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'ZOEPPRITZ' RATIONALIZED AND GENERALIZED TO ANISOTROPY

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ABSTRACT

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The Zoeppritz equations for isotropic plane wave reflection and transmission coefficients are cast in a form from which explicit solutions for the reflectivity and transmissivity matrices, as functions of slowness parallel to the reflecting plane, are derived in terms of four 2 × 2 submatrices of the Zoeppritz coefficient matrix. Two depend on the elastic properties of one medium, two on the properties of the other. These submatrices, called the 'impedance matrices' of the medium, are found for anisotropic media also, subject only to the condition that the medium has a mirror plane of symmetry parallel to the reflecting plane. Then the explicit solutions found for isotropic media in terms of the impedance matrices hold as well for anisotropic media. The two impedance matrices associated with a medium are also the building blocks for a simple construction of the propagator matrix for plane waves in a layer of that medium. The advantages of this approach to reflectivity are twofold. First, the solution is expressed in a form where the working of its component elements, and its commonality over a wide range of material behaviors is more manifest, and secondly, the programming necessary for the computation of reflection and transmission coefficients, and propagator matrices, is reduced to the evaluation of the impedance matrices, leading to highly modular, and easy to debug, programs.

KEY WORDS: elastic plane waves, reflection and transmission, impedance matrices, monoclinic anisotropy, propagator matrices.

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INTRODUCTION

It has long been an aim of geophysicists to be able to identify lithology by the nature of the reflection of seismic waves at interfaces and/or layers in the subsurface. This is the problem of amplitude vs. offset (AVO) analysis. Recently, with the advent of long offset surface arrays, walkaway VSP, and cross-well surveys, the problem has become more complex, since the earth is not isotropic, and the effects of anisotropy become particularly apparent in data collected over a wide angular aperture.

Even more significantly, three dimensional surveys now make it possible to evaluate the reflectivity of an interface as a function of azimuth, (AVA) when the media involved are no longer azimuthally isotropic. It thus becomes helpful to have a coherent unifying formulation of the forward problem of finding reflection and transmission coefficients at an interface between two elastic half-spaces when one or both of the media are anisotropic. Such a formulation from which emerges an explicit solution to the plane wave reflection and transmission problem - a matrix generalization of the scalar reflection and transmission coefficients - for a very broad range of anisotropic media is derived here. The solution is in terms of submatrices of the coefficient matrix of the Zoeppritz equations (1919) - which are merely equations expressing continuity of displacement and stress traction across the interface - extended to anisotropy. The formulation is valid as long as the media involved exhibit up-down symmetry relative to the interface between the media. In terms of anisotropy, this is equivalent to requiring that the media are at least monoclinic with a mirror plane of symmetry parallel to the reflecting plane.

The two dimensional plane strain case occurs whenever the displacements lie in the plane of propagation (qP and qS waves), and these wave types are uncoulpled from the anti-plane SH waves which have their displacements perpendicular to the plane of propagation (for a detailed discussion of SH wave reflectivity in anisotropic media, see Schoenberg and Costa, 1991). Such a plane, called a 'pure shear' plane, need not be a plane of symmetry for the medium (Dellinger, 1992). A medium for which the horizontal x_1,x_2-plane is a mirror plane of symmetry, while the vertical x_1,x_3-plane is a pure shear plane is almost but not quite an orthorhombic medium; it may have a non-zero value of c_{26}, in standard condensed notation for the elastic modulus matrix. However, as this doesn't affect the discussion of two-dimesional anisotropic reflectivity and transmissivity, it will be assumed for the two-dimensional anisotropic discussion, that the vertical plane of propagation as well as the horizontal plane are actually mirror planes of symmetry of both media, implying the media must be at least orthorhombic.

First the isotropic-isotropic case is considered, the solution to the Zoeppritz equations is expressed in terms of two 2×2 'impedance matrices' for each medium, and all reflection and transmission coefficients are expressed explicitly in terms of these impedance matrices. Then, for the anisotropic-anisotropic case, the solution for the reflection and transmission coefficients is seen to be identical, and the anisotropic reflectivity and transmissivity problem is thus reduced to evaluating the two impedance matrices for each of the media involved. These impedance matrices depend only on the density and the elastic moduli of the medium and the common horizontal slowness (Snell's law) of all the plane waves in the problem, but they entail finding the eigenvalues and eigenvectors of the Christoffel equations governing plane wave propagation in the medium.

The very interesting three dimensional case occurs when the propagation plane is not a mirror plane of symmetry, in which case the velocities,and ray directions associated with the plane waves need not lie in the plane of propagation, i.e., that plane defined by the horizontal slowness along the interface and the normal to said plane. In this case the impedance matrices are two 3×3 submatrices of the 6×6 Zoeppritz coefficient matrix generalized to anisotropic media in three dimensions, but the solution is explicitly expressed in exactly the same form.

The advantages of this formulation are a) heuristic, in that one can see what each term in the solution is, and how and why almost all anisotropic cases have a common solution, and b) it allows a modeler to make more modular the computer code needed for the computation of reflection coefficients and propagator matrices in full wave layered media programs (see for example, Kennett, 1974), or in dynamic ray tracing programs.

PLANE STRAIN WAVES IN ISOTROPIC MEDIA

Consider two isotropic half spaces separated by the horizontal x_1,x_2-plane, with the x_3-axis positive downward, and allow propagation in the vertical x_1,x_3-plane of waves associated with in-plane motion. For harmonic waves with radial frequency ω and horizontal phase slowness s_1, so that all waves interacting at the interface contain the phase factor exp $i\omega(s_1 x_1 - t)$, which will be suppressed but must be kept in mind, the possible plane waves are downward and upward compressional waves, denoted by P, and downward and upward shear waves (with motion in the vertical plane), denoted by S. The positive sense of each unit eigenvector, denoted by **e**, associated with a plane wave is chosen (in accord with Aki and Richards, 1980) such that its horizontal component has the same sign as that of the horizontal component of phase slowness, s_1, see Fig. 1.

