BIOT CODE 1D EXPLICIT

6/1/2022

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$$\left(\rho\left[\frac{u^{s,n+1}-2u^{s,n}+u^{s,n-1}}{(\Delta t)^2}\right], v^s\right)$$

$$+\left(\rho_f\left[\frac{u^{f,n+1}-2u^{f,n}+u^{f,n-1}}{(\Delta t)^2}\right], v^s\right)$$

$$+\left(\rho_f\left[\frac{u^{s,n+1}-2u^{s,n}+u^{s,n-1}}{(\Delta t)^2}\right], v^f\right)$$

$$+\left(g\left[\frac{u^{f,n+1}-2u^{f,n}+u^{f,n-1}}{(\Delta t)^2}\right], v^f\right)$$

$$+\left(\frac{\eta}{K}\left[\frac{u^{f,n+1}-u^{f,n-1}}{(2\Delta t)}\right], v^f\right) \qquad (1)$$

$$+\left(E\frac{\partial u^{s,n}}{\partial x}, \frac{\partial v^s}{\partial x}\right)+\left(\beta\frac{\partial u^{f,n}}{\partial x}, \frac{\partial v^s}{\partial x}\right)$$

$$+\left\langle D\left[\frac{u^{s,n+1}-u^{s,n-1}}{2\Delta t}\right]\cdot\nu, \left(\frac{u^{f,n+1}-u^{f,n-1}}{2\Delta t}\right)\cdot\nu, \begin{pmatrix} v^s\cdot\nu, \\ v^f\cdot\nu \end{pmatrix}\right\rangle$$

$$+\left(\beta\frac{\partial u^{s,n}}{\partial x}, \frac{\partial v^f}{\partial x}\right)+\left(M\frac{\partial u^{f,n}}{\partial x}, \frac{\partial v^f}{\partial x}\right)$$

$$=\left(f^{s,n}, v^s\right)+\left(f^{f,n}, v^f\right)$$

Take $v^s=\varphi^k$, $v^f=0$ in (1) $k=1,\ldots N_x+1$

$$LHS1(k)=\frac{1}{(\Delta t)^2}\left(\rho\, u^{s,n+1}, \varphi_k\right)+\frac{1}{(\Delta t)^2}\left(\rho_f\, u^{f,n+1}, \varphi_k\right)$$

$$+\frac{1}{2\Delta t}\left(D_{11}^{L} u_k^{s,n+1}+D_{12}^{L} u_k^{f,n+1}\right)\delta_{k1}+\frac{1}{2\Delta t}\left(D_{11}^{R} u_k^{s,n+1}+D_{12}^{R} u_k^{f,n+1}\right)\delta_{k,Nx+1}$$

$$=\frac{1}{(\Delta t)^2}\left(\rho\left[2u^{s,n}-u^{s,n-1}\right], \varphi_k\right)$$

$$+\frac{1}{(\Delta t)^2}\left(\rho_f\left[2u^{f,n}-u^{f,n-1}\right], \varphi_k\right) \qquad (2)$$

$$+\frac{1}{2\Delta t}\left[\left(D_{11}^{L} u_k^{s,n-1}+D_{12}^{L} u_k^{f,n-1}\right)\delta_{k1}+\left(D_{11}^{R} u_k^{s,n-1}+D_{12}^{R} u_k^{f,n-1}\right)\delta_{k,Nx+1}\right]$$

$$-\left(E\frac{\partial u^{s,n}}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)-\left(\beta\frac{\partial u^{f,n}}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)$$

$$+\left(f^{s,n}, \varphi_k\right)=RHS1(k), \quad k=1 \ldots Nx+1$$

$$u^{s,n}=\sum_{j=}^{Nx+1} u_j^{s,n}\varphi_j \qquad u^{f,n}=\sum_{j=1}^{Nx+1} u_j^{f,n}\varphi_j$$

For $k=2,\ldots N_x$ (page 2 TERMO CODE)

$$\frac{1}{(\Delta t)^2}\left(\rho\, u^{s,n+1}, \varphi_k\right)$$

$$=\frac{h}{2}\left(\rho_{k-1}+\rho_k\right)\frac{1}{(\Delta t)^2}u_k^{s,n+1} \qquad (3)$$

