphi_k\right)$$

$$\theta^{n}=\sum_{j=k-1}^{k+1}\theta^{n}_{j}\,\varphi_j$$

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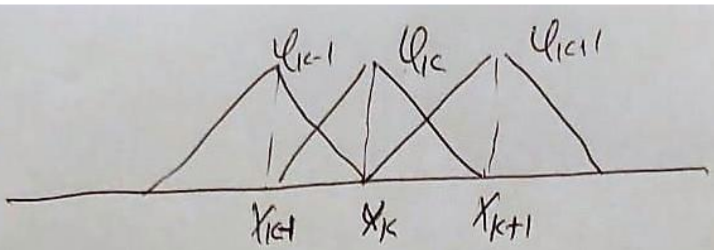

$$\int_{x_{k-1}}^{x_{k+1}} zc\left(\sum_{j=k-1}^{k+1}\theta_j^{n+1}\varphi_j\right)\varphi_k\,dx$$

$$= \int_{x_{k-1}}^{x_k} zc\,\theta_{k-1}^{n+1}\varphi_{k-1}\varphi_k\,dx + \int_{x_{k-1}}^{x_k} zc\,\theta_k^{n+1}\varphi_k\varphi_k\,dx$$

$$+ \int_{x_k}^{x_{k+1}} zc\,\cancel{\varphi_k}\,\theta_k^{n+1}\varphi_k\varphi_k\,dx + \int_{x_k}^{x_{k+1}} zc\,\theta_{k+1}^{n+1}\varphi_{k+1}\varphi_k\,dx$$

$$= \overbrace{(x_k-x_{k-1})}^{=h}\theta_{k-1}^{n+1}\left[\frac{\overbrace{(\varphi_{k-1}\varphi_k)(x_{k-1})}^{(zc)_{k-1}\;=0} + \overbrace{(\varphi_{k-1}\varphi_k)(x_k)}^{(zc)_k\;=0}}{2}\right]\cancel{(zc)_{k-1}}$$

$$+ \overbrace{(x_k-x_{k-1})}^{=h}\left[\frac{\overbrace{(\varphi_k\varphi_k)(x_{k-1})}^{=0\;[zc]_{k-1}} + \overbrace{(\varphi_k\varphi_k)(x_k)}^{=1\;(zc)_k}}{2}\right]\theta_k^{n+1}\cancel{(zc)_{k-1}}$$

$$+ \overbrace{(x_{k+1}-x_k)}^{=h}\left[\frac{\overbrace{(\varphi_k\varphi_k)(x_k)}^{=1\;(zc)_k} + \overbrace{(\varphi_k\varphi_k)(x_{k+1})}^{(zc)_{k+1}\;=0}}{2}\right]\theta_k^{n+1}\cancel{(zc)_k}$$

$$+ \overbrace{(x_{k+1}-x_k)}^{=h}\left[\overbrace{(\varphi_{k+1}\varphi_k)(x_k)}^{=0\;(zc)_k} + \overbrace{(\varphi_{k+1}\varphi_k)(x_{k+1})}^{=0\;(zc)_{k+1}}\right]^{2}\theta_{k+1}^{n+1}(zc)_k$$

$$= \frac{h}{2}\overset{(zc)_{k\cancel{-1}}}{}\theta_k^{n+1} + \overset{(zc)_k}{h}\frac{\theta_k^{n+1}}{2} = \frac{h}{2}\theta_k^{n+1}\left((zc)_{k\cancel{-1}} + (zc)_k\right)$$

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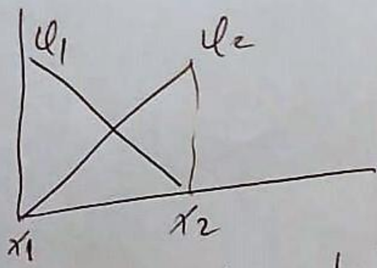
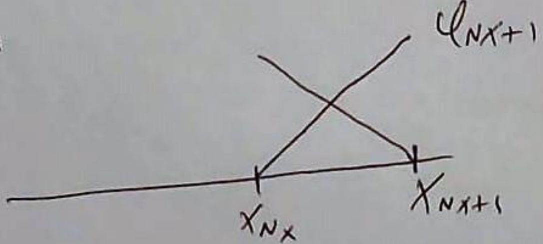
For $k=1$



$$\int_{x_1}^{x_2} \tau c\left(\sum_{j=1}^{2}\theta_j^{m+1}\varphi_j\right)\varphi_k = (\text{trapezoidal rule})$$

$$\cong (\tau c)_1 \int_{x_1}^{x_2}\theta_1^{m+1}\varphi_1\varphi_1 = \frac{h}{2}\theta_1^{m+1}(\tau c)_1$$

For $k = N_x+1$



$$\int_{x_{N_x}}^{x_{N_x+1}} \tau c\left(\sum_{j=N_x}^{N_x+1}\theta_j^{m+1}\varphi_j\right)\varphi_{N_x+1} \cong \frac{h}{2}\theta_{N_x+1}^{m+1}(\tau c)_{N_x+1}$$

Then

$$\frac{1}{(\Delta t)^2}(\tau c\,\theta^{m+1},\varphi_k) = \frac{h}{2}\theta_1^{m+1}\delta_{k1}(\tau c)_1\frac{1}{(\Delta t)^2}$$

$$+(\tau c)_{N_x+1}\frac{h}{2}\theta_{N_x+1}^{m+1}\delta_{k,N_x+1}\frac{1}{(\Delta t)^2} + \frac{h}{2}\frac{1}{(\Delta t)^2}\theta_k^{m+1}\left[\underbrace{(\tau c)_{k\cancel{-1}}+(\tau c)_k}_{k=2,\dots,N_x}\right]$$

$$\frac{1}{2\Delta t}(c\,\theta^{m+1},\varphi_k) = \frac{h}{2}\frac{1}{(2\Delta t)}\theta_1^{m+1}\delta_{k1}c_1 + \frac{1}{(2\Delta t)}\frac{h}{2}\theta_{N_x+1}^{m+1}\delta_{k,N_x+1}c_{N_x+1}^{m+1}$$

$$+\frac{h}{2}\theta_k^{m+1}\frac{1}{2\Delta t}\left[\underbrace{c_{k\cancel{-1}}+c_k}_{k=2,\dots,N_x}\right]$$

$$T_1 = \left(\gamma \frac{\partial \theta^n}{\partial x}, \frac{\partial \varphi_k}{\partial x} \right)$$

$$= \left(\gamma \sum_{j=k-1}^{k+1} \theta_j^n \frac{\partial \varphi_j}{\partial x}, \frac{\partial \varphi_k}{\partial x} \right) = \left(\begin{matrix} \text{1D BIOT.pdf} \\ \text{pag 11-12} \end{matrix} \right)$$

$$= -\frac{1}{h}(1-\delta_{k1})\gamma_{k-1}\theta_{k-1}^n$$

$$+ \left[\frac{1}{h}\gamma_{k-1}(1-\delta_{k1}) + \frac{1}{h}\gamma_k(1-\delta_{k,Nx+1}) \right] \theta_k$$

$$- \frac{1}{h}(1-\delta_{k,Nx+1})\gamma_k \theta_{k+1}^n .$$

$$T_2 = \frac{1}{2\Delta t} \langle \tau c v_\theta \theta^{n+1}, \varphi_k \rangle$$

$$= \frac{1}{2\Delta t}\left[(\tau c v_\theta)^1 \theta_k^{n+1} \delta_{k1} + (\tau c v_\theta)^R \theta_k^{n+1} \delta_{k,Nx+1} \right]$$

