**Approximation of Scalar Waves in the Space–Frequency Domain**

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*Technical Report # / May 1991*

Old Version

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

## Approxiamtion of Scalar Waves in the Space–Frequency Domain

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### §1. Introduction.

### §2. The Problem in the Time Domain.

Let $\Omega$ be an open unit cube in $\mathbb{R}^3$, and set $J = [0, \infty)$. We shall investigate the following model problem: find $u = u(\underline{x}, t)$ such that

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f, \qquad \Omega \times J, \tag{2.1.i}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial \underline{\nu}} = 0, \qquad \Gamma \times J, \tag{2.1.ii}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \qquad \Omega, \tag{2.1.iii}$$

where the constant $c$ denotes the wave speed, $\underline{\nu}$ the unit outward normal on $\Gamma = \partial\Omega$. The boundary condition (2.1.ii) is a standard first–order absorbing boundary condition, so that waves arriving normally at the boundary are absorbed completely. Indeed, by setting $f \equiv 0$ and taking the $L^2$–inner product of the differential equation (2.1.i) with $\frac{\partial u}{\partial t}$, we can obtain the following energy equation:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \left[ |\frac{\partial u}{\partial t}|^2 + c^2 |\nabla u|^2 \right] d\Omega + c \int_\Gamma |\frac{\partial u}{\partial t}|^2 d\Gamma = 0. \tag{2.2}$$

One of the physical implications of the non–negative boundary integral term in (2.2) would be that the sum of kinetic energy and potential energy will be absorbed on the boundary when the wave hits the boundary. For descriptions of various absorbing boundary conditions, see [.ce77.], [.em77.], [.hig86.], [.th86.].

Assume that the source function $f \in L^2(J; L^2(\Omega))$ satisfies the following decay rate in time: for a positive constant $\beta$,

$$\sup_{\underline{x} \in \Omega} |f(x,t)| \le B_0\, e^{-\beta t}, \qquad t \in J. \tag{2.4}$$

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where $H(t)$ and $\delta(x)$ denote the Heaviside and the Dirac $\delta$ distributions. In (8.1) $\omega_0 = 2\pi f_m$ rad $/msec$ with the main frequency $f_m$ which will be fixed as 10 $KHz$, and $\alpha = 0.79\ \omega_0/\pi$. The frequency spectrum $\widehat{f}_0(x,\omega)$ is given ([13]) by

$$\widehat{f}_0(x,\omega) = \frac{8\alpha\omega_0(\alpha - i\omega)}{[(\alpha-i\omega)^2+\omega_0^2]^2}\delta(x).$$

The wave simulation time is 2 $msec$. Denote by $\lambda$ the wave length. Then $\lambda = c/f_m = 0.2m$. Suppose that there are 25 grid points per wave length so that the mesh size $h = \lambda/25 = 0.008m$. In order to solve (2.1) using the finite difference method in $t$, the time step should satisfy $\Delta t \le h/c$ by the Courant–Friedrichs–Lewy stability condition. The largest such $\Delta t$ equals 0.004 $mssec$ and the number $N_{\text{time}}$ of time steps for the simulation time 2 $msec$ will be 500. Now let the domain size be 40 in wavelength unit. Then, $\Omega = (-4m, 4m)$. The number of grid points in $\Omega$ would then be 1001.

Let $L$ be the smallest integer that is a power of 2 not less than $N_{\text{time}}$. In our case $L = 512$. Let $m^*$ be a multiplication factor to increase the frequency resolution in solving the problem in the frequency domain.

Set $L^* = 2^{m^*}L$. The frequency step size $\Delta f$ is chosen as

$$\Delta f = 1/(\Delta t \cdot L^*) \cong 0.488/m^*\ KHz.$$

Also $\Delta\omega = 2\pi\Delta f \cong 3.068\ KHz$. ($f_{nyq} = 1/(2\Delta t)$ is called the Nyquist frequency so that $\widehat{f}(f_{nyq}+\omega) = \overline{\widehat{f}}(f_{nyq}-\omega)$.)

Given a continuous function $\varphi : \mathcal{R} \to \mathcal{C}$, consider $\{\varphi(n\Delta t)\}_{n=-\infty}^{\infty}$ and

$$\varphi^*(t) = \sum_{n=-\infty}^{\infty} \varphi(n\Delta t)\delta(t - n\Delta t). \tag{8.2}$$

The Fourier transform of (8.2) is given by

$$\widehat{\varphi^*}(\omega) = \sum_{n=-\infty}^{\infty} \varphi(n\Delta t)e^{-i\omega n\Delta t}. \tag{8.3}$$

For finite samples $\{\varphi(n\Delta t)\}_{n=0}^{L^*-1}$ assume that the spectrum $\widehat{\varphi}(\omega)$ is represented by $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$.

Motivated by (8.2) and (8.3), the discrete Fourier transform of the sequence $\{\varphi(n\Delta t)\}_{n=0}^{L^*-1}$ is defined as a sequence $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$ with

$$\begin{aligned}\widehat{\varphi}(k\Delta\omega) &= \sum_{n=0}^{L^*-1} \varphi(n\Delta t)\left(e^{-i\frac{2\pi}{N}}\right)^{nk} \\ &= \sum_{n=0}^{L^*-1} \varphi(n\Delta t)\left(e^{-i\Delta\omega\Delta t}\right)^{nk},\end{aligned} \tag{8.4}$$

Denote the Fourier transform of $v(\underline{x}, t)$ with respect to $t$ by

$$\widehat{v}(\underline{x}, \omega) = \int_{-\infty}^{\infty} v(\underline{x}, t) e^{-i\omega t} dt. \tag{3.1}$$

The Fourier inversion formula gives

$$v(\underline{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{v}(\underline{x}, \omega) e^{i\omega t} d\omega. \tag{3.2}$$

If $v(\underline{x}, t)$ is a real function, the Fourier transform $\widehat{v}(\underline{x}, \omega)$ of $v(\underline{x}, t)$ satisfies

$$\widehat{v}(\underline{x}, -\omega) = \overline{\widehat{v}}(\underline{x}, \omega), \text{ for all } \omega \in R. \tag{3.1*}$$

