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Wave propagation in thermo-poroelasticity: A finite-element approach

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Abstract We propose continuous and discrete-time finite-element (FE) methods to solve an initial boundary-value problem (IBVP) for the thermo-poroelasticity wave equation based on the combined Biot/Lord-Shulman (LS) theories to describe the porous and thermal effects, respectively. In particular, the LS model, that includes a Maxwell-Vernotte-Cattaneo (MVC) relaxation term, leads to a hyperbolic heat equation, thus avoiding infinite signal velocities. The FE methods are formulated on a bounded domain with absorbing boundary conditions at the artificial boundaries. The dynamical equations predict four propagation modes, namely, a fast P wave, a Biot slow wave, a thermal wave, and a shear wave. The spatial discretization uses globally continuous bilinear polynomials to represent the solid displacements and the temperature, while the vector part of the Raviart-Thomas Nedelec of zero order is used to represent the fluid displacements. First, a priori optimal error estimates are derived for the continuous-time FE method, and then explicit and implicit discrete-time FE methods are defined. The stability of the explicit FE method is analyzed and the stability constrain is derived. The al-

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gorithms can be useful for a better understanding of seismic waves in hydrocarbon reservoirs and crustal rocks, whose description is mainly based on the assumption of isothermal wave propagation.

1 Introduction

Thermoelasticity is the theory that couples the fields of deformation and temperature, where an elastic source gives rise to a temperature field and attenuation and a heat source induces anelastic deformations. The theory is useful in a variety of applications such as seismic attenuation in rocks and material science [1–3]. The theory might also be relevant in low-temperature physics, theories of shocks and vibrations and astrophysics.

The classical parabolic-type differential equations of thermoelasticity (non-porous) for the Fourier law of heat conduction were reported by Biot [4], but his theory has unphysical solutions, such as discontinuities and infinite velocities at high frequencies. Later, Lord and Shulman (LS) [5] overcame these problems by formulating a hyperbolic-type differential equations, introducing MVC relaxation times into the heat equation [6]. The thermoelasticity theory predicts and S wave and two P waves, an elastic wave and a thermal wave having similar characteristics to the fast and slow P waves of poroelasticity, respectively [3]. Zener work [1] already contains the concept of mode conversion from P wave to a thermal mode, e.g., he explains P-wave dissipation due to the presence of "microscopic stress inhomogeneities arise from imperfections, such as cavities, and from the elastic anisotropy of the individual crystallites", in the same way that the White model [7] describes attenuation in porous media due to mesoscopic-scale inhomogeneities (as P wave converted to Biot slow mode). Early works in geophysics worth to mention in this sense were conducted by Treitel [8], Savage [9], who obtained the P- and S-wave quality factors for empty round cavities or pores, and Armstrong [10], who considered a finely layered medium. Then, the subject has been neglected in practice till recent works by Carcione and co-workers who performed the first simulation of the thermal wave in the context of thermoelasticity and poro-thermoelasticity [3, 11–15]. In these works, the numerical simulation is performed with a direct method to compute the spatial derivatives, namely, the Fourier pseudospectral differential operator (e.g., [16]). The development of a new technique, based on the FEM algorithm, will provide a more flexible approach to represent the heterogeneities of the medium and will provide further crosscheck of both algorithms and the physics of wave propagation.

Santos and co-workers [17] prove the existence and uniqueness of the Biot/Lord-Shulman formulation in linear thermo-poroelastic isotropic media, with bounded domains under appropriate boundary and initial conditions. The analysis shows the existence of a unique solution, given in terms of displacements of the solid and fluid phases and temperature, and proves its regularity in the space and time variables. The FE spaces used for the spatial discretization of the IBVP are as follows. The components of the solid displacement vector and the temperature are represented by globally continuous piecewise bilinear functions. For the fluid phase, we use locally the vector part of the Raviart-Thomas-Nedelec space of zero order. First, we derive a variational formulation of the continuous-time FE IBVP problem and show the existence and uniqueness of the continuous-time FE solution. Then a priori error estimates are given, which are optimal for the FE spaces used and the assumed regularity of the solution. Finally, explicit and implicit discrete-time FE algorithms are defined, and the conditional stability of the explicit FE procedure is analyzed.

2 Model equations

We consider a porous medium saturated by a single phase, compressible viscous fluid and assume that the whole aggregate is isotropic. Let $\mathbf{u}^s = (u_i^s)$ and $\mathbf{u}^f = (u_i^f)$ denote the average displacement vectors of the solid and relative fluid phases, respectively and set $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$. Let $\boldsymbol{\varepsilon}(\mathbf{u}^s) = (\varepsilon_{ij}(\mathbf{u}^s))$ be the strain tensor of the solid. Also, let $\boldsymbol{\sigma}(\mathbf{u}, \theta) = (\sigma_{ij}(\mathbf{u}, \theta))$, and $p_f = p_f(\mathbf{u}, \theta)$ denote the stress tensor of the bulk material and the fluid pressure, respectively, with θ being the increment of temperature above a reference absolute temperature θ_0 for the state of zero stress and strain. The stress-strain relations are:

$$\sigma_{ij}(\mathbf{u}, \theta) = 2\mu\, \varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_u \nabla \cdot \mathbf{u}^s + B\nabla \cdot \mathbf{u}^f - \beta\, \theta), \tag{1}$$

$$p_f(\mathbf{u}, \theta) = -B\, \nabla \cdot \mathbf{u}^s - M\nabla \cdot \mathbf{u}^f + \beta_f \theta, \tag{2}$$

where μ is the wet- or dry-rock shear modulus, $\lambda_u = \lambda + \alpha^2 M$, $\alpha = 1 - \dfrac{K_m}{K_s}$, $M = \left(\dfrac{\alpha - \phi}{K_s} + \dfrac{\phi}{K_f}\right)^{-1}$, ϕ is the porosity, $B = \alpha\, M$, $\beta = \beta_m + \beta_f$, with λ_u being the Lamé coefficient of the fluid saturated frame and K_s, K_m and K_f denoting the bulk moduli of the grains, solid and fluid, respectively. The positive coupling coefficients β_m and β_f are the coefficients of thermoelasticity of the frame and fluid, respectively.

2.1 Dynamical equations

Let $\rho_b = (1-\phi)\rho_s + \phi\rho_f$ denote the mass density of the bulk material, with ρ_s and ρ_f being the mass densities of the grains and fluid, respectively. Let the positive definite matrix $\mathcal{P}$ and the nonnegative matrix $\mathcal{B}$ be defined by

$$\mathcal{P} = \begin{pmatrix} \rho_b I & \rho_f I \\ \rho_f I & gI \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0I & 0I \\ 0I & \frac{\eta}{\kappa} I \end{pmatrix}, \tag{3}$$

where I is the identity matrix in $R^{d\times d}$, with $d = 2, 3$, η is the fluid viscosity, κ is the permeability and $g = \dfrac{S\rho_f}{\phi}$, where S is the tortuosity.
Let us define the differential operator $\mathcal{L}(\mathbf{u},\theta) = \left(\nabla\cdot\boldsymbol{\sigma}(\mathbf{u},\theta), -\nabla p_f(\mathbf{u},\theta)\right)$. Then, Biot's dynamical equation taking into account temperature is

$$\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{\mathbf{u}}^f - \mathcal{L}(\mathbf{u},\theta) = \mathbf{f}. \tag{4}$$

Following Sharma [18] and Carcione et al [3], the generalized heat equation is

$$\begin{aligned} &\tau\, c\, \ddot{\theta} + c\, \dot{\theta} - \nabla\cdot(\gamma\nabla\theta) + (1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{u}}^s + \phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{u}}^f \\ &+\tau(1-\phi)\beta_m T_o\nabla\cdot\ddot{\mathbf{u}}^s + \tau\phi\beta_f T_o\nabla\cdot\ddot{\mathbf{u}}^f = -q. \end{aligned} \tag{5}$$

In (4)-(5) $\mathbf{f} = (\mathbf{f}^s, \mathbf{f}^f)$ is an external force and q is a heat source. Also, $\gamma = (1-\phi)\gamma_m + \phi\gamma_f$ is the bulk coefficient of heat conduction (or thermal conductivity), with γ_m and γ_f being the heat conduction of the frame and the fluid, respectively, and $c = (1-\phi)c_m + \phi c_f$, is the bulk specific heat of the unit volume in the absence of deformation and τ is a MVC relaxation time. These equations assume thermal equilibrium between the solid and the fluid, i.e., the temperature in both phases is the same. Thermal equilibrium is valid when the interstitial heat transfer coefficient between the solid and fluid is very large and the ratio of pore surface area to pore volume is sufficiently high. Here, we consider β_m, β_f, γ and c as parameters, obtained from experiments or from a specific theoretical model.

2.2 The initial boundary-value problem

The initial boundary-value problem is formulated in the 2D case (with obvious extension to the 3D case) for the case of thermal equilibrium in an open bounded domain Ω with piecewise smooth boundary and a time interval $J = (0, T)$ as follows: Find $(\mathbf{u}, \theta)$ satisfying (4)-(5) with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}^0 = (\mathbf{u}^{0,s}, \mathbf{u}^{0,f}), \qquad \dot{\mathbf{u}}(x,0) = \mathbf{u}^1 = (\mathbf{u}^{1,s}, \mathbf{u}^{1,f}), \qquad x\in\Omega, \tag{6}$$

$$\theta(x,0) = \theta^0, \quad \dot{\theta}(x,0) = \theta^1, \ x\in\Omega, \tag{7}$$

and absorbing boundary conditions

$$-\mathcal{G}_{\Gamma}(\mathbf{u},\theta) = \mathcal{D}\mathcal{S}(\dot{\mathbf{u}}), \qquad -\gamma\nabla\theta\cdot\boldsymbol{\nu} = \tau c v_\theta \dot{\theta} \quad x\in\Gamma, \quad t\in J, \tag{8}$$

where

$$\mathcal{G}(\mathbf{u},\theta) = \left(\boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}, \boldsymbol{\sigma\nu}\cdot\boldsymbol{\chi}, p_f\right)(\mathbf{u},\theta), \qquad \mathcal{S}(\dot{\mathbf{u}}) = \left(\dot{\mathbf{u}}^s\cdot\boldsymbol{\nu}, \dot{\mathbf{u}}^s\cdot\boldsymbol{\chi}, \dot{\mathbf{u}}^f\cdot\boldsymbol{\nu}\right) \tag{9}$$

In (8)-(9), $\boldsymbol{\nu}$ and $\boldsymbol{\chi}$ are the unit vector outer normal and unit vector tangent on Γ oriented counterclockwise. The matrix $\mathcal{D}$ is positive definite and $v_\theta = \sqrt{\gamma/(\tau c)}$ is the heat speed (e.g., Carcione et al.[12]).
An existence and uniqueness result for the solution of (4)-(7) with different boundary conditions than those in (8) is given in [17].

