A model for wave propagation in a porous medium saturated by a two-phase fluid

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A theory to describe the propagation of elastic waves in a porous medium saturated by a mixture of two immiscible, viscous, compressible fluids is presented. First, using the principle of virtual complementary work, the stress–strain relations are obtained for both anisotropic and isotropic media. Then the forms of the kinetic and dissipative energy density functions are derived under the assumption that the relative flow within the porous medium is of laminar type and obeys Darcy's law for two-phase flow in porous media. The equations of motion are derived, and a discussion of the different kinds of body waves that propagate in this type of medium is given. A theorem on the existence, uniqueness, and regularity of the solution of the equations of motion under appropriate initial and boundary conditions is stated.

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INTRODUCTION

Our purpose is to develop a model describing the propagation of waves in an elastic system composed of a porous solid saturated by two immiscible, compressible, viscous fluids. We develop our equations for the case of water and oil using the indices "w" and "o" to refer to the water and oil phases, but the formulation is valid for any wetting–nonwetting system in which the "w" refers to the wetting phase and the "o" to the nonwetting phase.

The relative flow of both fluids with respect to the rock frame will be considered to be of laminar type and to obey Darcy's law for two-phase flow in porous media. Such relative movements of the fluids produce energy losses which are included in the model by introducing a dissipation function in the Lagrangian formulation of the equations of motion. The assumption of laminar flow implies that we are considering wavelengths which are much larger than the average size of the pores or, equivalently, frequencies below a certain critical value.

Capillary pressure effects due to the pressure difference between the two fluids are also taken into account in our derivation. The theory of wave propagation in a porous medium saturated by a single-phase fluid was presented by Biot.[1–3] Burridge and Keller[4] gave an alternate derivation of Biot's equations via homogenization. Existence, uniqueness, and regularity of the solution of Biot's equations were established in Ref. 5, while some finite element methods for obtaining approximate solutions were discussed in Ref. 6.

The organization of the paper is as follows. In Sec. I we obtain the form of the strain energy density using the principle of virtual complementary work and derive stress–strain relations for both anisotropic and isotropic media. In Sec. II we determine expressions for the kinetic and dissipation energy densities and then state the Lagrangian form of the equations of motion, using as generalized coordinates the solid displacement and the oil and water relative displacement vectors. The expression for the dissipation function is obtained under the assumption that the relative flow within the porous medium takes place according to Darcy's law for two-phase flow in porous media.

In Sec. III we analyze the equations of motion for isotropic media with the object of determining the different kinds of waves which propagate in this type of medium. We find that there are five possible different body waves, three of them corresponding to compressional modes of propagation and the other two, of identical speed, associated with shear modes. This is an expected generalization of the single-phase theory of Biot. Analytical properties and attenuation effects for each of these waves will be given in a complementary publication.[7]

Finally, in Sec. IV we present results on the existence and uniqueness of the solution of the equations of motion derived in Sec. II under appropriate initial and boundary conditions.

I. THE STRESS–STRAIN RELATIONS

Let us consider a porous medium Ω saturated by a mixture of oil and water, and let $S_o = S_o(x)$ and $S_w = S_w(x)$ denote the oil and water saturations, respectively. We assume that the two phases completely saturate the porous part of Ω, which we shall denote by Ω_p, so that

$$S_o + S_w = 1.$$

Let $\phi = \phi(x)$ be the effective porosity in Ω and let $u^{s,T} = u^{s,T}(x)$, $\tilde{u}^{o,T} = \tilde{u}^{o,T}(x)$, and $\tilde{u}^{w,T} = \tilde{u}^{w,T}(x)$ denote the locally averaged solid, oil, and water displacements in Ω. The physical meaning of the variables $\tilde{u}^{o,T}$ and $\tilde{u}^{w,T}$ is the following. Let us consider a unit cube Q of bulk material. Then, for any face F of Q the quantity $\int_F \phi S_o \tilde{u}^{o,T}\cdot\nu \, d\sigma$ represents the amount of oil displaced through F, ν being the outer unit normal to F and $d\sigma$ the surface measure on F. A similar definition holds for $\tilde{u}^{w,T}$.

Let us consider an initial state of equilibrium about displacements $\bar{u}^s$, $\bar{\tilde{u}}^o$, and $\bar{\tilde{u}}^w$, and set

$$\tilde{u}^\theta = \tilde{u}^{\theta,T} - \bar{\tilde{u}}^\theta, \quad \theta = o,w,$$

$$u^s = u^{s,T} - \bar{u}^s.$$

Next, let $\tau_{ij} = \bar{\tau}_{ij} + \Delta\tau_{ij}$ and $\sigma_{ij} = \bar{\sigma}_{ij} + \Delta\sigma_{ij}$ be the total stress tensor in the bulk material and the stress tensor in the solid part of Ω, respectively, where $\Delta\tau_{ij}$ and $\Delta\sigma_{ij}$ represent changes in the corresponding stresses with respect to reference stresses $\bar{\tau}_{ij}$ and $\bar{\sigma}_{ij}$ associated with the initial equilibrium state. Similarly, let $p_o = \bar{p}_o + \Delta p_o$ and $p_w = \bar{p}_w + \Delta p_w$ be the oil and water pressures, Δp_o and Δp_w being increments in oil and water pressures with respect to given reference pressures $\bar{p}_o$ and $\bar{p}_w$ corresponding again to the initial equilibrium state. Recall the capillarity relation[8,9]:

$$p_c = p_c(S_o) = (\bar{p}_o + \Delta p_o) - (\bar{p}_w + \Delta p_w)$$
$$= p_c(\bar{S}_o) + \Delta p_o - \Delta p_w \geqslant 0, \qquad (1)$$

where p_c is assumed to depend only on the (oil) saturation and $\bar{S}_o$ denotes the saturation in the initial equilibrium state. Without loss of generality we can assume that $\bar{p}_w = 0$. Then, according to (1),

$$\bar{p}_o = p_c(\bar{S}_o).$$

Let

$$\Delta p_c = \Delta p_o - \Delta p_w, \qquad (2)$$

$$\Delta S_\theta = S_\theta - \bar{S}_\theta, \quad \theta = o,w. \qquad (3)$$

Note that, neglecting terms of the second order in ΔS_o,

$$\Delta p_c = p_c'(\bar{S}_o)\Delta S_o. \qquad (4)$$

Also set

$$\sigma_\theta = -\phi S_\theta p_\theta, \quad \theta = o,w,$$

$$\sigma = \sigma_o + \sigma_w.$$

Then,

$$\tau_{ij} = \sigma_{ij} + \delta_{ij}\sigma, \quad \delta_{ij} \text{ the Kronecker symbol.} \qquad (5)$$

Now, we shall derive the strain–stress relations for our system using the principle of virtual complementary work.[10] Let us consider a domain Ω of bulk material bounded by $\partial\Omega$. Assume that Ω is initially in static equilibrium under the action of surface forces $\bar{f}_i^\theta$, $\theta = s,o,w$, where $\bar{f}_i^\theta$ represents the force in the θ-part of $\partial\Omega$ per unit of surface area of bulk material (body forces such as gravity are ignored in our analysis). Thus,

$$\bar{f}_i^o = -\phi\bar{S}_o\bar{p}_o\delta_{ij}\nu_j, \quad \bar{f}_i^w = -\phi\bar{S}_w\bar{p}_w\delta_{ij}\nu_j = 0,$$

$$\bar{f}_i^s = \bar{\sigma}_{ij}\nu_j.$$

Now, consider a new system of surface forces f_i^θ superimposed on the original system $\bar{f}_i^\theta$ such that Ω remains in equilibrium under the action of the total surface forces

$$f_i^{\theta,T} = \bar{f}_i^\theta + f_i^\theta, \quad \theta = s,o,w. \qquad (6)$$

Since the fluids are at rest, all fluid pressures are constant on Ω. Hence,

$$\nabla p_o = \frac{\partial p_o}{\partial x_j} = 0, \quad \nabla p_w = 0,$$
$$\nabla\bar{p}_o = \nabla\bar{p}_w = 0. \qquad (7)$$

Since the total stress field is also in equilibrium,

$$\nabla\cdot\tau = \frac{\partial\tau_{ij}}{\partial x_j} = 0,$$
$$\nabla\cdot\bar{\tau} = 0. \qquad (8)$$

(Here and in what follows we use the Einstein summation convention; i.e., sum on repeated indices, except for those symbols indicating solid, oil, and water.)