For $k=1$

$$\frac{1}{(\Delta t)^2}\left(\rho\, u^{s,n+1}, \varphi_1\right)=\frac{1}{(\Delta t)^2}\frac{h}{2}\rho_1\, u_1^{s,n+1} \qquad (4)$$

For $k = N_x + 1$

$$\frac{1}{(\Delta t)^2}\left(\rho\, u^{s,n+1}, \varphi_{N_x+1}\right) = \frac{h}{2}\,\rho_{N_x}\,\frac{1}{(\Delta t)^2}\, u^{s,n+1}_{N_x+1} \quad (5)$$

Similarly, for $k = 2 -- N_x$

$$\frac{1}{(\Delta t)^2}\left(\rho_f\, u^{f,n+1}, \varphi_k\right) = \frac{h}{2}\left(\rho_{f,k} + \rho_{f,k-1}\right)\frac{1}{(\Delta t)^2}\, u^{f,n+1}_k \quad (6)$$

For $k = 1$

$$\frac{1}{(\Delta t)^2}\left(\rho_f\, u^{f,n+1}, \varphi_1\right) = \frac{h}{2}\,\rho_{f,1}\,\frac{1}{(\Delta t)^2}\, u^{f,n+1}_1 \quad (7)$$

For $k = N_x + 1$

$$\frac{1}{(\Delta t)^2}\left(\rho_f\, u^{f,n+1}, \varphi_{N_x+1}\right) = \frac{1}{(\Delta t)^2}\,\frac{h}{2}\,\rho_{f,N_x}\, u^{f,n+1}_{N_x+1} \quad (8)$$

Then, for $k = 1$

$$LHS1(1) = \left[\frac{1}{(\Delta t)^2}\frac{h}{2}\rho_1 + \frac{1}{(2\Delta t)} D^L_{11}\right] u^{s,n+1}_1 \quad (9)$$
$$\left. + \frac{1}{(\Delta t)^2}\frac{h}{2}\rho_{f,1} + \frac{1}{2\Delta t} D^L_{12}\right] u^{f,n+1}_1$$

For $k = N_x + 1$

$$LHS1(N_x+1) = \left[\frac{1}{(\Delta t)^2}\frac{h}{2}\rho_{Nx} + \frac{1}{(2\Delta t)}D_{11}^{R}\right]u^{s,n+1}_{Nx+1} \quad (10)$$

$$+ \left[\frac{1}{(\Delta t)^2}\frac{h}{2}\rho_{f,Nx} + \frac{1}{2\Delta t}D_{12}^{R}\right]u^{f,n+1}_{Nx+1}$$

For $k = 2, -\ Nx$

$$LHS1(k) = \frac{1}{(\Delta t)^2}\frac{h}{2}(\rho_{k-1} + \rho_k)\,u^{s,n+1}_{k} \quad (11)$$

$$+ \frac{1}{(\Delta t)^2}\frac{h}{2}(\rho_{f,k-1} + \rho_{f,k})\,u^{f,n+1}_{k}$$

Next we compute $RHS1(k)$:

for $k = 2, --, Nx$

$$RHS1(k) = \frac{1}{(\Delta t)^2} (e_{k-1} + e_k) \frac{h}{2} [2u_k^{s,n} - u_k^{s,n-1}]$$

$$+ \frac{1}{(\Delta t)^2} \frac{h}{2} (e_{f,k} + e_{f,k-1}) [2u_k^{f,n} - u_k^{f,n-1}]$$

$$- \left[-\frac{1}{h} E_{k-1} u_{k-1}^{s,n} + \frac{1}{h} (E_{k-1} + E_k) u_k^{s,n} \right.$$

$$\left. - \frac{1}{h} E_k u_{k+1}^{s,n} \right] \qquad (12)$$

$$- \left[-\frac{1}{h} B_{k-1} u_{k-1}^{f,n} + \frac{1}{h} (B_{k-1} + B_k) u_k^{f,n} \right.$$