3 A variational formulation

In order to obtain a variational formulation, we need to introduce some notation. For $\Omega\subset\mathbb{R}^2$ with boundary $\Gamma=\partial\Omega$, let $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote the $L^2(\Omega)$ and $L^2(\Gamma)$ inner products for scalar, vector, or matrix valued functions. Also, for $s\in\mathbb{R}$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Gamma}$ will denote the usual norms for the Sobolev space $H^s(\Omega)$ and $H^s(\Gamma)$, respectively. If $X=\Omega$ or $X=\Gamma$, the subscript X may be omitted such that $(\cdot,\cdot)=(\cdot,\cdot)_\Omega$, $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_\Gamma$, or $|\cdot|_s=|\cdot|_{s,\Gamma}$. Let

$$H(\mathrm{div};\Omega) = \{\mathbf{v}\in[L^2(\Omega)]^2 : \nabla\cdot\mathbf{v}\in L^2(\Omega)\},$$

provided with the norm $\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = \left[\|\mathbf{v}\|_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2\right]^{1/2}$. We also will refer to the space

$$H^1(\mathrm{div};\Omega) = \{\mathbf{v}\in[H^1(\Omega)]^2 : \nabla\cdot\mathbf{v}\in H^1(\Omega)\}.$$

The following known results will be used [19]

$$|\mathbf{v}\cdot\boldsymbol{\nu}|_{-1/2,\Gamma} \le C\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}, \tag{10}$$

$$|\mathbf{v}|_{0,\Gamma} \le C\|\mathbf{v}\|_{0,\Omega}^{1/2}\,\|\mathbf{v}\|_{1,\Omega}^{1/2} \le C\|\mathbf{v}\|_{1,\Omega}. \tag{11}$$

Here and in what follows, C denotes a generic constant that may take different values at different places. Also recall Korn's second inequality [20]

$$\int_\Omega \left[\sum_{i,j}(\varepsilon_{ij}(\mathbf{v}))^2\right] d\Omega + \|\mathbf{v}\|_0^2 \ge C\|\mathbf{v}\|_1^2. \tag{12}$$

Next, we introduce the space $\mathcal{V} = \left[H^1(\Omega)\right]^2 \times H(\mathrm{div};\Omega)$, provided with the natural norm

$$\|\mathbf{v}\|_{\mathcal{V}} = \left(\|\mathbf{v}^s\|_1^2 + \|\mathbf{v}^f\|_{H(\mathrm{div};\Omega)}^2\right)^{1/2}, \mathbf{v}^s\in[H^1(\Omega)]^2, \mathbf{v}^f\in H(\mathrm{div};\Omega).$$

Also, for any Banach space Y let

$$L^2(J,Y)=\{f:J\to Y:\|f\|_{J,Y}^2=\int_0^T\|f(t)\|_Y^2dt<\infty\},$$
$$L^\infty(J,Y)=\{f:J\to Y:\|f\|_{J,Y}^\infty=ess.sup_{t\in J}\|f(t)\|_Y\}.$$

To obtain a variational formulation of the initial boundary-value problem (4)-(8), multiply (4) by $\mathbf{v}^s$, (5) by $\mathbf{v}^f$ such that $\mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f)\in\mathcal{V}$, use integration by parts and the boundary conditions (8) to get

add here bry terms in the beta terms of biot equation

$$\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}}(x),\mathbf{v})+\left(\frac{\eta}{\kappa}\dot{u}^f,\mathbf{v}^f\right)+\Lambda(\mathbf{u},\mathbf{v})-(\beta\theta,\nabla\cdot\mathbf{v}^s)-\left(\beta_f\theta,\nabla\cdot\mathbf{v}^f\right)\\
&+\left\langle\mathbf{v}^s\cdot\boldsymbol{\nu},\beta\theta\right\rangle+\left\langle\mathbf{v}^f\cdot\boldsymbol{\nu},\beta_f\theta\right\rangle\\
&+\left(\tau\,c\,\ddot{\theta},w\right)+\left(c\,\dot{\theta},w\right)+(\gamma\nabla\theta,\nabla w)+((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{u}}^s,w)\\
&+\left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{u}}^f,w\right)+(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{u}}^s,w)+\left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{u}}^f,w\right)\\
&+\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{u}}),\mathcal{S}(\mathbf{v})\rangle+\left\langle\tau cv_\theta\dot{\theta},w\right\rangle\\
&=(\mathbf{f},\mathbf{v})-(q,w)\,,\quad \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{V}\times H^1(\Omega),\quad t\in J,
\end{aligned}\tag{13}$$

where $\Lambda(\mathbf{u},\mathbf{v})$ is the bilinear form

$$\Lambda(\mathbf{u},\mathbf{v})=\sum_{l,m}(\sigma_{lm}(\mathbf{u}),\varepsilon_{lm}(\mathbf{v}^s))=(\mathcal{E}\;\widetilde{\epsilon}(\mathbf{u}),\widetilde{\epsilon}(\mathbf{v}))\,.\tag{14}$$

In (14), the matrix $\mathcal{E}$ and the column vector $\widetilde{\varepsilon}(\mathbf{u})$ are defined by

$$\mathcal{E}=\begin{pmatrix}\lambda_u+2\mu&\lambda_u&B&0\\\lambda_u&\lambda_u+2\mu&B&0\\B&B&M&0\\0&0&0&4\mu\end{pmatrix},\quad\widetilde{\varepsilon}(\mathbf{u})=\begin{pmatrix}\varepsilon_{11}(\mathbf{u}^s)\\\varepsilon_{33}(\mathbf{u}^s)\\\nabla\cdot\mathbf{u}^f\\\varepsilon_{13}(\mathbf{u}^s)\end{pmatrix}.\tag{15}$$

The term $(\mathcal{E}\;\widetilde{\epsilon}(\mathbf{u}),\widetilde{\epsilon}(\mathbf{v}))$ in (14) is associated with the strain energy of the system, so that the symmetric matrix $\mathcal{E}$ must be positive definite. Furthermore, $\Lambda(\mathbf{u},\mathbf{v})\le C\|\mathbf{u}\|_{\mathcal{V}}\|\mathbf{v}\|_{\mathcal{V}}$.
Also, note that using (12), if $\lambda_*^{\mathcal{E}}$ is the minimum eigenvalue of $\mathcal{E}$, the following Gärding inequality holds:

$$\Lambda(\mathbf{v},\mathbf{v})\ge C_1\|\mathbf{v}\|_{\mathcal{V}}^2-\lambda_*^{\mathcal{E}}\|\mathbf{v}\|_0^2.\tag{16}$$

4 Finite-element formulations

We will find a FE solution of (13) as follows. Let $\mathcal{T}^h(\Omega)$ be a quasiregular nonoverlapping partition of Ω into rectangles Ω_j of diameter bounded by h such that $\overline{\Omega} = \cup_J^J \overline{\Omega}_j$. Let us denote by $\mathcal{W}^h(\Omega)$ the space of globally continuous piecewise bilinear polynomials to be used to approximate each component of the solid displacement $\mathbf{u}^s$ and the temperature θ. Also, let $\mathcal{V}^h(\Omega)$ be the vector part of the Raviart-Thomas-Nedelec space of zero order [21,22] used to approximate the fluid displacement vector $\mathbf{u}^f$. Then, let

$$\mathcal{Z}^h(\Omega) = \mathcal{W}^h(\Omega) \times \mathcal{W}^h(\Omega) \times \mathcal{V}^h(\Omega) \times \mathcal{W}^h(\Omega).$$

Next, let $\Pi : H^2(\Omega) \to \mathcal{W}^h(\Omega)$ be the interpolant operators associated with the space $\mathcal{W}^h$ and set $\Pi^{(2)} \equiv \Pi \times \Pi \to [\mathcal{W}^h(\Omega)]^2$. Let $Q : H^1(\mathrm{div};\Omega) \to \mathcal{V}^h(\Omega)$ be the projection defined by

$$\langle (Q\boldsymbol{\psi} - \boldsymbol{\psi}) \cdot \boldsymbol{\nu}, 1 \rangle_B = 0, \quad B = \Gamma_{jk} \text{ or } B = \Gamma_j.$$

The approximating properties of Π and Q are [21,**?**]

$$\|\boldsymbol{\varphi} - \Pi\boldsymbol{\varphi}\|_0 + h\|\boldsymbol{\varphi} - \Pi\varphi\|_1 \le Ch^2\|\boldsymbol{\varphi}\|_2, \quad \varphi \in H^2(\Omega) \tag{17}$$

$$\|\boldsymbol{\varphi} - (\Pi)^{(2)}\boldsymbol{\varphi}\|_0 + h\|\boldsymbol{\varphi} - (\Pi)^{(2)}\varphi\|_1 \le Ch^2\|\boldsymbol{\varphi}\|_2, \quad \varphi \in [H^2(\Omega)]^2 \tag{18}$$

$$\|\boldsymbol{\psi} - Q\boldsymbol{\psi}\|_0 \le Ch\|\boldsymbol{\psi}\|_1, \tag{19}$$

$$\|\nabla \cdot (\boldsymbol{\psi} - Q\boldsymbol{\psi})\|_0 \le Ch\left(\|\boldsymbol{\psi}\|_1 + \|\nabla \cdot \boldsymbol{\psi}\|_1\right) \quad \boldsymbol{\psi} \in H^1(\mathrm{div};\Omega). \tag{20}$$

4.1 Continuous-time finite-element procedure

Find $(\mathbf{U}(t), \Theta(t)) \in \mathcal{Z}^h(\Omega)$ such that

$$\begin{aligned}
&\left(\mathcal{P}\ddot{\mathbf{U}}, \mathbf{v}\right) + \left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{U}, \mathbf{v}) - (\beta\Theta, \nabla\cdot\mathbf{v}^s) - \left(\beta_f\Theta, \nabla\cdot\mathbf{v}^f\right) \\
&+ \left(\tau\, c\, \ddot{\Theta}, w\right) + \left(c\, \dot{\Theta}, w\right) + (\gamma\nabla\Theta, \nabla w) + \left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s, w\right) \\
&+ \left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f, w\right) + \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{U}}^s, w\right) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{U}}^f, w\right) \\
&+ \left\langle \mathcal{D}\mathcal{S}(\dot{\mathbf{U}}), \mathcal{S}(\mathbf{v})\right\rangle + \left\langle \tau c v_\theta \dot{\Theta}, w\right\rangle \\
&= (\mathbf{f}, \mathbf{v}) - (q, w)\,, \quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{Z}^h(\Omega), \quad t \in J.
\end{aligned} \tag{21}$$

In the next theorem we demonstrate the existence and uniqueness of the solution of problem (21).

Theorem 1 *Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in (3) are positive definite and semidefinite, respectively and that the matrix $\mathcal{E}$ in (15) is positive definite. Then, there exists a unique solution $(\mathbf{U},\Theta) \in \mathcal{Z}^h$ of the continuous-time FE procedure (21) and satisfies the inequality*

$$\begin{aligned}
&\|\mathbf{U}(t)\|^2_{L^\infty(J,\mathcal{V})} + \|\dot{\mathbf{U}}(t)\|^2_{L^\infty(J,\mathcal{V})} + \|\ddot{\mathbf{U}}(t)\|^2_{L^2(J,[L^2(\Omega)]^4)} \\
&+\|\Theta(t)\|^2_{L^2(J,L^2(\Omega))} + \|\dot{\Theta}(t)\|^2_{L^2(J,H^1(\Omega))} \\
&\le C\left(\|\mathbf{U}(0)\|^2_{\mathcal{V}} + \|\dot{\mathbf{U}}(0)\|^2_{\mathcal{V}} + \|\dot{\mathbf{U}}(0)\|^2_0 + \|\ddot{\mathbf{U}}(0)\|^2_0 + \|\dot{\Theta}(0)\|^2_0 + \|\Theta(0)\|^2_1\right) \\
&+ \le C\left(\|\mathbf{f}\|^2_{L^2(J,[L^2(\Omega)]^4)} + \|\dot{\mathbf{f}}\|^2_{L^2(J,[L^2(\Omega)]^4)} + +\|q(s)\|^2_{L^2(J,L^2(\Omega))}\right).
\end{aligned} \tag{22}$$

Proof Choose $\mathbf{v} = \dot{\mathbf{U}},\ w = \dot{\Theta}$ in (21) to obtain

$$\begin{aligned}
&\frac{1}{2}\left[\left(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}\right) + \Lambda(\mathbf{U},\mathbf{U}) + \left(\tau\, c\, \dot{\Theta},\dot{\Theta}\right) + \left(\gamma\nabla\Theta,\nabla\Theta\right)\right] + \left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right) \\
&-\left(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s\right) - \left(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f\right) + \left(c\,\dot{\Theta},\dot{\Theta}\right) \\
&+\left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}\right) + \left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}\right) + \left\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\right\rangle \\
&+\left\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\right\rangle + \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}\right) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta}\right) \\
&= \left(\mathbf{f}^s,\dot{\mathbf{U}}^s\right) + \left(\mathbf{f}^f,\dot{\mathbf{U}}^f\right) - \left(q,\dot{\Theta}\right), \quad t\in J.
\end{aligned} \tag{23}$$

To handle the last two terms in the left-hand side of (23), take time derivative in (21) and choose $w = 0$ to obtain

$$\begin{aligned}
&\left(\mathcal{P}\ddot{\mathbf{U}}^s,\mathbf{v}\right) + \Lambda(\dot{\mathbf{U}},\mathbf{v}) - \left(\beta\dot{\Theta},\nabla\cdot\mathbf{v}^s\right) - \left(\beta_f\dot{\Theta},\nabla\cdot\mathbf{v}^f\right) \\
&+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\mathbf{v}^f\right) + \left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\mathbf{v})\right\rangle = \left(\dot{\mathbf{f}^s},\mathbf{v}^s\right) + \left(\dot{\mathbf{f}^f},\mathbf{v}^f\right).
\end{aligned} \tag{24}$$

Choose $\mathbf{v}^s = \ddot{\mathbf{U}}^s, \mathbf{v}^f = 0$ in (24) to get

$$\begin{aligned}
&\left(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)\right) + \Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) - \left(\beta\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^s\right) \\
&+\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\right\rangle = \left(\dot{\mathbf{f}^s},\ddot{\mathbf{U}}^s\right).
\end{aligned} \tag{25}$$