Then the Temperature equation is

for $k=1$

$$\left[\frac{1}{(\Delta t)^2} \frac{h}{2} (\tau c)_1 + \frac{1}{(2\Delta t)} \frac{h}{2} c_1 + \frac{1}{(2\Delta t)} (\tau c v_\theta)^L \right] \theta_1^{n+1}$$

$$= \frac{1}{(\Delta t)^2} \frac{h}{2} (\tau c)_1 \left(2\theta_1^n - \theta_1^{n-1} \right) \qquad (3)$$

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$$+\frac{1}{2\Delta t}\frac{h}{2}c_1\theta_1^{n-1}-\left[\frac{1}{h}\gamma_1\theta_1^n-\frac{1}{h}\gamma_1\theta_2^n\right]$$

$$+\frac{1}{2\Delta t}(zc\,v_\theta)^1\theta_1^{n-1}-(g^n,\varphi_1)\equiv RHS_1^\theta \quad (3)$$

For $k=2, NX$

$$\frac{1}{(\Delta t)^2}\frac{h}{2}\left((zc)_{k-1}+(zc)_k\right)\theta_k^{n+1}$$

$$+\frac{1}{2\Delta t}\frac{h}{2}\left(c_{k-1}+c_k\right)\theta_k^{n+1}$$

$$=\frac{1}{(\Delta t)^2}\frac{h}{2}\left((zc)_{k-1}+(zc)_k\right)\left[2\theta_k^n-\theta_k^{n-1}\right]$$

$$+\frac{1}{2\Delta t}\frac{h}{2}\left(c_{k-1}+c_k\right)\theta_k^{n-1} \quad (4)$$

$$-\left[-\frac{1}{h}\gamma_{k-1}\theta_{k-1}^n+\frac{1}{h}(\gamma_{k-1}+\gamma_k)\theta_k^n\right.$$

$$\left.-\frac{1}{h}\gamma_k\theta_{k+1}^n\right]-(g^n,\varphi_k)$$

4/1/2022 (6)

For $k = Nx+1$:

$$\frac{1}{(\Delta t)^2}\frac{h}{2}(\tau c)_{Nx}\,\theta^{m+1}_{Nx+1} + \frac{1}{(2\Delta t)}\frac{h}{2}c_{Nx}\,\theta^{m+1}_{Nx+1}$$

$$+\frac{1}{(2\Delta t)}(\tau c V_\theta)^R\,\theta^{m+1}_{Nx+1}$$

$$= \frac{1}{(\Delta t)^2}\frac{h}{2}(\tau c)_{Nx}\left[2\theta^{m}_{Nx+1} - \theta^{m-1}_{Nx+1}\right]$$

$$+\frac{1}{2\Delta t}\frac{h}{2}c_{Nx}\,\theta^{m-1}_{Nx+1} \qquad (5)$$

$$-\left[-\frac{1}{h}\gamma_{Nx}\,\theta^{m}_{Nx} + \frac{1}{h}\gamma_{Nx}\,\theta^{m}_{Nx+1}\right]$$

$$+\frac{1}{(2\Delta t)}(\tau c V_\theta)^R\,\theta^{m-1}_{Nx+1} - (q^m, \varphi_{Nx+1})$$

Coupling terms in temperature code (7)

Add to RHS in equation (2) the terms (so in eq'n (3) for $K=1$, (4) for $K=2 \ldots Nx$ and eq'n (5) for $K=Nx+1$):

$$- \left((1-\phi)\beta_m T_0 \frac{\partial}{\partial x}\left[\frac{u^{s,n+1} - u^{s,n-1}}{2\Delta t}\right], \varphi_K \right)$$

$$- \left(\phi \beta_f T_0 \frac{\partial}{\partial x}\left[\frac{u^{f,n+1} - u^{f,n-1}}{2\Delta t}\right], \varphi_K \right) \qquad (6)$$

$$- \left(\tau (1-\phi)\beta_m T_0 \frac{\partial}{\partial x}\left[\frac{u^{s,n+1} - 2u^{s,n} + u^{s,n-1}}{(\Delta t)^2}\right], \varphi_K \right)$$

$$- \left(\tau \phi \beta_f T_0 \frac{\partial}{\partial x}\left[\frac{u^{f,n+1} - 2u^{f,n} + u^{f,n-1}}{(\Delta t)^2}\right], \varphi_K \right)$$

$$\equiv -T_{101} - T_{100} - T_{103} - T_{102} \qquad (7)$$

Set $D^n u^f = \dfrac{u^{f,n+1} - u^{f,n-1}}{2\Delta t}$,

$D^n u^s = \dfrac{u^{s,n+1} - u^{s,n-1}}{2\Delta t}$, $D^{2,n} u^s = \dfrac{u^{s,n+1} - 2u^{s,n} + u^{s,n-1}}{(\Delta t)^2}$ (8)

Then using (32) p. 17 of manuscript BIOT CODE EXPLICIT:

$$(\phi \beta_f T_0 \frac{\partial}{\partial x} D^n u^f, \varphi_k) = T_{100}$$

$$= T_0 \phi_{k-1} \beta_f (1-\delta_{k1}) \left[(D^n u^f)_k - (D^n u^f)_{k-1} \right] \frac{1}{2}$$

$$+ T_0 \phi_k \beta_f (1-\delta_{k,Nx+1}) \left[D^n u^f_{k+1} - D^n u^f_k \right] \frac{1}{2}$$

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$$= \Bigg\{ T_0 \phi_{k-1} \beta_f (1-\delta_{k1}) \left[\left(\frac{u_k^{f,n+1} - u_k^{f,n-1}}{2\Delta t} \right) - \left(\frac{u_{k-1}^{f,n+1} - u_{k-1}^{f,n-1}}{2\Delta t} \right) \right]$$

$$+ T_0 \phi_k \beta_f (1-\delta_{k,Nx+1}) \left[\left(\frac{u_{k+1}^{f,n+1} - u_{k+1}^{f,n-1}}{2\Delta t} \right) - \left(\frac{u_k^{f,n+1} - u_k^{f,n-1}}{2\Delta t} \right) \right] \Bigg\} \frac{1}{2}$$