Next, note that it follows from (1) and (7) that in the initial equilibrium state ($\Delta p_o = \Delta p_w = 0$),

$$p_c'(\bar{S}_o)\nabla\bar{S}_o = \nabla\bar{p}_o - \nabla\bar{p}_w = 0.$$

Since it is the case that $p_c' > 0$, we see that

$$\nabla\bar{S}_o = 0. \qquad (9)$$

Let $W^* = W^*(\Delta\tau_{ij},\Delta p_o,\Delta p_w,\Delta p_c)$ be the complementary strain energy density and

$$\mathscr{V}^* = \int_\Omega W^* \, dx - \int_{\partial\Omega} (f_i^s u_i^s + f_i^o \tilde{u}_i^o + f_i^w \tilde{u}_i^w) d\sigma \qquad (10)$$

be the complementary energy. Then, according to the complementary energy theorem,[10] of all generalized tensor fields $(\Delta\tau_{ij},\Delta p_o,\Delta p_w,\Delta p_c)$ satisfying the equilibrium conditions (7) and (8) and the constraint (2), the actual one is distinguished by being an extremum (minimum) of the complementary energy $\mathscr{V}^*$. We shall include the condition (2) by introducing a Lagrange multiplier. Thus, if

$$J = \mathscr{V}^* + \int_\Omega \lambda(\Delta p_o - \Delta p_w - \Delta p_c) dx,$$

the complementary energy theorem implies that

$$\delta J = 0 = \int_\Omega \delta W^* \, dx - \int_{\partial\Omega} (\delta f_i^s u_i^s + \delta f_i^o \tilde{u}_i^o + \delta f_i^w \tilde{u}_i^w) d\sigma$$
$$+ \int_\Omega [\delta\lambda(\Delta p_o - \Delta p_w - \Delta p_c)$$
$$+ \lambda(\delta\Delta p_o - \delta\Delta p_w - \delta\Delta p_c)] dx. \qquad (11)$$

The equation

$$\int_\Omega \delta W^* \, dx = \int_{\partial\Omega} (\delta f_i^s u_i^s + \delta f_i^o \tilde{u}_i^o + \delta f_i^w \tilde{u}_i^w) d\sigma$$
$$- \int_\Omega \{\delta\lambda(\Delta p_o - \Delta p_w - \Delta p_c)$$
$$+ \lambda(\delta\Delta p_o - \delta\Delta p_w - \delta\Delta p_c)\} dx \qquad (12)$$

states the principle of virtual complementary work for our system.

Next, neglecting terms of the second order and using (3) and (4), we see that

$$f_i^{o,T} = -\phi S_o p_o \delta_{ij} \nu_j$$
$$= -\phi(\bar{S}_o + \Delta S_o)(\bar{p}_o + \Delta p_o)\delta_{ij}\nu_j$$
$$\cong -\phi \bar{S}_o \bar{p}_o \delta_{ij}\nu_j - \phi(\bar{S}_o \, \Delta p_o + \bar{p}_o \, \Delta S_o)\delta_{ij}\nu_j$$
$$= \bar{f}_i^o - \phi(\bar{S}_o \, \Delta p_o + [p_c(\bar{S}_o)/p_c'(\bar{S}_o)]\Delta p_c)\delta_{ij}\nu_j,$$
$$f_i^{w,T} = -\phi S_w p_w \delta_{ij}\nu_j \tag{13}$$
$$\cong \bar{f}_i^w - \phi \bar{S}_w \Delta p_w \delta_{ij}\nu_j.$$

Similarly, using (4) and (5),

$$f_i^{s,T} = \sigma_{ij}\nu_j$$
$$= [\tau_{ij} + \delta_{ij}\phi(S_o p_o + S_w p_w)]\nu_j$$
$$\cong (\bar{\tau}_{ij} + \delta_{ij}\phi \bar{S}_o \bar{p}_o)\nu_j$$
$$+ [\Delta\tau_{ij} + \delta_{ij}\delta(\bar{S}_o \, \Delta p_o + \bar{S}_w \, \Delta p_w + \bar{p}_o \, \Delta S_o)]\nu_j$$
$$= \bar{f}_i^s + \{\Delta\tau_{ij} + \delta_{ij}\phi(\bar{S}_o \, \Delta p_o + \bar{S}_w \, \Delta p_w$$
$$+ [p_c(\bar{S}_o)/p_c'(\bar{S}_o)]\Delta p_c)\}\nu_j. \tag{14}$$

Set

$$\beta = p_c(\bar{S}_o)/p_c'(\bar{S}_o).$$

Then it follows from (6), (13), and (14) that

$$\delta f_i^o = -\phi(\bar{S}_o \delta \, \Delta p_o + \beta \delta \, \Delta p_c)\delta_{ij}\nu_j,$$
$$\delta f_i^w = -\phi \bar{S}_w \delta \, \Delta p_w \delta_{ij}\nu_j,$$
$$\delta f_i^s = [\delta \, \Delta\tau_{ij} + \delta_{ij}\phi(\bar{S}_o \delta \, \Delta p_o + \bar{S}_w \delta \, \Delta p_w + \beta\delta \, \Delta p_c)]\nu_j.$$

Thus (11) becomes

$$\delta J = \int_\Omega \delta W^* \, dx - \int_{\partial\Omega} [u_i^s \delta \, \Delta\tau_{ij} \, \nu_j - u_i^o \bar{S}_o \delta \, \Delta p_o \, \delta_{ij}\nu_j$$
$$- u_i^w \bar{S}_w \delta \, \Delta p_w \, \delta_{ij}\nu_j - u_i^o \beta\delta \, \Delta p_c \, \delta_{ij}\nu_j] d\sigma$$
$$+ \int_\Omega \{[\delta\lambda(\Delta p_o - \Delta p_w - \Delta p_c)$$
$$+ \lambda(\delta \, \Delta p_o - \delta \, \Delta p_w - \delta \, \Delta p_c)]\} dx = 0, \tag{15}$$

where

$$u^o = \phi(\tilde{u}^o - u^s)$$

and

$$u^w = \phi(\tilde{u}^w - u^s)$$

are the oil and water displacements relative to the solid frame. Let

$$\epsilon_{ij}(u^s) = \frac{1}{2}\left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i}\right)$$

be the strain tensor in the solid part of Ω. Also, set

$$\xi^\theta = -\nabla \cdot u^\theta, \quad \theta = o,w.$$

In the case of uniform porosity, $\bar{S}_o \xi^o$ and $\bar{S}_w \xi^w$ measure the amounts of oil and water entering or leaving a unit cube of bulk material. Also, since Ω remains in equilibrium, the generalized virtual stresses $\delta\Delta\tau_{ij}$, $\delta\Delta p_o$, and $\delta\Delta p_w$ satisfy the equilibrium conditions (7) and (8), so that

$$\frac{\partial}{\partial x_j}\delta\Delta\tau_{ij} = 0, \quad \nabla(\delta\Delta p_o) = \nabla(\delta\Delta p_w) = 0, \tag{16}$$

which obviously imply that

$$\nabla(\delta\Delta p_c) = 0. \tag{17}$$

Next, applying the Gauss theorem to transform the surface integral in the right-hand side of (15), we see that

$$\delta J = \int_\Omega \delta W^* \, dx - \int_\Omega \Big[\epsilon_{ij}\delta \, \Delta\tau_{ij} + (\bar{S}_o \xi^o - \lambda)\delta \, \Delta p_o$$
$$+ (\bar{S}_w \xi^w + \lambda)\delta \, \Delta p_w + (\beta\xi^o + \lambda)\delta \, \Delta p_c$$
$$+ (\Delta p_o - \Delta p_w - \Delta p_c)\delta\lambda$$
$$+ u_i^s \frac{\partial}{\partial x_j}\delta \, \Delta\tau_{ij} - u_i^o \frac{\partial}{\partial x_i}(\bar{S}_o \delta \, \Delta p_o + \beta\delta \, \Delta p_c)$$
$$- u_i^w \frac{\partial}{\partial x_i}(\bar{S}_w \delta\Delta p_w)\Big] dx = 0. \tag{18}$$

Now, using (9) and the equilibrium conditions (16) and (17), (18) becomes

$$\delta J = \int_\Omega \delta W^* \, dx - \int_\Omega [\epsilon_{ij}\delta\Delta\tau_{ij} + (\bar{S}_o \xi^o - \lambda)\delta\Delta p_o$$
$$+ (\bar{S}_w \xi^w + \lambda)\delta\Delta p_w + (\beta\xi^o + \lambda)\delta\Delta p_c$$
$$+ (\Delta p_o - \Delta p_w - \Delta p_c)\delta\lambda \,] dx$$
$$\equiv \int_\Omega \delta F(\Delta\tau_{ij}, \Delta p_o, \Delta p_w, \Delta p_c, \lambda) dx = 0. \tag{19}$$