Also, the choice $\mathbf{v}^s = 0, \mathbf{v}^f = \ddot{\mathbf{U}}^f$ in (24) yields

$$\begin{aligned}
&\left(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)\right) + \Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) - \left(\beta_f\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^f\right) + \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right) \\
&+\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\right\rangle = \left(\dot{\mathbf{f}^f},\ddot{\mathbf{U}}^f\right).
\end{aligned} \tag{26}$$

Set

$$C_{m,\beta} = \inf_{x\in\Omega}\left(\frac{\tau(1-\phi)\beta_m\theta_0}{\beta}\right), C_{f,\beta} = \inf_{x\in\Omega}\left(\tau\phi\theta_0\right), C_\beta = \min(C_{m,\beta},C_{f,\beta})$$

Then, from (25)

$$\begin{aligned}&\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}\right)\geq C_\beta(\beta\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^s)\\&=C_\beta\left[\left(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)\right)+\Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0))+\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\right\rangle-\left(\dot{\mathbf{f}^s},\ddot{\mathbf{U}}^s\right)\right].\end{aligned}\tag{27}$$

Also, from (26)

$$\begin{aligned}&\left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta}\right)\geq C_\beta(\beta_f\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^f)\\&=C_\beta\left[\left(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)\right)+\Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f))+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)+\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\right\rangle-\left(\dot{\mathbf{f}^f},\ddot{\mathbf{U}}^f\right)\right].\end{aligned}\tag{28}$$

Next after algebraic manipulations, setting $\widehat{\mathcal{P}}=C_\beta\mathcal{P}$ we get

$$C_\beta\left(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)\right)+C_\beta\left(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)\right)=\left(\widehat{\mathcal{P}}\ddot{\mathbf{U}},\ddot{\mathbf{U}}\right).\tag{29}$$

Also, if $\widehat{\Lambda}(\dot{\mathbf{U}},\ddot{\mathbf{U}})=C_\beta\Lambda(\dot{\mathbf{U}},\ddot{\mathbf{U}})=\left(C_\beta\mathcal{E}\ \widetilde{\epsilon}(\dot{\mathbf{U}}),\widetilde{\epsilon}(\ddot{\mathbf{U}})\right)$ we see that

$$C_\beta\Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0))+C_\beta\Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f))=\widehat{\Lambda}(\dot{\mathbf{U}},\ddot{\mathbf{U}}).\tag{30}$$

Furthermore,

$$C_\beta\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\right\rangle+C_\beta\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\right\rangle=\left\langle C_\beta\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\right\rangle.$$

Hence,

$$\begin{aligned}&\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}\right)+\left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta}\right)\\&\geq\frac{1}{2}\frac{d}{dt}\left[\left(\widehat{\mathcal{P}}\ddot{\mathbf{U}},\ddot{\mathbf{U}}\right)+\widehat{\Lambda}(\dot{\mathbf{U}},\dot{\mathbf{U}})\right]+C_\beta\left[\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\right\rangle\right.\\&\left.+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)-\left(\dot{\mathbf{f}^s},\ddot{\mathbf{U}}^s\right)-\left(\dot{\mathbf{f}^f},\ddot{\mathbf{U}}^f\right)\right].\end{aligned}\tag{31}$$

Using (31) in (23) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}\right)+\left(\widehat{\mathcal{P}}\ddot{\mathbf{U}},\ddot{\mathbf{U}}\right)+\Lambda(\mathbf{U},\mathbf{U})+\widehat{\Lambda}(\dot{\mathbf{U}},\dot{\mathbf{U}})+\left(\tau\, c\,\dot{\Theta},\dot{\Theta}\right)+(\gamma\nabla\Theta,\nabla\Theta)\right]\\&+\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)-\left(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s\right)-\left(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f\right)+\left(c\,\dot{\Theta},\dot{\Theta}\right)\\&+\left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}\right)+\left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}\right)+\left\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\right\rangle\\&+\left\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\right\rangle+C_\beta\left[\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\right\rangle\right.\\&\left.+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)-\left(\dot{\mathbf{f}^s},\ddot{\mathbf{U}}^s\right)-\left(\dot{\mathbf{f}^f},\ddot{\mathbf{U}}^f\right)\right],\quad t\in J.\\&\leq\frac{1}{2}\left[\left(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}\right)+\Lambda(\mathbf{U},\mathbf{U})+\left(\tau\, c\,\dot{\Theta},\dot{\Theta}\right)+(\gamma\nabla\Theta,\nabla\Theta)\right]+\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)\\&-\left(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s\right)-\left(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f\right)+\left(c\,\dot{\Theta},\dot{\Theta}\right)\\&+\left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}\right)+\left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}\right)+\left\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\right\rangle\\&+\left\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\right\rangle+\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}\right)+\left(\tau\phi\beta_f\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta}\right)\\&=\left(\mathbf{f}^s,\mathbf{U}^s\right)+\left(\mathbf{f}^f,\mathbf{U}^f\right)-\left(q,\dot{\Theta}\right),\quad t\in J.\end{aligned}\tag{32}$$

Next note that using the Gärding inequality (16), we can choose ζ_1, ζ_2 to define the bilinear forms

$$\Lambda_{\zeta_1}(\mathbf{v},\mathbf{v}) = \Lambda(\mathbf{v},\mathbf{v}) + \zeta_1(\mathbf{v},\mathbf{v}), \quad \widehat{\Lambda}_{\zeta_2}(\mathbf{v},\mathbf{v}) = \widehat{\Lambda}(\mathbf{v},\mathbf{v}) + \zeta_2(\mathbf{v},\mathbf{v})$$

such that Λ_{ζ_1} and Λ_{ζ_2} are $\mathcal{V}$-coercive, i.e.

$$\Lambda_{\zeta_1}(\mathbf{v},\mathbf{v}) \ge C_2\|\mathbf{v}\|^2_{\mathcal{V}}, \quad \widehat{\Lambda}_{\zeta_2}(\mathbf{v},\mathbf{v}) \ge C_3\|\mathbf{v}\|^2_{\mathcal{V}}. \tag{33}$$

Thus, adding to (32) the inequalities

$$\zeta\frac{d}{dt}\|\mathbf{u}\|_0^2 \le \zeta\left(\|\mathbf{U}\|_0^2 + \|\dot{\mathbf{U}}\|_0^2\right), \, \zeta = \zeta_1, \zeta_2, \quad \frac{d}{dt}\|\gamma^{1/2}\theta\|_0^2 \le \left(\|\gamma^{1/2}\theta\|_0^2 + \|\gamma^{1/2}\dot{\theta}\|_0^2\right)$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}) + \left(\widehat{\mathcal{P}}\ddot{\mathbf{U}},\ddot{\mathbf{U}}\right) + \Lambda_{\zeta_1}(\mathbf{U},\mathbf{U}) + \widehat{\Lambda}_{\zeta_2}(\dot{\mathbf{U}},\dot{\mathbf{U}}) + \|(\tau\, c)^{1/2}\ \dot{\Theta}\|_0^2 + \|\gamma^{1/2}\Theta\|_1^2\right] \\
&+ \left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right) + \left(c\,\dot{\Theta},\dot{\Theta}\right) + \left\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\right\rangle + \left\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\right\rangle \\
&+C_\beta\left[\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\right\rangle + \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)\right] \\
&\le C\left(\|\mathbf{f}\|_0^2 + \|q\|_0^2 + \|\mathbf{U}\|_0^2 + \|\dot{\mathbf{U}}\|_0^2 + \|\Theta\|_0^2 + \|\dot{\Theta}\|_0^2\right) + \left(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s\right) + \left(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f\right) \\
&- \left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}\right) - \left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}\right) + C_\beta\left[\left(\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s\right) + \left(\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f\right)\right], \, t\in J.
\end{aligned} \tag{34}$$

Next we will integrate in time (34). Using that

$$\begin{aligned}
&\left|\int_0^t \left(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s\right)(s)\right| + \left|\int_0^t \left(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f\right)(s)ds\right| \le C\int_0^t\left[\|\theta(s)\|_0^2 + \|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2\right]ds, \\
&\left|\int_0^t \left((1-\phi)\beta_m\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}\right)(s)ds\right| + \left|\int_0^t \left(\phi\beta_f\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}\right)(s)ds\right| \\
&\le C\int_0^t\left[\|\dot{\Theta}(s)\|_0^2 + \|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2\right]ds, \\
&\left|\int_0^t \left(\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s\right)(s)ds\right| + \left|\int_0^t \left(\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f\right)(s)ds\right| \le C\int_0^t\left[\|\dot{\mathbf{f}}(s)\|_0^2 + \|\ddot{\mathbf{U}}(s)\|_0^2\right]ds,
\end{aligned}$$

integration in time of (34) and (33) yields

$$\begin{aligned}
&\left(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}\right)(t) + \left(\widehat{\mathcal{P}}\ddot{\mathbf{U}},\ddot{\mathbf{U}}\right)(t) + \|\mathbf{U}\|_{\mathcal{V}}^2 + \|\dot{\mathbf{U}}\|_{\mathcal{V}}^2 + \|(\tau\, c)^{1/2}\ \dot{\Theta}(t)\|_0^2 + \|\gamma^{1/2}\Theta(t)\|_1^2 \\
&+ \int_0^t\left[\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)(s) + \left(c\,\dot{\Theta},\dot{\Theta}\right)(s) + \left\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\right\rangle(s)\right. \\
&\left.+\left\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\right\rangle(s) + C_\beta\left(\left\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\right\rangle(s) + \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)(s)\right)\right]ds \\
&\le C\int_0^t\left(\|\mathbf{f}(s)\|_0^2 + \|q(s)\|_0^2\right)ds \\
&+C\left(\|\mathbf{U}(0)\|_{\mathcal{V}}^2 + \|\dot{\mathbf{U}}(0)\|_{\mathcal{V}}^2 + \|\dot{\mathbf{U}}(0)\|_0^2 + \|\ddot{\mathbf{U}}(0)\|_0^2 + \|\dot{\Theta}(0)\|_0^2 + \|\Theta(0)\|_1^2\right) \\
&+C\int_0^t\left(\|\mathbf{U}(s)\|_0^2 + \|\dot{\mathbf{U}}(s)\|_0^2 + \|\ddot{\mathbf{U}}(s)\|_0^2 + \|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2 + \|\Theta(s)\|_0^2 + \|\dot{\Theta}(s)\|_0^2\right), \, t\in J.
\end{aligned} \tag{35}$$

Since all integral terms in the left-hand side of (35) are non-negative and the matrices $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positive definite, apply Gronwall's lemma in (34) to obtain the conclusion of Theorem 1.

4.2 A priori error estimates

From (13) and (21) we see that $\mathbf{Eu} = (\mathbf{Eu}^s, \mathbf{Eu}^f) = (\mathbf{u}^s - \mathbf{U}^s, \mathbf{u}^f - \mathbf{U}^f)$ and $\mathbf{E}\theta = \theta - \Theta$ satisfy the equation

$$
\begin{aligned}
&(\mathcal{P}\mathbf{E\ddot{u}}, \mathbf{v}) + \left(\frac{\eta}{\kappa}\mathbf{E\dot{u}}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{Eu}, \mathbf{v}) - (\beta\mathbf{E}\theta, \nabla\cdot\mathbf{v}^s) - \left(\beta_f\mathbf{E}\theta, \nabla\cdot\mathbf{v}^f\right)\\
&+ \left(\tau\, c\ \mathbf{E}\ddot{\theta}, w\right) + \left(c\ \mathbf{E}\dot{\theta}, w\right) + (\gamma\nabla\mathbf{E}\theta, \nabla w) + ((1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{u}^s, w)\\
&+ \left(\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\dot{u}^f, w\right) + (\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\ddot{u}^s, w) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\ddot{u}^f, w\right)\\
&+ \langle\mathcal{D}\mathcal{S}(\mathbf{E\dot{u}}), \mathcal{S}(\mathbf{v})\rangle + \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, w\right\rangle = 0, \quad t\in J.
\end{aligned}
\tag{36}
$$