Let the upper incident isotropic medium occupying $x_3 < 0$ be specified by density ϱ, compressional speed α, and shear speed β. Every plane of an isotropic medium is a mirror plane of symmetry, so clearly, this medium has up-down symmetry. The velocity field in the incident halfspace due to incident and reflected P and S plane harmonic waves for $x_3 < 0$ can be written,

$$\begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = i_P \begin{bmatrix} \alpha s_1 \\ \alpha s_{3_P} \end{bmatrix} \exp i\omega s_{3_P} x_3 + r_P \begin{bmatrix} \alpha s_1 \\ -\alpha s_{3_P} \end{bmatrix} \exp -i\omega s_{3_P} x_3$$

$$+ i_S \begin{bmatrix} \beta s_{3_S} \\ -\beta s_1 \end{bmatrix} \exp i\omega s_{3_S} x_3 + r_S \begin{bmatrix} \beta s_{3_S} \\ \beta s_1 \end{bmatrix} \exp -i\omega s_{3_S} x_3 \ , \qquad (1)$$

where

$$s_{3_P} = \sqrt{(\alpha^{-2} - s_1^2)} \ , \qquad s_{3_S} = \sqrt{(\beta^{-2} - s_1^2)} \ .$$

The vectors associated with each of these waves are unit vectors, chosen in accord with the above mentioned sign convention assuming s_1 is positive.The coefficients i_P and i_S are the incident compressional and shear wave amplitudes, respectively, the only two possible incident plane strain waves at the particular frequency ω and horizontal slowness s_1. Of course, one of these being zero means there is only a single incident wave, P or S. The coefficients r_P and r_S are the reflected compressional and shear wave amplitudes.

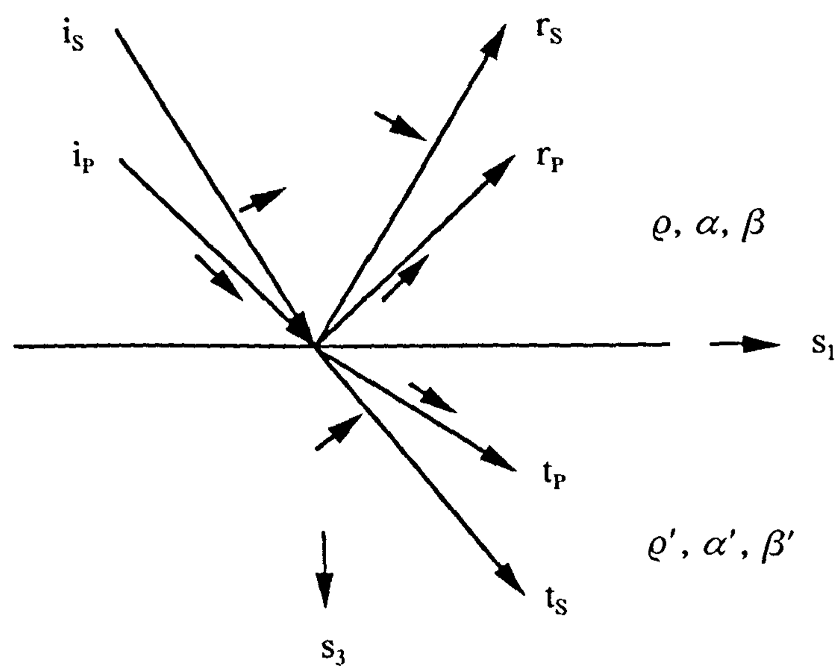

Fig. 1. Slowness vectors of the waves making up the full wave-field at horizontal slowness s_1 at a plane interface. Unit polarization vectors are the small arrows; the letters subscripted with P or S are the wave amplitudes.

From the assumed motion, the stress may be found from the isotropic stress-strain law which may be put in the form of a stress-velocity relation. Velocities have been used instead of displacements to eliminate any frequency dependence in the derivation. The plane strain stress traction on a constant x_3 plane has components $\sigma_{33} \equiv \sigma_3$ and $\sigma_{13} \equiv \sigma_5$. Condensed notation is used here even for the isotropic case to better smooth the below discussed generalization to anisotropy. The isotropic stress-velocity law applied to the velocity field (1) gives

$$\begin{bmatrix} \sigma_3 \\ \sigma_5 \end{bmatrix} = i_P \begin{bmatrix} -\varrho\alpha\Gamma \\ -2\varrho\alpha\beta^2 s_1 s_{3_P} \end{bmatrix} \exp i\omega s_{3_P} x_3 + r_P \begin{bmatrix} -\varrho\alpha\Gamma \\ 2\varrho\alpha\beta^2 s_1 s_{3_P} \end{bmatrix} \exp -i\omega s_{3_P} x_3$$

$$+ i_S \begin{bmatrix} 2\varrho\beta^3 s_1 s_{3_S} \\ -\varrho\beta\Gamma \end{bmatrix} \exp i\omega s_{3_S} x_3 + r_S \begin{bmatrix} 2\varrho\beta^3 s_1 s_{3_S} \\ \varrho\beta\Gamma \end{bmatrix} \exp -i\omega s_{3_S} x_3 , \quad (2)$$

$$\Gamma = 1 - 2\beta^2 s_1^2 .$$

Equations (1) and (2) may be reordered to yield the following matrix form for velocity and traction components in the incident medium,

$$\mathbf{b}_X(x_3) \equiv \begin{bmatrix} v_1 \\ \sigma_3 \end{bmatrix}_{x_3} = \mathbf{X}\,[\Lambda(x_3)\mathbf{i} + \Lambda^{-1}(x_3)\mathbf{r}] ,$$

$$\mathbf{b}_Y(x_3) \equiv \begin{bmatrix} \sigma_5 \\ v_3 \end{bmatrix}_{x_3} = \mathbf{Y}\,[\Lambda(x_3)\mathbf{i} - \Lambda^{-1}(x_3)\mathbf{r}] , \quad (3)$$

where

$$\mathbf{i} = \begin{bmatrix} i_P \\ i_S \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_P \\ r_S \end{bmatrix}, \quad \Lambda(x_3) = \begin{bmatrix} \exp i\omega s_{3_P} x_3 & 0 \\ 0 & \exp i\omega s_{3_S} x_3 \end{bmatrix},$$

and

$$\mathbf{X} = \begin{bmatrix} \alpha s_1 & \beta s_{3_S} \\ -\varrho\beta\Gamma & 2\varrho\beta^3 s_1 s_{3_S} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2\varrho\alpha\beta^2 s_1 s_{3_P} & -\varrho\beta\Gamma \\ \alpha s_{3_P} & -\beta s_1 \end{bmatrix}. \quad (4)$$

Matrices **X** and **Y** are the impedance matrices of an isotropic elastic medium, and depend only on ϱ, α, β and horizontal slowness s_1. Rows of the impedance matrices are either dimensionless (rows corresponding to velocity components) or have dimension *impedance* (rows corresponding to traction components). The determinants of the impedance matrices are

$$|\mathbf{X}| = \varrho\alpha\beta s_{3_S}, \qquad |\mathbf{Y}| = \varrho\alpha\beta s_{3_P}, \tag{5}$$

which necessarily also have dimension *impedance.*

Let the transmitting isotropic medium occupying $x_3 > 0$ be specified by density ϱ', compressional speed α', and shear speed β'. Every quantity then referring to this medium will be a primed quantity. However in the lower medium, assume there is no upward wave incident on the $x_3 = 0$ interface from below, but only downward transmitted waves. Then the velocities and stresses may be written, by analogy to (3) and (4) with $\mathbf{i}$ replaced by $\mathbf{t}$ and $\mathbf{r}$ set identically to $\mathbf{O}$, as

$$\begin{aligned} \mathbf{b}_{X'}(x_3) &\equiv \begin{bmatrix} v_1 \\ \sigma_3 \end{bmatrix}_{x_3} = \mathbf{X}'\mathbf{\Lambda}'(x_3)\mathbf{t}, \\ \mathbf{b}_{Y'}(x_3) &\equiv \begin{bmatrix} \sigma_5 \\ v_3 \end{bmatrix}_{x_3} = \mathbf{Y}'\mathbf{\Lambda}'(x_3)\mathbf{t}, \end{aligned} \tag{6}$$

where

$$\mathbf{t} = \begin{bmatrix} t_P \\ t_S \end{bmatrix},$$

t_P and t_S being the transmitted compressional and shear wave amplitudes, respectively. $\mathbf{X}'$ and $\mathbf{Y}'$ are the same matrix functions of parameters as described in (4) but with the primed parameters of the lower transmitting medium.