Since δW^* (or dW^*) is an exact differential in the generalized stresses $(\Delta\tau_{ij}, \Delta p_o, \Delta p_w, \Delta p_c)$, δF is also an exact differential of the same variables and λ, and from (19) we deduce that

$$\frac{\partial F}{\partial \Delta\tau_{ij}} = \frac{\partial W^*}{\partial \Delta\tau_{ij}} - \epsilon_{ij} = 0, \tag{20a}$$
$$\frac{\partial F}{\partial \Delta p_o} = \frac{\partial W^*}{\partial \Delta p_o} - \bar{S}_o \xi^o + \lambda = 0, \tag{20b}$$
$$\frac{\partial F}{\partial \Delta p_w} = \frac{\partial W^*}{\partial \Delta p_w} - \bar{S}_w \xi^w - \lambda = 0, \tag{20c}$$
$$\frac{\partial F}{\partial \Delta p_c} = \frac{\partial W^*}{\partial \Delta p_c} - \beta\xi^o - \lambda = 0, \tag{20d}$$
$$\frac{\partial F}{\partial \lambda} = \Delta p_o - \Delta p_w - \Delta p_c = 0. \tag{20e}$$

Hence,

$$\delta W^* = \epsilon_{ij}\delta \, \Delta\tau_{ij} + (\bar{S}_o \xi^o - \lambda)\delta \, \Delta p_o$$
$$+ (\bar{S}_w \xi^w + \lambda)\delta \, \Delta p_w + (\beta\xi^o + \lambda)\delta \, \Delta p_c. \tag{21}$$

The fact that δW^* is an exact differential also implies that

$$\frac{\partial^2 W^*}{\partial\Delta\tau_{ij} \, \partial\Delta\tau_{kl}} = \frac{\partial^2 W^*}{\partial\Delta\tau_{kl} \, \partial\Delta\tau_{ij}},$$
$$\frac{\partial^2 W^*}{\partial\Delta\tau_{ij} \, \partial\Delta p_\theta} = \frac{\partial^2 W^*}{\partial\Delta p_\theta \, \partial\Delta\tau_{ij}}, \quad \theta = o,w,c, \tag{22}$$
$$\frac{\partial^2 W^*}{\partial\Delta p_\alpha \, \partial\Delta p_\theta} = \frac{\partial^2 W^*}{\partial\Delta p_\theta \, \partial\Delta p_\alpha}, \quad \alpha,\theta = o,w,c,\alpha \neq \theta.$$

If we restrict ourselves to linear stress–strain relations, W^* is a quadratic, positive-definite function of the generalized stress components $(\Delta\tau_{ij}, \Delta p_o, \Delta p_w, \Delta p_c)$. The strain–

stress relations for an anisotropic medium can be written in the form

$$\epsilon_{ij} = A_{ijkl}\Delta\tau_{kl} + P_{1ij}\Delta p_o + P_{2ij}\Delta p_w + P_{3ij}\Delta p_c,$$
$$\bar{S}_o\xi^o - \lambda = P_{1ij}\,\Delta\tau_{ij} + Q_1\,\Delta p_o + Q_4\,\Delta p_w + Q_5\,\Delta p_c,$$
$$\bar{S}_w\xi^w + \lambda = P_{2ij}\,\Delta\tau_{ij} + Q_4\,\Delta p_o + Q_2\,\Delta p_w + Q_6\,\Delta p_c,$$
$$\beta\xi^o + \lambda = P_{3ij}\,\Delta\tau_{ij} + Q_5\,\Delta p_o + Q_6\,\Delta p_w + Q_3\,\Delta p_c. \quad (23)$$

From (23) and the symmetry of the stress tensor τ_{ij} we see that the following conditions must hold:

$$A_{ijkl} = A_{ijlk} = A_{lkij},$$
$$P_{lij} = P_{lji}, \quad l = 1,2,3.$$

Let us consider the linear, isotropic case. Following Refs. 1 and 3, W^* is a quadratic, positive-definite form in the invariants $\Delta\tau = \Delta\tau_{11} + \Delta\tau_{22} + \Delta\tau_{33}$, Δp_o, Δp_w, Δp_c, and

$$I_2 = 2[(\Delta\tau_{12})^2 + (\Delta\tau_{21})^2 + (\Delta\tau_{13})^3 + (\Delta\tau_{31})^2 + (\Delta\tau_{23})^2 + (\Delta\tau_{32})^2 - 2\Delta\tau_{11}\Delta\tau_{22} - 2\Delta\tau_{11}\Delta\tau_{33} - 2\Delta\tau_{22}\Delta\tau_{33}].$$

Thus,

$$W^* = \tfrac{1}{2}[H(\Delta\tau)^2 + (1/4N)I_2 - 2P_1\Delta\tau\Delta p_o - 2P_2\Delta\tau\Delta p_w - 2P_3\Delta\tau\Delta p_c + 2Q_4\Delta p_o\Delta p_w + 2Q_5\Delta p_o\Delta p_c + 2Q_6\Delta p_w\Delta p_c + Q_1(\Delta p_o)^2 + Q_2(\Delta p_w)^2 + Q_3(\Delta p_c)^2].$$

Using (20) and (21), we see that the strain–stress relations are given by

$$\epsilon_{ij} = (1/2N)\Delta\tau_{ij} + \delta_{ij}(D\Delta\tau - P_1\Delta p_o - P_2\Delta p_w - P_3\Delta p_c), \quad (24a)$$
$$\bar{S}_o\xi^o - \lambda = -P_1\Delta\tau + Q_1\Delta p_o + Q_4\Delta p_w + Q_5\Delta p_c, \quad (24b)$$
$$\bar{S}_w\xi^w + \lambda = -P_2\Delta\tau + Q_4\Delta p_o + Q_2\Delta p_w + Q_6\Delta p_c, \quad (24c)$$
$$\beta\xi^o + \lambda = -P_3\Delta\tau + Q_5\Delta p_o + Q_6\Delta p_w + Q_3\Delta p_c, \quad (24d)$$

where

$$D = H - 1/2N.$$

Let $W = W(\epsilon_{ij},\xi^o,\xi^w,\lambda)$ be the strain energy of the system; it is an exact differential in its variables. Since we are considering only linear strain–stress relations, $W = W^*$.[10] From (21) we see that

$$W = \tfrac{1}{2}[\epsilon_{ij}\Delta\tau_{ij} + (\bar{S}_o\xi^o - \lambda)\Delta p_o + (\bar{S}_w\xi^w + \lambda)\Delta p_w + (\beta\xi^o + \lambda)\Delta p_c] \quad (25)$$

and

$$\delta W = \Delta\tau_{ij}\delta\epsilon_{ij} + (\bar{S}_o\Delta p_o + \beta\Delta p_c)\delta\xi^o + \bar{S}_w\Delta p_w\delta\xi^w + [\Delta p_c - (\Delta p_o - \Delta p_w)]\delta\lambda. \quad (26)$$

Note that, when the constraint (2) is imposed on the system, (25) becomes

$$W = \tfrac{1}{2}\{\epsilon_{ij}\Delta\tau_{ij} + \xi^o[(\bar{S}_o + \beta)\Delta p_o - \beta\Delta p_w] + \xi^w\bar{S}_w\Delta p_w\}$$
$$= \tfrac{1}{2}[\epsilon_{ij}\Delta\tau_{ij} + \xi^o(\bar{S}_o + \beta)\Delta p_o + (\bar{S}_w\xi^w - \beta\xi^o)\Delta p_w]. \quad (27)$$

Next, set

$$Y = [(\Delta\tau_{ij})_{1<i,j<3}, \Delta p_o, \Delta p_w, p_c]^t,$$
$$Z = [(\epsilon_{ij})_{1<i,j<3}, \bar{S}_o\xi^o - \lambda, \bar{S}_w\xi^w + \lambda, \beta\xi^o + \lambda]^t.$$