Choose $\mathbf{v}^s = \mathbf{E\dot{u}}^s + \Pi^{(2)}\mathbf{\dot{u}}^s - \mathbf{\dot{u}}^s$, $\mathbf{v}^f = \mathbf{E\dot{u}}^f + Q\mathbf{\dot{u}}^f - \mathbf{\dot{u}}^f$, $w = \mathbf{E}\dot{\theta} + \Pi\dot{\theta} - \dot{\theta}$ in (36) to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(\mathcal{P}\mathbf{E\dot{u}}, \mathbf{E\dot{u}}) + \Lambda(\mathbf{Eu}, \mathbf{Eu}) + \left(\tau\, c\ \mathbf{E}\dot{\theta}, \mathbf{E}\dot{\theta}\right) + (\gamma\nabla\mathbf{E}\theta, \nabla\mathbf{E}\theta)\right]\\
&+ \left(\frac{\eta}{\kappa}\mathbf{E\dot{u}}^f, \mathbf{E\dot{u}}^f\right) - (\beta\mathbf{E}\theta, \nabla\cdot\mathbf{E\dot{u}}^s) - \left(\beta_f\mathbf{E}\theta, \nabla\cdot\mathbf{E\dot{u}}^f\right)\\
&+ \left(c\ \mathbf{E}\dot{\theta}, \mathbf{E}\dot{\theta}\right) + \langle\mathcal{D}\mathcal{S}(\mathbf{E\dot{u}}), \mathcal{S}(\mathbf{E\dot{u}})\rangle + \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, \mathbf{E}\dot{\theta}\right\rangle\\
&+ \left((1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E\dot{u}}^s, \mathbf{E}\dot{\theta}\right) + \left(\phi\beta_f\theta_0\nabla\cdot\mathbf{E\dot{u}}^f, \mathbf{E}\dot{\theta}\right) + \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E\ddot{u}}^s, \mathbf{E}\dot{\theta}\right)\\
&+ \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E\dot{u}}^f, \mathbf{E}\dot{\theta}\right) \qquad (37)\\
&= \left(\mathcal{P}\mathbf{E\ddot{u}}, (\mathbf{\dot{u}}^s - \Pi^{(2)}\mathbf{\dot{u}}^s, \mathbf{\dot{u}}^f - Q\mathbf{\dot{u}}^f)\right) + \left(\frac{\eta}{\kappa}\mathbf{E\dot{u}}^f, \mathbf{\dot{u}}^f - Q\mathbf{\dot{u}}^f\right) + \Lambda(\mathbf{Eu}, (\mathbf{\dot{u}}^s - \Pi^{(2)}\mathbf{\dot{u}}^s, \mathbf{\dot{u}}^f - Q\mathbf{\dot{u}}^f))\\
&- \left(\beta\mathbf{E}\theta, \nabla\cdot(\mathbf{\dot{u}}^s - \Pi^{(2)}\mathbf{\dot{u}}^s)\right) - \left(\beta_f\mathbf{E}\theta, \nabla\cdot\ (Q\mathbf{\dot{u}}^f - \mathbf{\dot{u}}^f)\right)\\
&+ \left(\tau\, c\ \mathbf{E}\ddot{\theta}, \Pi\dot{\theta} - \dot{\theta}\right) + \left(c\ \mathbf{E}\dot{\theta}, \dot{\theta} - \Pi\dot{\theta}\right) + \left(\gamma\nabla\mathbf{E}\theta, \nabla(\Pi\dot{\theta} - \dot{\theta})\right) + \left((1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E\dot{u}}^s, \Pi\dot{\theta} - \dot{\theta}\right)\\
&+ \left(\phi\beta_f\theta_0\nabla\cdot\mathbf{E\dot{u}}^f, \Pi\dot{\theta} - \dot{\theta}\right) + \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E\ddot{u}}^s, \Pi\dot{\theta} - \dot{\theta}\right) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E\ddot{u}}^f, \Pi\dot{\theta} - \dot{\theta}\right)\\
&+ \left\langle\mathcal{D}\mathcal{S}(\mathbf{E\dot{u}}), \mathcal{S}(\mathbf{\dot{u}}^s - \Pi^{(2)}\mathbf{\dot{u}}^s, \mathbf{\dot{u}}^f - Q\mathbf{\dot{u}}^f)\right\rangle + \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, \Pi\dot{\theta} - \dot{\theta}\right\rangle, \quad t\in J.
\end{aligned}
$$

The last two terms in the left-hand side of (37) can be handled by taking time derivative in (36) and choosing $w = 0$, $\mathbf{v}^s = \mathbf{E\ddot{u}}^s, \mathbf{v}^f = 0$ and $\mathbf{v}^s = 0, \mathbf{v}^f = \mathbf{E\ddot{u}}^f$ in the resulting equation. Then the argument leading to (31) yields the inequality

$$
\begin{aligned}
&\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E\ddot{u}}^s, \dot{\Theta}\right) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E\ddot{u}}^f, \dot{\Theta}\right) \qquad (38)\\
&\geq \frac{1}{2}\frac{d}{dt}\left[\left(\widehat{\mathcal{P}}\mathbf{E\ddot{u}}, \mathbf{E\ddot{u}}\right) + \widehat{\Lambda}(\mathbf{E\dot{u}}, \mathbf{E\dot{u}})\right] + C_\beta\left[\langle\mathcal{D}\mathcal{S}(\mathbf{E\ddot{u}}), \mathcal{S}(\mathbf{E\ddot{u}})\rangle + \left(\frac{\eta}{\kappa}\mathbf{E\ddot{u}}^f, \mathbf{E\ddot{u}}^f\right)\right]
\end{aligned}
$$

Use (38) in (37) and add to the resulting equation the inequalities

$$
\zeta\frac{d}{dt}\|\mathbf{Eu}\|_0^2 \leq \zeta\left(\|\mathbf{Eu}\|_0^2 + \|\mathbf{E\dot{u}}\|_0^2\right), \ \zeta = \zeta_1, \zeta_2,\ \frac{d}{dt}\|\gamma^{1/2}\mathbf{E}\theta\|_0^2 \leq \left(\|\gamma^{1/2}\mathbf{E}\theta\|_0^2 + \|\gamma^{1/2}\mathbf{E}\dot{\theta}\|_0^2\right)
$$

to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}}) + \left(\widehat{\mathcal{P}}\mathbf{E}\ddot{\mathbf{u}},\mathbf{E}\ddot{\mathbf{u}}\right) + \Lambda_{\zeta_1}(\mathbf{E}\mathbf{u},\mathbf{E}\mathbf{u}) + \widehat{\Lambda}_{\zeta_2}(\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}})\\
&+\left(\tau\, c\,\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\right) + (\gamma\mathbf{E}\theta,\mathbf{E}\theta) + (\gamma\nabla\mathbf{E}\theta,\nabla\mathbf{E}\theta)\Big]\\
&+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\mathbf{u}}^f\right) + \left(c\,\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\right) + \langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle + \left\langle\tau c v_\theta\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\right\rangle \qquad (39)\\
&\le\left(\|\mathbf{E}\mathbf{u}\|_0^2 + \|\mathbf{E}\dot{\mathbf{u}}\|_0^2 + \|\mathbf{E}\theta\|_0^2 + \|\mathbf{E}\dot{\theta}\|_0^2\right) + (\beta\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s) + \left(\beta_f\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f\right)\\
&-\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\mathbf{E}\dot{\theta}\right) - \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\theta}\right)\\
&+\left(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f)\right) + \left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f\right)\\
&+\Lambda(\mathbf{E}\mathbf{u},(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f))\\
&-\left(\beta\mathbf{E}\theta,\nabla\cdot(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s)\right) - \left(\beta_f\mathbf{E}\theta,\nabla\cdot(Q\dot{\mathbf{u}}^f - \dot{\mathbf{u}}^f)\right)\\
&+\left(\tau\, c\,\mathbf{E}\ddot{\theta},\Pi\dot{\theta}-\dot{\theta}\right) + \left(c\,\mathbf{E}\dot{\theta},\dot{\theta}-\Pi\dot{\theta}\right) + \left(\gamma\nabla\mathbf{E}\theta,\nabla(\Pi\dot{\theta}-\dot{\theta})\right) + \left((1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right)\\
&+\left(\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta}\right) + \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right) + \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta}\right)\\
&+\left\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f)\right\rangle + \left\langle\tau c v_\theta\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta}\right\rangle, \quad t\in J.
\end{aligned}
$$

Next we get estimates for the time integrals of the terms in the right-hand side of (39). First,

$$
\begin{aligned}
&\left|\int_0^t(\beta\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s)(s)ds\right| + \left|\int_0^t\left(\beta_f\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f\right)(s)ds\right| \qquad (40)\\
&\le C\left(\int_0^t\|\mathbf{E}\theta(s)\|_0^2ds + \int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left|\int_0^t\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\mathbf{E}\dot{\theta}\right)(s)ds\right| + \left|\int_0^t\left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\theta}\right)(s)ds\right|\\
&\le C\left(\int_0^t\|\mathbf{E}\dot{\theta}(s)\|_0^2ds + \int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds\right). \qquad (41)
\end{aligned}
$$

Next, using the approximating properties of Π in (17)

$$
\begin{aligned}
&\left|\int_0^t\left((1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right)(s)ds\right| + \left|\int_0^t\left(\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta}\right)(s)ds\right|\\
&\le C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds + h^4\|\dot{\theta}(s)\|_2^2ds\right) \qquad (42)\\
&\le C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds + h^4\|\dot{\theta}\|_{L^2(J,H^2(\Omega))}^2\right).
\end{aligned}
$$

Also, using (19)

$$
\left|\int_0^t\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f\right)(s)ds\right| \le C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}^f(s)\|_0^2ds + h^2\|\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}^2\right) \qquad (43)
$$

Next,

$$
\begin{aligned}
&\left|\int_0^t \Lambda(\mathbf{E}\mathbf{u},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))(s)ds\right| \qquad (44)\\
&\leq C\left(\|\mathbf{E}\mathbf{u}\|_{\mathcal{V}}\|\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s\|_1+\|\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f\|_{H(\mathrm{div};\Omega)}\right)\\
&\leq C\left(\int_0^t\|\mathbf{E}\mathbf{u}(s)\|_{\mathcal{V}}^2ds+h^2(\|\dot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)}+\|\dot{\mathbf{u}}^f\|^2_{L^2(J,[H^1(\Omega)]^2)}+\|\nabla\cdot\dot{\mathbf{u}}^f\|^2_{L^2(J,H^1(\Omega))})\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_0^t\left(\beta\mathbf{E}\theta,\nabla\cdot(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s)\right)(s)ds\right|+\left|\int_0^t\left(\beta_f\mathbf{E}\theta,\nabla\cdot(Q\dot{\mathbf{u}}^f-\dot{\mathbf{u}}^f)\right)(s)ds\right| \quad (45)\\
&\leq C\left(\int_0^t\|\mathbf{E}\theta(s)\|_0^2ds+h^2\left[\|\dot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)}+(\|\nabla\cdot\dot{\mathbf{u}}^f\|^2_{L^2(J,H^1(\Omega))}\right]\right),
\end{aligned}
$$

$$
\left|\int_0^t\left(c\ \mathbf{E}\dot{\theta},\dot{\theta}-\Pi\dot{\theta}\right)(s)ds\right|\leq C\left(\int_0^t\|\mathbf{E}\dot{\theta}(s)\|_0^2ds+h^4\|\dot{\theta}\|^2_{L^2(J,H^2(\Omega))}\right), \quad (46)
$$

and

$$
\left|\int_0^t\left(\gamma\nabla\mathbf{E}\theta,\nabla(\Pi\dot{\theta}-\dot{\theta})\right)(s)ds\right|\leq C\left(\int_0^t\|\mathbf{E}\theta\|_1^2(s)ds+h^2\|\dot{\theta}\|_{L^2(J,H^2(\Omega))}\right). (47)
$$

The terms on second time derivatives of $\mathbf{Eu}$ and $\mathbf{E}\theta$ in the right-hand side of (39) can be bounded using integration by parts in time as follows. First,

$$
\begin{aligned}
&\left|\int_0^t\left(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)(s)ds\right)\right| \qquad (48)\\
&=\left|\left(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\right)(t)-\left(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\right)(0)\right.\\
&\left.-\int_0^t\left(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\ddot{\mathbf{u}}^s-\Pi^{(2)}\ddot{\mathbf{u}}^s,\ddot{\mathbf{u}}^f-Q\ddot{\mathbf{u}}^f)\right)(s)ds\right|\\
&\leq\epsilon\|\mathbf{E}\dot{\mathbf{u}}(t)\|_0^2+C\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_0^2ds+C\left(\|\mathbf{E}\dot{\mathbf{u}}(0)\|_0^2+h^4\|\dot{\mathbf{u}}^s\|^2_{L^\infty(J,[H^2(\Omega)]^2)}\right.\\
&\left.+h^2\|\dot{\mathbf{u}}^f\|^2_{L^\infty(J,[H^1(\Omega)]^2)}+h^4\|\ddot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)}+h^2\|\ddot{\mathbf{u}}^f\|^2_{L^2(J,[H^1(\Omega)]^2)}\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left|\int_0^t\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right)(s)ds\right| \qquad (49)\\
&=\left|\tau(1-\phi)\beta_m\theta_0\left[\left(\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right)(t)-\left(\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta}\right)(0)\right]\right.\\
&\left.-\int_0^t\left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\ddot{\theta}-\ddot{\theta}\right)(s)ds\right|\leq\epsilon\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s(t)\|_0^2\\
&+C\left(\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s(0)\|_0^2+\int_0^t\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s\|_0^2(s)ds\right)+Ch^4\left(\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))}+\|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}\right).
\end{aligned}
$$