At the 'welded' interface $x_3 = 0$, the velocity and stress traction components are continuous and $\mathbf{\Lambda} = \mathbf{\Lambda}' = \mathbf{I}$, the identity matrix, yielding the Zoeppritz equations in the following matrix form,

$$\begin{aligned} \mathbf{X}(\mathbf{i} + \mathbf{r}) &= \mathbf{X}'\mathbf{t}, \\ \mathbf{Y}(\mathbf{i} - \mathbf{r}) &= \mathbf{Y}'\mathbf{t}. \end{aligned} \tag{7}$$

Assuming for now that **X** and **Y** are invertible (singularity occurs at a horizontal slowness for which at least one of the reflected waves is a grazing wave, i.e., has a horizontal group velocity), (7) may be written

$$\mathbf{(i + r) = X^{-1}X't},$$
$$\mathbf{(i - r) = Y^{-1}Y't}. \tag{8}$$

Then addition and substraction give

$$2\mathbf{i} = \mathbf{(X^{-1}X' + Y^{-1}Y')t},$$
$$2\mathbf{r} = \mathbf{(X^{-1}X' - Y^{-1}Y')t}, \tag{9}$$

which can be solved, assuming $\mathbf{(X^{-1}X' + Y^{-1}Y')}$ is invertible (singularity here occurs at a horizontal slowness for which an interface wave, e.g., a Stoneley wave, exists), first for **t** and then for **r**, yielding

$$\mathbf{t} \equiv \mathbf{Ti} = 2\mathbf{(X^{-1}X' + Y^{-1}Y')^{-1}i},$$
$$\mathbf{r} \equiv \mathbf{Ri} = \mathbf{(X^{-1}X' - Y^{-1}Y')(X^{-1}X' + Y^{-1}Y')^{-1}i}. \tag{10}$$

R and **T** are the reflection and transmission matrices and this is the explicit solution of the generalized Zoeppritz equations for all s_1, s_2 when neither **X**, **Y** nor $\mathbf{(X^{-1}X' + Y^{-1}Y')}$ is singular.

Note that the solution to the Zoeppritz equations, **T** and **R** from (10), can be written without $\mathbf{Y}^{-1}$, in case **Y** is singular. Straight forward matrix manipulations yield

$$\mathbf{T} = 2\,\mathbf{Y'^{-1}Y(X^{-1}X'Y'^{-1}Y + I)^{-1}},$$
$$\mathbf{R} = \mathbf{(X^{-1}X'Y'^{-1}Y + I)^{-1}(X^{-1}X'Y'^{-1}Y - I)}. \tag{11}$$

Similarly, **T** and **R** from (10), can be written without $\mathbf{X}^{-1}$, in case **X** is singular. In the same manner, one finds that

$$\mathbf{T} = 2\,\mathbf{X}'^{-1}\mathbf{X}(\mathbf{I} + \mathbf{Y}^{-1}\mathbf{Y}'\mathbf{X}'^{-1}\mathbf{X})^{-1},$$

$$\mathbf{R} = (\mathbf{I} + \mathbf{Y}^{-1}\mathbf{Y}'\mathbf{X}'^{-1}\mathbf{X})^{-1}(\mathbf{I} - \mathbf{Y}^{-1}\mathbf{Y}'\mathbf{X}'^{-1}\mathbf{X}). \tag{12}$$

Alternate solutions to (7) may be found assuming that $\mathbf{X}'$ and $\mathbf{Y}'$ are invertible (singularity corresponds to grazing waves in the transmitting medium), in which case (7) may be written

$$\mathbf{X}'^{-1}\mathbf{X}(\mathbf{i} + \mathbf{r}) = \mathbf{t},$$

$$\mathbf{Y}'^{-1}\mathbf{Y}(\mathbf{i} - \mathbf{r}) = \mathbf{t}. \tag{13}$$

Then elimination of $\mathbf{t}$ from (13), solving for $\mathbf{r} \equiv \mathbf{Ri}$, and then solving for $\mathbf{t} \equiv \mathbf{Ti}$ from either the first or second of (13) give

$$\mathbf{R} = (\mathbf{Y}'^{-1}\mathbf{Y} + \mathbf{X}'^{-1}\mathbf{X})^{-1}(\mathbf{Y}'^{-1}\mathbf{Y} - \mathbf{X}'^{-1}\mathbf{X}),$$

$$\mathbf{T} = 2\,\mathbf{X}'^{-1}\mathbf{X}(\mathbf{Y}'^{-1}\mathbf{Y} + \mathbf{X}'^{-1}\mathbf{X})^{-1}\mathbf{Y}'^{-1}\mathbf{Y} \tag{14}$$

$$= 2\,\mathbf{Y}'^{-1}\mathbf{Y}(\mathbf{Y}'^{-1}\mathbf{Y} + \mathbf{X}'^{-1}\mathbf{X})^{-1}\mathbf{X}'^{-1}\mathbf{X}.$$

An imaginary value for any of the vertical slownesses implies that the corresponding wave is inhomogeneous, or evanescent. The downward wave decays in the $+x_3$ direction, $s_{3_P} = i\sqrt{(s_1^2 - \alpha^{-2})}$ for a P wave and/or $s_{3_S} = i\sqrt{(s_1^2 - \beta^{-2})}$ for a S wave, similarly in the primed medium. The upward wave decays in the $-x_3$ direction. The complex eigenvector merely means the two components are out of phase for an evanescent wave. A suitable sign convention for the choice of eigenvectors can then be stated as follows: compute the eigenvectors algebraically in terms of the real values of s_3 for the pre-critical case, and carry the formulation over to the post-critical case by using the complex values of s_3 in the same calculation. This yields, for an evanescent P wave, a positive real 1-component and a positive imaginary 3-component; for an evanescent S wave, a positive imaginary 1-component and a negative real 3-component. Specifically, the components of $\mathbf{R}$ and $\mathbf{T}$ as calculated from (10), (11), (12) or (14), have the meaning

$$\mathbf{R} = \begin{bmatrix} R_{PP} & R_{PS} \\ R_{SP} & R_{SS} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_{PP} & T_{PS} \\ T_{SP} & T_{SS} \end{bmatrix}, \tag{15}$$

where the first subscript denotes the type of reflected or transmitted wave and the second subscript denotes the type of incident wave.