Analogously, if $\widehat{v}(\underline{x}, -\omega) = \overline{\widehat{v}}(\underline{x}, \omega)$ holds for all $\omega$, the Fourier inversion formula (3.2) takes the form

$$v(\underline{x}, t) = \frac{1}{\pi} Re \int_{0}^{\infty} \widehat{v}(\underline{x}, \omega) e^{i\omega t} d\omega. \tag{3.2*}$$

Assume that $f = u = 0$ for $t < 0$. By taking the Fourier transform of equations (2.1), the following set of elliptic problems is obtained: for each $\omega$, find $\widehat{u}(\underline{x}, \omega)$ such that

$$-\omega^2 \widehat{u} - c^2 \Delta \widehat{u} = \widehat{f}, \quad \Omega, \tag{3.3.i}$$

$$-c\frac{\partial \widehat{u}}{\partial \nu} = i\omega \widehat{u}, \quad \Gamma. \tag{3.3.ii}$$

Since the source $f(\underline{x}, t)$ is a real function, an application of (3.1*) to the equations (3.3) leads to $\widehat{u}(\underline{x}, -\omega) = \overline{\widehat{u}}(\underline{x}, \omega)$. Therefore, after finding solutions $\widehat{u}(\underline{x}, \omega)$ of Problem (3.3) for all $\omega \geq 0$, we can compute the solution $u(\underline{x}, t)$ in the time domain using the Fourier inversion formula (3.2*). Below in this section are given investigations of Problem (3.3) for each frequency $\omega$.

If $\omega = 0$, Problem (3.3) becomes a Neumann boundary value problem and therefore a solution exists and is unique up to an additive constant so long as

$$\int_{\Omega} \widehat{f}(\underline{x}, 0) d\Omega = 0. \tag{3.4}$$

It then follows from (3.1) that the mean value of the source should vanish; i.e.,

$$\int_{\Omega} \int_{0}^{\infty} f(\underline{x}, t) dt \, d\Omega = 0. \tag{3.5}$$

Since we want to take the inverse Fourier transform of solutions $\widehat{u}(\cdot, \omega)$ back into the time domain, we need to specify the solution $\widehat{u}(\cdot, \omega)$ for each frequency, in particular, for $\omega = 0$. Therefore the restriction (3.5) on the source is necessary. Let the indication of picking up a good additive constant for $\widehat{u}(\cdot, 0)$ be deferred to the end of this section.

REMARK 3.1. *For instance, if*

$$f(\underline{x},t) = \frac{\partial}{\partial t} g(\underline{x},t),$$

*with* $g(\underline{x},t) = 0$ *for* $t \leq 0$ *and* $t \geq T$, *then the condition (3.5) is evidently satisfied. Also, the source function*

$$f(\underline{x},t) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} g_i(\underline{x},t),$$

*with* $g_i \in C_0^1(\Omega)$ *for each* $t \in (0,T)$ *and* $g_i = 0$ *for* $t \leq 0$ *and* $t \geq T$ *satisfies the condition (3.5). These two cases cover most cases. In particular, if* $f(\underline{x},t) = F(\underline{x})g'(t)$, *with* $g = 0$ *for* $t \leq 0$ *and* $t \geq T$, $\widehat{f}(\underline{x},0) \equiv 0$ *on* $\Omega$. *Similarly,* $f(\underline{x},t) = G(t)\nabla \cdot F(\underline{x})$, *with* $G = 0$ *for* $t \leq 0$ *and* $t \geq T$ *and* $F \in [C_0^1(\Omega)]^3$ *leads again to* $\int_\Omega \widehat{f}(\underline{x},0)d\Omega = 0$.

Now consider the case when $\omega \neq 0$. Define a sesquilinear form $\Lambda(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\Lambda(v,w) = -\omega^2(v,w) + c^2(\nabla v, \nabla w) + i\omega c\langle v,w\rangle_\Gamma, \tag{3.6}$$

for $v,w \in H^1(\Omega)$. A weak formulation of Problem (3.3) is then given as follows: find $\widehat{u}(\cdot,\omega) \in H^1(\Omega)$ such that

$$\Lambda(\widehat{u},v) = (\widehat{f},v), \qquad v \in H^1(\Omega). \tag{3.7}$$

The uniqueness of Problem (3.3) can be established:

THEOREM 3.1. *Suppose that* $\Omega$ *is an open bounded set with piecewise smooth boundary* $\Gamma = \partial\Omega$. *Then, the solution of Problem* (3.3) *is unique for each frequency* $\omega \neq 0$.

PROOF: Let $\omega \neq 0$ be fixed. It is enough to show that $\widehat{u} \equiv 0$ is the only solution of (3.3) for $\widehat{f} \equiv 0$. Therefore set $\widehat{f} \equiv 0$. The choice of $v = \widehat{u}$ in (3.7) yields

$$-\omega^2\|\widehat{u}\|_0^2 + c^2\|\nabla\widehat{u}\|_0^2 + i\omega c|\widehat{u}|_{0,\Gamma}^2 = 0.$$

Hence $\widehat{u} = 0$ on $\Gamma$. Again from (3.3.ii) it follows that

$$\widehat{u} = \frac{\partial\widehat{u}}{\partial\underline{\nu}} = 0 \text{ on } \Gamma. \tag{3.8}$$

Problem (3.3.i), (3.8) can be regarded as a Cauchy problem. By the Cauchy–Kovalevski theorem and the Holmgrem theorem, there exists a unique solution, which is analytic, in a subdomain $\Omega_0 \subset \Omega$ with an analytic portion $\partial\Omega_0 \cap \Gamma$ of $\Gamma$. The analytic solution is $\widehat{u} \equiv 0$ on $\Omega_0$. Now, unique continuation ([* *]) shows $\widehat{u} \equiv 0$ on $\Omega$, which completes the proof. ∎

In order to establish the existence of the solution of Problem (3.3), consider the dual problem of (3.3) for $\omega \neq 0$; i.e, find $z$ fulfilling

$$-\omega^2 z - c^2 \Delta z = 0, \qquad \Omega, \tag{3.9.i}$$

$$c\frac{\partial z}{\partial \underline{\nu}} = i\omega z, \qquad \Gamma. \tag{3.9.ii}$$