$$\left. - \frac{1}{h} B_k u_{k+1}^{f,n} \right] + (f^{s,n}, \varphi_k)$$

For $k=1$,

$$RHS1(1) = \frac{1}{(\Delta t)^2} \frac{h}{2} e_1 [2u_1^{s,n} - u_1^{s,n-1}]$$

$$+ \frac{1}{(\Delta t)^2} \frac{h}{2} e_{f,1} [2u_1^{f,n} - u_1^{f,n-1}] \qquad (13)$$

$$- \left[\frac{1}{h} E_1 u_1^{s,n} - \frac{1}{h} E_1 u_2^{s,n} \right]$$

$$- \left[\frac{1}{h} B_1 u_1^{f,n} - \frac{1}{h} B_1 u_2^{f,n} \right]$$

$$+ \frac{1}{2\Delta t} D_{11}^L u_1^{s,n-1} + \frac{1}{2\Delta t} D_{12}^L u_1^{f,n-1}$$

FOR $K = N_x + 1$

$$RHS1(N_x+1) = \frac{1}{(\Delta t)^2} \frac{h}{2} \rho_{N_x} \left[2 u^{s,n}_{N_x+1} - u^{s,n-1}_{N_x+1} \right]$$

$$+ \frac{1}{(\Delta t)^2} \frac{h}{2} \rho_{f,N_x} \left[2 u^{f,n}_{N_x+1} - u^{f,n-1}_{N_x+1} \right]$$

$$- \left[-\frac{1}{h} E_{N_x} u^{s,n}_{N_x} + \frac{1}{h} E_{N_x} u^{s,n}_{N_x+1} \right] \quad (14)$$

$$- \left[-\frac{1}{h} B_{N_x} u^{f,n}_{N_x} + \frac{1}{h} B_{N_x} u^{f,n}_{N_x} \right]$$

$$+ \frac{1}{2\Delta t} \left(D^R_{11} u^{s,n-1}_{N_x+1} + D^R_{12} u^{f,n-1}_{N_x+1} \right)$$

Next take $v^s = 0$ $v^f = \varphi_K$ in (1), $K = 1, \dots, N_x+1$

$$
\begin{aligned}
LHS2(k) = {} & \frac{1}{(\Delta t)^2}\left(\rho_f \, u^{s,n+1}, \varphi_k\right) \\
& + \frac{1}{(\Delta t)^2}\left(g \, u^{f,n+1}, \varphi_k\right) + \frac{1}{2\Delta t}\left(\frac{\eta}{k} \, u^{f,n+1}, \varphi_k\right) \\
= {} & \frac{1}{(\Delta t)^2}\left(\rho_f \left[2u^{s,n} - u^{s,n-1}\right], \varphi_k\right) \\
& + \frac{1}{(\Delta t)^2}\left(g \left[2u^{f,n} - u^{f,n-1}\right], \varphi_k\right) \\
& + \frac{1}{2\Delta t}\left(\frac{\eta}{k} \, u^{f,n-1}, \varphi_k\right) - \left(B \frac{\partial u^{f,n}}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right) \\
& - \left(M \frac{\partial u^{f,n}}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right) + \left(f^{f,n}, \varphi_k\right) \\
= {} & RHS2(k)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
LHS2(K) = {} & \frac{1}{(\Delta t)^2}\left(\rho_f\, u^{s,n+1}, \varphi_K\right) \\
& + \frac{1}{(\Delta t)^2}\left(g\, u^{f,n+1}, \varphi_K\right) + \left(\frac{\eta}{K}\, u^{f,n+1}, \varphi_K\right)\frac{1}{2\Delta t} \\
& + \frac{1}{2\Delta t}\left(D^L_{21}\, u^{s,n+1}_K + D^L_{22}\, u^{f,n+1}_K\right)\delta_{K1} \\
& + \frac{1}{2\Delta t}\left(D^R_{21}\, u^{s,n+1}_K + D^R_{22}\, u^{f,n+1}_K\right)\delta_{K,Nx+1} \\
= {} & \frac{1}{(\Delta t)^2}\left(\rho_f\left[2u^{s,n} - u^{s,n-1}\right], \varphi_K\right) \qquad (16) \\
& + \frac{1}{(\Delta t)^2}\left(g\left[2u^{f,n} - u^{f,n-1}\right], \varphi_K\right) \\
& + \frac{1}{2\Delta t}\left(\frac{\eta}{K}\, u^{f,n-1}, \varphi_K\right) - \left(B\,\frac{\partial u^{f,n}}{\partial x}, \frac{\partial \varphi_K}{\partial x}\right) \\
& - \left(M\,\frac{\partial u^{f,n}}{\partial x}, \frac{\partial \varphi_K}{\partial x}\right) + \left(f^{f,n}, \varphi_K\right) \\
& \cancel{RHS2(K)} \qquad \cancel{K = 2,\ Nx+1} \\
& + \frac{1}{2\Delta t}\left(D^L_{21}\, u^{s,n-1}_K + D^L_{22}\, u^{f,n-1}_K\right)\delta_{K1} \\
& + \frac{1}{2\Delta t}\left(D^R_{21}\, u^{s,n-1}_K + D^R_{22}\, u^{f,n-1}_K\right)\delta_{K,Nx+1},
\end{aligned}
$$