Note that a homogeneous (as opposed to evanescent) incident wave at a critical angle for the incident medium can only occur when $s_1 = 1 / \alpha$, i.e., when a shear wave is incident at the incident P critical horizontal slowness. In this special case, $s_{3_P} = 0$, $\mathbf{Y}$ is singular and the solution (assuming $\alpha' \neq \alpha$ so $\mathbf{Y}'$ is not singular) is given by (11). Since, at the P critical horizontal slowness, $s_{3_P} = 0$, $Y_{11} = Y_{21} = 0$ and it may be seen that $\mathbf{R}$ and $\mathbf{T}$ are of the form

$$\mathbf{R} = \begin{bmatrix} -1 & R_{PS} \\ 0 & R_{SS} \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} 0 & T_{PS} \\ 0 & T_{SS} \end{bmatrix}.$$

If there were an incident grazing P wave at grazing incidence, it would be exactly canceled by the reflected grazing P wave giving a null effect, and reflected and transmitted waves are due only to the incident S wave at the P critical horizontal slowness.

Thus it has been shown how reflectivity and transmissivity for isotropic media can be calculated using the submatrix form of the Zoeppritz equations. However the real power will be seen to be in the application to anisotropic media, where the problem reduces to finding the corresponding $\mathbf{X}$ and $\mathbf{Y}$ matrices for each anisotropic medium.

PLANE STRAIN WAVES IN ANISOTROPIC MEDIA

For wave propagation to consist only of waves which are associated with in-plane motion, uncoupled from any motion normal to the vertical plane, the vertical plane must be at least a mirror plane of symmetry, and as mentioned above, there must be a horizontal plane of symmetry for up-down symmetry. If a medium has two such symmetry planes it is at least orthorhombic. Consider propagation in the vertical x_1,x_3-plane through orthorhombic media aligned so that their natural coordinate planes, the planes of mirror symmetry, are parallel to the coordinate planes. A tetragonal or cubic medium aligned along the coordinate planes, a hexagonal medium with its symmetry axis along either of the three coordinate axes, or an isotropic medium is a special case of an aligned orthorhombic medium, and thus is automatically included in the discussion of this section.

An orthorhombic medium, in condensed notation, has a 6×6 elastic modulus matrix of the form

$$
c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{16}
$$

The stress-velocity relations prescribed by (16) applied to plane strain velocity, $v_1(x_1,x_3)$, $v_3(x_1,x_3)$, gives the following equations of motion on the particle velocity components v_1 and v_3,

$$
\begin{aligned} c_{11}v_{1,11} + c_{55}v_{1,33} + (c_{55}+c_{13})v_{3,13} &= \varrho\ddot{v}_1 , \\ (c_{55}+c_{13})v_{1,13} + c_{55}v_{3,11} + c_{33}v_{3,33} &= \varrho\ddot{v}_3 , \end{aligned} \tag{17}
$$

Thus plane strain motion depends on only four of nine elastic moduli, c_{11}, c_{33}, c_{55} and c_{13}. Substitution of a harmonic plane wave of unit amplitude

$$
\begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_3 \end{bmatrix} \exp i\omega(s_1x_1 + s_3x_3 - t) , \tag{18}
$$

into (17) gives the plane strain Christoffel equations on the allowable slowness vectors and their associated eigenvector components e_1, e_3,

$$
\begin{bmatrix} c_{11}s_1^2 + c_{55}s_3^2 - \varrho & (c_{55} + c_{13})s_1s_3 \\ (c_{55} + c_{13})s_1s_3 & c_{55}s_1^2 + c_{33}s_3^2 - \varrho \end{bmatrix} \begin{bmatrix} e_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}
$$

Given s_1, the vanishing of the determinant for the existence of a non-trivial solution gives a biquadratic equation on s_3,

$$
\begin{aligned} & c_{33}c_{55}(s_3^2)^2 + [\{c_{55}(c_{11} + c_{33}) + E^2\}s_1^2 \\ & - \varrho(c_{33} + c_{55})]s_3^2 + (c_{11}s_1^2 - \varrho)(c_{55}s_1^2 - \varrho) = 0 , \end{aligned} \tag{20}
$$

where E^2, the key anisotropy parameter which determines the deviation of the qP slowness surface from an ellipse and the qS surface from a circle (see for example Helbig and Schoenberg, 1987), is given by

$$E^2 = (c_{11} - c_{55})(c_{33} - c_{55}) - (c_{13} + c_{55})^2 \,.$$

The biquadratic equation has two solutions, and for real values of s_3^2, let $s_{3_P}^2 < s_{3_S}^2$. The four roots are denoted by $+s_{3_P}$, $-s_{3_P}$, $+s_{3_S}$ and $-s_{3_S}$ where + denotes a downward wave, − an upward wave. A downward wave is characterized by a downward pointing group velocity vector or the root having a positive imaginary part if it is non-real. Note that $s_{3_P}^2$ and $s_{3_S}^2$ may, in some special circumstances, not be real, in which case, the difference between the two downward roots are the sign of the real part. Lets arbitrarily specify the 'negative real part' root to be the P root, the 'positive real part' root, the S root. Each root has its associated unit eigenvectors $\mathbf{e}_P^+$, $\mathbf{e}_P^-$, $\mathbf{e}_S^+$ and $\mathbf{e}_S^-$, respectively, with the 1-component of all eigenvectors taken as positive for $s_1 > 0$ which implies that the 3-component is of opposite sign for corresponding downward and upward waves.