Then we can rewrite the strain–stress relations (24) in the equivalent form

$$Z = EY,$$

where $E \in R^{12\times 12}$ is a symmetric, positive-definite matrix. Let $\lambda_*(A)$ denote the minimum eigenvalue of A for any matrix $A \in R^{m\times m}$. Then, it follows from (25) that (with $[\cdot,\cdot]_e$ and $\|\cdot\|_e$ denoting the usual Euclidean inner product and norm in R^n, respectively)

$$W = \tfrac{1}{2}[EY,Y]_e \geqslant [\lambda_*(E)/2]\|Y\|_e^2 = [\lambda_*(E)/2]\Big((\Delta p_o)^2 + (\Delta p_w)^2 + (\Delta p_c)^2 + \sum_{i,j}(\Delta\tau_{ij})^2\Big) \geqslant [\lambda_*(E)/2]\Big((\Delta p_o)^2 + (\Delta p_w)^2 + \sum_{i,j}(\Delta\tau_{ij})^2\Big), \quad (28)$$

and

$$W = \tfrac{1}{2}[E^{-1}Z,Z]_e \geqslant [\lambda_*(E^{-1})/2]\|Z\|_e^2 = [\lambda_*(E^{-1})/2]\Big((\bar{S}_o\xi^o)^2 + (\bar{S}_w\xi^w)^2 + (\beta\xi^o)^2 + 3\lambda^2 + 2\lambda[(\beta - \bar{S}_o)\xi^o + \bar{S}_w\xi^w] + \sum_{i,j}[\epsilon_{ij}(u^s)]^2\Big) \geqslant C_1\Big((\xi^o)^2 + (\xi^w)^2 + \sum_{i,j}[\epsilon_{ij}(u^s)]^2\Big); \quad (29)$$

$C_1 > 0$, since $\bar{S}_o$, $\bar{S}_w$, and β are positive and are bounded away from zero. To obtain the strain–stress relations for the constrained system [i.e., the system under the constraint (2)] we eliminate the Lagrange multiplier λ from (24) and use (2) to see that

$$\epsilon_{ij} = (1/2N)\Delta\tau_{ij} + \delta_{ij}(D\Delta\tau - F_1\Delta p_o - F_2\Delta p_w), \quad (30a)$$
$$(\bar{S}_o + \beta)\xi^o = -F_1\Delta\tau + H_1\Delta p_o + H_3\Delta p_w, \quad (30b)$$
$$\bar{S}_w\xi^w - \beta\xi^o = -F_2\Delta\tau + H_3\Delta p_o + H_2\Delta p_w, \quad (30c)$$

where

$$F_1 = P_1 + P_3, \quad F_2 = P_2 - P_3,$$
$$H_1 = Q_1 + Q_3 + 2Q_5, \quad H_3 = Q_4 + Q_6 - Q_3 - Q_5,$$
$$H_2 = Q_2 + Q_3 - 2Q_6.$$

Thus, we see that in our constrained system ϵ_{ij}, $(\bar{S}_o + \beta)\xi^o$ and $\bar{S}_w\xi^w - \beta\xi^o$ play the role of generalized strains, which are linear functions of the stresses $\Delta\tau_{ij}$, Δp_o, and Δp_w [cf. (27)]. Next, set

$$\widetilde{Y} = [(\Delta\tau_{ij})_{1<i,j<3}, \Delta p_o, \Delta p_w]^t,$$
$$\widetilde{Z} = [(\epsilon_{ij})_{1<i,j<3}, (\bar{S}_o + \beta)\xi^o, \bar{S}_w\xi^w - \beta\xi^o]^t.$$

Then, (30) can be written in the form

$$\widetilde{Z} = \widetilde{E}\widetilde{Y}, \tag{31}$$

where $\widetilde{E} \in R^{11\times 11}$ is a symmetric matrix. Combining (2), (25), (27), and (28) shows that

$$W = \tfrac{1}{2}[\widetilde{E}\widetilde{Y}, \widetilde{Y}]_e$$
$$= \tfrac{1}{2}[EY, Y]_e$$
$$\geqslant \frac{\lambda_*(E)}{2}\Big((\Delta p_o)^2 + (\Delta p_w)^2 + \sum_{i,j}(\Delta\tau_{ij})^2\Big).$$

Thus, the matrix $\widetilde{E}$ in (31) is positive definite. Also, combining (2), (25), (27), and (29) gives

$$W = \tfrac{1}{2}[\widetilde{Z}, (\widetilde{E})^{-1}\widetilde{Z}]_e$$
$$= \tfrac{1}{2}[Z, E^{-1}Z]_e$$
$$\geqslant C_1\Big((\xi^o)^2 + (\xi^w)^2 + \sum_{i,j}[\epsilon_{ij}(u^s)]^2\Big). \tag{32}$$

A calculation shows that the matrix $(\widetilde{E})^{-1}$ has the same structure as $\widetilde{E}$, which allows us to write the stress–strain relations

$$\widetilde{Y} = (\widetilde{E})^{-1}\widetilde{Z} \tag{33}$$

in the form

$$\Delta\tau_{ij} = 2N\epsilon_{ij} + \delta_{ij}[\lambda_c e - R_1(\bar{S}_o + \beta)\xi^o - R_2(\bar{S}_w\xi^w - \beta\xi^o)], \tag{34a}$$

$$\Delta p_o = -R_1 e + J_1(\bar{S}_o + \beta)\xi^o + J_3(\bar{S}_w\xi^w - \beta\xi^o), \tag{34b}$$

$$\Delta p_w = -R_2 e + J_3(\bar{S}_o + \beta)\xi^o + J_2(\bar{S}_w\xi^w - \beta\xi^o), \tag{34c}$$

where

$$e = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}.$$

From (34) we see immediately that the generalized stresses $\Delta\tau_{ij}$, $(\bar{S}_o + \beta)\Delta p_o - \beta\Delta p_w$, and $\bar{S}_w\Delta p_w$ appearing in the expression for the strain energy W given in (27) can be written in the form

$$\Delta\tau_{ij} = 2N\epsilon_{ij} + \delta_{ij}(\lambda_c e - B_1\xi^o - B_2\xi^w), \tag{35a}$$

$$(\bar{S}_o + \beta)\Delta p_o - \beta\Delta p_w = -B_1 e + M_1\xi^o + M_3\xi^w, \tag{35b}$$

$$\bar{S}_w\Delta p_w = -B_2 e + M_3\xi^o + M_2\xi^w, \tag{35c}$$

where

$$B_1 = (\bar{S}_o + \beta)R_1 - \beta R_2, \quad B_2 = \bar{S}_w R_2,$$
$$M_1 = (\bar{S}_o + \beta)^2 J_1 - 2\beta(\bar{S}_o + \beta)J_3 + \beta^2 J_2,$$
$$M_2 = \bar{S}_w^2 J_2, \quad M_3 = \bar{S}_w[(\bar{S}_o + \beta)J_3 - \beta J_2].$$

Set

$$\hat{Y} = [(\Delta\tau_{ij})_{1<i,j<3}, (\bar{S}_o + \beta)\Delta p_o - \beta\Delta p_w, \bar{S}_w\Delta p_w],$$
$$\hat{Z} = [(\epsilon_{ij})_{1<i,j<3}, \xi^o, \xi^w].$$

In matrix form (35) becomes

$$\hat{Y} = \hat{E}\hat{Z}.$$

On the other hand, we see by (33) that $\hat{E} = L(\widetilde{E})^{-1}L^t$, where

$$L = \begin{bmatrix} I & 0 \\ 0 & \hat{L} \end{bmatrix}, \quad \hat{L} = \begin{bmatrix} \bar{S}_o + \beta & \beta \\ 0 & \bar{S}_w \end{bmatrix},$$

I being the identity matrix in $R^{9\times 9}$. Since $\det \hat{L} > 0$, $\hat{E}$ is also symmetric and positive-definite.

The coefficients in the right-hand side of (34) or (35) should be determined by performing the analogues of the jacketed and unjacketed compressibility tests as described in Ref. 2. This problem will be the subject of a complementary publication.[7]

Next, we shall obtain some relations satisfied by the potential energy $\mathscr{V}$ of the system; these will be useful in deriving the equations of motion. Recall that $\mathscr{V}$ is defined by the equation

$$\mathscr{V} = \int_\Omega W\,dx - \int_{\partial\Omega}(f_i^s u_i^s + f_i^o\tilde{u}_i^o + f_i^w\tilde{u}_i^w)d\sigma. \tag{36}$$

Since $W = W^*$, obviously

$$\mathscr{V} = \mathscr{V}^*. \tag{37}$$

Also, transforming the surface integral in the right-hand side of (36) into a volume integral in the usual fashion, we can always write $\mathscr{V}$ in the form

$$\mathscr{V} = \int_\Omega \mathscr{V}_d\,dx, \tag{38}$$

where $\mathscr{V}_d$ denotes the potential energy density of the system.