The adjoint boundary condition (3.9.ii) to the boundary condition (3.3.ii) relative to the operator $-\omega^2 - c^2\Delta$ is obtained by integrations by parts twice and by using (3.3.ii):

$$(-\omega^2 \widehat{u} - c^2 \Delta \widehat{u}, z) = (\widehat{u}, -\omega^2 z - c^2 \Delta z) + c\langle \widehat{u}, -i\omega z + c\frac{\partial z}{\partial \underline{\nu}}\rangle.$$

By replacing $\omega$ by $-\omega$ in (3.9), the uniqueness of Problem (3.9) follows from Theorem 3.1. Therefore, if $\Omega$ is smooth and bounded, the existence and uniqueness of Problem (3.3) is obtained by the argument in Chapter 10 of [. schechter77.]. Now assume that $\Omega$ is a domain with piecewise analytic boundary $\Gamma$ so that the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Due to the continuity of the trace operator

$$H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma) : v \mapsto v|_\Gamma,$$

the sesquilinear form $\Lambda(\cdot,\cdot)$ is continuous, that is,

$$|\Lambda(v,w)| \leq C\|v\|_1\|w\|_1, \qquad v, w \in H^1(\Omega). \tag{3.10}$$

Also, the following Gårding's inequality holds:

$$|\Lambda(v,v)| \geq \frac{c^2}{\sqrt{2}}\|v\|_1^2 - \frac{\omega^2 + c^2}{\sqrt{2}}\|v\|_0^2, \qquad v \in H^1(\Omega). \tag{3.11}$$

Indeed, for $v \in H^1(\Omega)$,

$$\begin{aligned}\sqrt{2}|\Lambda(v,v)| &\geq |\operatorname{Re}\Lambda(v,v)| + |\operatorname{Im}\Lambda(v,v)| \\ &\geq \operatorname{Re}\Lambda(v,v) \\ &= -\omega^2\|v\|_0^2 + c^2\|\nabla v\|_0^2 \\ &= c^2\|v\|_1^2 - (\omega^2 + c^2)\|v\|_0^2.\end{aligned}$$

Introduce the sesquilinear form

$$\Lambda^0(v,w) = \Lambda(v,w) + \frac{\omega^2 + c^2}{\sqrt{2}}(v,w).$$

Then (3.7) takes the form

$$\Lambda^0(\widehat{u}, v) = \left(\widehat{f} + \frac{\omega^2 + c^2}{\sqrt{2}}\widehat{u}, v\right), \qquad v \in H^1(\Omega). \tag{3.12}$$

Owing to (3.10) and (3.11), $\Lambda^0(\cdot,\cdot)$ is continuous and $H^1$–coercive. Therefore the Lax–Milgram's theorem applies to Problem (3.12): there exists a solution operator

$$T^0 : L^2(\Omega) \to H^1(\Omega)$$

such that for every $F \in L^2(\Omega)$

$$\Lambda^0(T^0 F, v) = (F, v), \qquad v \in H^1(\Omega). \tag{3.13}$$

By the $H^1$–coercivity of $\Lambda^0$, the choice of $v = T^0 F$ in (3.13) shows

$$\begin{aligned} \frac{c^2}{\sqrt{2}}\|T^0 F\|_1^2 &\leq |\Lambda^0(T^0 F, T^0 F)| = |(F, T^0 F)| \\ &\leq \|F\|_0 \|T^0 F\|_1, \end{aligned}$$

which yields

$$\|T^0\|_{\mathcal{L}(L^2(\Omega); H^1(\Omega))} \leq \frac{\sqrt{2}}{c^2}.$$

Hence $T^0 : L^2(\Omega) \to H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Recall that we seek $\widehat{u} \in H^1(\Omega)$ such that

$$\widehat{u} = T^0 \left(\widehat{f} + \frac{\omega^2 + c^2}{\sqrt{2}}\widehat{u}\right)$$

or

$$\widehat{u} - \frac{\omega^2 + c^2}{\sqrt{2}} T^0 \widehat{u} = T^0 \widehat{f}. \tag{3.14}$$

By the Riesz–Schauder theory of the Fredholm alternative, either Problem (3.14) has a solution for every $T^0 \widehat{f} \in L^2(\Omega)$ or Problem (3.14) has a nonzero solution for $T^0 \widehat{f} = 0$.

Since $\widehat{f} \in L^2(\Omega)$, the solution $\widehat{u}$ of (3.14) belongs to $H^1(\Omega)$. But the uniqueness theorem for (3.3) implies (3.14) does not have a nonzero solution $\widehat{u}$ for the case of $T^0 \widehat{f} = 0$. This means that for every $\widehat{f} \in L^2(\Omega)$ there exists a solution $\widehat{u} \in H^1(\Omega)$ of (3.14) and hence of (3.12). Thus we have proved:

THEOREM 3.2. *Suppose $\Omega$ is a smooth domain or a domain with piecewise analytic boundary $\Gamma$ so that the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then there exists a solution $\widehat{u} \in H^1(\Omega)$ of Problem (3.7), for $\omega \neq 0$.*

As a corollary of Theorems 3.1 and 3.2,

COROLLARY 3.3. *If $\Omega$ is a unit cube, then there exists a unique solution $\widehat{u}(\cdot,\omega)$ of Problem (3.3) for $\omega \neq 0$.*

REMARK 3.2. *For piecewise polygonal or piecewise analytical boundaries, the following trace theorem holds:*

$$|g|_{0,\Gamma} \leq C\|g\|_{0,\Omega}^{1/2}\|g\|_{1,\Omega}^{1/2}.$$

*Therefore, (3.10) follows without introducing the space $H^{\frac{1}{2}}(\Gamma)$. It is also trivial(directly) that, for our domain $\Omega$, $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact; only elementary Fourier analysis is needed.*

Consider the solution operator $T(\omega) : L^2(\Omega) \to H^2(\Omega)$ given by $T(\omega)\widehat{f} = \widehat{u}(\cdot,\omega)$ for Problem (3.3). Let $C(\omega) = \|T(\omega)\|_{\mathcal{L}(L^2(\Omega);H^2(\Omega))}$. Assume that for each $\omega \neq 0$, $0 < C(\omega) < \infty$. Then we prove that $C(\omega)$ is a continuous function of $\omega$.