$$K = 1 \,\_\_\, N_x + 1$$

For $K = 1$

$$LHS2(1) = \left[\frac{1}{(\Delta t)^2}\frac{h}{2} e_{f,1} + \frac{1}{2\Delta t} D^{L}_{21}\right] u_1^{s,n+1}$$

$$+ \left[\frac{1}{(\Delta t)^2}\frac{h}{2} g_1 + \frac{1}{2\Delta t} D^{L}_{22} + \frac{1}{2\Delta t}\left(\frac{\eta}{K}\right)_1 \frac{h}{2}\right] u_1^{f,n+1} \qquad (17)$$

For $K = Nx+1$

$$LHS2(Nx+1) = \left[\frac{1}{(\Delta t)^2}\frac{h}{2} e_{f,Nx} + \frac{1}{2\Delta t} D^{R}_{21}\right] u_{Nx+1}^{s,n+1}$$

$$+ \left[\frac{1}{(\Delta t)^2}\frac{h}{2} g_{Nx} + \frac{1}{2\Delta t} D^{R}_{22} + \frac{1}{2\Delta t}\left(\frac{\eta}{K}\right)_{Nx} \frac{h}{2}\right] u_{Nx+1}^{f,n+1} \qquad (18)$$

For $K = 2, \ldots Nx$

$$LHS2(K) = \frac{1}{(\Delta t)^2}\frac{h}{2}\left(e_{f,K-1} + e_{f,K}\right) u_K^{s,n+1} \qquad (19)$$

$$+ \left[\frac{1}{(\Delta t)^2}\frac{h}{2}\left(g_{K-1} + g_K\right) + \frac{1}{2\Delta t}\left[\left(\frac{\eta}{K}\right)_{K-1} + \left(\frac{\eta}{K}\right)_K\right]\frac{h}{2}\right] u_K^{f,n+1}$$

Next we compute $RHS2(K)$, $K = 2, \ldots, Nx$

$$RHS2(k) = \frac{1}{(\Delta t)^2}\left(\rho_{f,k-1}+\rho_{f,k}\right)\frac{h}{2}\left(2u_k^{s,n}-u_k^{s,n-1}\right)$$

$$+\frac{1}{(\Delta t)^2}\left(g_{k-1}+g_k\right)\frac{h}{2}\left(2u_k^{f,n}-u_k^{f,n-1}\right)$$

$$+\frac{1}{2\Delta t}\frac{h}{2}\left[\left(\frac{\eta}{k}\right)_{k-1}+\left(\frac{\eta}{k}\right)_k\right]u_k^{f,n-1}$$

$$-\left[-\frac{1}{h}B_{k-1}u_{k-1}^{s,n}+\frac{1}{h}\left(B_{k-1}+B_k\right)u_k^{s,n}\right.$$

$$\left.-\frac{1}{h}B_k\, u_{k+1}^{s,n}\right] \qquad (20)$$

$$-\left[-\frac{1}{h}M_{k-1}u_{k-1}^{f,n}+\frac{1}{h}\left(M_{k-1}+M_k\right)u_k^{f,n}\right.$$