Let us consider a perturbation of the system from the equilibrium state. Using (21), the argument leading to (18) shows that

$$\delta\mathscr{V}^* = \int_\Omega \delta W^*\,dx - \int_{\partial\Omega}(u_i^s\delta f_i^s + \tilde{u}_i^o\delta f_i^o + \tilde{u}_i^w\delta f_i^w)d\sigma$$
$$= \int_\Omega\Big(-u_i^s\delta\frac{\partial\Delta\tau_{ij}}{\partial x_j} + u_i^o\delta\frac{\partial}{\partial x_i}(\bar{S}_o\Delta p_o + \beta\Delta p_c) + u_i^w\delta\frac{\partial}{\partial x_i}(\bar{S}_w\Delta p_w) + \lambda\delta[\Delta p_c - (\Delta p_o - \Delta p_w)]\Big)dx$$
$$= \int_\Omega \delta\mathscr{V}_d^*\,dx,$$

with $\mathscr{V}_d^*$ denoting the complementary potential energy density. Thus,

$$\delta\mathscr{V}_d^* = -u_i^s\delta\frac{\partial\Delta\tau_{ij}}{\partial x_j} + u_i^o\delta\frac{\partial}{\partial x_i}(\bar{S}_o\Delta p_o + \beta\Delta p_c) + u_i^w\delta\frac{\partial}{\partial x_i}(\bar{S}_w\Delta p_w) + \lambda\delta[\Delta p_c - (\Delta p_o - \Delta p_w)].$$

Hence, (37), (38), and the assumption that $\delta\mathscr{V}_d^*$ is an exact differential imply that

$$\mathscr{V}_d = \mathscr{V}_d^* = -u_i^s \frac{\partial \Delta\tau_{ij}}{\partial x_j} + u_i^o \frac{\partial}{\partial x_i}(\bar{S}_o \Delta p_o + \beta \Delta p_c) + u_i^w \frac{\partial}{\partial x_i}(\bar{S}_w \Delta p_w) + \lambda[\Delta p_c - (\Delta p_o - \Delta p_w)].$$

Note that, when the constraint (2) is imposed on the system, the last term in the right-hand side above disappears. Also, if u_i^s, u_i^o, and u_i^w are chosen as generalized coordinates to describe our system, the hypothesis that the system is conservative gives us the relations

$$\frac{\partial \mathscr{V}_d}{\partial u_i^s} = -\frac{\partial \Delta\tau_{ij}}{\partial x_j}, \tag{39a}$$

$$\frac{\partial \mathscr{V}_d}{\partial u_i^o} = \frac{\partial}{\partial x_i}(\bar{S}_o \Delta p_o + \beta \Delta p_c) = \frac{\partial}{\partial x_i}[\bar{S}_o \Delta p_o + p_c(\bar{S}_o)\Delta S_o], \tag{39b}$$

$$\frac{\partial \mathscr{V}_d}{\partial u_i^w} = \frac{\partial}{\partial x_i}(\bar{S}_w \Delta p_w), \quad 1 \leqslant i \leqslant 3. \tag{39c}$$

II. THE EQUATIONS OF MOTION

Set $u = (u_i^s, u_i^o, u_i^w) = (u_j)$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 9$. The u_j's will be chosen to be the generalized coordinates or state variables to describe the evolution of the fluid–solid system. In order to obtain the Lagrangian form of the equations of motion we need to compute the kinetic energy density T and the dissipation energy density function $\mathscr{D}$. Let us take a unit cube Q of bulk material and let Q_p denote the porous part of Q. Let ρ_θ, $\theta = s,o,w$, be the mass densities of solid, oil, and water, respectively, and let

$$\rho_1 = (1 - \phi)\rho_s$$

be the mass of solid per unit volume of bulk material.

Let v_i^o and v_i^w denote the relative microvelocity field of each particle of oil and water, respectively. Since the relative flow inside the pores is assumed to be of laminar type, the following linear relations must hold:

$$v_i^o = a_{ij}\frac{\partial u_j^o}{\partial t} + b_{ij}\frac{\partial u_j^w}{\partial t}, \quad v_i^w = c_{ij}\frac{\partial u_j^o}{\partial t} + d_{ij}\frac{\partial u_j^w}{\partial t}. \tag{40}$$

Now, we observe that, in $Q \setminus Q_p$, the kinetic energy is given by

$$\frac{1}{2}\int_{Q \setminus Q_p} \rho_s \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t}\, dx = \frac{1}{2}\rho_1 \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^3}{\partial t}.$$

Thus, since the amount of mass of oil in Q_p is given by $\phi\rho_o S_o$ and that of water by $\phi\rho_w S_w$, the kinetic energy T in the cube Q is given by the expression

$$T = \frac{1}{2}\rho_1 \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t} + \frac{1}{2}\rho_o S_o \int_{Q_p}\left(\frac{\partial u_i^s}{\partial t} + v_i^o\right)\left(\frac{\partial u_i^s}{\partial t} + v_i^o\right) dx + \frac{1}{2}\rho_w S_w \int_{Q_p}\left(\frac{\partial u_i^s}{\partial t} + v_i^w\right)\left(\frac{\partial u_i^s}{\partial t} + v_i^w\right) dx. \tag{41}$$

Let us compute each of the integral terms above. First, let

$$\rho_2 = \phi(\rho_o S_o + \rho_w S_w)$$

be the mass of fluid per unit volume of bulk material. Then, since $\partial u_i^s/\partial t$ is constant on Q,

$$\frac{1}{2}\left(\rho_o S_o \int_{Q_p} \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t}\, dx + \rho_w S_w \int_{Q_p} \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t}\, dx\right) = \frac{1}{2}\rho_2 \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t}.$$

Also, since $\partial u_i^\theta/\partial t$ is obtained by averaging v_i^θ over the cube Q, we see that

$$\rho_\theta S_\theta \int_{Q_p} \frac{\partial u_i^s}{\partial t} v_i^\theta\, dx = \rho_\theta S_\theta \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^\theta}{\partial t}, \quad \theta = o,w.$$

Next, since $\partial u_i^o/\partial t$ is constant over Q_p, using (40) we obtain

$$\rho_o S_o \int_{Q_p} v_k^o v_k^o\, dx = S_o\left(m_{1_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^o}{\partial t} + m_{2_{ij}} \frac{\partial u_i^w}{\partial t}\frac{\partial u_j^w}{\partial t} + 2m_{3_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^w}{\partial t}\right),$$

where

$$m_{1_{ij}} = \rho_o \int_{Q_p} a_{ki} a_{kj}\, dx,$$

$$m_{2_{ij}} = \rho_o \int_{Q_p} b_{ki} b_{kj}\, dx,$$

$$m_{3_{ij}} = \rho_o \int_{Q_p} a_{ki} b_{kj}\, dx.$$

Similarly,

$$\rho_w S_w \int_{Q_p} v_k^w v_k^w\, dx = S_w\left(q_{1_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^o}{\partial t} + q_{2_{ij}} \frac{\partial u_i^w}{\partial t}\frac{\partial u_j^w}{\partial t} + 2q_{3_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^w}{\partial t}\right),$$

with

$$q_{1_{ij}} = \rho_w \int_{Q_p} c_{ki} c_{kj}\, dx,$$

$$q_{2_{ij}} = \rho_w \int_{Q_p} d_{ki} d_{kj}\, dx,$$

$$q_{3_{ij}} = \rho_w \int_{Q_p} d_{ki} c_{kj}\, dx.$$

Thus, the kinetic energy density T in (41) takes the final form

$$T = \frac{1}{2}\rho \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^s}{\partial t} + \rho_o S_o \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^o}{\partial t} + \rho_w S_w \frac{\partial u_i^s}{\partial t}\frac{\partial u_i^w}{\partial t} + \frac{1}{2} g_{1_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^o}{\partial t} + \frac{1}{2} g_{2_{ij}} \frac{\partial u_i^w}{\partial t}\frac{\partial u_j^w}{\partial t} + g_{3_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^w}{\partial t}, \tag{42}$$

where

$$\rho = \rho_1 + \rho_2,$$

$$g_1 = S_o m_1 + S_w q_1,$$

$$g_2 = S_o m_2 + S_w q_2,$$

$$g_3 = S_o m_3 + S_w q_3,$$

where ρ is the mass density of the bulk material. The matrices corresponding to a_{ij}, b_{ij}, c_{ij}, and d_{ij} must be such that g_3 is symmetric.

Now, we proceed to compute the form of the dissipation energy density function $\mathscr{D}$. Recall that dissipation has been assumed to depend only on the relative motion between the fluids and the rock frame. Also, it is known[11] that $\mathscr{D}$ can be written as as quadratic form in the relative velocities $\partial u^o/\partial t$, $\partial u^w/\partial t$. Thus, ignoring the friction effects between the oil and water phases, we can write $\mathscr{D}$ in the form

$$\mathscr{D} = \frac{1}{2}\left(\mu_o r_{o_{ij}} \frac{\partial u_i^o}{\partial t}\frac{\partial u_j^o}{\partial t} + \mu_w r_{w_{ij}} \frac{\partial u_i^w}{\partial t}\frac{\partial u_j^w}{\partial t}\right), \quad (43)$$

μ_o and μ_w being the oil and water viscosities, respectively.