Similarly

$$((1-\phi) \beta_m T_0 D^n u^s, \varphi_k) = T_{101}$$

$$= \Bigg\{ (1-\phi_{k-1}) \beta_m T_0 (1-\delta_{k1}) \left[\frac{u_k^{s,n+1} - u_k^{s,n-1}}{2\Delta t} - \left(\frac{u_{k-1}^{s,n+1} - u_{k-1}^{s,n-1}}{2\Delta t} \right) \right]$$

$$+ (1-\phi_k) \beta_m T_0 (1-\delta_{k,Nx+1}) \left[\left(\frac{u_{k+1}^{s,n+1} - u_{k+1}^{s,n-1}}{2\Delta t} \right) - \left(\frac{u_k^{s,n+1} - u_k^{s,n-1}}{2\Delta t} \right) \right] \Bigg\} \frac{1}{2}$$

$$(\tau\phi\beta_f T_0 D^{2,m} u^f, \varphi_k) = T_{102}$$

$$= \Big\{ \tau_{k-1}\phi_{k-1}\beta_f T_0 (1-\delta_{k1}) \left[(D^{2,m} u^f)_k - (D^{2,n} u^f)_{k-1} \right] + \tau_k \phi_k \beta_f T_0 (1-\delta_{k,Nx+1}) \left[(D^{2,m} u^f)_{k+1} - (D^{2,m} u^f)_k \right] \Big\} \frac{1}{2}$$

$$= \Big\{ \tau_{k-1}\phi_{k-1}\beta_f T_0 (1-\delta_{k1}) \left[\underbrace{\frac{u_k^{f,n+1} - 2u_k^{f,m} + u_k^{f,m-1}}{(\Delta t)^2}}_{D^{2,f}_{m,k}} - \underbrace{\frac{u_{k-1}^{f,n+1} - 2u_{k-1}^{f,m} + u_{k-1}^{f,m-1}}{(\Delta t)^2}}_{D^{2,f}_{m,k-1}} \right]$$

$$+ \tau_k \phi_k \beta_f T_0 (1-\delta_{k,Nx+1}) \left[\underbrace{\frac{u_{k+1}^{f,n+1} - 2u_{k+1}^{f,n} + u_{k+1}^{f,n-1}}{(\Delta t)^2}}_{D^{2,f}_{m,k+1}} - \underbrace{\frac{u_k^{f,n+1} - 2u_k^{f,n} + u_k^{f,n-1}}{(\Delta t)^2}}_{D^{2,f}_{m,k}} \right] \Big\} \frac{1}{2}$$

$$(\tau(1-\phi)\beta_m T_0 D^{2,m} u^s, \varphi_k) = T_{103}$$

$$= \Big\{ \tau_{k-1}(1-\phi_{k-1})\beta_m T_0 (1-\delta_{k1}) \left[(D^{2,m} u^s)_k - (D^{2,n} u^s)_{k-1} \right] + \tau_k (1-\phi_k) \beta_m T_0 (1-\delta_{k,Nx+1}) \left[(D^{2,m} u^s)_{k+1} - (D^{2,n} u^s)_k \right] \Big\} \frac{1}{2}$$

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$$= \Bigg\{ \tau_{k-1}\,(1-\phi_{k-1})\,\beta_m T_0\,(1-\delta_{k1}) \cdot \left[\underbrace{\frac{u_k^{s,n+1} - 2u_k^{s,n} + u_k^{s,n-1}}{(\Delta t)^2}}_{D^{2,s}_{m,k}} - \left(\underbrace{\frac{u_{k-1}^{s,n+1} - 2u_{k-1}^{s,n} + u_{k-1}^{s,n-1}}{(\Delta t)^2}}_{D^{2,s}_{m,km1}} \right) \right]$$

$$+ \tau_k(1-\phi_k)\,\beta_m T_0\,(1-\delta_{kNk+1}) \cdot$$

$$\left[\underbrace{\frac{u_{k+1}^{s,n+1} - 2u_{k+1}^{s,n} + u_{k+1}^{s,n-1}}{(\Delta t)^2}}_{D^{2,s}_{m,kp1}} - \left(\underbrace{\frac{u_k^{s,n+1} - 2u_k^{s,n} + u_k^{s,n-1}}{(\Delta t)^2}}_{D^{2,s}_{m,k}} \right) \right] \Bigg\} \frac{1}{2}$$

DENUEVO CALCULO T100-T103 (10-1)

$$T_{100} = \left(\phi \beta_f T_0 \frac{\partial}{\partial x}\left[\frac{u^{f,n+1} - u^{f,n-1}}{2\Delta t}\right], \varphi_k\right)$$

$$= \frac{1}{2\Delta t}\left[\left(\phi \beta_f T_0 \frac{\partial}{\partial x} u^{f,n+1}, \varphi_k\right) - \left(\phi \beta_f T_0 \frac{\partial}{\partial x} u^{f,n-1}, \varphi_k\right)\right] \quad (8)$$

$$= T_{100}^{A} - T_{100}^{B}$$

$$T_{100}^{A} = \frac{1}{2\Delta t} \beta_f T_0 \left(\phi \sum_{j=1}^{N_x+1} u_j^{f,n+1} \frac{\partial}{\partial x}\varphi_j, \varphi_k\right) \quad (9)$$

$$= \frac{1}{2\Delta t} \beta_f T_0 \frac{1}{2}\left[\left(u_k^{f,n+1} - u_{k-1}^{f,n+1}\right)(1-\delta_{k,1})\phi_{k-1} + \phi_k\left(u_{k+1}^{f,n+1} - u_k^{f,n+1}\right)(1-\delta_{k,N_x+1})\right]$$

For $K=1$

$$T_{100}^{A}\Big|_{K=1} = \frac{1}{2\Delta t} \beta_f T_0 \frac{1}{2}\left(u_2^{f,n+1} - u_1^{f,n+1}\right)\phi_1 \quad (10)$$

For $K = N_x+1$

$$T_{100}^{A}\Big|_{K=N_x+1} = \frac{1}{2\Delta t} \beta_f T_0 \frac{1}{2}\left(u_{N_x+1}^{f,n+1} - u_k^{f,n+1}\right)\phi_{N_x} \quad (11)$$

For $K = 2, \ldots N_x$

$$T^{A}_{100}(K) = \beta_f T_0 \frac{1}{2\Delta t}\Big[\phi_{K-1}\big(U^{f,n+1}_{K} - U^{f,n+1}_{K-1}\big) + \phi_K\big(U^{f,n+1}_{K+1} - U^{f,n+1}_{K}\big)\Big] \quad (12)$$

$$= \beta_f T_0 \frac{1}{2\Delta t}\Big[U^{f,n+1}_{K}\,(\phi_{K-1} - \phi_K)$$

Similarly

$$T^{3}_{100}\Big|_{K=1} = \frac{1}{2\Delta t}\phi_1 \beta_f T_0 \frac{1}{2}\big(U^{f,n-1}_{2} - U^{f,n-1}_{1}\big) \quad (13)$$

$$T^{B}_{100}\Big|_{K=NX+1} = \frac{1}{2\Delta t}\phi_{NX}\beta_f T_0 \frac{1}{2}\big(U^{f,n-1}_{NX+1} - U^{f,n-1}_{NX}\big) \quad (14)$$