$$\left.-\frac{1}{h}M_k\, u_{k+1}^{f,n}\right]+\left(f^{f,n},\varphi_k\right)$$

For $k=1$

$$RHS2(1) = \frac{1}{(\Delta t)^2}\frac{h}{2}\rho_{f,1}\left(2u_1^{s,n}-u_1^{s,n-1}\right)$$

$$+\frac{1}{(\Delta t)^2}\frac{h}{2}g_1\left(2u_1^{f,n}-u_1^{f,n-1}\right)+\frac{1}{2\Delta t}\frac{h}{2}\left(\frac{\eta}{k}\right)_1 u_1^{f,n-1}$$

$$-\left[\frac{1}{h}B_1u_1^{s,n}-\frac{1}{h}B_1u_2^{s,n}\right] \qquad (21)$$

$$-\left[\frac{1}{h}M_1u_1^{f,n}-\frac{1}{h}M_1u_2^{f,n}\right]$$

$$+\frac{1}{2\Delta t}D_{21}^{L}u_1^{s,n-1}+\frac{1}{2\Delta t}D_{22}^{L}u_1^{f,n-1}$$

FOR $K = N_x + 1$

$$
\begin{aligned}
RHS2(N_x+1) = {} & \frac{1}{(\Delta t)^2}\frac{h}{2}\rho_{f,N_x}\left(2u^{s,m}_{N_x+1} - u^{s,m-1}_{N_x+1}\right) \\
& + \frac{1}{(\Delta t)^2}\frac{h}{2}\,g_{N_x}\left(2u^{f,m}_{N_x+1} - u^{f,m-1}_{N_x+1}\right) \\
& + \frac{1}{2\Delta t}\frac{h}{2}\left(\frac{m}{k}\right)_{N_x} u^{f,m-1}_{N_x+1} \\
& - \left[-\frac{1}{h}B_{N_x}u^{s,m}_{N_x} + \frac{1}{h}B_{N_x}u^{s,m}_{N_x+1}\right] \qquad (22) \\
& - \left[-\frac{1}{h}M_{N_x}u^{f,m}_{N_x} + \frac{1}{h}M_{N_x}u^{f,m}_{N_x+1}\right] \\
& + \frac{1}{2\Delta t}\left(D^R_{21}u^{s,m-1}_{N_x+1} + D^R_{22}u^{f,m-1}_{N_x+1}\right)
\end{aligned}
$$

— ∘ —

For $k=1$, set

Set (from eq's (9) and (17))

$$a_{11} = \frac{1}{(\Delta t)^2}\frac{h}{2}e_1 + \frac{1}{2\Delta t}D^L_{11}$$

$$a_{12} = \frac{1}{(\Delta t)^2}\frac{h}{2}e_{f,1} + \frac{1}{2\Delta t}D^L_{12} \qquad (23)$$

$$a_{21} = \frac{1}{(\Delta t)^2}\frac{h}{2}e_{f1} + \frac{1}{2\Delta t}D^L_{21}$$

$$a_{22} = \frac{1}{(\Delta t)^2}\frac{h}{2}g_1 + \frac{1}{2\Delta t}D^L_{22} + \frac{1}{2\Delta t}\frac{h}{2}\left(\frac{n}{k}\right)_{1}$$

$(a_{12} = a_{21})$

Then (9) and (23) can be written as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u_1^{s,n+1} \\ u_1^{f,n+1} \end{pmatrix} = \begin{pmatrix} RHS1(1) \\ RHS2(1) \end{pmatrix} \qquad (24)$$

with $RHS(1)$, $RHS2(1)$ given by (13) and (21). By Cramer's Rule

~~Next,~~ we solve (24)

$$u_1^{s,n+1} = \begin{pmatrix} RHS1(1) & a_{12} \\ RHS2(1) & a_{22} \end{pmatrix} \Big/ \underbrace{(a_{11}a_{22} - a_{12}^2)}_{\det A} \qquad (25)$$

$$u_1^{f,n+1} = \begin{pmatrix} a_{11} & RHS1(1) \\ a_{21} & RHS2(1) \end{pmatrix} \Big/ \det A \qquad (26)$$