THEOREM 3.4. *The elliptic regularity coefficient $C(\omega)$ of Problem ((3.3) is a continuous function of $\omega$ for $\omega \neq 0$.*

PROOF: Consider Problem (3.3) for $\omega_1 \neq 0$, $\omega_2 \neq 0$, $\omega_1 \neq \omega_2$. Set $\widehat{d} = \widehat{u}_1(\cdot,\omega_1) - \widehat{u}_2(\cdot,\omega_2)$ and take the differences of equations in (3.3) for $\omega = \omega_1$ and for $\omega = \omega_2$. Then,

$$\begin{aligned}
&\left[-\omega_1^2 - c^2\Delta\right]\widehat{d} = (\omega_2^2 - \omega_1^2)\widehat{u}(\cdot,\omega_2), \quad \Omega,\\
&\left[\frac{\partial}{\partial\nu} + i\omega_1\right]\widehat{d} = i(\omega_2 - \omega_1)\widehat{u}(\cdot,\omega_2), \quad \Gamma.
\end{aligned}$$

Hence, by the elliptic regularity,

$$\begin{aligned}
\|\widehat{d}\|_2 &\leq C(\omega_1)\left[|\omega_2^2 - \omega_1^2| \cdot \|\widehat{u}(\cdot,\omega_2)\|_0 + |\omega_2 - \omega_1| \cdot |\widehat{u}(\cdot,\omega_2)|_{\frac{1}{2},\Gamma}\right]\\
&\leq C(\omega_1)C(\omega_2)\left(|\omega_1 + \omega_2| + 1\right) \cdot |\omega_2 - \omega_1|\,\|f\|_0.
\end{aligned}$$

In particular,

$$\begin{aligned}
\|\widehat{u}(\cdot,\omega_2)\|_2 &\leq \|\widehat{u}(\cdot,\omega_1)\|_2 + C(\omega_1)C(\omega_2)\left(|\omega_1 + \omega_2| + 1\right) \cdot |\omega_2 - \omega_1|\,\|f\|_0\\
&\leq \left[C(\omega_1) + C(\omega_1)C(\omega_2)\left(|\omega_1 + \omega_2| + 1\right) \cdot |\omega_2 - \omega_1|\right]\|f\|_0.
\end{aligned}$$

By the definition of $C(\omega)$, we have

$$C(\omega_2) \leq C(\omega_1) + C(\omega_1)C(\omega_2)\left(|\omega_1 + \omega_2| + 1\right) \cdot |\omega_2 - \omega_1|.$$

Similarly,

$$C(\omega_1) \le C(\omega_2) + C(\omega_1)C(\omega_2)\left(|\omega_1+\omega_2|+1\right)\cdot|\omega_2-\omega_1|$$

which implies

$$\left|\frac{1}{C(\omega_1)} - \frac{1}{C(\omega_2)}\right| \le \left(|\omega_1+\omega_2|+1\right)\,|\omega_2-\omega_1|.$$

Therefore, $C(\omega)$ is a continuous function of $\omega$. ∎

REMARK 3.3. *The elliptic regularity coefficient $C(\omega)$ may blow up near $\omega = 0$. This might restrict the choice of the source function $f(\underline{x},t)$ in Problem 2.1 so that the Fourier transform $\widehat{f}(\underline{x},\omega)$ vanishes near $\omega = 0$. However, this restriction seems to be not necessarily as our numerical example in §6 shows.*

Now let us indicate on the choice of the additive constant for $\widehat{u}(\underline{x},0)$ for every $\underline{x}\in\Omega$. Due to the Fourier inversion formula (3.2), the initial condition (3.1.iii) yields to the condition

$$\int_{-\infty}^{\infty} \widehat{u}(\underline{x},\omega)d\omega = 0. \tag{3.15}$$

Since there is a unique solution $\widehat{u}(\cdot,\omega)$ for each $\omega \ne 0$, the value $\widehat{u}(\underline{x},0)$ for every $\underline{x}\in\Omega$ should be uniquely determined such that (3.15) is satisfied. Indeed, the initial condition (3.1.*iii*) requires more than (3.15);

$$\int_{-\infty}^{\infty} \omega\widehat{u}(\underline{x},\omega)d\omega = 0, \tag{3.16}$$

although how it should contribute to finding the values $\widehat{u}(\underline{x},\omega)$ is open.

## §4. Finite Element Procedures.

Let $0 < h \le 1$ be a parameter and $\mathcal{T}_h$ be a quasiregular partitions of $\Omega$ into simplicies or 3–rectangles with diameter bounded by $h$. Choose a standard finite element subspace $V_h$ of $H^1(\Omega)$ associated with $\mathcal{T}_h$ such that, for integers $1 \le k \le \overline{k} \le m \le \overline{m}$, for any $v \in H^m(\Omega)$,

$$\inf_{\chi\in V_h} \sum_{j=0}^{k} h^j\|v-\chi\|_j \le Ch^m\|v\|_m. \tag{4.1}$$

Here $C$ is a constant independent of $h$ and $v$.

Fix a sufficiently high frequency $\omega^* > 0$. Recalling the formula (3.2), we shall first approximate the solution $\widehat{u}(\omega)$ by $\widehat{u}_h(\omega)$ in $V_h$ for $|\omega| \le \omega^*$, and by 0 for $|\omega| > \omega^*$. Let a positive integer $N$ be a parameter and $\Delta\omega = \frac{\omega^*}{N+\frac{1}{2}}$ a frequency step. Then, for $\omega = \pm j\Delta\omega$, $j = 1,\cdots,N$, a Galerkin approximation $\widehat{u}_h(\omega)$ to $\widehat{u}(\omega)$ of Problem (3.7) is defined as an element $\in V_h$ such that

$$\Lambda(\widehat{u}_h(\omega),v) = (\widehat{f},v), \qquad v\in V_h. \tag{4.2}$$