Corresponding to (1) and (2), we have for the velocity and traction components in the incident medium

$$\begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = i_P \begin{bmatrix} e_{P_1} \\ e_{P_3} \end{bmatrix} \exp i\omega s_{3_P} x_3 + r_P \begin{bmatrix} e_{P_1} \\ -e_{P_3} \end{bmatrix} \exp -i\omega s_{3_P} x_3$$
$$+ i_S \begin{bmatrix} e_{S_1} \\ e_{S_3} \end{bmatrix} \exp i\omega s_{3_S} x_3 + r_S \begin{bmatrix} e_{S_1} \\ -e_{S_3} \end{bmatrix} \exp -i\omega s_{3_S} x_3 \,, \qquad (21)$$

and

$$\begin{bmatrix} \sigma_3 \\ \sigma_3 \end{bmatrix} = i_P \begin{bmatrix} -(c_{13}s_1e_{P_1} + c_{33}s_{3_P}e_{P_3}) \\ -c_{55}(s_1e_{P_3} + s_{3_P}e_{P_1}) \end{bmatrix} \exp i\omega s_{3_P} x_3$$
$$+ r_P \begin{bmatrix} -(c_{13}s_1e_{P_1} + c_{33}s_{3_P}e_{P_3}) \\ c_{55}(s_1e_{P_3} + s_{3_P}e_{P_1}) \end{bmatrix} \exp -i\omega s_{3_P} x_3$$
$$+ i_S \begin{bmatrix} -(c_{13}s_1e_{S_1} + c_{33}s_{3_S}e_{S_3}) \\ -c_{55}(s_1e_{S_3} + s_{3_S}e_{S_1}) \end{bmatrix} \exp i\omega s_{3_S} x_3$$
$$+ r_S \begin{bmatrix} -(c_{13}s_1e_{S_1} + c_{33}s_{3_S}e_{S_3}) \\ c_{55}(s_1e_{S_3} + s_{3_S}e_{S_1}) \end{bmatrix} \exp -i\omega s_{3_S} x_3 \,, \qquad (22)$$

with e_{P_1} and e_{S_1} taken to be positive; the velocity field in the transmitting medium, $x_3 > 0$, consists only of downward propagating waves, with coefficients t_P and t_S instead of i_P and i_S respectively, and, as above for the isotropic case, it can be written exactly as for the incident medium using primed parameters s'_{3_P}, s'_{3_S}, e'_P and e'_S, instead of unprimed parameters, with e'_{P_1} and e'_{S_1} taken to be positive. Thus the interface conditions are four continuity equations, which, when written as the continuity of two 'vectors' $\mathbf{b}_X$ and $\mathbf{b}_Y$ shown in (3), give the plane strain generalized Zoeppritz equations, (7), with

$$\mathbf{X} = \begin{bmatrix} e_{P_1} & e_{S_1} \\ -(c_{13}s_1e_{P_1} + c_{33}s_{3_P}e_{P_3}) & -(c_{13}s_1e_{S_1} + c_{33}s_{3_S}e_{S_3}) \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} -c_{55}(s_1e_{P_3} + s_{3_P}e_{P_1}) & -c_{55}(s_1e_{S_3} + s_{3_S}e_{S_1}) \\ e_{P_3} & e_{S_3} \end{bmatrix}, \tag{23}$$

and $\mathbf{X}'$ and $\mathbf{Y}'$ are the same, except with primed parameters replacing unprimed parameters. Matrices $\mathbf{X}$ and $\mathbf{Y}$ depend only on density ϱ, the four stiffness moduli, c_{11}, c_{33}, c_{55}, c_{13}, and the horizontal slowness s_1. The dimensions of the various terms are the same as the corresponding terms in the isotopic case. The determinants of the impedance matrices are given by

$$|\mathbf{X}| = c_{33}\begin{vmatrix} e_{P_1} & e_{S_1} \\ -s_{3_P}e_{P_3} & -s_{3_S}e_{S_3} \end{vmatrix}, \quad |\mathbf{Y}| = c_{55}\begin{vmatrix} -s_{3_P}e_{P_1} & -s_{3_S}e_{S_1} \\ e_{P_3} & e_{S_3} \end{vmatrix}, \tag{24}$$

The solution to (7) from the previous section on isotopic media applies as well to this case of plane strain waves in anisotropic media when $\mathbf{X}$ and $\mathbf{Y}$ are given by (23).

THREE DIMENSIONAL WAVES IN ANISOTROPIC MEDIA

In the general three dimensional case, there are three down and three up solutions for s_3 for each value of horizontal slowness specified by s_1, s_2. Thus, reflectivity, transmissivity, and impedance matrices are 3×3. For up-down symmetry, there must be a horizontal plane of symmetry, i.e., the medium must be at least monoclinic. In condensed notation the 6×6 elastic modulus matrix has the form

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} , \tag{25}$$

Note that the case of an orthorhombic medium whose vertical symmetry planes are at a non-zero azimuthal angle from the vertical coordinates planes (still with a horizontal symmetry plane) is included in this discussion; the 13 non-zero stiffnesses of (25) are derivable from the 9 stiffnesses of (16) by rotating the coordinates the required azimuthal angle. Substitution of a harmonic plane wave of unit amplitude, now with velocity and slowness not lying in any particular plane

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \exp i\omega(s_1x_1 + s_2x_2 + s_3x_3 - t) , \tag{26}$$

into the equations of motion for a monoclinic medium gives the three dimensional Christoffel equations on the allowable slowness vectors and their associated eigenvectors,

$$[\Gamma(s) - \rho I]\mathbf{e} = 0 , \tag{27}$$

where $\mathbf{\Gamma(s)}$ is given by

$$\begin{bmatrix} c_{11}s_1^2 + c_{66}s_2^2 + c_{55}s_3^2 + 2c_{16}s_1s_2 & c_{16}s_1^2 + c_{26}s_2^2 + c_{45}s_3^2 + A_{12}s_1s_2 & A_{13}s_1s_3 + A_{45}s_2s_3 \\ c_{16}s_1^2 + c_{26}s_2^2 + c_{45}s_3^2 + A_{12}s_1s_2 & c_{66}s_1^2 + c_{22}s_2^2 + c_{44}s_3^2 + 2c_{26}s_1s_2 & A_{45}s_1s_3 + A_{23}s_2s_3 \\ A_{13}s_1s_3 + A_{45}s_2s_3 & A_{45}s_1s_3 + A_{23}s_2s_3 & c_{55}s_1^2 + c_{44}s_2^2 + c_{33}s_3^2 + 2c_{45}s_1s_2 \end{bmatrix}$$

with

$$A_{23} \equiv c_{44} + c_{23},\ A_{13} \equiv c_{55} + c_{13},\ A_{12} \equiv c_{66} + c_{12},\ A_{45} \equiv c_{36} + c_{45}.$$

Given s_1, s_2, the vanishing of the determinant for the existence of a non-trivial solution gives a bicubic equation on s_3 with six solutions denoted by $\pm\sqrt{s_{3_P}^2}$, $\pm\sqrt{s_{3_S}^2}$, $\pm\sqrt{s_{3_T}^2}$, with associated unit eigenvectors $\mathbf{e}_P^\pm$, $\mathbf{e}_S^\pm$, $\mathbf{e}_T^\pm$, respectively. Here subscript T denotes the tertiary or third wave and as a convention, specify, for real $s^2_{3_P}$, $s^2_{3_S}$, and $s^2_{3_T}$,

$$s^2_{3_P} < s^2_{3_S} < s^2_{3_T} \,. \tag{28}$$

For 'mildly' anisotropic media, the P will denote quasi-P, the S, the first arriving quasi-S, and the T, the last arriving quasi-S.