Proceeding similarly

$$\left|\int_0^t \left(\tau\phi\beta_f\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f, \Pi\dot{\theta}-\dot{\theta}\right)(s)ds\right| \le \epsilon\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(t)\|_0^2 \tag{50}$$

$$+C\left(\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(0)\|_0^2 + \int_0^t \|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(\tau)\|_0^2 d\tau + h^4\left[\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))} + \|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}\right]\right).$$

Next,

$$\left|\int_0^t \left(\tau\, c\, \mathbf{E}\ddot{\theta}, \Pi\dot{\theta}-\dot{\theta}\right)(s)ds\right| \tag{51}$$

$$= \left|\left(\tau\, c\, \mathbf{E}\dot{\theta}, \Pi\dot{\theta}-\dot{\theta}\right)(t) - \left(\tau\, c\, \mathbf{E}\dot{\theta}, \Pi\dot{\theta}-\dot{\theta}\right)(0) - \int_0^t \left(\tau\, c\, \mathbf{E}\dot{\theta}, \Pi\ddot{\theta}-\ddot{\theta}\right)(s)ds\right|$$

$$\le \epsilon\|\mathbf{E}\dot{\theta}\|_0^2(t) + C\left(\|\mathbf{E}\dot{\theta}\|_0^2(0) + \int_0^t \|\mathbf{E}\dot{\theta}\|_0^2(s)ds + h^4\left[\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))} + \|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}\right]\right).$$

Also, using the approximating properties of the projections Π, Q in (17)- (19) and (11) we get

$$\left|\int_0^t \left\langle \mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}), \mathcal{S}(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s, \dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f)\right\rangle(s)ds\right| \tag{52}$$

$$\le \epsilon\int_0^t \langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}), \mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds$$

$$+C\left(\int_0^t \left\langle S(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s, \dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f), S(\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s, \dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f)\right\rangle(s)ds\right)$$

$$\le \epsilon\int_0^t \langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}), \mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds + \sum_j\left[\int_0^t |\dot{\mathbf{u}}^s - \Pi^{(2)}\dot{\mathbf{u}}^s|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds\right.$$

$$\left.+\int_0^t |\left(\dot{\mathbf{u}}^f - Q\dot{\mathbf{u}}^f\right)\cdot\boldsymbol{\nu}|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds\right] \le \epsilon\int_0^t \langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}), \mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds$$

$$+\sum_j\left[\int_0^t h^3\|\dot{\mathbf{u}}^s\|^2_{2,\Omega_j}(s)ds + h^2\int_0^t |\dot{\mathbf{u}}^f\cdot\boldsymbol{\nu}|^2_{1,\Gamma\cap\partial\Omega_j}(s)ds\right]$$

$$\le \epsilon\int_0^t \langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}), \mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds + \left[\int_0^t h^3\|\dot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)} + h^2\|\dot{\mathbf{u}}^f\|^2_{L^2(J,[H^{3/2}(\Omega)]^2)}\right]$$

In a similar fashion,

$$\left|\int_0^t \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, \Pi\dot{\theta}-\dot{\theta}\right\rangle(s)ds\right| \le \epsilon\int_0^t \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, \mathbf{E}\dot{\theta}\right\rangle(s)ds + C\int_0^t \sum_j |\Pi\dot{\theta}-\dot{\theta}|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds$$

$$\le \epsilon\int_0^t \left\langle\tau c v_\theta\mathbf{E}\dot{\theta}, \mathbf{E}\dot{\theta}\right\rangle(s)ds + Ch^3\|\dot{\theta}\|^2_{L^2(J,H^2(\Omega))}. \tag{53}$$

Thus, integrate in time the inequality (39) and absorb the ϵ terms in (40)-(53) in the left-hand side of (39). Then apply Gronwall's lemma in the resulting equation and use that $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positive definite, and Λ, $\widehat{\Lambda}$ are $\mathcal{V}$-coercive to obtain

$$
\begin{aligned}
&\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)} + \|\mathbf{E}\ddot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)} + \|\mathbf{E}\mathbf{u}\|_{L^\infty(J,\mathcal{V})} + \|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,\mathcal{V})} \\
&+\|\mathbf{E}\theta\|_{L^\infty(J,H^1(\Omega))} + \|\mathbf{E}\dot{\theta}\|_{L^\infty(J,L^2(\Omega))} \\
&\le C\Big(\|\mathbf{E}\dot{\mathbf{u}}(0)\|_0^2 + \|\mathbf{E}\ddot{\mathbf{u}}(0)\|_0^2 + \|\mathbf{E}\mathbf{u}(0)\|_{\mathcal{V}}^2 + \|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}}^2 \\
&+\|\mathbf{E}\dot{\theta}(0)\|_0^2 + \|\mathbf{E}\theta(0)\|_1^2 \\
&+h\Big[\|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)} + \|\dot{\mathbf{u}}^f\|_{L^\infty(J,[H^1(\Omega)]^2)} + \|\dot{\mathbf{u}}^f\|_{L^2(J,[H^{3/2}(\Omega)]^2)} + \|\nabla\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))} \\
&+\|\ddot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)} + \|\ddot{\mathbf{u}}^f\|_{L^2(J,[H^1(\Omega)]^2)} + \|\dot{\theta}\|_{L^\infty(J,H^2(\Omega))} + \|\ddot{\theta}\|_{L^2(J,H^2(\Omega))}\Big]\Big)
\end{aligned}
\tag{54}
$$

The error at $t = 0$ in (54) can be estimated by defining the FE initial conditions as follows. First, take $\mathbf{U}(0), \dot{\mathbf{U}}(0) \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h$ such that

$$
\Lambda_{\zeta_1}(\mathbf{u}^0 - \mathbf{U}(0), \mathbf{v}) = \Lambda_{\zeta_1}(\mathbf{E}\mathbf{u}(0), \mathbf{v}) = 0, \quad \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h, \tag{55}
$$

$$
\Lambda_{\zeta_1}(\mathbf{u}^1 - \dot{\mathbf{U}}(0), \mathbf{v}) = \Lambda_{\zeta_1}(\mathbf{E}\dot{\mathbf{u}}(0), \mathbf{v}) = 0, \quad \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h. \tag{56}
$$

Choose $\mathbf{v} = \mathbf{E}\mathbf{u}(0) + (\Pi^{(2)}\mathbf{u}^{0,s} - \mathbf{u}^{0,s}, Q\mathbf{u}^{0,f} - \mathbf{u}^{0,f})$ in (55) and use the $\mathcal{V}$-coercivity of Λ_{ζ_1} and the approximating properties of $\Pi^{(2)}$ and Q in (17)-(20) to get

$$
\begin{aligned}
&C_4\|\mathbf{E}\mathbf{u}(0)\|_{\mathcal{V}}^2 \le \Lambda_{\zeta_1}(\mathbf{E}\mathbf{u}(0), (\Pi^{(2)}\mathbf{u}^{0,s} - \mathbf{u}^{0,s}, Q\mathbf{u}^{0,f} - \mathbf{u}^{0,f})) \\
&\le C_5 h\|\mathbf{E}\mathbf{u}(0)\|_{\mathcal{V}}\left(\|\mathbf{u}^{0,s}\|_2 + \|\mathbf{u}^{0,f}\|_1 + \|\nabla\cdot\mathbf{u}^{0,f}\|_1\right).
\end{aligned}
\tag{57}
$$

Thus,

$$
\|\mathbf{E}\mathbf{u}(0)\|_{\mathcal{V}} \le Ch\left(\|\mathbf{u}^{0,s}\|_2 + \|\mathbf{u}^{0,f}\|_1 + \|\nabla\cdot\mathbf{u}^{0,f}\|_1\right). \tag{58}
$$

Similarly, by choosing $\mathbf{v} = \mathbf{E}\dot{\mathbf{u}}(0) + (\Pi^{(2)}\dot{\mathbf{u}}^{1,s} - \mathbf{u}^{1,s}, Q\mathbf{u}^{1,f} - \mathbf{u}^{1,f})$ in (56) we obtain

$$
\|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}} \le Ch\left(\|\mathbf{u}^{1,s}\|_2 + \|\mathbf{u}^{1,f}\|_1 + \|\nabla\cdot\mathbf{u}^{1,f}\|_1\right). \tag{59}
$$

To get a bound for the term $\|\mathbf{E}\ddot{\mathbf{u}}(0)\|_0$ in(54) we assume that the initial value problem (4)-(5) with the initial conditions (6) and the boundary condition (8) satisfies the regularity inequality

$$
\|\mathbf{u}^s\|_2 + \|\mathbf{u}^f\|_1 + \|\nabla\cdot\mathbf{u}^f\|_1 + \|\theta\|_2 \le C\left(\|f\|_0 + \|q\|_0\right). \tag{60}
$$

We also assume that (60) holds for time derivatives of $\mathbf{u}$ and θ. Thus, at $t = 0$ we have

$$
\|\ddot{\mathbf{u}}^s(0)\|_2 + \|\ddot{\mathbf{u}}^f(0)\|_1 + \|\nabla\cdot\ddot{\mathbf{u}}^f(0)\|_1 + \|\theta(0)\|_2 \le C\left(\|\ddot{f}(0)\|_0 + \|\ddot{q}(0)\|_0\right). \tag{61}
$$

Hence, defining $\ddot{\mathbf{U}}(0)$ by the equation

$$
\Lambda_{\zeta_1}(\ddot{\mathbf{u}}(0) - \ddot{\mathbf{U}}(0), \mathbf{v}) = \Lambda_{\zeta_1}(\mathbf{E}\ddot{\mathbf{u}}(0), \mathbf{v}) = 0, \quad \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h, \tag{62}
$$

the choice $\mathbf{v} = \mathbf{E}\ddot{\mathbf{u}}(0) + (\Pi^{(2)}\ddot{\mathbf{u}}^s(0) - \ddot{\mathbf{u}}^s(0), Q\ddot{\mathbf{u}}^f(0) - \ddot{\mathbf{u}}^f(0))$ in (62) yields the bound

$$\|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}} \le Ch \left(\|\ddot{u}^s(0)\|_2 + \|\ddot{u}^f(0)\|_1 + \|\nabla \cdot \ddot{u}^f(0)\|_1 + \|\ddot{\theta}(0)\|_2 \right) \tag{63}$$
$$\le Ch \left((\|\ddot{f}(0)\|_0 + \|\ddot{q}(0)\|_0 \right).$$

For the temperature variables, take $\Theta(0), \dot{\Theta}(0) \in \mathcal{W}^h$ such that

$$\mathbf{E}\theta(0) = \theta^0 - \Theta(0), \quad \mathbf{E}\dot{\theta}(0) = \theta^1 - \dot{\Theta}(0)$$

satisfy the relations

$$(\gamma \mathbf{E}\theta(0), w) + (\gamma \nabla \mathbf{E}\theta(0), \nabla w) = 0, w \in \mathcal{W}^h, \tag{64}$$
$$\left(\gamma \mathbf{E}\dot{\theta}(0), w\right) + \left(\gamma \nabla \mathbf{E}\dot{\theta}(0), \nabla w\right) = 0, w \in \mathcal{W}^h. \tag{65}$$

Since

$$C_6 \|\mathbf{E}\theta(0)\|_1^2 \le (\gamma \mathbf{E}\theta(0), \mathbf{E}\theta(0)) + (\gamma \nabla \mathbf{E}\theta(0), \nabla \mathbf{E}\theta(0)),$$

choose $w = \mathbf{E}\theta(0) + \Pi\theta^0 - \theta^0$ in (64) to obtain

$$C_5 \|\mathbf{E}\theta(0)\|_1^2 \le \left(\gamma \mathbf{E}\theta(0), \theta^0 - \Pi\theta^0\right) + \left(\gamma \nabla \mathbf{E}\theta(0), \nabla(\theta^0 - \Pi\theta^0)\right) \le C_7 h \|\mathbf{E}\theta(0)\|_1 \|\theta^0\|_2,$$

so that

$$\|\mathbf{E}\theta(0)\|_1 \le Ch\|\theta^0\|_2. \tag{66}$$

Similarly, the choice $w = \mathbf{E}\dot{\theta}(0) + \Pi\theta^1 - \theta^1$ in (65) yields the inequality

$$\|\mathbf{E}\dot{\theta}(0)\|_1 \le Ch\|\theta^1\|_2. \tag{67}$$

The bounds(58)-(67) in (54) imply the validity of the following theorem.