Next, we shall relate the matrices $R_o = (r_{o_{ij}})$ and $R_w = (r_{w_{ij}})$ to the relative permeability functions $k_{ro} = k_{ro}(S_o)$ and $k_{rw} = k_{rw}(S_o)$ and to the absolute permeability matrix $K = [k_{ij}(x)]$ using Darcy's law for two-phase flow in porous media. The symbols S_{ro} and S_{rw} will denote the residual oil and water saturations, respectively. Note that in the range $0 \leqslant S_o \leqslant S_{ro}$ the oil is not allowed to move relative to the solid, and similarly, the water is not allowed any relative motion for $S_w \in [0, S_{rw}]$. Since in our model both fluid phases are allowed to move, we have implicitly assumed that

$$S_{ro} < S_o < 1 - S_{rw}. \quad (44)$$

The reader is referred to Refs. 12, 8, 13, and 9 for detailed discussions of multiphase flow in porous media.

Next, recall that in the absence of generalized external forces, the fundamental relation between forces and state variables can be expressed in the form[11]

$$\frac{\partial \mathscr{V}_d}{\partial u_i} + \frac{\partial \mathscr{D}}{\partial \dot{u}_i} = 0, \quad 1 \leqslant i \leqslant 9. \quad (45)$$

Hence, using (39b), (39c), (43), and (45), we obtain the relations

$$\mu_o R_o \frac{\partial u^o}{\partial t} = -\nabla[\bar{S}_o \Delta p_o + p_c(\bar{S}_o)\Delta S_o], \quad (46a)$$

$$\mu_w R_w \frac{\partial u^w}{\partial t} = -\nabla(\bar{S}_w \Delta p_w). \quad (46b)$$

Ignoring gravity forces, we can write Darcy's law for our system in the form[8,9,13]

$$\frac{\partial}{\partial t}(S_o u^o) = -K\frac{k_{ro}}{\mu_o}\nabla \Delta p_o, \quad (47a)$$

$$\frac{\partial}{\partial t}(S_w u^w) = -K\frac{k_{rw}}{\mu_w}\nabla \Delta p_w, \quad (47b)$$

while conservation of mass of each phase is given by the relations

$$\frac{\partial(\phi\rho_o S_o)}{\partial t} = \nabla\cdot\left(\rho_o K\frac{k_{ro}}{\mu_o}\nabla \Delta p_o\right), \quad (48a)$$

$$\frac{\partial(\phi\rho_w S_w)}{\partial t} = \nabla\cdot\left(\rho_w K\frac{k_{rw}}{\mu_w}\nabla \Delta p_w\right), \quad (48b)$$

Using Eqs. (48), we can analyze the time and spatial dependence of the saturation for the wavelengths under consideration. For that purpose, let λ be a characteristic wave length and let $\bar{p}_o$ and $\bar{S}_o$ be the reference oil pressure and the reference saturation. Let us assume for the moment that all coefficients in (48a) are constant and that the permeability K is scalar. Then, the change of variables $x' = x/\lambda$, $t' = t/\tau$ in (48a) shows that the characteristic time τ for a significant change in oil saturation takes the form

$$\tau = \phi \bar{S}_o \mu_o \lambda^2 / \bar{p}_o k_{ro} K.$$

An evaluation of the expression above for common values of the variables involved and for wavelengths of the order of one to ten centimeters gives us values of τ of the order of a tenth of a second, which is at least three orders of magnitude greater than the time at which significant changes in oil and water pressures are expected. Thus, we shall assume that S_o (and S_w) are independent of time, which in turn implies that $\nabla \Delta S_o$ is much smaller than $\nabla \Delta p_o$. This fact and (9) allow us to rewrite (46) in the form

$$\mu_o R_o \frac{\partial u^o}{\partial t} = -\bar{S}_o \nabla \Delta p_o,$$
$$\mu_w R_w \frac{\partial u^w}{\partial t} = -\bar{S}_w \nabla \Delta p_w. \quad (49)$$

Also, using the hypothesis of time independence of the saturation and neglecting terms containing a factor ΔS_θ, $\theta = o,w$, we see that

$$\frac{\partial}{\partial t}(S_\theta u^\theta) = \frac{\partial}{\partial t}[(\bar{S}_\theta + \Delta S_\theta)u^\theta] \cong \bar{S}_\theta \frac{\partial u^\theta}{\partial t}, \quad \theta = o,w.$$

Hence, (47) becomes

$$\bar{S}_\theta \frac{\partial u^\theta}{\partial t} = -K\frac{k_{r_\theta}}{\mu_\theta}\nabla \Delta p_\theta, \quad \theta = o,w. \quad (50)$$

From (49) and (50) we conclude that

$$R_\theta = [(\bar{S}_\theta)^2 / k_{r_\theta}]K^{-1}, \quad \theta = o,w, \quad (51)$$

which gives us the desired expressions for the matrices R_o and R_w in (43).

The Lagrange formulation of the equations of motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_i}\right) + \frac{\partial \mathscr{D}}{\partial \dot{u}_i} = -\frac{\partial \mathscr{V}_d}{\partial u_i}, \quad 1 \leqslant i \leqslant 9.$$

Thus, using (39), (42), (43), and (51), the assumption of time independence for the saturation and the linearizing argument given in (50) to drop terms containing factors of ΔS_o in $d/dt(\partial T/\partial \dot{u}_i)$, we obtain the equations

$$\rho \frac{\partial^2 u_i^s}{\partial t^2} + \rho_o \bar{S}_o \frac{\partial^2 u_i^o}{\partial t^2} + \rho_w \bar{S}_w \frac{\partial^2 u_i^w}{\partial t^2} = \frac{\partial \Delta \tau_{ij}}{\partial x_j}, \quad (52a)$$

$$\rho_o \bar{S}_o \frac{\partial^2 u_i^s}{\partial t^2} + \bar{g}_{1_{ij}} \frac{\partial^2 u_j^o}{\partial t^2} + \bar{g}_{3_{ij}} \frac{\partial^2 u_j^w}{\partial t^2} + (\bar{S}_o)^2 \frac{\mu_o}{k_{r_o}}(K^{-1})_{ij}\frac{\partial u_j^o}{\partial t} = -\frac{\partial}{\partial x_i}[(\bar{S}_o + \beta)\Delta p_o - \beta \Delta p_w], \quad (52b)$$

$$\rho_w \overline{S}_w \frac{\partial^2 u_i^s}{\partial t^2} + \overline{g}_{3_{ij}} \frac{\partial^2 u_j^o}{\partial t^2} + \overline{g}_{2_{ij}} \frac{\partial^2 u_j^w}{\partial t^2} + (\overline{S}_w)^2 \frac{\mu_w}{k_{r_w}} (K^{-1})_{ij} \frac{\partial u_j^w}{\partial t} = -\frac{\partial}{\partial x_i} (\overline{S}_w \Delta p_w), \quad i=1,2,3. \tag{52c}$$

III. CLASSIFICATION OF THE WAVES IN ISOTROPIC MEDIA

In the isotropic case (35) shows that Eqs. (52) take the form

$$\rho \frac{\partial^2 u_i^s}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 u_i^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 u_i^w}{\partial t^2} = \frac{\partial}{\partial x_j} [2N\epsilon_{ij} + \delta_{ij} (\lambda_c e - B_1 \xi^o - B_2 \xi^w)], \tag{53a}$$

$$\rho_o \overline{S}_o \frac{\partial^2 u_i^s}{\partial t^2} + \overline{g}_1 \frac{\partial^2 u_i^o}{\partial t^2} + \overline{g}_3 \frac{\partial^2 u_i^w}{\partial t^2} + (\overline{S}_o)^2 \frac{\mu_o}{kk_{r_o}} \frac{\partial u_i^o}{\partial t} = -\frac{\partial}{\partial x_i} (-B_1 e + M_1 \xi^o + M_3 \xi^w), \tag{53b}$$

$$\rho_o \overline{S}_w \frac{\partial^2 u_i^s}{\partial t^2} + \overline{g}_3 \frac{\partial^2 u_i^o}{\partial t^2} + \overline{g}_2 \frac{\partial^2 u_i^w}{\partial t^2} + (\overline{S}_w)^2 \frac{\mu_w}{kk_{r_w}} \frac{\partial u_i^w}{\partial t} = -\frac{\partial}{\partial x_i} (-B_2 e + M_3 \xi^o + M_2 \xi^w), \quad i=1,2,3. \tag{53c}$$

The matrices $\overline{g}_1$, $\overline{g}_2$, $\overline{g}_3$, and K have been taken as scalars: $\overline{g}_{1ij}, = \overline{g}_1 \delta_{ij}$, $\overline{g}_{2ij} = \overline{g}_2 \delta_{ij}$, $\overline{g}_{3ij} = \overline{g}_3 \delta_{ij}$, and $K_{ij} = k\delta_{ij}$.