Motivated by (3.15), $\widehat{u}_h(0)$ is calculated by

$$\widehat{u}_h(0) = -2\sum_{j=1}^{N} Re\widehat{u}_h(j\Delta\omega).$$

The time domain solution $u$ of Problem (2.1) will then be approximated by

$$W_h^N(t) = \frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{u}_h(j\Delta\omega)e^{ij\Delta\omega\cdot t}\Delta\omega.$$

A natural question is then: what is the convergence rate of

$$|||u(x,t) - W_h^N(x,t)||| \to 0, \qquad \text{as} \qquad h \to 0, \qquad N \to \infty \tag{4.3}$$

with an appropriate norm $|||\cdot|||$? In order to answer this question we begin by looking at the spatial discretization:

THEOREM 4.1. *For each frequency $\omega \neq 0$, the approximate solution $\widehat{u}_h(\omega)$ of (4.2) to the solution $\widehat{u}(\omega)$ of Problem (3.7) satisfies the optimal order error estimate*

$$\|\widehat{u}(\omega) - \widehat{u}_h(\omega)\|_0 \leq C(\omega)h^2\|\widehat{f}(\omega)\|_0. \tag{4.4}$$

PROOF: First, an error equation is obtained from (3.7) and (4.2):

$$\Lambda(\widehat{u} - \widehat{u}_h, v) = 0, \qquad v \in V_h. \tag{4.4}$$

By the Gårding's inequality (3.11) and the inequality (3.10), the above error equation (4.4) yields the following: for every $v \in V_h$,

$$\begin{aligned} C_1\|\widehat{u} - \widehat{u}_h\|_1^2 - C_2\|\widehat{u} - \widehat{u}_h\|_0^2 &\leq \Lambda(\widehat{u} - \widehat{u}_h, \widehat{u} - \widehat{u}_h) \\ &= \Lambda(\widehat{u} - \widehat{u}_h, \widehat{u} - v) \\ &\leq C_3\|\widehat{u} - \widehat{u}_1\|_1\|\widehat{u} - v\|_1. \end{aligned} \tag{4.5}$$

Dividing (4.5) by $\|\widehat{u} - \widehat{u}_h\|_1$ and taking the infimum of $\|\widehat{u} - v\|_1$ over $v \in V_h$, we get

$$\|\widehat{u} - \widehat{u}_h\|_1 \leq C_4\|\widehat{u} - \widehat{u}_h\|_0 + C_5 h\|\widehat{u}\|_2 \tag{4.6}$$

as an application of (4.1) with $k = 1$ and $m = 2$. In order to bound the term $\|\widehat{u} - \widehat{u}_h\|_0$ in terms of $\|\widehat{u} - \widehat{u}_h\|_1$ we shall use the duality argument ([.bo73.], [.nit70.], [.sch74.]) as follows.

Consider the dual problem

$$-\omega^2\varphi - c^2\Delta\varphi = \widehat{u} - \widehat{u}_h, \quad \Omega, \tag{4.7.i}$$

$$c\frac{\partial\varphi}{\partial\underline{\nu}} = i\omega\varphi, \quad \Gamma. \tag{4.7.ii}$$

Define a sesquilinear form $\Lambda^*(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\Lambda^*(v,w) = -\omega^2(v,w) + c^2(\nabla v, \nabla w) - i\omega c\langle v, w\rangle_\Gamma.$$

Then a weak formulation of Problem (4.7) is given by

$$\Lambda^*(\varphi, v) = (\widehat{u} - \widehat{u}_h, v), \qquad v \in V_h,$$

which is equivalent to

$$\Lambda(v, \varphi) = (v, \widehat{u} - \widehat{u}_h), \qquad v \in V_h. \tag{4.8}$$

According to (4.4), (3.10) the choice $v = \widehat{u} - \widehat{u}_h$ in (4.8) shows that

$$\begin{aligned} \|\widehat{u} - \widehat{u}_h\|_0^2 &\le |\Lambda(\widehat{u} - \widehat{u}_h, \varphi - w)| \\ &\le C_3\|\widehat{u} - \widehat{u}_h\|_1\|\varphi - w\|_1, \end{aligned} \tag{4.9}$$

for arbitrary $w \in V_h$. By taking the infimum on the last term in (4.9) over $w \in V_h$, another application of (4.1) and an elliptic regularity for Problem (4.7) imply

$$\begin{aligned} \|\widehat{u} - \widehat{u}_h\|_0^2 &\le C_6 h\|\widehat{u} - \widehat{u}_h\|_1\|\varphi\|_2 \\ &\le C_7 h\|\widehat{u} - \widehat{u}_h\|_1\|\widehat{u} - \widehat{u}_h\|_0, \end{aligned}$$

or

$$\|\widehat{u} - \widehat{u}_h\|_0 \le C_7 h\|\widehat{u} - \widehat{u}_h\|_1. \tag{4.10}$$

Combining (4.6) and (4.10) with a sufficiently small $h$ so that $\frac{1}{2} \le 1 - C_4C_7h$, we have

$$\|\widehat{u} - \widehat{u}_h\|_0 \le C_8 h^2\|\widehat{u}\|_2. \tag{4.11}$$

By an elliptic regularity for Problem (3.3), we get (4.4). This completes our proof. ∎

Now, we turn to the question (4.3) about the convergence rate of the approximate solution $W_h^N$ to $u(\cdot, t)$ for a fixed time $t$.

THEOREM 4.2. *Assume that (2.4) and (2.6) hold for sufficiently large $k$. Then, for $t \in J$, the following error estimate holds*

$$
\begin{aligned}
\|u(\cdot,t) - W_h^N(\cdot,t)\|_{L^2(\Omega)}(t) \le C \Bigg[ \left\| \frac{\partial^k u}{\partial t^k} \right\|_{L^2(J,L^2(\Omega))} \left( \frac{1}{(2k-1)(\omega^*)^{2k-1}} \right)^{\frac{1}{2}} \\
+ \left(\frac{\omega^*}{N}\right)^2 \left\{ \|\tau^2 u\|_{L^2(J,L^2(\Omega))} + t\|\tau u\|_{L^2(J,L^2(\Omega))} + t^2 \|u\|_{L^2(J,L^2(\Omega))} \right\} \\
+ h^2 \omega^* \, \|\widehat{f}\|_{L^\infty_\omega(-\infty,\infty,L^2(\Omega))} \Bigg].
\end{aligned}
\tag{4.12}
$$