Theorem 2 *Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in* (3) *are positive definite and semidefinite, respectively and that the matrix $\mathcal{E}$ in* (15) *is positive definite. Then, the solution $(\mathbf{U}, \Theta) \in \mathcal{Z}^h$ of the FE procedure* (21) *satisfies the a priori error estimate*

$$\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)} + \|\mathbf{E}\ddot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)} + \|\mathbf{E}\mathbf{u}\|_{L^\infty(J,\mathcal{V})} + \|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,\mathcal{V})} \tag{68}$$
$$+ \|\mathbf{E}\theta\|_{L^\infty(J,H^1(\Omega))} + \|\mathbf{E}\dot{\theta}\|_{L^\infty(J,L^2(\Omega))}$$
$$\le Ch \Big(\|\mathbf{u}^{0,s}\|_2 + \|\mathbf{u}^{0,f}\|_1 + \|\nabla \cdot \mathbf{u}^{0,f}\|_1 + \|\mathbf{u}^{1,s}\|_2 + \|\mathbf{u}^{1,f}\|_1 + \|\nabla \cdot \mathbf{u}^{1,f}\|_1$$
$$+ \|\theta^0\|_2 + |\theta^1\|_2 + \|\ddot{f}(0)\|_0 + \|\ddot{q}(0)\|_0$$
$$+ \|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)} + \|\dot{\mathbf{u}}^f\|_{L^\infty(J,[H^1(\Omega)]^2)} + \|\dot{\mathbf{u}}^f\|_{L^2(J,[H^{3/2}(\Omega)]^2)} + \|\nabla \dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}$$
$$+ \|\ddot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)} + \|\ddot{\mathbf{u}}^f\|_{L^2(J,[H^1(\Omega)]^2)} + \|\dot{\theta}\|_{L^\infty(J,H^2(\Omega))} + \|\ddot{\theta}\|_{L^2(J,H^2(\Omega))} \Big).$$

5 Time stepping procedures

Let

$$\partial^2\mathbf{U}^n = \frac{\mathbf{U}^{n+1}-2\mathbf{U}^n+\mathbf{U}^{n-1}}{\Delta t^2}, \quad \partial\mathbf{U}^n = \frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}, \quad D_t\mathbf{U}^n = \frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}$$
$$\partial^{2,*}\mathbf{U}^n = \frac{\mathbf{U}^{n+2}-2\mathbf{U}^n+\mathbf{U}^{n-2}}{(2\Delta t)^2} = \frac{1}{2\Delta t}\left[\partial\mathbf{U}^{n+1}-\partial\mathbf{U}^{n-1}\right] = \frac{1}{2}\left[D_t\partial\mathbf{U}^n + D_t\partial\mathbf{U}^{n-1}\right]$$
$$D_t^2\mathbf{U}^n = \frac{\mathbf{U}^{n+2}-\mathbf{U}^{n-2}}{4\Delta t} == \frac{1}{2}\left[\partial\mathbf{U}^{n+1}+\partial\mathbf{U}^{n-1}\right].$$

An implicit time discretization of (13) is: Find $(\mathbf{U}^n, \Theta^n) \in \mathcal{Z}^h$ such that

$$\begin{aligned}
&\Big(\mathcal{P}\partial^2\mathbf{U}^n, \mathbf{v}\Big) + \Big(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n}, \mathbf{v}^f\Big) + \Lambda(\frac{1}{2}(\mathbf{U}^{n+1}+\mathbf{U}^{n-1}), \mathbf{v}) - (\beta\Theta^n, \nabla\cdot\mathbf{v}^s) - \Big(\beta_f\Theta^n, \nabla\cdot\mathbf{v}^f\Big)\\
&+\Big(\tau\, c\,\partial^2\Theta^n, w\Big) + (c\,\partial\Theta^n, w) + \Big(\gamma\nabla(\frac{1}{2}(\theta^{n+1}+\theta^{n-1}), \nabla w\Big) + ((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n}, w)\\
&+\Big(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n}, w\Big) + \Big(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, w\Big) + \Big(\tau\phi\beta_f\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{f,n}, w\Big)\\
&+\big\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n), \mathcal{S}(\mathbf{v})\big\rangle + \big\langle\tau c v_\theta\partial\Theta^n, w\big\rangle \qquad (69)\\
&= (\mathbf{f}^n, \mathbf{v}) - (q^n, w)\,, \quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{Z}^h(\Omega), \; n = 1, 2, \cdots, M,
\end{aligned}$$

where $M\Delta t = T$ and T is the final time to compute the solution.
Also, an explicit time discretization of (13) can be stated as follows: Find $(\mathbf{U}^n, \Theta^n) \in \mathcal{Z}^h$ such that

$$\begin{aligned}
&\Big(\mathcal{P}\partial^2\mathbf{U}^n, \mathbf{v}\Big) + \Big(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n}, \mathbf{v}^f\Big) + \Lambda(\mathbf{U}^n, \mathbf{v}) - (\beta\Theta^n, \nabla\cdot\mathbf{v}^s) - \Big(\beta_f\Theta^n, \nabla\cdot\mathbf{v}^f\Big) \qquad (70)\\
&+\Big(\tau\, c\,\partial^2\Theta^n, w\Big) + (c\,\partial\Theta^n, w) + (\gamma\nabla\Theta^n, \nabla w) + ((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n}, w)\\
&+\Big(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n}, w\Big) + \Big(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, w\Big) + \Big(\tau\phi\beta_f\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{f,n}, w\Big)\\
&+\big\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n), \mathcal{S}(\mathbf{v})\big\rangle + \big\langle\tau c v_\theta\partial\Theta^n, w\big\rangle\\
&= (\mathbf{f}^n, \mathbf{v}) - (q^n, w)\,, \quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{Z}^h(\Omega), \; n = 1, 2, \cdots, M.
\end{aligned}$$

5.1 Conditional stability of the discrete FE procedurel (70)

Choose $\mathbf{v} = \partial\mathbf{U}^n = (\partial\mathbf{U}^{s,n}, \partial\mathbf{U}^{f,n})$, $w = \partial\Theta^n$ in (70) to get

$$\begin{aligned}
&\Big(\mathcal{P}\partial^2\mathbf{U}^n, \partial\mathbf{U}^n\Big) + \Big(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n}, \partial\mathbf{U}^{f,n}\Big) + \Lambda(\mathbf{U}^n, \partial\mathbf{U}^n) - (\beta\Theta^n, \nabla\cdot\partial\mathbf{U}^{s,n}) - \Big(\beta_f\Theta^n, \nabla\cdot\partial\mathbf{U}^{f,n}\Big)\\
&+\Big(\tau\, c\,\partial^2\Theta^n, \partial\Theta^n\Big) + (c\,\partial\Theta^n, \partial\Theta^n) + (\gamma\nabla\Theta^n, \nabla\partial\Theta^n) + ((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n}, \partial\Theta^n)\\
&+\Big(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n}, \partial\Theta^n\Big) + \Big(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\Big) + \Big(\tau\phi\beta_f\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{f,n}, \partial\Theta^n\Big)\\
&+\big\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n), \mathcal{S}(\partial\mathbf{U}^n)\big\rangle + \big\langle\tau c v_\theta\partial\Theta^n, \partial\Theta^n\big\rangle \qquad (71)\\
&= (\mathbf{f}^n, \partial\mathbf{U}^n) - (q^n, \partial\Theta^n)\,, \quad n = 1, 2, \cdots, M.
\end{aligned}$$

Next, use the identities

$$
\begin{aligned}
2\Delta t \Lambda(\mathbf{U}^n, \partial\mathbf{U}^n) = \frac{1}{2}\Big[&\Lambda(\mathbf{U}^{n+1}, \mathbf{U}^{n+1}) - \Lambda(\mathbf{U}^{n-1}, \mathbf{U}^{n-1}) + \Lambda(\mathbf{U}^n - \mathbf{U}^{n-1}, \mathbf{U}^n - \mathbf{U}^{n-1}) \\
&-\Lambda(\mathbf{U}^{n+1} - \mathbf{U}^n, \mathbf{U}^{n+1} - \mathbf{U}^n)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\Delta t\,(\gamma\nabla\Theta^n, \nabla\partial\Theta^n) = \frac{1}{2}\Big[\Big(\gamma\nabla\Theta^{n+1}, \nabla\Theta^{n+1}\Big) - \Big(\gamma\nabla\Theta^{n-1}, \nabla\Theta^{n-1}\Big) \\
&+ \Big(\gamma\nabla(\Theta^n - \Theta^{n-1}), \nabla(\Theta^n - \Theta^{n-1}\Big) - \Big(\gamma\nabla(\Theta^{n+1} - \Theta^n), \nabla(\Theta^{n+1} - \Theta^n\Big)\Big],
\end{aligned}
$$

and add to (71) the inequalities

$$
\begin{aligned}
&\frac{\zeta_1}{4\Delta t}\Big[\|\mathbf{U}^{n+1}\|_0^2 - \|\mathbf{U}^{n-1}\|_0^2\Big] \le \frac{\zeta_1}{4}\Big(\|\mathbf{U}^{n+1}\|_0^2 + \|\mathbf{U}^{n-1}\|_0^2 + \|D_t\mathbf{U}^n\|_0^2 + \|D_t\mathbf{U}^{n-1}\|_0^2\Big), \\
&\frac{1}{4\Delta t}\Big(\gamma\Theta^{n+1}, \Theta^{n+1}\Big) - \Big(\gamma\Theta^{n-1}, \Theta^{n-1}\Big) \le C\Big(\|\Theta^{n+1}\|_0^2 + \|\Theta^{n-1}\|_0^2 + \|D_t\Theta^n\|_0^2 + \|D_t\Theta^{n-1}\|_0^2\Big)
\end{aligned}
$$

to get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big[(\mathcal{P}D_t\mathbf{U}^n, D_t\mathbf{U}^n) - \Big(\mathcal{P}D_t\mathbf{U}^{n-1}, D_t\mathbf{U}^{n-1}\Big)\Big] \qquad (72) \\
&+\frac{1}{4\Delta t}\Big[\Lambda_{\zeta_1}(\mathbf{U}^{n+1}, \mathbf{U}^{n+1}) - \Lambda_{\zeta_1}(\mathbf{U}^{n-1}, \mathbf{U}^{n-1})\Big] \\
&+\frac{1}{4\Delta t}\Big[\Lambda(\mathbf{U}^n - \mathbf{U}^{n-1}, \mathbf{U}^n - \mathbf{U}^{n-1}) - \Lambda(\mathbf{U}^{n+1} - \mathbf{U}^n, \mathbf{U}^{n+1} - \mathbf{U}^n)\Big] \\
&+\frac{1}{2\Delta t}\Big[(\tau c D_t\Theta^n, D_t\Theta^n) - \Big(\tau c D_t\Theta^{n-1}, D_t\Theta^{n-1}\Big)\Big] \\
&+\frac{1}{4\Delta t}\Big(\|\gamma^{1/2}\Theta^{n+1}\|_1^2 - \|\gamma^{1/2}\Theta^{n-1}\|_1^2\Big) \\
&+\frac{1}{4\Delta t}\Big[\Big(\gamma\nabla(\Theta^n - \Theta^{n-1}), \nabla(\Theta^n - \Theta^{n-1}\Big) - \Big(\gamma\nabla(\Theta^{n+1} - \Theta^n), \nabla(\Theta^{n+1} - \Theta^n\Big)\Big] \\
&+\Big(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n}, \partial\mathbf{U}^{f,n}\Big) + (c\,\partial\Theta^n, \partial\Theta^n) + \big\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n), \mathcal{S}(\partial\mathbf{U}^n)\big\rangle + \big\langle\tau c v_\theta\partial\Theta^n, \partial\Theta^n\big\rangle \\
&+\Big(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\Big) + \Big(\tau\phi\beta_f\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{f,n}, \partial\Theta^n\Big) \\
&\le C\Big(\|\mathbf{U}^{n+1}\|_0^2 + \|\mathbf{U}^{n-1}\|_0^2 + \|D_t\mathbf{U}^n\|_0^2 + \|D_t\mathbf{U}^{n-1}\|_0^2 \\
&+\|\Theta^{n+1}\|_0^2 + \|\Theta^{n-1}\|_0^2 + \|D_t\Theta^n\|_0^2 + \|D_t\Theta^{n-1}\|_0^2\Big) \\
&+(\mathbf{f}^n, \partial\mathbf{U}^n) - (q^n, \partial\Theta^n) + (\beta\Theta^n, \nabla\cdot\partial\mathbf{U}^{s,n}) + \Big(\beta_f\Theta^n, \nabla\cdot\partial\mathbf{U}^{f,n}\Big) \\
&-((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n}, \partial\Theta^n) - \Big(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n}, \partial\Theta^n\Big), \qquad n = 1, \cdots M.
\end{aligned}
$$