Now in three dimensions, having found down and up going eigenvalues and their associated eigenvectors, the velocity components (corresponding to (21)) in the incident medium are

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = i_P \begin{bmatrix} e_{P_1} \\ e_{P_2} \\ e_{P_3} \end{bmatrix} \exp i\omega s_{3_P} x_3 + r_P \begin{bmatrix} e_{P_1} \\ e_{P_2} \\ -e_{P_3} \end{bmatrix} \exp -i\omega s_{3_P} x_3$$
$$+ i_S \begin{bmatrix} e_{S_1} \\ e_{S_2} \\ e_{S_3} \end{bmatrix} \exp i\omega s_{3_S} x_3 + r_S \begin{bmatrix} e_{S_1} \\ e_{S_2} \\ -e_{S_3} \end{bmatrix} \exp -i\omega s_{3_S} x_3$$
$$+ i_T \begin{bmatrix} e_{T_1} \\ e_{T_2} \\ e_{T_3} \end{bmatrix} \exp i\omega s_{3_T} x_3 + r_T \begin{bmatrix} e_{T_1} \\ e_{T_2} \\ -e_{T_3} \end{bmatrix} \exp -i\omega s_{3_T} x_3 \,, \tag{29}$$

and the components of the traction on a constant x_3 surface (corresponding to (21)) are found from the stress-velocity law for monoclinic media, the elastic modulus matrix for which is given by (25). The sign convention for the choice of the sense of the eigenvectors can be generalized to the case when both the horizontal slowness and the eigenvector no longer lie in the 1,3-plane, by stating it as follows: the positive sense of the eigenvector is chosen such that $\mathbf{e} \cdot \mathbf{s}_H > 0$, where $\mathbf{s}_H \equiv s_1\mathbf{e}_1 + s_2\mathbf{e}_2$ is the horizontal part of the phase slowness vector.

The velocity field in the transmitting medium, $x_3 > 0$, consists only of downward propagating waves, with coefficients t_P, t_S and t_T instead of i_P, i_S and i_T, respectively, and, as above, it can be written exactly as for the incident medium using primed parameters s'_{3_P}, s'_{3_S}, s'_{3_T}, e'_P, e'_S, and e'_T, instead of unprimed parameters. Thus the continuity equations at the interface $x_3 = 0$ of the three components of velocity and the three components of traction may be expressed as the continuity, across the interface $x_3 = 0$, of two three-component 'vectors',

$$\mathbf{b}_X \equiv \begin{bmatrix} v_1 \\ v_2 \\ \sigma_3 \end{bmatrix} \,, \quad \mathbf{b}_Y \equiv \begin{bmatrix} \sigma_5 \\ \sigma_4 \\ v_3 \end{bmatrix} \,, \tag{30}$$

which gives the three dimensional generalized Zoeppritz equations, (7), with

$$\mathbf{i} = \begin{bmatrix} i_P \\ i_S \\ i_T \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_P \\ r_S \\ r_T \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_P \\ t_S \\ t_T \end{bmatrix},$$

and

$$\mathbf{X} = \begin{bmatrix} e_{P_1} & e_{S_1} & e_{T_1} \\ e_{P_2} & e_{S_2} & e_{T_2} \\ \begin{array}{c} -(c_{13}e_{P_1} + c_{36}e_{P_2})s_1 \\ -(c_{23}e_{P_2} + c_{36}e_{P_1})s_2 \\ -c_{33}e_{P_3}s_{3_P} \end{array} & \begin{array}{c} -(c_{13}e_{S_1} + c_{36}e_{S_2})s_1 \\ -(c_{23}e_{S_2} + c_{36}e_{S_1})s_2 \\ -c_{33}e_{S_3}s_{3_S} \end{array} & \begin{array}{c} -(c_{13}e_{T_1} + c_{36}e_{T_2})s_1 \\ -(c_{23}e_{T_2} + c_{36}e_{T_1})s_2 \\ -c_{33}e_{T_3}s_{3_T} \end{array} \end{bmatrix},$$

$$(31)$$

$$\mathbf{Y} = \begin{bmatrix} \begin{array}{c} -(c_{55}s_1 + c_{45}s_2)e_{P_3} \\ -(c_{55}e_{P_1} + c_{45}e_{P_2})s_{3_P} \end{array} & \begin{array}{c} -(c_{55}s_1 + c_{45}s_2)e_{S_3} \\ -(c_{55}e_{S_1} + c_{45}e_{S_2})s_{3_S} \end{array} & \begin{array}{c} -(c_{55}s_1 + c_{45}s_2)e_{T_3} \\ -(c_{55}e_{T_1} + c_{45}e_{T_2})s_{3_T} \end{array} \\ \begin{array}{c} -(c_{45}s_1 + c_{44}s_2)e_{P_3} \\ -(c_{45}e_{P_1} + c_{44}e_{P_2})s_{3_P} \end{array} & \begin{array}{c} -(c_{45}s_1 + c_{44}s_2)e_{S_3} \\ -(c_{45}e_{S_1} + c_{44}e_{S_2})s_{3_S} \end{array} & \begin{array}{c} -(c_{45}s_1 + c_{44}s_2)e_{T_3} \\ -(c_{45}e_{T_1} + c_{44}e_{T_2})s_{3_T} \end{array} \\ e_{P_3} & e_{S_3} & e_{T_3} \end{bmatrix}.$$

$\mathbf{X}'$ and $\mathbf{Y}'$ are the same as above, except with primed parameters replacing unprimed parameters. Matrices $\mathbf{X}$ and $\mathbf{Y}$ depend on density ϱ, all thirteen non-zero stiffness moduli, and the horizontal slowness s_1,s_2, as can be seen from the Christoffel equations (27). However, if s_2 were to equal zero, for example, the impedance matrices would not depend on c_{22}, c_{12} or c_{26}. As above, terms in rows corresponding to velocity are dimensionless, while terms in rows corresponding to stress have dimension *impedance*. The determinants of the impedance matrices are given by

$$|\mathbf{X}| = c_{33} \begin{vmatrix} e_{P_1} & e_{S_1} & e_{T_1} \\ e_{P_2} & e_{S_2} & e_{T_2} \\ -s_{3_P}e_{P_3} & -s_{3_S}e_{S_3} & -s_{3_T}e_{T_3} \end{vmatrix},$$

$$|\mathbf{Y}| = (c_{55}\,c_{44} - c_{45}^2) \begin{vmatrix} s_3\,e_{P_1} & s_3\,e_{S_1} & s_3\,e_{T_1} \\ s_{3_P}e_{P_2} & s_{3_S}e_{S_2} & s_{3_T}e_{T_2} \\ e_{P_3} & e_{S_3} & e_{T_3} \end{vmatrix}, \qquad (32)$$

and note that in three dimensions, $|\mathbf{Y}|$ has dimension *impedance*[2]. The coefficients, c_{33} of $|\mathbf{X}|$ and $c_{55}c_{44} - c_{45}^2$ of $|\mathbf{Y}|$, are invariant with respect to rotation of the coordinate system about the x_3-axis. The discussion of the solution to (7) for the isotropic case applies as well to this case of monoclinic media when **X** and **Y** are given by (31), although now inversions of 3×3 matrices are required.