For $k = N_x+1$ set (eq's (10) and (18))

$$
\left.
\begin{aligned}
Q_{11} &= \frac{1}{(\Delta t)^2}\frac{h}{2} e_{Nx} + \frac{1}{2\Delta t} D^R_{11} \\
Q_{12} &= \frac{1}{(\Delta t)^2}\frac{h}{2} e_{f,Nx} + \frac{1}{2\Delta t} D^R_{12} \\
Q_{21} &= \frac{1}{(\Delta t)^2}\frac{h}{2} e_{f,Nx} + \frac{1}{2\Delta t} D^R_{21} \\
Q_{22} &= \frac{1}{(\Delta t)^2}\frac{h}{2} g_{Nx} + \frac{1}{2\Delta t} D^R_{22} + \frac{1}{2\Delta t}\frac{h}{2}\left(\frac{\eta}{k}\right)_{Nx}
\end{aligned}
\right\} \quad (24)
$$

Then (10) and (18) are

$$
\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}
\begin{pmatrix} u^{s,n+1}_{Nx+1} \\ u^{f,n+1}_{Nx+1} \end{pmatrix}
=
\begin{pmatrix} RHS1(N_x+1) \\ RHS2(N_x+1) \end{pmatrix} \quad (25)
$$

Thus

$$
u^{s,n+1}_{Nx+1} = \begin{pmatrix} RHS1(N_x+1) & Q_{12} \\ RHS2(Nx+1) & Q_{22} \end{pmatrix} \Big/ \det A \qquad (26)
$$

$$
u^{f,n+1}_{Nx+1} = \begin{pmatrix} Q_{11} & RHS1(Nx+1) \\ Q_{12} & RHS2(Nx+1) \end{pmatrix} \Big/ \det A \qquad (27)
$$

For $k = 2, \ldots, N_x$ (eq's (11) and (19))

$$\left.\begin{aligned}
a_{11} &= \frac{1}{(\Delta t)^2}\frac{h}{2}\left(e_{k-1}+e_k\right)\\
a_{12} &= \frac{1}{(\Delta t)^2}\frac{h}{2}\left(e_{f,k-1}+e_{f,k}\right)\\
a_{21} &= \frac{1}{(\Delta t)^2}\frac{h}{2}\left(e_{f,k-1}+e_{f,k}\right)\\
a_{22} &= \frac{1}{(\Delta t)^2}\frac{h}{2}\left(g_{k-1}+g_k\right) + \frac{h}{2}\left[\left(\frac{\eta}{k}\right)_{k-1}+\left(\frac{\eta}{k}\right)_{k}\right]\frac{1}{2\Delta t}
\end{aligned}\right\} \quad (28)$$

Then (11) and (19) can be written as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} u_k^{s,n+1} \\ u_k^{f,n+1} \end{pmatrix} = \begin{pmatrix} RHS1(k) \\ RHS2(k) \end{pmatrix} \quad (29)$$

Thus,

$$u_k^{s,n+1} = \begin{pmatrix} RHS1(k) & a_{12} \\ RHS2(k) & a_{22} \end{pmatrix} / \det A \quad (30)$$

$$u_k^{f,n+1} = \begin{pmatrix} a_{11} & RHS1(k) \\ a_{12} & RHS2(k) \end{pmatrix} / \det A \quad (31)$$

Coupling term in BIOT equation code: 15

Add ~~in equation~~ the term 16/1/2022

$+ (\beta \nabla \theta^m, v^s) + (\beta_f \nabla \theta^m, v^f)$

in the RHS of equation (1).

Assume β, β_f constants or $\nabla\beta, \nabla\beta_f$ in L^∞.

The code is written for uniform β, β_f, β_m

This terms go to the RHS in ~~eq~~ equation (2)