PROOF: Set $\omega_j = j\Delta\omega = j\frac{\omega^*}{N+\frac{1}{2}}$ and $\widehat{\xi}(\underline{x},\omega_j) = \widehat{u}_h(\underline{x},\omega_j) - \widehat{u}(\underline{x},\omega_j)$. Then,

$$
\begin{aligned}
u(\underline{x},t) &- W_h^N(\underline{x},t) \equiv u(t) - W_h^N(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\omega)e^{i\omega t}d\omega - \frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{u}(\omega_j)e^{i\omega_j t}\Delta\omega \\
&\quad + \left[ -\frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{\xi}(\omega_j)e^{i\omega_j t}\Delta\omega \right] \\
&= \frac{1}{2\pi}\int_{|\omega|>\omega^*} \widehat{u}(\omega)e^{i\omega t}d\omega + \left[ \frac{1}{2\pi}\int_{|\omega|\le\omega^*} \widehat{u}(\omega)e^{i\omega t}d\omega \right. \\
&\quad \left. -\frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{u}(\omega_j)e^{i\omega_j t}\Delta\omega \right] + \left[ -\frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{\xi}(\omega_j)e^{i\omega_j t}\Delta w \right] \\
&\equiv E_1(x,t) + E_2(x,t) + E_3(x,t).
\end{aligned}
$$

By the Cauchy–Schwarz's inequality and Parseval's identity, it follow that

$$
\begin{aligned}
\int_\Omega |E_1(\underline{x},t)|^2 d\Omega &= \int_\Omega \left| \frac{1}{2\pi}\int_{|\omega|>\omega^*} \widehat{u}(\omega)e^{i\omega t}d\omega \right|^2 d\Omega \\
&\le \int_\Omega \frac{1}{4\pi^2} \left[ \int_{|\omega|>\omega^*} \omega^{2k}|\widehat{u}(\omega)|^2 d\omega \int_{|\omega|>\omega^*} \frac{1}{\omega^{2k}} d\omega \right] d\Omega \\
&\le \int_\Omega \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left| \frac{\partial^k u}{\partial t^k}(t) \right|^2 dt\, d\Omega \frac{2}{(2k-1)(\omega^*)^{2k-1}} \\
&= \frac{1}{2\pi^2(2k-1)(\omega^*)^{2k-1}} \left\| \frac{\partial^k u}{\partial t^k} \right\|^2_{L^2(J,L^2(\Omega))}.
\end{aligned}
$$

Therefore,

$$\|E_1(\cdot,t)\|_{L^2(\Omega)} \le \left\|\frac{\partial^k u}{\partial t^k}\right\|_{L^2(J,L^2(\Omega))} \left[\frac{1}{2\pi^2(2k-1)(\omega^*)^{2k-1}}\right]^{\frac{1}{2}}. \tag{4.13}$$

By Corollary 2.2,

$$\left\|\frac{\partial^k u}{\partial t^k}\right\|_{L^2(J,L^2(\Omega))} < \infty.$$

Next,

$$\|E_2(\cdot,t)\|^2_{L^2(\Omega)} = \frac{1}{4\pi^2}\int_\Omega \left[\int_{-\omega^*}^{\omega^*} \widehat{u}(\omega)e^{i\omega t}d\omega - \sum_{j=-N}^{N} \widehat{u}(\omega_j)e^{i\omega_j t}\Delta\omega\right]^2 d\Omega$$

$$\le C(\Delta\omega)^4 \int_\Omega \left\|\frac{\partial^2[\widehat{u}(\omega)e^{i\omega t}]}{\partial\omega^2}\right\|^2_{L^2(-\omega^*,\omega^*)} d\Omega$$

$$= C(\Delta\omega)^4 \int_\Omega \left\|\left\{-\widehat{t^2u(t)} + 2(-\widehat{it)u}(t)\,(it) + \widehat{u(t)}\,(it)^2\right\} e^{i\omega t}\right\|^2_{L^2(-\omega^*,\omega^*)} d\Omega$$

$$\le C(\Delta\omega)^4 \int_\Omega \|\widehat{t^2u(t)}\|^2_{L^2_\omega(-\infty,\infty)} + \|\widehat{tu(t)}\|^2_{L^2_\omega(-\infty,\infty)}t^2 + \|\widehat{u(t)}\|^2_{L^2_\omega(-\infty,\infty)}t^4\Big] d\Omega$$

$$= C(\Delta\omega)^4 \left[\|t^2u\|^2_{L^2(J,L^2(\Omega))} + t^2\|tu\|^2_{L^2(J,L^2(\Omega))} + t^4\|u\|^2_{L^2(J,L^2(\Omega))}\right],$$

where $L^2_\omega(-\infty,\infty)$ denotes the $L^2$–space in the frequency domain and the last equality is again due to the Parseval's identity.

Recall that $\|t^j u\|_{L^2(J,L^2(\Omega))} < \infty$ by Theorem 2.1. Hence,

$$\begin{aligned}\|E_2(\cdot,t)\|^2_{L^2(\Omega)} \le C(\Delta\omega)^2 \big[&\|\tau^2 u\|_{L^2(J,L^2(\Omega))} + t\|\tau u\|_{L^2(J,L^2(\Omega))} \\ &+t^2\|u\|_{L^2(J,L^2(\Omega))}\big].\end{aligned} \tag{4.14}$$

Finally, by Theorem 4.1,

$$\begin{aligned}\|E_3(\cdot,t)\|_{L^2(\Omega)} &= \left\|\frac{1}{2\pi}\sum_{j=-N}^{N} \widehat{\xi}(\omega_j)e^{i\omega_j t}\Delta\omega\right\|_{L^2(\Omega)} \\ &\le \frac{1}{2\pi}\sum_{j=-N}^{N} \|\widehat{\xi}(\omega_j)\|_{L^2(\Omega)}\Delta\omega \\ &\le \frac{1}{2\pi}\sum_{j=-N}^{N} C(\omega_j)h^2\|\widehat{f}(\omega_j)\|_{L^2(\Omega)}\Delta\omega \\ &\le C\cdot \sup_{-\omega^*\le\omega<\omega^*} C(\omega)\cdot h^2\omega^*\|\widehat{f}\|_{L^\infty_\omega(-\infty,\infty,L^2(\Omega))}.\end{aligned} \tag{4.15}$$

Combining (4.13), (4.14), (4.15) , we obtain the estimate (4.12). ∎

## §5. The Proof of Theorem 2.1.

In this section we shall prove Theorem 2.1 using the concepts of geometric optics. It is enough to prove the theorem for $m = 0$ since if Theorem 2.1 holds for $m = 0$, then, for $m > 0$, the constant $C_m$ and a smaller constant $\alpha$ can be chosen accordingly.