LAYER PROPAGATORS FROM IMPEDANCE MATRICES

The propagator matrix for an up-down symmetric anisotropic layer can also be found in terms of the impedance matrices of the medium. Consider an anisotropic layer, in general three dimensional, occupying the region $x_{3_t} < x_3 < x_{3_b}$. At horizontal slowness specified by s_1, s_2, let the layer have impedance matrices **X** and **Y**.

Let **d** denote the vector of complex amplitudes of the downward ($+x_3$ propagating) waves and **u** denote the vector of complex amplitudes of the upward ($-x_3$ propagating) waves, with the vertical phase term of each wave calculated relative to $x_3 = 0$, see Fig. 2. For an up-down symmetric layer, the velocity components and stress traction components can be written in terms of **d** and **u** in the following matrix form,

$$\mathbf{b}_X(x_3) \equiv \begin{bmatrix} v_1 \\ v_2 \\ \sigma_3 \end{bmatrix}_{x_3} = \mathbf{X}[\Lambda(x_3)\mathbf{d} + \Lambda^{-1}(x_3)\mathbf{u}] ,$$

$$\mathbf{b}_Y(x_3) \equiv \begin{bmatrix} \sigma_5 \\ \sigma_4 \\ v_3 \end{bmatrix}_{x_3} = \mathbf{Y}[\Lambda(x_3)\mathbf{d} - \Lambda^{-1}(x_3)\mathbf{u}] , \tag{33}$$

where

$$\Lambda(x_3) = \begin{bmatrix} \exp i\omega s_{3_P} x_3 & 0 & 0 \\ 0 & \exp i\omega s_{3_S} x_3 & 0 \\ 0 & 0 & \exp i\omega s_{3_T} x_3 \end{bmatrix}$$

s_{3_P}, s_{3_S} and s_{3_T} are the three vertical slownesses for downward propagating waves. Because of up-down symmetry, the vertical slownesses of the upward propagating waves are just the negatives of these. A P-wave is a primary wave, an S-wave is a secondary wave and a T-wave denotes a tertiary, or third, wave, because of the fact that in anisotropic media, the polarization of each wave can be very different from the longitudinal and shear waves in isotropic media. Of course, in the two dimensional case, i.e., in a vertical plane of symmetry of the medium, there will be no third wave and all the matrices of (33) will be 2×2 matrices. The two equations (33) can be combined, yielding

$$\begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_Y \end{bmatrix}_{x_3} = \begin{bmatrix} X\Lambda & X\Lambda^{-1} \\ Y\Lambda & -Y\Lambda^{-1} \end{bmatrix}_{x_3} \begin{bmatrix} \mathbf{d} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} X & X \\ Y & -Y \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}_{x_3} \begin{bmatrix} \mathbf{d} \\ \mathbf{u} \end{bmatrix} \equiv U(x_3) \begin{bmatrix} \mathbf{d} \\ \mathbf{u} \end{bmatrix} . \quad (34)$$

The vector composed of **d** and **u** can be written in terms of the vector composed of $\mathbf{b}_X$ and $\mathbf{b}_Y$ at $x_3 = x_{3_t}$, the top of the layer,

$$\begin{bmatrix} \mathbf{d} \\ \mathbf{u} \end{bmatrix} = {}_2U^{-1}(x_{3_t}) \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_Y \end{bmatrix}_{x_{3_t}} \quad (35)$$

and substitution of (35) into (34) at $x_3 = x_{3_b}$, the bottom of the layer, gives

$$\begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_Y \end{bmatrix}_{x_{3_b}} = U(x_{3_b})U^{-1}(x_{3_t}) \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_Y \end{bmatrix}_{x_{3_t}} \quad (35)$$

$$= \tfrac{1}{2} \begin{bmatrix} X & X \\ Y & -Y \end{bmatrix} \begin{bmatrix} \Lambda(x_{3_b}-x_{3_t}) & 0 \\ 0 & \Lambda^{-1}(x_{3_b}-x_{3_t}) \end{bmatrix} \begin{bmatrix} X^{-1} & Y^{-1} \\ X^{-1} & -Y^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_X \end{bmatrix}_{x_{3_t}}$$

$$\equiv Q(x_{3_b}-x_{3_t}) \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_Y \end{bmatrix}_{x_{3_t}} . \quad (36)$$

Q is the propagator for the layer, relating the velocity and traction at the bottom to the velocity and traction at the top.

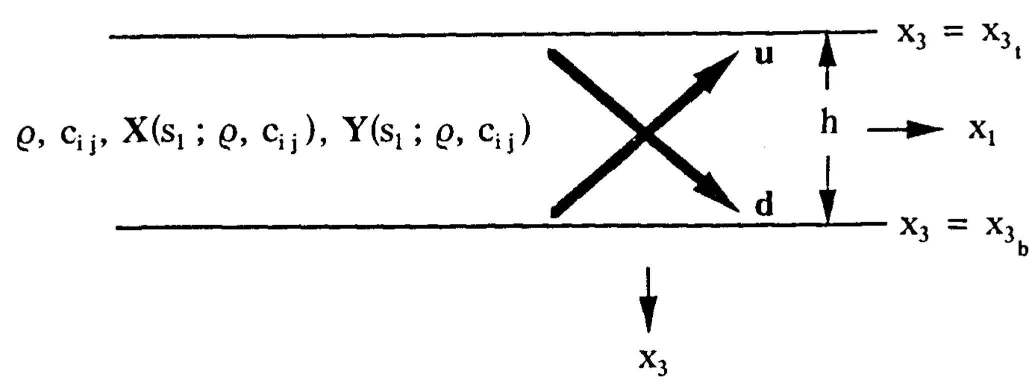

Fig. 2. A schematic diagram of the down waves, the heavy arrow **d**, and the up waves, the heavy arrow **u**, in a layer of thickness h.