For $K=2,-NX$:

$$T^{B}_{100}(K) = \beta_f T_0 \frac{1}{2\Delta t}\Big[\phi_{K-1}\big(U^{}_{K} - U^{f,n-1}_{K-1}\big) + \phi_K\big(U^{f,n-1}_{K+1} - U^{f,n-1}_{K}\big)\Big]$$

Then $T100\big|_{K=1} = \overset{A}{T100}\big|_{K=1} - \overset{B}{T100}\big|_{K=1} =$

$$T100\Big|_{K=1} = \frac{1}{2\Delta t}\phi_1 \beta_f T_0 \frac{1}{2}\Big[U^{f,n+1}_{2} - U^{f,n+1}_{1} - \big(U^{f,n-1}_{2} - U^{f,n-1}_{1}\big)\Big]$$

$$= \phi_1 \beta_f T_0 \frac{1}{2}\Big[\underbrace{\frac{U^{f,n+1}_{2} - U^{f,n-1}_{2}}{2\Delta t}}_{D_n^{f}(2)} - \underbrace{\frac{U^{f,n+1}_{1} - U^{f,n-1}_{1}}{2\Delta t}}_{D_m^{f}(1)}\Big]$$

$$\boxed{T_{100}\Big|_{K=1} = \phi_1 \beta_f T_0 \frac{1}{2}\Big[D_m^{f}(2) - D_m^{f}(1)\Big]} \quad (15)$$

(11)

$$T_{100}\Big|_{k=Nx+1} = \phi_{Nx}\,\beta_f T_0 \frac{1}{2\Delta t}\frac{1}{2}\Big[\Big(U^{f,n+1}_{Nx+1} - U^{f,n-1}_{Nx}\Big) - \Big(U^{f,n-1}_{Nx+1} - U^{f,n-1}_{Nx}\Big)\Big]$$

$$= \phi_{Nx}\,\beta_f T_0 \frac{1}{2}\Bigg[\underbrace{\frac{U^{f,n+1}_{Nx+1} - U^{f,n-1}_{Nx+1}}{2\Delta t}}_{D^f_m(Nx+1)} - \underbrace{\left(\frac{U^{f,n-1}_{Nx+1} - U^{f,n-1}_{Nx}}{2\Delta t}\right)}_{D^f_m(Nx)}\Bigg]$$

$$\boxed{T_{100}\Big|_{k=Nx+1} = \phi_{Nx}\,\beta_f T_0 \frac{1}{2}\Big[D^f_m(Nx+1) - D^f_m(Nx)\Big] \quad (16)}$$

$$T_{100}(k) = \beta_f T_0 \frac{1}{2\Delta t}\frac{1}{2}\Big[\phi_{k-1}\Big(U^{f,n+1}_{k} - U^{f,n+1}_{k-1}\Big) + \phi_k\Big(U^{f,n+1}_{k+1} - U^{f,n+1}_{k}\Big)\Big]$$

$$-\frac{1}{2}\beta_f T_0 \frac{1}{2\Delta t}\Big[\phi_{k-1}\Big(U^{f,n-1}_{k} - U^{f,n-1}_{k-1}\Big) + \phi_k\Big(U^{f,n-1}_{k+1} - U^{f,n-1}_{k}\Big)\Big]$$

$$T_{100}(k) = \frac{1}{2\Delta t}\beta_f T_0 \frac{1}{2}\left\{\phi_{k-1}\left[\frac{\left(U^{f,n+1}_{k} - U^{n-1}_{k}\right)}{D^{f}_{m,k}} - \frac{\left(U^{n+1}_{k-1} - U^{n-1}_{k-1}\right)}{D^{f}_{n,k-1}}\right] + \phi_k\left[\frac{\left(U^{f,n+1}_{k+1} - U^{f,n-1}_{k+1}\right)}{D^{f}_{m,k+1}} - \frac{\left(U^{n+1}_{k} - U^{n-1}_{k}\right)}{D^{f}_{m,k}}\right]\right\}$$

Then

$$\boxed{\begin{aligned} T_{100}(k) \doteq \beta_f T_0 \frac{1}{2}\Big[&\phi_{k-1}\left(D^{f}_{m,k} - D^{f}_{m,k-1}\right) \\ &+ \phi_k\left(D^{f}_{m,k+1} - D^{f}_{m,k}\right)\Big]\end{aligned}} \quad (17)$$

[For ϕ constant, reduces to

$$T_{100}(k) = \frac{1}{2}\beta_f T_0 \phi\left[D^{f}_{m,k+1} - D^{f}_{m,k}\right]$$

where $D^{f}_{m,k+1} = \left(U^{f,n+1}_{k+1} - U^{f,n-1}_{k+1}\right)/2\Delta t$

$D^{f}_{m,k} = \left(U^{f,n+1}_{k} - U^{f,n-1}_{k}\right)/2\Delta t$

Similarly:

$$T_{101}(k) = (1-\phi)\beta_m T_0 \frac{\partial}{\partial x}\left[\frac{U^{s,n+1} - U^{s,n-1}}{2\Delta t}\right], U_k)$$

$$= T^{A}_{101} - T^{B}_{101}$$

(13)

$$T_{101}\Big|_{k=1} = (1-\phi_1)\beta_m T_0 \frac{1}{2}\left[D^s_m(2) - D^s_m(1)\right] \quad (18)$$

where $D^s_m(k) = \left(u_k^{s,n+1} - u_k^{s,n-1}\right)/2\Delta t$

$$T_{101}\Big|_{k=NX+1} = (1-\phi_{NX})\beta_m T_0 \frac{1}{2}\left[D^s_{m,NX+1} - D^s_{m,NX}\right] \quad (19)$$

$\cancel{T_{101}(k) = \beta_m T_0}$

$$T_{101}(k) = \beta_m T_0 \frac{1}{2}\Big[(1-\phi_{k-1})\left(D^s_{m,k} - D^s_{m,k-1}\right) + (1-\phi_k)\left(D^s_{m,k+1} - D^s_{m,k}\right)\Big] \quad (20)$$