PROOF: Given an incident wave with the incident angle $\phi$, let $R_\phi$ be the reflection coefficient at $\Gamma$ so that

$$u(\underline{x}, t) = e^{i(\omega t - \underline{\kappa} \cdot \underline{x})} + R_\phi e^{i(\omega t - \underline{\kappa}^1 \cdot \underline{x})}$$

is a local solution of (2.1.i) and (2.1.ii) at $\Gamma$. Here, $\omega$ denotes the frequency, $\underline{\kappa}$ the incident wave vector such that $\underline{\kappa} \cdot \underline{\nu} = |\underline{\kappa}| \cdot |\underline{\nu}| \cos\phi = |\underline{\kappa}| \cos\phi$, and $\underline{\kappa}'$ the reflected wave vector of $\underline{\kappa}$ at $\Gamma$. $R_\phi$ is then explicitly given by

$$R_\phi = -\frac{1 - \cos\phi}{1 + \cos\phi}.$$

Since the wave speed is constant, the orthogonal trajectories of a set of wave fronts are locally straight lines with the direction $\underline{\kappa}$, the constant incident wave vector at the boundary $\Gamma$, until the wave fronts hit the boundary. Such orthogonal trajectories still remain as straight lines with the direction $\underline{\kappa}'$, the reflected wave vector of $\underline{\kappa}$ at $\Gamma$, after the wave fronts are reflected at $\Gamma$ and before they hit $\Gamma$ again. We shall call each such such trajectory as a *ray* originated at $\underline{x}_0$ at time $t_0$ if it is generated by a point source $\delta(\underline{x} - \underline{x}_0)\delta(t - t_0)$.

Let $\Pi(\underline{x}, t; s)$, $0 \le s < t$, be the set of all possible rays toward $\underline{x}$ originated at some point $\underline{y} \in \Omega$ at time $s$ whose arc lengths along the rays from time $s$ to time $t$ are equal to $c(t-s)$.

Regard $\mathbb{R}^3$ as the periodic array of unit cube. Then $\Pi(\underline{x}, t; s)$ can be interpreted as the set of *centripetal rays* toward the point $\underline{x}$ whose origins are points $\underline{y}^*$ on the sphere

$$S(\underline{x}; c(t-s)) = \left\{ \underline{y}^* \in \mathbb{R}^3 \,|\, |\underline{y}^* - \underline{x}| = c(t-s) \right\}.$$

If two rays in $\Pi(\underline{x}, t; s)$ agrees for $(t - \varepsilon, t)$, then they are identical. Therefore, there is a $1-1$ correspondence between rays in $\Pi(\underline{x}, t; s)$ and the set of all centripetal rays toward $\underline{x}$ with arc length $c(t-s)$. The mapping sending the starting point $y^*$ on $S(\underline{x}, c(t-s))$ of a centripetal ray to the starting point $\underline{y} \in \Omega$ of the corresponding ray in $\Pi(\underline{x}, t; s)$ is surjective, but not necessarily injective.

For $|\underline{x} - \underline{y}^*| = c(t-s)$, define $N(\underline{x}, t; \underline{y}^*, s) = N(\underline{x}; \underline{y}^*)$ to be the number of the intersection of the centripetal ray from $\underline{y}^*$ toward $\underline{x}$ with the lattice of the periodic extension of $\Gamma$. It is obvious that the number $N(\underline{x}, t; \underline{y}^*, s)$ is the same as the number of reflections at $\Gamma$ of the ray arriving at $\underline{x}$ at time $t$ which is originated at $\underline{y} \in \Omega$ at time $s$.

Next consider *the characteristic traveling time*, $T_\chi$, for a domain with the velocity $c$ to be the infemum time required for all rays starting from any fixed point in $\Omega$ to hit the boundary $\Gamma$ at least once. Then $T_\chi = \operatorname{diam} \Omega / c$. Since $\Omega$ is a unit cube, $T_\chi = \sqrt{3}/c$.

Now let us reformulate Problem (2.1) as follows. Since Boundary Condition (2.1.ii) is not perfectly absorbing, the reflected waves generated at $(\underline{y}, s) \in \operatorname{supp} f$ will affect on the behavior of the waves at a given observation point as a parasite source with a characteristic traveling time $T_X$. Conveniently, such a parasite source can be thought as a function $f^*(\underline{y}^*, s)$ where $\underline{y}^* \in S(\underline{x}, c(t-s))$. The effect at $(\underline{x}, t)$ by the parasite source $f^*(\underline{y}^*, s)$ is then given by

$$f^*(\underline{y}^*, s) = R(\underline{x}, t; \underline{y}^*, s) f(\underline{y}, s) \tag{5.1}$$

where $R(\underline{x}, t; \underline{y}^*, s)$ is the reduction factor due to the absorption at $\Gamma$, or equivalently, at the lattice of the walls in the periodic extension of the unit cube. From (2.3),

$$R(\underline{x}, t; \underline{y}^*, s) = \begin{cases} 1, & \text{if } N(\underline{x}, t; \underline{y}^*, s) = 0, \\ \prod_{j=1}^{N(\underline{x}, t; \underline{y}^*, s)} \left(-\frac{1-\cos\phi_j}{1+\cos\phi_j}\right), & \text{if } N(\underline{x}, t; \underline{y}^*, s) > 0, \end{cases} \tag{5.2}$$

where $\phi_j = \phi_j(\underline{x}, t; \underline{y}^*, s)$, $1 \le j \le N(\underline{x}, t; \underline{y}^*, s)$ is the $j$th incident angle at $\Gamma$ of the wave from $y$ to $x$ along a ray inside $\Omega$.