Letting the layer thickness $x_{3_b} - x_{3_t} = h$, and noting that

$$[\Lambda(h) + \Lambda^{-1}(h)]/2 = \begin{bmatrix} \cos \omega s_{3_P} h & 0 & 0 \\ 0 & \sin \omega s_{3_S} h & 0 \\ 0 & 0 & \sin \omega s_{3_T} h \end{bmatrix} \equiv C(h),$$

$$[\Lambda(h) - \Lambda^{-1}(h)]/2 = i \begin{bmatrix} \sin \omega s_{3_P} h & 0 & 0 \\ 0 & \sin \omega s_{3_S} h & 0 \\ 0 & 0 & \sin \omega s_{3_T} h \end{bmatrix} \equiv iS(h), \tag{37}$$

one finds the simple to write, simple to program, form of the propagator matrix,

$$Q(h) = \begin{bmatrix} \mathbf{X}\ C(h)\ \mathbf{X}^{-1} & i\mathbf{X}\ S(h)\ \mathbf{Y}^{-1} \\ i\mathbf{Y}\ S(h)\ \mathbf{X}^{-1} & \mathbf{Y}\ C(h)\ \mathbf{Y}^{-1} \end{bmatrix} . \tag{38}$$

Note that if the layer is thin, i.e. for propagation such that $\omega h s_3 \ll 1$ for all waves in the problem,

$$Q(h) \approx \begin{bmatrix} \mathbf{I} & i\omega h\mathbf{X}\ \mathbf{s}\ \mathbf{Y}^{-1} \\ i\omega h\mathbf{Y}\ \mathbf{s}\ \mathbf{X}^{-1} & \mathbf{I} \end{bmatrix} , \tag{39}$$

where

$$\mathbf{s} \equiv \begin{bmatrix} s_{3_P} & 0 & 0 \\ 0 & s_{3_S} & 0 \\ 0 & 0 & s_{3_T} \end{bmatrix} .$$

This approximate propagator matrix (39) is the order ω expansion of the solution of the system of the first order differential equations on velocity and traction governing harmonic wave propagation in the medium, which has the form

$$\partial/\partial x_3 \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_X \end{bmatrix} = i\omega \begin{bmatrix} \mathbf{0} & \mathbf{XsY}^{-1} \\ \mathbf{YsX}^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{b}_X \\ \mathbf{b}_X \end{bmatrix} . \tag{40}$$

Thus, independent calculation of the differential equations (from the equations of motion and the stress-velocity equations) yields for $\mathbf{XsY}^{-1}$ and $\mathbf{YsX}^{-1}$, respectively, for the two-dimensional (orthorhombic in plane symmetry) case,

$$\mathbf{XsY}^{-1} = \begin{bmatrix} -1/c_{55} & -s_1 \\ -s_1 & -\varrho \end{bmatrix} , \ \mathbf{YsX}^{-1} = \begin{bmatrix} -\varrho + (c_{11} - c_{13}^2/c_{33}) s_1^2 & -s_1 c_{13}/c_{33} \\ -s_1 c_{13}/c_{33} & -1/c_{33} \end{bmatrix} ; \tag{41}$$

and for the three-dimensional (monoclinic, up-down symmetric) case

$$
\mathbf{XsY^{-1}} = \begin{bmatrix} -c_{44}/(c_{44}c_{55}-c_{45}^2) & c_{45}/(c_{44}c_{55}-c_{45}^2) & -s_1 \\ c_{45}/(c_{44}c_{55}-c_{45}^2) & -c_{55}/(c_{44}c_{55}-c_{45}^2) & -s_2 \\ -s_1 & -s_2 & -\varrho \end{bmatrix},
$$

$$
\mathbf{YsX^{-1}} = \begin{bmatrix} -\varrho + d_{11}s_1^2 + 2d_{16}s_1s_2 + d_{66}s_2^2 & d_{16}s_1^2 + (d_{66}+d_{12})s_1s_2 + d_{26}s_2^2 & -(s_1c_{13}+s_2c_{36})/c_{33} \\ d_{16}s_1^2 + (d_{66}+d_{12})s_1s_2 + d_{26}s_2^2 & -\varrho + d_{66}s_1^2 + 2d_{26}s_1s_2 + d_{22}s_2^2 & -(s_1c_{36}+s_2c_{23})/c_{33} \\ -(s_1c_{13}+s_2c_{36})/c_{33} & -(s_1c_{36}+s_2c_{23})/c_{33} & -1/c_{33} \end{bmatrix}. \quad (42)
$$

with

$$
d_{ij} \equiv c_{ij} - c_{i3}c_{3j}/c_{33}, \quad i,j = 1,2, \text{ or } 6 .
$$

For all cases of up-down symmetry, $\mathbf{XsY^{-1}}$ and $\mathbf{YsX^{-1}}$ are symmetric matrices. This form gives the approximate propagator matrices without the necessity of evaluating the components of matrix **s** or their associated eigenvectors.

Further if this thin layer is itself made up of n layers of width h_i, numbered from 1 to n from the top, so the total thickness is $H = \sum_{i=1}^{n} h_i$, the propagator of this set of n very thin layers is now given by

$$
\mathbf{Q}(H) \approx \begin{bmatrix} \mathbf{I} & i\omega H < \mathbf{XsY^{-1}} > \\ i\omega H < \mathbf{XsY^{-1}} > & \mathbf{I} \end{bmatrix}, \quad (43)
$$

where $< \cdot >$ denotes the thickness weighted average.

CONCLUSIONS

This paper has been meant to be a guide through the labyrinth of reflectivity from anisotropic media. A general and efficient formulation for reflection and transmission of plane waves at an interface between elastic media has been derived for anisotropic media, which allows for more straightforward reflectivity and transmissivity calculations in both two and three dimensions. For all cases the formulae for reflection and transmission matrices are identical to those given in (10), or alternatively in (11), (12) or (14). Thus all anisotropic reflectivity problems when the media have up-down symmetry have the same explicit solution in terms of the impedance matrices, which can also be used for the convenient generation of propagator matrices for anisotropic layers. Note that at a fluid-anisotropic solid interface, the reflection and transmission coefficients can be calculated merely by letting the impedance matrices for the fluid medium be those for an isotropic medium and letting the shear speed approach zero. Similarly, reflection coefficients from a free surface of an anisotropic medium can be calculated by letting the impedance matrices of the transmitting medium be those of anisotropic medium and letting the density $\varrho' \to 0$ and/or $\alpha', \beta' \to 0$.

This note is not meant to imply that everything about the forward problem is settled, or that reflection and transmission coefficients between anisotropic media have not been computed previously (see for example Rokhlin et al., 1989). However, there are still many aspects of the problem that are not clear, especially in three dimensions. These include 1) the effects of being at or near the repeated roots of the slowness equation, i.e.,the vanishing of the determinant of the matrix $\mathbf{\Gamma} - \varrho\mathbf{I}$ in (27), for the s_3^2, 2) precisely how to treat triplicating regions, particularly when these regions overlap with those regions of critical reflection or transmission, and 3) how to categorize the entire range of post-critical behavior possible in three dimensions. The power of the formalism is that the entire reflection behavior of an anisotropic medium is encapsulated in its two impedance matrices taken as functions of horizontal slowness. Hopefully, it will help in the tremendous work ahead in determining the information present in amplitude vs. angle and azimuth data, and in extracting that information.

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