Next set

$$\cancel{D_t} \; D_t^{n} u_k = \frac{u_k^{n+1} - u_k^{n}}{\Delta t} = (D_t^{n} u)_k$$

so that

$$D_2 u_k^n = \frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{(\Delta t)^2} = \frac{1}{\Delta t}\left[D_t^n u_k - D_t u_k^{n-1}\right]$$

and

$$T_{103} = \frac{1}{\Delta t}\left(\tau\phi\beta_f T_0 \frac{\partial}{\partial x}\left[D_t u^{f,n} - D_t u^{f,n-1}\right], \varphi_k\right)$$

$$= \frac{1}{\Delta t}\left(\tau\phi\beta_f T_0 \frac{\partial}{\partial x} D_t^n u^f, \varphi_k\right) \qquad (21)$$

$$- \frac{1}{\Delta t}\left(\tau\phi\beta_f T_0 \frac{\partial}{\partial x} D_t^{n-1}, \varphi_k\right) = T_{103}^A - T_{103}^B$$

$$T_{103}^A = \frac{1}{\Delta t}\left(\tau\phi\beta_f T_0 \sum_{j=1}^{Nx+1} (D_t^n u^f)_j \varphi_j , \varphi_k\right)$$

$$\cancel{= \frac{1}{\Delta t}\left(\tau\phi\beta_f T_0 \sum_j \left(\frac{u_j^{f,n+1} - u_j^{f,n}}{\Delta t}\right), \varphi_k\right)}$$

$$= \frac{1}{(\Delta t)} \beta_f T_0 \frac{1}{2}\Big[\left(D_t^n u_k^f - D_t^n u_{k-1}^f\right)\phi_{k-1}(1-\delta_{k1})\tau_{k-1}$$
$$+\left(D_t^n u_{k+1}^{f} - D_t^n u_k^f\right)\phi_k(1-\delta_{k,Nx+1})\tau_k$$

For $k=1$

$$T_{103}^{A}\Big|_{k=1} = \frac{1}{(\Delta t)} \beta_f T_0 \frac{1}{2}\left(D_t^{m} u_2^{f} - D_t^{m} u_1^{f}\right)\phi_1 \tau_1 \quad (22)$$

For $k = N_x+1$

$$T_{103}^{A}\Big|_{k=N_x+1} = \frac{1}{\Delta t} \beta_f T_0 \frac{1}{2}\left(D_t^{m} u_{N_x+1}^{f} - D_t^{m} u_{N_x}^{f}\right) \phi_{N_x} \tau_{N_x} \quad (23)$$

For $k = 2 \ldots N_x$

$$T_{103}^{A}(k) = \frac{1}{\Delta t} \beta_f T_0 \frac{1}{2}\Big[\phi_{k-1}\tau_{k-1}\left(D_t^{m} u_k^{f} - D_t^{m} u_{k-1}^{f}\right) \quad (24)$$
$$+ \phi_k \tau_k \left(D_t^{m} u_{k+1}^{f} - D_t^{m} u_k^{f}\right)\Big]$$

Similarly

$$T_{103}^{B}\Big|_{k=1} = \frac{1}{2}\frac{1}{\Delta t} \beta_f T_0 \phi_1 \tau_1 \left(D_t^{m-1} u_2^{f} - D_t^{m-1} u_1^{f}\right) \quad (25)$$

$$T_{103}^{B}\Big|_{k=N_x+1} = \frac{1}{\Delta t} \beta_f T_0 \frac{1}{2}\left(D_t^{m-1} u_{N_x+1}^{f} - D_t^{m-1} u_{N_x}^{f}\right)\phi_{N_x} \tau_{N_x} \quad (26)$$

$$T_{103}^{B}(k) = \frac{1}{\Delta t} \beta_f T_0 \frac{1}{2}\Big[\phi_{k-1}\tau_{k-1}\left(D_t^{m-1} u_k^{f} - D_t^{m-1} u_{k-1}^{f}\right) \quad (27)$$
$$+ \phi_k \tau_k \left(D_t^{m-1} u_{k+1}^{f} - D_t^{m-1} u_k^{f}\right)\Big]$$

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Then, for $k=1$

$$T_{103}\Big|_{k=1} = \frac{1}{2}\frac{1}{\Delta t}\beta_f T_0 \overset{\tau_1}{\cancel{\phi_1}}\left(D_t^n u_2^f - D_t^n u_1^f - \left(D_t^{n-1} u_2^f - D_t^{n-1} u_1^f\right)\right)$$

$$= \overset{\tau_1}{\phi_1}\frac{1}{\Delta t}\beta_f T_0 \frac{1}{2}\left[\left(D_t^n u_2^f - D_t^{n-1} u_2^f\right) - D_t^n u_1^f - D_t^{n-1} u_1^f\right)\right]$$

$$\boxed{= \tau_1 \phi_1 \beta_f T_0 \frac{1}{2}\left[D_2^n u_2^f - D_2^n u_1^f\right]} \qquad (28)$$

For $k = N_x+1$

$$T_{103}\Big|_{k=N_x+1} = \frac{1}{2}\frac{1}{\Delta t}\beta_f T_0 \overset{\tau_{Nx}}{\phi_{Nx}}\left[D_t^n u_{Nx+1}^f - D_t^n u_{Nx}^f - \left(D_t^{n-1} u_{Nx+1}^f - D_t^{n-1} u_{Nx}^f\right)\right]$$

$$= \frac{1}{\Delta t}\phi_{Nx}\beta_f T_0 \frac{\tau_{Nx}}{2}\left[D_t^n u_{Nx+1}^f - D_t^{n-1} u_{Nx+1}^f - \left(D_t^n u_{Nx}^f - D_t^{n-1} u_{Nx}^f\right)\right]$$

$$\boxed{= \tau_{Nx} \phi_{Nx} \beta_f T_0 \frac{1}{2}\left[D_2^n u_{Nx+1}^f - D_2^n u_{Nx}^f\right]} \qquad (29)$$

For $k = 2, \ldots, N_x$

$$T_{103}(k) = \frac{1}{2}\frac{1}{\Delta t}\beta_f T_0 \Big[\phi_{k-1}\left(D_t^n u_k^f - D_t^n u_{k-1}^f\right)^{\tau_{k-1}} + \phi_k^{\tau_k}\left(D_t^n u_{k+1}^f - D_t^n u_k^f\right)\Big]$$

$$- \frac{1}{2}\frac{1}{\Delta t}\beta_f T_0 \Big[\phi_{k-1}^{\tau_{k-1}}\left(D_t^{n-1} u_k^f - D_t^{n-1} u_{k-1}^f\right) + \phi_k^{\tau_k}\left(D_t^{n-1} u_{k+1}^f - D_t^{n-1} u_k^f\right)\Big]$$

$$= \frac{1}{2}\frac{1}{\Delta t}\beta_f T_0 \Big[\phi_{k-1}\Big[\left(D_t^n u_k^f - D_t^{n-1} u_k^f\right)^{\tau_{k-1}} - \left(D_t^n u_{k-1}^f - D_t^{n-1} u_{k-1}^f\right)\Big]$$

$$+ \tau_k \phi_k \Big[\left(D_t^n u_{k+1}^f - D_t^{n-1} u_{k+1}^f\right) - \left(D_t^n u_k^f - D_t^{n-1} u_k^f\right)\Big]$$

$$\boxed{\begin{aligned} &= \frac{1}{2}\beta_f T_0 \Big[\tau_{k-1}\phi_{k-1}\left(D_2^n u_k^f - D_2^n u_{k-1}^f\right) \\ &\qquad + \tau_k \phi_k \left(D_2^n u_{k+1}^f - D_2^n u_k^f\right) \\ &= T_{103}(k) \qquad (30) \end{aligned}}$$