Next, we obtain estimates for the last two terms in the left-hand side of (72) with an argument similar to that in the continuous-time case. A discrete-time form of (24) is

$$
\begin{aligned}
&\Big(\mathcal{P}\partial^2(\partial\mathbf{U}^n), \mathbf{v}\Big) + \Big(\frac{\eta}{\kappa}\partial^{2,*}\mathbf{U}^{f,n}, \mathbf{v}^f\Big) + \Lambda(D_t^2\mathbf{U}^n, \mathbf{v}) - (\beta\partial\Theta^n, \nabla\cdot\mathbf{v}^s) - \Big(\beta_f\partial\Theta^n, \nabla\mathbf{v}^f\Big) \\
&\quad + \Big\langle\mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^n), \mathcal{S}(\mathbf{v})\Big\rangle = (\partial\mathbf{f}^{s,n}, \mathbf{v}^s) + \Big(\partial\mathbf{f}^{f,n}, \mathbf{v}^f\Big), \quad n = 1, 2, \cdots, M. \qquad (73)
\end{aligned}
$$

Choose $\mathbf{v}^s = \partial^{2,*}\mathbf{U}^{s,n}, \mathbf{v}^f = 0$ and $\mathbf{v}^s = 0, \mathbf{v}^f = \partial^{2,*}\mathbf{U}^{f,n}$ in (73) to get the equations

$$\begin{aligned}
&\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (\partial^{2,*}\mathbf{U}^{s,n}, 0)\right) + \Lambda(D_t^2\mathbf{U}^n, (\partial^{2,*}\mathbf{U}^{s,n}, 0)) - \left(\beta\partial\Theta^n, \nabla\partial^{2,*}\mathbf{U}^{s,n}\right) \\
&\quad + \left\langle \mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0), \mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0)\right\rangle = \left(\mathbf{f}^{s,n}, \partial^{2,*}\mathbf{U}^{s,n}\right), \qquad (74) \\
&\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (0, \partial^{2,*}\mathbf{U}^{f,n})\right) + \left(\frac{\eta}{\kappa}\partial^{2,*}\mathbf{U}^{f,n}, \partial^{2,*}\mathbf{U}^{f,n}\right) + \Lambda(D_t^2\mathbf{U}^n, (0, \partial^{2,*}\mathbf{U}^{f,n})) \\
&\quad - \left(\beta_f\partial\Theta^n, \nabla\partial^{2,*}\mathbf{U}^{f,n}\right) + \left\langle \mathcal{D}\mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n}), \mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n})\right\rangle \qquad (75) \\
&= \left(\mathbf{f}^{f,n}, \partial^{2,*}\mathbf{U}^{f,n}\right), n = 1, 2, \cdots, M.
\end{aligned}$$

Then using (74) and (75), we obtain the estimates

$$\begin{aligned}
&\qquad \left(\tau(1-\phi)\beta_m\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\right) \qquad (76) \\
&\geq\; C_\beta\left(\beta\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\right) = C_\beta\Big[\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (\partial^{2,*}\mathbf{U}^{s,n}, 0)\right) \\
&\quad +\Lambda(D_t^2\mathbf{U}^n, (\partial^{2,*}\mathbf{U}^{s,n}, 0)) + \left\langle \mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0), \mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0)\right\rangle - \left(\mathbf{f}^{s,n}, \partial^{2,*}\mathbf{U}^{s,n}\right)\Big], \\
&\qquad \left(\tau\phi\beta_f\theta_0\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\right) \qquad (77) \\
&\geq\; C_\beta\left(\beta\nabla\cdot\partial^{2,*}\mathbf{U}^{s,n}, \partial\Theta^n\right) = C_\beta\Big[\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (0, \partial^{2,*}\mathbf{U}^{f,n})\right) \\
&\quad + \left(\frac{\eta}{\kappa}\partial^{2,*}\mathbf{U}^{f,n}, \partial^{2,*}\mathbf{U}^{f,n}\right) + \Lambda(D_t^2\mathbf{U}^n, (0, \partial^{2,*}\mathbf{U}^{f,n})) \\
&\quad + \left\langle \mathcal{D}\mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n}), \mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n})\right\rangle - \left(\mathbf{f}^{f,n}, \partial^{2,*}\mathbf{U}^{f,n}\right)\Big].
\end{aligned}$$

Next, a calculation shows that

$$\begin{aligned}
&C_\beta\left\langle \mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0), \mathcal{S}(\partial^{2,*}\mathbf{U}^{s,n}, 0)\right\rangle + C_\beta\left\langle \mathcal{D}\mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n}), \mathcal{S}(0, \partial^{2,*}\mathbf{U}^{f,n})\right\rangle \\
&\qquad = C_\beta\left\langle \mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^n), \mathcal{S}(\partial^{2,*}\mathbf{U}^n)\right\rangle, \qquad (78)
\end{aligned}$$

$$\begin{aligned}
&C_\beta\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (\partial^{2,*}\mathbf{U}^{s,n}, 0)\right) + C_\beta\left(\mathcal{P}\partial^2(\partial\mathbf{U}^n), (0, \partial^{2,*}\mathbf{U}^{f,n})\right) \qquad (79) \\
&= \left(\widehat{\mathcal{P}}(\partial^2(\partial\mathbf{U}^n), \partial^{2,*}\mathbf{U}^n\right) = \frac{1}{2\Delta t}\left[\left(\widehat{\mathcal{P}}D_t(\partial\mathbf{U}^n), D_t(\partial\mathbf{U}^n)\right) - \left(\widehat{\mathcal{P}}D_t(\partial\mathbf{U}^{n-1}), D_t(\partial\mathbf{U}^{n-1})\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
&C_\beta\Lambda(D_t^2\mathbf{U}^n, (\partial^{2,*}\mathbf{U}^{s,n}, 0)) + C_\beta\Lambda(D_t^2\mathbf{U}^n, (0, \partial^{2,*}\mathbf{U}^{f,n})) \qquad (80) \\
&= \widehat{\Lambda}(D_t^2\mathbf{U}^n, \partial^{2,*}\mathbf{U}^n) = \frac{1}{4\Delta t}\left[\widehat{\Lambda}(\partial\mathbf{U}^{n+1}, \partial\mathbf{U}^{n+1}) - \widehat{\Lambda}(\partial\mathbf{U}^{n-1}, \partial\mathbf{U}^{n-1})\right],
\end{aligned}$$

Then, use (76)-(80) in the last two terms of the left-hand side of (72), add the inequality

$$\begin{aligned}
\frac{\zeta_2}{4\Delta t}\left[\|\partial\mathbf{U}^{n+1}\|_0^2 - \|\partial\mathbf{U}^{n-1}\|_0^2\right] &\leq C\left(\|\partial^{2,*}\mathbf{U}^n\|_0^2 + \|\partial\mathbf{U}^{n+1}\|_0^2 + \|\partial\mathbf{U}^{n-1}\|_0^2\right) \\
&\leq C\left(\|D_t(\partial\mathbf{U}^n)\|_0^2 + \|D_t(\partial\mathbf{U}^{n-1})\|_0^2 + \|\partial\mathbf{U}^{n+1}\|_0^2 + \|\partial\mathbf{U}^{n-1}\|_0^2\right),
\end{aligned}$$

multiply the resulting equation by Δt and sum form $n=1$ to $n=N$, $N \le M$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\left[\left(\mathcal{P}D_t\mathbf{U}^N, D_t\mathbf{U}^N\right) - \left(\mathcal{P}D_t\mathbf{U}^0, D_t\mathbf{U}^0\right)\right] \\
&+\frac{1}{2}\left[\left(\widehat{\mathcal{P}}D_t\partial\mathbf{U}^N, D_t\partial\mathbf{U}^N\right) - \left(\widehat{\mathcal{P}}D_t\partial\mathbf{U}^0, D_t\partial\mathbf{U}^0\right)\right] \\
&+\frac{1}{4}\left[\Lambda_{\zeta_1}(\mathbf{U}^{N+1},\mathbf{U}^{N+1}) + \Lambda_{\zeta_1}(\mathbf{U}^{N},\mathbf{U}^{N}) - \Lambda_{\zeta_1}(\mathbf{U}^{1},\mathbf{U}^{1}) - \Lambda_{\zeta_1}(\mathbf{U}^{0},\mathbf{U}^{0})\right] \\
&+\frac{1}{4}\left[\widehat{\Lambda}_{\zeta_2}(\partial\mathbf{U}^{N+1},\partial\mathbf{U}^{N+1}) + \widehat{\Lambda}_{\zeta_2}(\partial\mathbf{U}^{N},\partial\mathbf{U}^{N}) - \widehat{\Lambda}_{\zeta_2}(\partial\mathbf{U}^{1},\partial\mathbf{U}^{1}) - \widehat{\Lambda}_{\zeta_2}(\partial\mathbf{U}^{0},\partial\mathbf{U}^{0})\right] \\
&+\frac{1}{4}\left[\Lambda(\mathbf{U}^{N+1},\mathbf{U}^{N+1}) + \Lambda(\mathbf{U}^{N},\mathbf{U}^{N}) - \Lambda(\mathbf{U}^{1},\mathbf{U}^{1}) - \Lambda(\mathbf{U}^{0},\mathbf{U}^{0})\right] \\
&-\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^N, D_t\mathbf{U}^N) + \frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^0, D_t\mathbf{U}^0) + \frac{1}{2}\left[\left(\tau c D_t\Theta^N, D_t\Theta^N\right) - \left(\tau c D_t\Theta^0, D_t\Theta^0\right)\right] \\
&+\frac{1}{4}\left(\|\gamma^{1/2}\Theta^{N+1}\|_1^2 + \|\gamma^{1/2}\Theta^{N}\|_1^2 - \|\gamma^{1/2}\Theta^{0}\|_1^2 - \|\gamma^{1/2}\Theta^{1}\|_1^2\right) \\
&-\frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^N\|_1^2 + \frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^0\|_1^2 \\
&+\sum_{n=1}^{n=N}\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n}, \mathbf{U}^{f,n}\right)\Delta t + \sum_{n=1}^{n=N}\left(c\ \partial\Theta^n, \partial\Theta^n\right)\Delta t + C_\beta\sum_{n=1}^{n=N}\left\langle \mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^n), \mathcal{S}(\partial^{2,*}\mathbf{U}^n)\right\rangle \Delta t \\
&+\sum_{n=1}^{n=N} C_\beta\left(\frac{\eta}{\kappa}\partial^{2,*}\mathbf{U}^{f,n}, \partial^{2,*}\mathbf{U}^{f,n}\right) + \sum_{n=1}^{n=N}\left\langle \tau c v_\theta \partial\Theta^n, \partial\Theta^n\right\rangle\ \Delta t \\
&\le C\sum_{n=1}^{n=N}\Big(\|\mathbf{f}^n\|_0^2 + \|q^n\|_0^2 + \|\mathbf{U}^{n+1}\|_0^2 + \|\mathbf{U}^{n-1}\|_0^2 + \|D_t\mathbf{U}^n\|_0^2 + \|D_t\mathbf{U}^{n-1}\|_0^2 + \|D_t(\partial\mathbf{U}^n)\|_0^2 \\
&\qquad\qquad + \|\Theta^{n+1}\|_0^2 + \|\Theta^{n-1}\|_0^2 + \|D_t\Theta^n\|_0^2 + \|D_t\Theta^{n-1}\|_0^2 \\
&\qquad\qquad + \|D_t(\partial\mathbf{U}^n)\|_0^2 + \|D_t(\partial\mathbf{U}^{n-1})\|_0^2 + \|\partial\mathbf{U}^{n+1}\|_0^2 + \|\partial\mathbf{U}^{n-1}\|_0^2\Big)\Delta t \\
&+\sum_{n=1}^{n=N}\left(\beta\Theta^n, \nabla\cdot\partial\mathbf{U}^{s,n}\right)\Delta t + \sum_{n=1}^{n=N}\left(\beta_f\Theta^n, \nabla\cdot\partial\mathbf{U}^{f,n}\right)\Delta t \\
&-\sum_{n=1}^{n=N}\left((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n}, \partial\Theta^n\right)\Delta t - \sum_{n=1}^{n=N}\left(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n}, \partial\Theta^n\right)\Delta t.
\end{aligned}
\tag{81}
$$