Here, we can assume constant coefficients. After some algebraic manipulations, we can write the equations above in the vector form

$$\rho \frac{\partial^2 u^s}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 u^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 u^w}{\partial t^2} = N\Delta u^s + \nabla [(\lambda_c + N)e + B_1 \epsilon^o + B_2 \epsilon^w], \tag{54a}$$

$$\rho_o \overline{S}_o \frac{\partial^2 u^s}{\partial t^2} + \overline{g}_1 \frac{\partial^2 u^o}{\partial t^2} + \overline{g}_3 \frac{\partial^2 u^w}{\partial t^2} + (\overline{S}_o)^2 \frac{\mu_o}{kk_{r_o}} \frac{\partial u^o}{\partial t} = \nabla [B_1 e + M_1 \epsilon^o + M_3 \epsilon^w], \tag{54b}$$

$$\rho_w \overline{S}_w \frac{\partial^2 u^s}{\partial t^2} + \overline{g}_3 \frac{\partial^2 u^o}{\partial t^2} + \overline{g}_2 \frac{\partial^2 u^w}{\partial t^2} + (\overline{S}_w)^2 \frac{\mu_w}{kk_{r_w}} \frac{\partial u^w}{\partial t} = \nabla [B_2 e + M_3 \epsilon^o + M_2 \epsilon^w], \tag{54c}$$

where

$$\epsilon^o = \nabla \cdot u^o, \quad \epsilon^w = \nabla \cdot u^w.$$

To obtain the equation governing the propagation of dilatational waves, we apply the divergence operator to the relations above. In doing so, we get the equations

$$\Delta [Ge + B_1 \epsilon^o + B_2 \epsilon^w] = \rho \frac{\partial^2 e}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 \epsilon^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 \epsilon^w}{\partial t^2},$$

$$\Delta [B_1 e + M_1 \epsilon^o + M_3 \epsilon^w] = \rho_o \overline{S}_o \frac{\partial^2 e}{\partial t^2} + \overline{g}_1 \frac{\partial^2 \epsilon^o}{\partial t^2} + \overline{g}_3 \frac{\partial^2 \epsilon^w}{\partial t^2} + (\overline{S}_o)^2 \frac{\mu_o}{kk_{r_o}} \frac{\partial \epsilon^o}{\partial t},$$

$$\Delta [B_2 e + M_3 \epsilon^o + M_2 \epsilon^w] = \rho_w \overline{S}_w \frac{\partial^2 e}{\partial t^2} + \overline{g}_3 \frac{\partial^2 \epsilon^o}{\partial t^2} + \overline{g}_2 \frac{\partial^2 \epsilon^w}{\partial t^2} + (\overline{S}_w)^2 \frac{\mu_w}{kk_{r_w}} \frac{\partial \epsilon^w}{\partial t}, \tag{55}$$

with

$$G = \lambda_c + 2N.$$

Now, assume a plane compressional wave of angular frequency c and wave number $a_r + ia_i$ traveling in the x_1 direction in the form

$$e = q_1^{(a)} e^{i(ax_1 + ct)},$$
$$\epsilon^o = q_2^{(a)} e^{i(ax_1 + ct)},$$
$$\epsilon^w = q_3^{(a)} e^{i(ax_1 + ct)}.$$

Thus, the wave has phase velocity $v = c/|a_r|$ and attenuation coefficient a_i. Set $\alpha = c/a = \alpha_r + i\alpha_i$. Then, substitution in (55) gives us the vector equation

$$\widetilde{\mathscr{B}} q^{(a)} = \alpha^2 (\widetilde{\mathscr{A}} q^{(a)} - i\widetilde{\mathscr{C}} q^{(a)}), \tag{56}$$

where

$$q^{(a)} = (q_1^{(a)}, q_2^{(a)}, q_3^{(a)}),$$

$$\widetilde{\mathscr{C}} = \mathrm{diag}\left(0, \frac{(\overline{S}_o)^2}{c} \frac{\mu_o}{kk_{r_o}}, \frac{(\overline{S}_w)^2}{c} \frac{\mu_w}{kk_{r_w}}\right),$$

and $\widetilde{\mathscr{A}} \in R^{3\times3}$ and $\widetilde{\mathscr{B}} \in R^{3\times3}$ have the forms

$$\widetilde{\mathscr{B}} = \begin{bmatrix} G & B_1 & B_2 \\ B_1 & M_1 & M_3 \\ B_2 & M_3 & M_2 \end{bmatrix}, \quad \widetilde{\mathscr{A}} = \begin{bmatrix} \rho & \rho_o \overline{S}_o & \rho_w \overline{S}_w \\ \rho_o \overline{S}_o & \overline{g}_1 & \overline{g}_3 \\ \rho_w \overline{S}_w & \overline{g}_3 & \overline{g}_2 \end{bmatrix}.$$

Next, we observe that since $\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{A}}$ are associated with the strain and kinetic energies, they are positive definite, while the diagonal matrix $\widetilde{\mathscr{C}}$ is obviously nonnegative. Thus, any solution α^2 of the generalized eigenvalue problem (56) satisfies the condition

$$\mathrm{Re}(\alpha^2) > 0, \quad \mathrm{Im}(\alpha^2) \geqslant 0.$$

Let $(\alpha^{(j)})^2, j = \mathrm{I, II, III}$, be the solutions of (56). Using the relation

$$a_r^{(j)} + ia_i^{(j)} = c/\alpha^{(j)} = c(\alpha_r^{(j)} - i\alpha_i^{(j)})/|\alpha^{(j)}|^2,$$

we choose $\alpha^{(j)}$ such that $\alpha_i^{(j)} \leqslant 0$, so that $a_i^{(j)} \geqslant 0$ and we have the physically meaningful solution. The corresponding phase velocities are given by

$$v^{(j)} = c/|a_r^{(j)}|.$$

Let us consider the purely elastic case (i.e., $\mu_o = \mu_w = 0$). Multiplying both sides of (56) by $\widetilde{\mathscr{A}}^{-1/2}$, we deduce the relation

$$M\overline{q}^\alpha = \alpha^2 \overline{q}^\alpha,$$

with $M = \widetilde{\mathscr{A}}^{-1/2} \widetilde{\mathscr{B}} \widetilde{\mathscr{A}}^{-1/2}$ and $\overline{q}^\alpha = \widetilde{\mathscr{A}}^{-1/2} q^\alpha$. Let $\overline{q}^{\alpha_l}$ be the set of orthonormal eigenvectors associated with the symmetric, positive-definite matrix M. The fact that the $\overline{q}^{\alpha_p}$s are orthogonal implies that

$$[\widetilde{\mathscr{A}} q^{\alpha_i}, q^{\alpha_j}]_e = 0, \quad i \neq j. \tag{57}$$

The orthogonality relation (57) is analogous to the one derived by Biot[1] for the single phase case. For realistic values of the parameters in $\widetilde{\mathscr{B}}$, $\widetilde{\mathscr{A}}$, and $\widetilde{\mathscr{C}}$ the three compressional wave phase velocities and attenuations are different.[7]

Now, we proceed to analyze the rotational waves. Let $\omega^\theta = \operatorname{curl}(u^\theta)$, $\theta = s,o,w$. Then, applying the curl operator to (54), we obtain the equations

$$\rho \frac{\partial^2 \omega^s}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 \omega^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 \omega^w}{\partial t^2} = N \Delta \omega^s, \quad (58a)$$

$$\rho_o \overline{S}_o \frac{\partial^2 \omega^s}{\partial t^2} + \overline{g}_1 \frac{\partial^2 \omega^o}{\partial t^2} + \overline{g}_3 \frac{\partial^2 \omega^w}{\partial t^2} + (\overline{S}_o)^2 \frac{\mu_o}{kk_{r_o}} \frac{\partial \omega^o}{\partial t} = 0, \quad (58b)$$

$$\rho_w \overline{S}_w \frac{\partial^2 \omega^s}{\partial t^2} + \overline{g}_3 \frac{\partial^2 \omega^o}{\partial t^2} + \overline{g}_2 \frac{\partial^2 \omega^w}{\partial t^2} + (\overline{S}_w)^2 \frac{\mu_w}{kk_{r_w}} \frac{\partial \omega^w}{\partial t} = 0. \quad (58c)$$