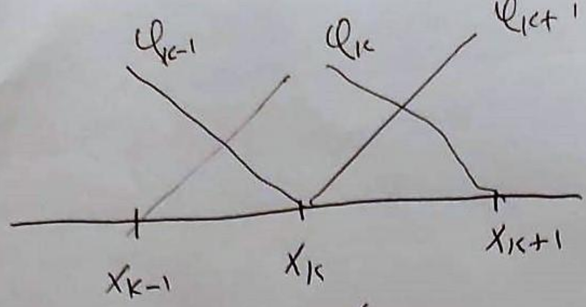
$$(\beta \nabla \theta^m, \varphi_k) = \left(\beta \sum_{j=k-1}^{k+1} \theta_j^m \frac{\partial \varphi_j}{\partial x}, \varphi_k\right)$$

$$= \beta \int_{x_{k-1}}^{x_k} \theta^m_{k-1} \frac{\partial \varphi_{k-1}}{\partial x} \varphi_k \, dx$$

$$+ \beta \int_{x_{k-1}}^{x_k} \theta^m_k \frac{\partial \varphi_k}{\partial x} \varphi_k \, dx + \beta \int_{x_k}^{x_{k+1}} \theta^m_k \frac{\partial \varphi_k}{\partial x} \varphi_k \, dx$$

$$+ \beta \int_{x_k}^{x_{k+1}} \theta^m_{k+1} \frac{\partial \varphi_{k+1}}{\partial x} \varphi_k \, dx$$

$$= I_1 + I_2 + I_3 + I_4$$



$$\varphi_k = \begin{cases} \frac{x - x_{k-1}}{h}, & (x_{k-1}, x_k) \\ 1 - \frac{x - x_k}{h}, & (x_k, x_{k+1}) \end{cases}$$

$$I_1 = \beta \theta^m_{k-1} \left(-\frac{1}{h}\right) \int_{x_{k-1}}^{x_k} \left(\frac{x - x_{k-1}}{h}\right) dx \qquad u = \frac{x - x_{k-1}}{h}, \quad du = \frac{dx}{h}$$

$$= \beta \theta^m_{k-1} \left(-\frac{1}{h}\right) \int_0^1 u \, h \, du =$$

$$= \beta \theta^m_{k-1} (-1) \left. \frac{u^2}{2} \right|_0^1 = -\frac{\beta}{2} \theta^m_{k-1}$$

$$I_2 = \beta \int_{x_{k-1}}^{x_k} \theta^m_{k} \left(\frac{1}{h}\right) \left(\frac{x - x_{k-1}}{h}\right) dx = \frac{\beta}{2} \theta^m_{k}$$

$$I_3 = \beta \int_{x_k}^{x_{k+1}} \theta^m_k \left(-\frac{1}{h}\right) \Big(1 - \underbrace{\frac{x - x_k}{h}}_{u}\Big) dx \qquad u = \frac{x - x_k}{h}$$

$$= \beta \theta^m_k \left(-\frac{1}{h}\right) \int_0^1 (1 - u) \, h \, du = \beta \theta^m_k (-1) \left. \left(u - \frac{u^2}{2}\right) \right|_0^1$$

$$= -\frac{\beta}{2} \theta^m_k$$

$$I_4 = \beta\,\theta^n_{k+1} \int_{x_k}^{x_{k+1}} \left(\frac{1}{h}\right)\Big(1 - \underbrace{\frac{x - x_k}{h}}_{u}\Big)\, dx$$

$$= \beta\,\theta^n_{k+1} \frac{1}{h}\int_0^1 (1-u)\, h\, du = \frac{\beta}{2}\,\theta^n_{k+1}$$

Then

$$(\beta \nabla\theta^n, \varphi_k) = \frac{\beta}{2}\Big[(\theta^n_k - \theta^n_{k-1})(1-\delta_{k1}) + (\theta^n_{k+1} - \theta^n_k)(1-\delta_{k,Nx+1})\Big] \quad (32)$$

For $k=1$,

$$(\beta\nabla\theta^n, \varphi_1) = \frac{\beta}{2}(\theta^n_2 - \theta^n_1) \quad (33)$$

For $k = Nx+1$)

$$(\beta\nabla\theta^n, \varphi_{Nx+1}) = \frac{\beta}{2}\left(\theta^n_{Nx+1} - \theta^n_{Nx}\right) \quad (34)$$

For $k=2, \ldots, Nx$

$$(\beta\nabla\theta^n, \varphi_k) = \frac{\beta}{2}\left(\theta^n_{k+1} - \theta^n_{k-1}\right) \quad (35)$$

Now (33), (34) and (35) are used in RHS1 in equations (12)-(13)-(14) —