Let us get bounds of the last four terms in the right-hand side of (81). First,

$$
\begin{aligned}
&\left|\sum_{n=1}^{n=N}\left(\beta\Theta^n, \nabla\cdot\partial\mathbf{U}^{s,n}\right)\Delta t\right| + \left|\sum_{n=1}^{n=N}\left(\beta_f\Theta^n, \nabla\cdot\partial\mathbf{U}^{f,n}\right)\Delta t\right| \\
&\le C\sum_{n=1}^{n=N}\left(\|\Theta^n\|_0^2 + \|\nabla\cdot\partial\mathbf{U}^{s,n}\|_0^2 + \|\nabla\cdot\partial\mathbf{U}^{f,n}\|_0^2\right)\Delta t \le C\sum_{n=1}^{n=N}\left(\|\Theta^n\|_0^2 + \|\partial\mathbf{U}^n\|_{\mathcal{V}}^2\right)\Delta t.
\end{aligned}
\tag{82}
$$

Also,

$$
\begin{aligned}
&\left|\sum_{n=1}^{n=N}\left((1-\phi)\beta_m\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n\right)\Delta t\right| + \left|\sum_{n=1}^{n=N}\left(\phi\beta_f\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n\right)\Delta t\right| \qquad (83)\\
&\le C\sum_{n=1}^{n=N}\left(\left(\|\nabla\cdot\partial\mathbf{U}^{s,n}\|_0+\|\nabla\cdot\partial\mathbf{U}^{f,n}\|_0\right)\left(\|D_t\Theta^n\|_0+\|D_t\Theta^{n-1}\|_0\right)\right)\Delta t\\
&\le C\sum_{n=1}^{n=N}\left(\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2+\|\partial\mathbf{U}^n\|_{\mathcal{V}}^2\right)\Delta t.
\end{aligned}
$$

Using the bounds (82)-(83) in (81) yields the inequality

$$
\begin{aligned}
&\left(\mathcal{P}D_t\mathbf{U}^N,D_t\mathbf{U}^N\right)+\left(\widehat{\mathcal{P}}D_t\partial\mathbf{U}^N,D_t\partial\mathbf{U}^N\right) \qquad (84)\\
&+\Lambda_\zeta(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda_\zeta(\mathbf{U}^N,\mathbf{U}^N)+\widehat{\Lambda}_{\zeta_1}(\partial\mathbf{U}^{N+1},\partial\mathbf{U}^{N+1})+\widehat{\Lambda}_{\zeta_1}(\partial\mathbf{U}^N,\partial\mathbf{U}^N)\\
&+\Lambda(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda(\mathbf{U}^N,\mathbf{U}^N)-\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^N,D_t\mathbf{U}^N)\\
&+\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^0,D_t\mathbf{U}^0)+\left(\tau c D_t\Theta^N,D_t\Theta^N\right)+\|\gamma^{1/2}\Theta^{N+1}\|_1^2+\|\gamma^{1/2}\Theta^N\|_1^2\\
&-\frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^N\|_1^2+\frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^0\|_1^2+\sum_{n=1}^{n=N}\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\partial\mathbf{U}^{f,n}\right)\Delta t\\
&+\sum_{n=1}^{n=N}\left(c\,\partial\Theta^n,\partial\Theta^n\right)\Delta t+C_\beta\sum_{n=1}^{n=N}\left\langle\mathcal{D}\mathcal{S}(\partial^{2,*}\mathbf{U}^n),\mathcal{S}(\partial^{2,*}\mathbf{U}^n)\right\rangle\Delta t\\
&+\sum_{n=1}^{n=N}C_\beta\left(\frac{\eta}{\kappa}\partial^{2,*}\mathbf{U}^{f,n},\partial^{2,*}\mathbf{U}^{f,n}\right)+\sum_{n=1}^{n=N}\left\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\right\rangle\ \Delta t\\
&\le C\Bigg(\|\mathbf{U}^0\|_{\mathcal{V}}^2+\|\mathbf{U}^1\|_{\mathcal{V}}^2+\|D_t\mathbf{U}^0\|_0^2+\|\partial\mathbf{U}^0\|_{\mathcal{V}}^2+\|\partial\mathbf{U}^1\|_{\mathcal{V}}^2\\
&\qquad+\|\Theta^0\|_1^2+\|\Theta^1\|_1^2+\|D_t\Theta^0\|_0^2+\sum_{n=1}^{n=N}\left(\|\mathbf{f}^n\|_0^2+\|q^n\|_0^2\right)\Delta t\Bigg)\\
&+C\sum_{n=1}^{n=N}\Big(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2\\
&+\|\Theta^{n+1}\|_0^2+\|\Theta^n\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2+\|\partial\mathbf{U}^n\|_{\mathcal{V}}^2\\
&+\|D_t(\partial\mathbf{U}^n)\|_0^2+\|D_t(\partial\mathbf{U}^{n-1})\|_0^2+\|\partial\mathbf{U}^{n+1}\|_0^2+\|\partial\mathbf{U}^{n-1}\|_0^2\Big)\Delta t.
\end{aligned}
$$

Since $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positive definite and Λ_ζ and $\widehat{\Lambda}_{\zeta_1}$ are $\mathcal{V}$-coercive, use that the last five terms in the left-hand side of (84) are nonnegative and apply Gronwall's lemma in (84) to conclude that

$$\|\mathcal{P}^{1/2} D_t \mathbf{U}^N\|_0^2 + \|\widehat{\mathcal{P}}^{1/2} D_t \partial \mathbf{U}^N\|_0^2 + \|\mathbf{U}^{N+1}\|_{\mathcal{V}}^2 + \|\mathbf{U}^N\|_{\mathcal{V}}^2 + \|\partial \mathbf{U}^{N+1}\|_{\mathcal{V}}^2 + \|\partial \mathbf{U}^N\|_{\mathcal{V}}^2$$

$$+\Lambda(\mathbf{U}^{N+1}, \mathbf{U}^{N+1}) + \Lambda(\mathbf{U}^N, \mathbf{U}^N) - \frac{(\Delta t)^2}{4} \Lambda(D_t \mathbf{U}^N, D_t \mathbf{U}^N) \tag{85}$$

$$+ \left(\tau c D_t \Theta^N, D_t \Theta^N\right) + \|\gamma^{1/2} \Theta^{N+1}\|_1^2 + \|\gamma^{1/2} \Theta^N\|_1^2 - \frac{(\Delta t)^2}{4} \|\gamma^{1/2} D_t \Theta^N\|_1^2$$

$$\leq C \Big(\|\mathbf{U}^0\|_{\mathcal{V}}^2 + \|\mathbf{U}^1\|_{\mathcal{V}}^2 + \|D_t \mathbf{U}^0\|_0^2 + \|\partial \mathbf{U}^0\|_{\mathcal{V}}^2 + \|\partial \mathbf{U}^1\|_{\mathcal{V}}^2$$
$$+\|\Theta^0\|_1^2 + \|\Theta^1\|_1^2 + \|D_t \Theta^0\|_0^2 + \sum_{n=1}^{n=N} \left(\|\mathbf{f}^n\|_0^2 + \|q^n\|_0^2 \right) \Delta t \Big) \qquad 1 \leq N \leq M.$$

Next, note that there exist constants C_8, C_9 independent of h such that the following inverse hypothesis hold:

$$\Lambda(\mathbf{U}^N, \mathbf{U}^N) \leq \lambda^*(\mathcal{E}) \|\widetilde{\epsilon}(D_t \mathbf{U}^n)\|_0^2 \leq C_8^2 h^{-2} \|(D_t \mathbf{U}^n)\|_0^2, \tag{86}$$

$$\|\gamma^{1/2} D_t \Theta^N\|_1^2 \leq \gamma^* C_9^2 h^{-2} \|(D_t \Theta^n)\|_0^2.$$

In (86) the constants C_8 and C_9 have a factor that measures the quasiuniformity of $\mathcal{T}^h$, and $\lambda^*(\mathcal{E})$ and γ^* denote the maximum eigenvalue of $\mathcal{E}$ and the maximum value of γ, respectively. Let $\lambda_*(\mathcal{P})$ and $(\tau c)_*$ be the minimum eigenvalue of $\mathcal{P}$ and the minimum value of (τc). Hence

$$\begin{aligned}
&\|\mathcal{P}^{1/2} D_t \mathbf{U}^N\|_0^2 - \frac{(\Delta t)^2}{4} \Lambda(D_t \mathbf{U}^N, D_t \mathbf{U}^N) \\
&\geq \left(\lambda_*(\mathcal{P}) - C_8^2 h^{-2} \frac{(\Delta t)^2}{4} \lambda^*(\mathcal{E}) \right) \|D_t \mathbf{U}^N\|_0^2 \geq \frac{1}{2} \lambda_*(\mathcal{P}) \|D_t \mathbf{U}^N\|_0^2, \\
&\|(\tau c)^{1/2} D_t \Theta^N\|_0^2 - \frac{(\Delta t)^2}{4} \|\gamma^{1/2} \Theta^N\|_1^2 \\
&\geq \left((\tau c)_* - \frac{(\Delta t)^2}{4} C_9^2 \gamma^* h^{-2} \right) \|D_t \Theta^n\|_0^2 \geq \frac{1}{2} (\tau c)_* \|D_t \Theta^n\|_0^2,
\end{aligned} \tag{87}$$

provided that Δt and h satisfy the stability constrain

$$\Delta t \leq \min \left(h \, \frac{\sqrt{2}}{C_8} \left(\frac{\lambda_*(\mathcal{P})}{\lambda^*(\mathcal{E})} \right)^{1/2}, h \, \frac{\sqrt{2}}{C_9} \left(\frac{(\tau c)_*}{\gamma^*} \right)^{1/2} \right). \tag{88}$$

Thus we conclude the validity of the following Theorem:

Theorem 3 *Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in* (3) *are positive definite and semidefinite, respectively, and that the matrix $\mathcal{E}$ in* (15) *is positive definite. Also, assume that*

Δt and h satisfy the stability constrain (88)*. Then, there exists a unique solution* $(\mathbf{U}^n, \Theta^n) \in \mathcal{Z}^h$ *of the discrete time explicit FE procedure* (70) *which satisfies the estimate*

$$\begin{aligned}&\max_{1\le N\le M}\Big(\|D_t\mathbf{U}^N\|_0^2 + \|D_t\partial\mathbf{U}^N\|_0^2 + \|\mathbf{U}^N\|_{\mathcal{V}}^2 \\ &\qquad\qquad + \|\partial\mathbf{U}^N\|_{\mathcal{V}}^2 + \|D_t\Theta^N\|_0^2 + \|\Theta^N\|_1^2\Big) \\ &\le C\Big(\|\mathbf{U}^0\|_{\mathcal{V}}^2 + \|\mathbf{U}^1\|_{\mathcal{V}}^2 + \|D_t\mathbf{U}^0\|_0^2 + \|\partial\mathbf{U}^0\|_{\mathcal{V}}^2 + \|\partial\mathbf{U}^1\|_{\mathcal{V}}^2 \\ &+ \|\Theta^0\|_1^2 + \|\Theta^1\|_1^2 + \|D_t\Theta^0\|_0^2 + \sum_{n=1}^{n=M}\Big(\|\mathbf{f}^n\|_0^2 + \|q^n\|_0^2\Big)\Delta t\Big).\end{aligned} \tag{89}$$

Remark Under the hypothesis of Theorem 3, it can be shown that the time discrete implicit procedure (69) is unconditionally stable and has a unique solution satisfying the estimate (89).

Remark Note that the first and second time derivatives in the formulation of the time-discrete implicit and explicit FE procedures (69) and (70) are discretized with errors on the order of $(\Delta t)^2$. Thus, the arguments for obtaining a priori error estimates for the time-continuous FE procedure (21) can be used to conclude that the a priori errors associated with those discrete-time FE methods are on the order of $(\Delta t)^2 + h$.

6 Conclusions

We solve the initial boundary-value problem associated with the thermo-poroelasticity wave equation by applying continuous and discrete-time finite-element methods. A priori error estimates are derived, which are optimal for the assumed regularity of the solution. Furthermore, we present explicit and implicit discrete-time FE methods and analyze the stability of the explicit formulation and establish stability constrains. The proposed algorithms overcome the limitations of isothermal wave propagation.

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