Again, let us consider a plane rotational wave of angular frequency c and wave number $a^{(s)} = a_r^{(s)} + i a_i^{(s)}$ traveling in the x_1-direction in the form

$$\omega^s = q_1^{(a^{(s)})} e^{i(a^{(s)} x_1 + ct)},$$
$$\omega^o = q_2^{(a^{(s)})} e^{i(a^{(s)} x_1 + ct)},$$
$$\omega^w = q_3^{(a^{(s)})} e^{i(a^{(s)} x_1 + ct)}.$$

Substitution in (58b) and (58c) yields the relations

$$q_2^{(a^{(s)})} = -\frac{g_2^* \rho_o \overline{S}_o - \overline{g}_3 \rho_w \overline{S}_w}{g_1^* g_2^* - \overline{g}_3^2} q_1^{(a^{(s)})},$$

$$q_3^{(a^{(s)})} = -\frac{g_1^* \rho_w \overline{S}_w - \overline{g}_3 \rho_o \overline{S}_o}{g_1^* g_2^* - \overline{g}_3^2} q_1^{(a^{(s)})},$$

where

$$g_1^* = \overline{g}_1 - [i(\overline{S}_o)^2/c](\mu_o/kk_{r_o}),$$
$$g_2^* = \overline{g}_2 - [i(\overline{S}_w)^2/c](\mu_w/kk_{r_w}).$$

Using the expressions above in (58a) we obtain the equation

$$\left(\frac{c}{a^{(s)}}\right)^2 = N\left(\rho - \frac{\rho_o \overline{S}_o (g_2^* \rho \overline{S}_o - \overline{g}_3 \rho_w \overline{S}_w) + \rho_w \overline{S}_w (g_1^* \rho_w \overline{S}_w - \overline{g}_3 \rho_o \overline{S}_o)}{g_1^* g_2^* - \overline{g}_3^2}\right)^{-1}, \quad (59)$$

which allows us to determine the shear wave phase velocity $v^{(s)} = c/|a_r^{(s)}|$ and attenuation coefficient $a_i^{(s)}$.

In a complementary publication[7] we analyze the behaviour of the different types of waves described here as they depend upon the different parameters involved.

IV. EXISTENCE AND UNIQUENESS RESULTS

Let the positive-definite mass matrix $\mathscr{A} \in R^{9\times 9}$ and the non-negative dissipation matrix $\mathscr{C} \in R^{9 \times 9}$ be defined by

$$\mathscr{A} = \begin{bmatrix} \rho I & \rho_o \overline{S}_o I & \rho_w \overline{S}_w I \\ \rho_o \overline{S}_o I & \overline{g}_1 & \overline{g}_3 \\ \rho_w \overline{S}_w I & \overline{g}_3 & \overline{g}_2 \end{bmatrix},$$

$$\mathscr{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{(\overline{S}_o)^2 \mu_o}{k_{r_o}} K^{-1} & 0 \\ 0 & 0 & \dfrac{(\overline{S}_w)^2 \mu_w}{k_{r_w}} K^{-1} \end{bmatrix},$$

I being the identity matrix in $R^{3\times 3}$. Also, let

$$\mathscr{L}(u) = \{\nabla \cdot \Delta\tau(u), -\nabla[(\overline{S}_o + \beta)\Delta p_o(u) - \beta \Delta p_w(u)], -\nabla[\overline{S}_w \Delta p_w(u)]\}.$$

Then, in the nonhomogeneous case, the vector form of the equations of motion (52) is given by

$$\mathscr{A}\frac{\partial^2 u}{\partial t^2} + \mathscr{C}\frac{\partial u}{\partial t} - \mathscr{L}(u) = F(x,t), \quad (x,t) \in \Omega \times (0,T) \equiv \Omega \times J. \quad (60)$$

Let us impose the initial conditions

$$u(x,0) = u^0, \quad x \in \Omega,$$
$$\frac{\partial u}{\partial t}(x,0) = v^0, \quad x \in \Omega, \quad (61)$$

and the boundary conditions

$$\Delta\tau\nu = -g(x,t), \quad (x,t) \in \partial\Omega \times J,$$
$$(\overline{S}_o + \beta)\Delta p_o - \beta \Delta p_w = \gamma_o(x,t), \quad (x,t) \in \partial\Omega \times J, \quad (62)$$
$$\overline{S}_w \Delta p_w = \gamma_w(x,t), \quad (x,t) \in \partial\Omega \times J.$$

Now, we shall introduce a weak form of problem (60)–(62). For $n \geqslant 1$, let $(\cdot,\cdot)$ denote the inner product in $[L^2(\Omega)]^n$, and let $\langle \cdot,\cdot \rangle$ the inner product in $[L^2(\partial\Omega)]^n$. Let ν denote the outer normal to $\partial\Omega$. Also, let

$$H(\operatorname{div},\Omega) = \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\},$$
$$V = [H^1(\Omega)]^3 \times H(\operatorname{div},\Omega) \times H(\operatorname{div},\Omega),$$

provided with the natural norm

$$\|v\|_V = [\|v_1\|_1^2 + \|v_2\|^2_{H(\operatorname{div},\Omega)} + \|v_3\|^2_{H(\operatorname{div},\Omega)}]^{1/2},$$
$$v = (v_1, v_2, v_3) \in V.$$

The weak form of (60) is found by testing Eq. (60) against $v \in V$ and consists of finding a map $u: J \to V$ such that

$$\left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right) + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right) + \Lambda(u,v) + \langle g, v_1 \rangle + \langle v_2 \cdot \nu, \gamma_o \rangle + \langle v_3 \cdot \nu, \gamma_w \rangle = (F, v), \quad v \in V, \quad t \in J,$$

where $\Lambda(\cdot,\cdot)$ is the symmetric, bilinear form on V defined by

$$\Lambda(v,w) = (\Delta\tau_{ij}(v),\epsilon_{ij}(w_1)) - ((\bar{S}_o + \beta)\Delta p_o(v) - \beta\Delta p_w(v),\nabla\cdot w_2) - ((\bar{S}_w \Delta p_w)(v),\nabla\cdot w_3).$$

Note that it follows from (27), (32), and Korn's second inequality[14–16] that

$$\Lambda(v,v) = 2\int_\Omega W(\epsilon_{ij}(v_1),\nabla\cdot v_2,\nabla\cdot v_3)dx \geqslant C_2\|v\|_V^2 - C_3\|v\|_0^2, \quad v\in V. \tag{63}$$

Next, let

$$P_r^2 = \left\|\frac{\partial^r g}{\partial t^r}\right\|^2_{L^\infty(J,[H^{-1/2}(\partial\Omega)]^3)} + \left\|\frac{\partial^{r+1} g}{\partial t^{r+1}}\right\|^2_{L^2\{J,[H^{-1/2}(\partial\Omega)]^3\}} + \left\|\frac{\partial^r \gamma_o}{\partial t^r}\right\|^2_{L^\infty[J,H^{1/2}(\partial\Omega)]} + \left\|\frac{\partial^{r+1} \gamma_o}{\partial t^{r+1}}\right\|^2_{L^2[J,H^{1/2}(\partial\Omega)]} + \left\|\frac{\partial^r \gamma_w}{\partial t^r}\right\|^2_{L^\infty[J,H^{1/2}(\partial\Omega)]} + \left\|\frac{\partial^{r+1} \gamma_w}{\partial t^{r+1}}\right\|^2_{L^2[J,H^{1/2}(\partial\Omega)]} + \left\|\frac{\partial^r F}{\partial t^r}\right\|^2_{L^2\{J,[L^2(\Omega)]^9\}},$$

$$Q^2 = \|u^0\|_2^2 + \|v^0\|_1^2 + \|F(x,0)\|_0^2 + 1.$$

We can state the main theorem on the existence and uniqueness of the solution of problem (60)–(62). Its proof will be omitted since it is very similar to that of the corresponding theorem in (Ref. 5) in the single-phase case.

Theorem 5.1: Let F, g, γ_o, γ_w, u^o, and v^o be given and such that $P_o < \infty$, $P_1 < \infty$, and $Q < \infty$. Then, there exists a unique solution $u(x,t)$ of (60)–(62) such that u, $\partial u/\partial t \in L^\infty(J,V)$ and $\partial^2 u/\partial t^2 \in L^\infty\{J,[L^2(\Omega)]^9\}$.

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