Similarly

$$T102\big|_{K=1} = (1-\phi_1)\beta_m T_0 \frac{\tau_1}{2}\left[D_2^n u_2^s - D_2^n u_1^s\right] \tag{31}$$

$$T102\big|_{K=N_x+1} = (1-\phi_{N_x})\beta_m T_0 \frac{\tau_{N_x}}{2}\left[D_2^n u_{N_x+1}^s - D_2^n u_{N_x}^s\right] \tag{32}$$

$$T102(K) = \frac{1}{2}\beta_m T_0 \Big[\tau_{K-1}(1-\phi_{K-1})\left(D_2^n u_K^s - D_2^n u_{K-1}^s\right) + \tau_K (1-\phi_K)\left(D_2^n u_{K+1}^s - D_2^n u_K^s\right)\Big] \tag{33}$$

$$T_{100}\Big|_{K=1} = \phi_1 \beta_f T_0 \frac{1}{2} \frac{1}{2\Delta t}\Big[\left(u_2^{f,n+1} - u_1^{f,n+1}\right) - \cancel{\frac{1}{2\Delta t}}\left(u_2^{f,n-1} - u_1^{f,n-1}\right)\Big]$$

$$\equiv T_{100}\Big|_{K=1}^{n+1} - T_{100}\Big|_{K=1}^{n-1} \qquad (34)$$

$$T_{100}\Big|_{K=Nx+1} = \phi_{Nx} \beta_f T_0 \frac{1}{2} \frac{1}{2\Delta t}\Big[\left(u_{Nx+1}^{f,n+1} - u_{Nx}^{f,n+1}\right) - \left(u_{Nx+1}^{f,n-1} - u_{Nx}^{f,n-1}\right)\Big] \qquad (35)$$

$$\equiv T100\Big|_{K=Nx}^{n+1} - T100\Big|_{K=Nx}^{n-1}$$

For $K=2,\ldots Nx$

$$T_{100}\Big|_{K} = \frac{1}{2\Delta t} T_0 \beta_f \frac{1}{2}\Big\{ \phi_{K-1}\left(u_K^{f,n+1} - u_{K-1}^{f,n+1}\right) - \phi_{K-1}\left(u_K^{f,n-1} - u_{K-1}^{f,n-1}\right) + \phi_K\left(u_{K+1}^{f,n+1} - u_K^{f,n+1}\right) - \phi_K\left(u_{K+1}^{f,n-1} - u_K^{f,n-1}\right)\Big\}$$

$$= \frac{1}{2\Delta t} T_0 \beta_f \frac{1}{2}\Big\{ \phi_{K-1}\left(u_K^{f,n+1} - u_{K-1}^{f,n+1}\right) + \phi_K\left(u_{K+1}^{f,n+1} - u_K^{f,n+1}\right) - \phi_{K-1}\left(u_K^{f,n-1} - u_{K-1}^{f,n-1}\right) - \phi_K\left(u_{K+1}^{f,n-1} - u_K^{f,n-1}\right)\Big\}$$

$$\equiv T_{100}^{n+1}(k) - T_{100}^{n-1}(k)$$

$$T_{100}^{n+1}(k) = \frac{1}{2\Delta t} T_0 \beta_f \frac{1}{2}\left[\phi_{k-1}\left(u_k^{f,n+1} - u_{k-1}^{f,n+1}\right) + \phi_k\left(u_{k+1}^{f,n+1} - u_k^{f,n+1}\right)\right] \tag{36}$$

$$T_{100}^{n-1}(k) = \frac{1}{2\Delta t} T_0 \beta_f \frac{1}{2}\left[\phi_{k-1}\left(u_k^{f,n-1} - u_{k-1}^{f,n-1}\right) + \phi_k\left(u_{k+1}^{f,n-1} - u_k^{f,n-1}\right)\right]$$

Similarly:

$$T_{101}\Big|_{k=1} = (1-\phi_1)\beta_m T_0 \frac{1}{2}\frac{1}{2\Delta t}\left[\left(u_2^{s,n+1} - u_{\cancel{2}1}^{s,n+1}\right) - \frac{1}{2\Delta t}\left(u_2^{s,n-1} - u_1^{s,n-1}\right)\right]$$

$$\equiv T101\Big|_{k=1}^{n+1} - T101\Big|_{k=1}^{n-1}$$

$$T101\Big|_{k=N_x+1} = (1-\phi_{N_x})\beta_m T_0 \frac{1}{2}\frac{1}{2\Delta t}\left[\left(u_{N_x+1}^{s,n+1} - u_{N_x}^{s,n+1}\right) - \left(u_{N_x+1}^{s,n-1} - u_{N_x}^{s,n-1}\right)\right]$$

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$$T_{101}(K) = T_{101}^{m+1}(K) - T_{101}^{m-1}(K)$$

$$T_{101}^{m+1}(K) = \frac{1}{2\Delta t} T_0 \, \beta_m T_0 \frac{1}{2}\Big[(1-\phi_{K-1})\big(U_K^{s,m+1} - U_{K-1}^{s,m+1}\big) + (1-\phi_K)\big(U_{K+1}^{s,m+1} - U_K^{s,m+1}\big)\Big] \quad (37)$$

$$T_{101}^{m-1}(K) = \frac{1}{2\Delta t} T_0 \, \beta_m T_0 \frac{1}{2}\Big[(1-\phi_{K-1})\big(U_K^{s,m-1} - U_{K-1}^{s,m-1}\big) + (1-\phi_K)\big(U_{K+1}^{s,m-1} - U_K^{s,m-1}\big)\Big] \quad (38)$$

$$T103 = T103^{A} - T103^{B}$$

$$T103^{A}\Big|_{K=1} = \tau_1 \phi_1 T_0 \frac{1}{2}\frac{1}{\Delta t}\left(D_t^m u_2^f - D_t^m u_1^f\right)$$

$$= \tau_1 \phi_1 T_0 \frac{1}{\Delta t}\frac{1}{2}\left(\frac{U_2^{f,m+1} - U_2^{f,m}}{\Delta t} - \frac{U_1^{f,m+1} - U_1^{f,m}}{\Delta t}\right)$$

$$T103\Big|_{K=1} = \tau_1 \phi_1 \beta_f T_0 \frac{1}{2}\left[D_2^m u_2^f - D_2^m u_1^f\right] \quad (39)$$

$$T103\Big|_{k=1} = \tau_1 \phi_1 \beta_f T_0 \frac{1}{2} \left[\left(\frac{u_2^{f,n+1} - 2u_2^{f,n} + u_2^{f,n-1}}{(\Delta t)^2} \right) - \left(\frac{u_1^{f,n+1} - 2u_1^{f,n} + u_1^{f,n-1}}{(\Delta t)^2} \right) \right]$$

$$= \tau_1 \phi_1 \beta_f T_0 \frac{1}{2} \frac{1}{(\Delta t)^2} \left\{ \left[\left(u_2^{f,n+1} - u_1^{f,n+1} \right) \right] - \left[\left(2u_2^{f,n} - u_2^{f,n-1} \right) - \left(2u_1^{f,n} - u_1^{f,n-1} \right) \right] \right\} \quad (40)$$

$$\cancel{\tau_1 \phi_1 \beta_f T_0 \tfrac{1}{2}} \equiv T103\Big|_{k=1}^{n+1} - T103\Big|_{k=1}^{n,n-1}$$

$$T103\Big|_{k=Nx+1} = \tau_{Nx} \phi_{Nx} \beta_f T_0 \frac{1}{2} \left[\left(\frac{u_{Nx+1}^{f,n+1} - 2u_{Nx+1}^{f,n} + u_{Nx+1}^{f,n-1}}{(\Delta t)^2} \right) - \left(\frac{u_{Nx}^{f,n+1} - 2u_{Nx}^{f,n} + u_{Nx}^{f,n-1}}{(\Delta t)^2} \right) \right]$$

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$$T103\Big|_{K=NX+1} = \tau_{NX}\phi_{NX}\beta_f T_0 \frac{1}{(\Delta t)^2}\frac{1}{2}\left\{\left[\left(u^{f,n+1}_{NX+1} - u^{f,n+1}_{NX}\right)\right.\right.$$

$$\left.\left. - \left[\left(2u^{f,n}_{NX+1} - u^{f,n-1}_{NX+1}\right) - \left(2u^{f,n}_{NX} - u^{f,n-1}_{NX}\right)\right]\right]\right\}$$

$$\equiv T103\Big|^{n+1}_{K=NX+1} - T103\Big|^{n,n-1}_{K=NX+1} \qquad (41)$$

FOR $K = 2 \ldots NX$,

$$T103(K) = \frac{1}{2}\beta_f T_0\left\{\left[\tau_{K-1}\phi_{K-1}\left(\frac{u^{f,n+1}_{K} - 2u^{f,n}_{K} + u^{f,n-1}_{K}}{(\Delta t)^2}\right)\right.\right.$$

$$\left. - \left(\frac{u^{f,n+1}_{K-1} - 2u^{f,n}_{K-1} + u^{f,n-1}_{K-1}}{(\Delta t)^2}\right)\right]$$

$$\left. + \tau_K\phi_K\left[\left(\frac{u^{f,n+1}_{K+1} - 2u^{f,n}_{K+1} + u^{f,n-1}_{K+1}}{(\Delta t)^2}\right) - \left(\frac{u^{f,n+1}_{K} - 2u^{f,n}_{K} + u^{f,n-1}_{K}}{(\Delta t)^2}\right)\right]\right\}$$

$$
\begin{aligned}
T103(k) = {} & \frac{1}{2}\beta_f T_0 \frac{1}{(\Delta t)^2}\Big[\tau_{k-1}\phi_{k-1}\left(u_k^{f,n+1} - u_{k-1}^{f,n+1}\right) \\
& + \tau_k \phi_k \left(u_{k+1}^{f,n+1} - u_k^{f,n+1}\right) \\
& - \frac{1}{2}\beta_f T_0 \frac{\tau_{k-1}\phi_{k-1}}{(\Delta t)^2}\Big[\left(2u_k^{f,n} - u_k^{f,n-1}\right) \\
& \qquad + \left(2u_{k-1}^{f,n} - u_{k-1}^{f,n-1}\right)\Big] \qquad (42) \\
& - \frac{1}{2}\beta_f T_0 \frac{1}{(\Delta t)^2}\tau_k\phi_k \Big[\left(2u_{k+1}^{f,n} - u_{k+1}^{f,n-1}\right) \\
& \qquad + \left(2u_k^{f,n} - u_k^{f,n-1}\right)\Big] \\
\equiv {} & T103^{n+1}(k) - T03^{(n,n-1)}(k)
\end{aligned}
$$

Similarly (from (33))

$$T102\Big|_{k=1} = \tau_1(1-\phi_1)\beta_m T_0 \frac{1}{2}\frac{1}{(\Delta t)^2}\Big\{\Big[\left(u_2^{s,n+1} - u_1^{s,n+1}\right)$$

$$- \left[\left(2u_2^{s,n} - u_2^{s,n-1}\right) - \left(2u_1^{s,n} - u_1^{s,n-1}\right)\right]\Big\}$$

$$T102\Big|_{k=Nx+1} = \tau_{Nx}(1-\phi_{Nx})\beta_m T_0 \frac{1}{2}\frac{1}{(\Delta t)^2}\Big\{\left(u_{Nx+1}^{s,n+1} - u_{Nx}^{s,n+1}\right)$$

$$- \left[\left(2u_{Nx+1}^{s,n} - u_{Nx+1}^{s,n-1}\right) - \left(2u_{Nx}^{s,n} - u_{Nx}^{s,n-1}\right)\right]\Big\}$$

$$\equiv T102\Big|_{k=Nx+1}^{n+1} - T102\Big|_{k=Nx+1}^{n,\,n-1}$$

$$T102(K) = \frac{1}{2}\beta_m T_0 \frac{1}{(\Delta t)^2}\Big[\tau_{K-1}(1-\phi_{K-1})\left(u^{s,n+1}_{K} - u^{s,n+1}_{K-1}\right)$$

$$+\tau_K(1-\phi_K)\left(u^{s,n+1}_{K+1} - u^{s,n+1}_{K}\right)$$

$$-\frac{1}{2}\beta_m T_0 \frac{1}{(\Delta t)^2}\tau_{K-1}(1-\phi_{K-1})\Big[\left(2u^{s,n}_{K} - u^{s,n-1}_{K}\right)$$

$$\overset{-}{+}\left(2u^{s,n}_{K-1} - u^{s,n-1}_{K-1}\right)\Big]$$

$$-\frac{1}{2}\beta_m T_0 \frac{1}{(\Delta t)^2}\tau_{K}(1-\phi_{K})\Big[\left(2u^{s,n}_{K+1} - u^{s,n-1}_{K+1}\right)$$

$$\overset{-}{+}\left(2u^{s,n}_{K} - u^{s,n-1}_{K}\right)\Big]$$

$$\equiv T102^{n+1}_{(K)} - T102^{n,n-1}_{(K)}$$

~~The coefficients of $u_1^{f,n+1}$, $u_2^{f,n+1}$ in (34) go to the LHS ($T100|^{n+1}_{K=1}$ defines the coff. in the LHS — Similarly $T100|_{K=N_X}$ defines the~~

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~~Coefficients of~~ $\cancel{u^{f,n+1}_{Nx+1}}$, $\cancel{u^{f,n+1}_{Nx}}$.

~~The coef. in~~ $\cancel{T100(k)^{n+1}}$ ~~define coefficients~~

~~for~~ $\cancel{u^{f,n+1}_{k-1}}$, $\cancel{u^{f,n+1}_{k}}$, $\cancel{u^{f,n+1}_{k+1}}$ ~~in the LHS.~~

~~Similarly for~~ $\cancel{u^{s,n+1}_{1}}$, $\cancel{u^{s,n+1}_{2}}$

$\cancel{u^{s,n+1}_{Nx}}$ $\cancel{u^{s,n+1}_{Nx+1}}$, $\cancel{u^{s,n+1}_{k-1}}$ $\cancel{u^{s,n+1}_{k}}$, $\cancel{u^{s,n+1}_{k+1}}$ --

CHANGES IN RHS of TEMPERATURE EQUATION:

(28)

EQ'N (3) p2x (5) TEMPERATURE CODE

MANUSCRIPT 4/1/22: Add to

RHS1$^{\theta}$ the TERM

$$-T_{100}^{m+1}\Big|_{K=1} + T100\Big|_{K=1}^{m-1} = \frac{1}{2\Delta t}\left(U_2^{f,m-1} - U_1^{f,m-1}\right)\phi_1\beta_f T_0 \frac{1}{2} - \text{Idem}(m+1)$$

For $K = 2, - NX$ Add to RHS of EQ'N (4)

$$-T_{100}^{m+1}(K) + T100^{m-1}(K) = \frac{1}{2\Delta t} T_0 \beta_f \frac{1}{2}\left[\phi_{K-1}\left(U_K^{f,m-1} - U_{K-1}^{f,m-1}\right) + \phi_K\left(U_{K+1}^{f,m-1} - U_K^{f,m-1}\right)\right]$$

$$- \text{Idem}(m+1)$$

For $K = N_x + 1$ Add to RHS of EQ'N (5)

$$T100\Big|_{K=N_x+1}^{m-1} = \phi_{Nx}\beta_f T_0 \frac{1}{2}\frac{1}{2\Delta t}\left(U_{Nx+1}^{f,m-1} - U_{Nx}^{f,m-1}\right)$$

$$-T_{100}^{m+1}\Big|_{K=Nx+1} \quad - \text{Idem}(m+1)$$

Similarly for T101 _ T103:

Add to RHS1$^{\theta}$

$$T101\Big|_{K=1}^{m-1} = (1-\phi_1)\beta_s T_0 \frac{1}{2}\frac{1}{2\Delta t}\left(U_2^{s,m-1} - U_1^{s,m-1}\right)$$

$$-T_{101}^{m+1}\Big|_{K=1} \quad - \text{Idem}(m+1)$$

Add to RHS of EQN (4) (29)

$$T101(k)^{n-1} = \frac{1}{2\Delta t} T_0 \frac{1}{2}\beta_m \Big[(1-\phi_{k-1})(u_k^{s,n-1} - u_{k-1}^{s,n-1})$$

$$-T_{101}^{n+1}(k)$$

$$+ (1-\phi_k)(u_{k+1}^{s,n-1} - u_k^{s,n-1})\Big]$$

$$- \text{Idem}(n+1)$$

Add to RHS of EQ'N (5)

$$T101\Big|_{k=Nx+1}^{n-1} = (1-\phi_{Nx})\beta_m T_0 \frac{1}{2}\frac{1}{2\Delta t}\Big[(u_{Nx+1}^{s,n-1} - u_{Nx}^{s,n-1})\Big]$$

$$-T_{101}\Big|_{k=Nx+1}^{n+1} \qquad - \text{Idem}(n+1)$$

From T103: add to EQN' (3)

$$T103\Big|_{k=1}^{n,n-1} = z_1\phi_1\beta_f T_0 \frac{1}{2}\frac{1}{(\Delta t)^2}\Big[(2u_2^{f,n} - u_2^{f,n-1}) - (2u_1^{f,n} - u_1^{f,n-1})\Big]$$

$$-T_{103}\Big|_{k=1}^{n+1} \qquad - \text{Idem}(n+1)$$

Add to RHS of EQ'N (4)

$$\cancel{T103(k)} \quad T103(k)^{n,n-1} - T_{103}^{n+1}(k)$$

$$= \frac{1}{2}\beta_f T_0 \, z_{k-1}\phi_{k-1}\frac{1}{(\Delta t)^2}$$

$$\Big[(2u_k^{f,n} - u_k^{f,n-1}) + (2u_{k-1}^{f,n} - u_{k-1}^{f,n-1})\Big]$$

$$+\frac{1}{2}\beta_f T_0 z_k\phi_k\frac{1}{(\Delta t)^2}\Big[(2u_{k+1}^{f,n} - u_{k+1}^{f,n-1}) - (2u_k^{f,n} - u_k^{f,n-1})\Big]$$

$$- \text{Idem}(n+1)$$

Add to RHS of EQN (5) (30)

$$T103\Big|_{k=Nx+1}^{n,n-1} = \tau_{Nx}\phi_{Nx}\beta_f T_0 \frac{1}{(\Delta t)^2}\frac{1}{2}\Big[\left(2u_{Nx+1}^{f,n} - u_{Nx+1}^{f,n-1}\right) - \left(2u_{Nx}^{f,n} - u_{Nx}^{f,n-1}\right)\Big]$$

$$-T_{103}\Big|_{Nx+1}^{n+1} \qquad - \text{Idem}(n+1)$$

From T102:

Add to RHS of Eq'N (3)

$$T102\Big|_{k=1}^{n,n-1} = \tau_1(1-\phi_1)\beta_m T_0 \frac{1}{2}\frac{1}{(\Delta t)^2}\Big[\left(2u_2^{s,n} - u_2^{s,n-1}\right) - \left(2u_1^{s,n} - u_1^{s,n-1}\right)\Big]$$

$$-T_{102}^{n+1}\Big|_{k=1} \qquad - \text{Idem}(n+1)$$

Add to RHS of Eq'n (4)

$$T102(k)^{n,n-1} = \frac{1}{2}\beta_m T_0\, \tau_{k-1}(1-\phi_{k-1})\frac{1}{(\Delta t)^2}\Big[\left(2u_k^{s,n} - u_k^{s,n-1}\right) + \left(2u_{k-1}^{s,n} - u_{k-1}^{s,n-1}\right)\Big]$$

$$-T_{102}^{n+1}(k)$$

$$+\frac{1}{2}\beta_m T_0\, \tau_k(1-\phi_k)\frac{1}{(\Delta t)^2}\Big[\left(2u_{k+1}^{s,n} - u_{k+1}^{s,n-1}\right) + \left(2u_k^{s,n} - u_k^{s,n-1}\right)\Big]$$

$$- \text{Idem}(n+1)$$

Add to RHS of EQ'N (5) 31

$$T102\Big|^{M,n-1}_{K=Nx+1} = Z_{Nx}\,(1-\phi_{Nx})\beta_m T_0 \frac{1}{2}\frac{1}{(\Delta t)^2}\Big\{$$

$$-T102^{n+1}\Big|_{k=Nx+1}\Big[\left(2u^{s,n}_{Nx+1} - u^{s,n-1}_{Nx+1}\right) - \left(2u^{s,n}_{Nx} - u^{s,n-1}_{Nx}\right)\Big]$$

$$-\,\mathrm{Iderm}(n+1)$$