Finally let an integer $\lambda$ be a geometric factor such that for any ray hitting $\Gamma$ $\lambda$ times must hit $\Gamma$ at least once with the incident angle $\ge \pi/4$. Then

$$N(\underline{x}, t; \underline{y}^*, s) \ge \frac{t-s}{\lambda T_X} - 1.$$

Therefore

$$\begin{aligned} |R(\underline{x}, t; \underline{y}^*, s)| &\le \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{\frac{t-s}{\lambda T_X} - 1} \\ &\le C\, e^{-\frac{1}{\lambda T_X}(t-s)}. \end{aligned} \tag{5.3}$$

Now we interpret Problem (2.1) as follows: find $u^*(\underline{x}, t)$, $(\underline{x}, t) \in \mathbb{R}^3 \times J$, such that

$$\frac{\partial^2 u^*}{\partial t^2} - c^2 \Delta u^* = f^*, \qquad \mathbb{R}^3 \times J, \tag{5.4.i}$$

$$u^*\Big|_{t=0} = \frac{\partial u^*}{\partial t}\Big|_{t=0} = 0, \qquad \mathbb{R}^3 \tag{5.4.ii}$$

where $f^*$ is given by (5.1).

The solution $u^*$ of (5.4) has the following representation ([.john81.]), for $f^* \in C^2(\mathbb{R}^3 \times J)$,

$$u^*(\underline{x}, t) = \frac{1}{4\pi c^2} \int_0^t \frac{dt}{t-s} \int_{|\underline{y}^* - \underline{x}| = c(t-s)} f^*(\underline{y}^*, s) d\, S_{y^*}. \tag{5.5}$$

If $\underline{x} \in \Omega$, then

$$u^*(\underline{x}, t) = u(\underline{x}, t), \qquad t \in J.$$

Due to (5.1), (5.3), (5.5), and integrations by parts twice,

$$\begin{aligned}
|u(\underline{x}, t)| &= |u^*(\underline{x}, t)| \\
&\leq \frac{1}{4\pi c^2} \int_0^t \frac{dt}{t-s} \int_{|\underline{y}^* - \underline{x}| = c(t-s)} C\, e^{-\frac{1}{\lambda T_X}(t-s)} |f(\underline{y}^*, s)| d\, S_{y^*} \\
&\leq C \int_0^t (t-s) e^{-\frac{1}{\lambda T_X}(t-s)} e^{-\beta s} ds. \\
&= C \left[ e^{-\frac{1}{\lambda T_X} t} \left\{ 1 + \left( \beta - \frac{1}{\lambda T_X} \right) t \right\} + e^{-\beta t} \right] \\
&\leq C\, e^{-\alpha t},
\end{aligned}$$

where $\alpha$ is a positive number strictly less than $min\{\frac{1}{\lambda T_X}, \beta\}$. This completes the proof of Theorem 2.1.

**§6. A Numerical Example.**

As a numerical example of frequency domain treatment consider $\Omega$ as an open interval centered at origin and fix $c = 2m/msec$. The size of the domain $\Omega$ will be determined later. The source function will be slightly modified from the following rapidly decaying sinusoidal function

$$f_0(x, t) = 4\alpha t e^{-\alpha t} \sin \omega_0 t\ H(t) \delta(x), \tag{6.1}$$

where $H(t)$ and $\delta(x)$ denote the Heaviside and the Dirac $\delta$ distributions. In (6.1) $\omega_0 = 2\pi f_m\ rad/msec$ with the main frequency $f_m$ which will be fixed as 10 $KHz$, and $\alpha = 0.79\ \omega_0/\pi$. The frequency spectrum $\widehat{f_0}(x, \omega)$ is given ([.tr79.]) by

$$\widehat{f_0}(x, \omega) = \frac{8\alpha\omega_0(\alpha - i\omega)}{[(\alpha - i\omega)^2 + \omega_0^2]^2} \delta(x).$$

The wave simulation time is 2 $msec$. Denote by $\lambda$ the wave length. Then $\lambda = c/f_m = 0.2m$. Suppose that there are 25 grid points per wave length so that the mesh size $h = \lambda/25 = 0.008m$. In order to solve (2.1) using the finite difference method in $t$, the time step should satisfy $\Delta t \leq h/c$ by the Courant – Friedrichs – Lewy stability condition. The largest such $\Delta t$ equals 0.004 $mssec$ and the number $N_{\text{time}}$ of time steps for the simulation time 2 $msec$ will be 500. Now let the domain size be 40 in wavelength unit. Then, $\Omega = (-4m, 4m)$. The number of grid points in $\Omega$ would then be 1001.

Let $L$ be the smallest integer that is a power of 2 not less than $N_{\text{time}}$. In our case $L = 512$. Let $m^*$ be a multiplication factor to increase the frequency resolution in solving the problem in the frequency domain.

Set $L^* = 2^{m^*} L$. The frequency step size $\Delta f$ is chosen as

$$\Delta f = 1/(\Delta t \cdot L^*) \cong 0.488/m^* \ KHz.$$

Also $\Delta\omega = 2\pi\Delta f \cong 3.068 \ KHz$. ($f_{nyq} = 1/(2\Delta t)$ is called the Nyquist frequency so that $\widehat{f}(f_{nyq} + \omega) = \overline{\widehat{f}}(f_{nyq} - \omega)$.)

Given a continuous function $\varphi : \mathbb{R} \to \mathbb{C}$, consider $\{\varphi(n\Delta t)\}_{n=-\infty}^{\infty}$ and

$$\varphi^*(t) = \sum_{n=-\infty}^{\infty} \varphi(n\Delta t)\delta(t - n\Delta t). \tag{6.2}$$

The Fourier transform of (6.2) is given by

$$\widehat{\varphi^*}(\omega) = \sum_{n=-\infty}^{\infty} \varphi(n\Delta t)e^{-i\omega n\Delta t}. \tag{6.3}$$

For finite samples $\{\varphi(n\Delta t)\}_{n=0}^{L^*-1}$ assume that the spectrum $\widehat{\varphi}(\omega)$ is represented by $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$.

Motivated by (6.2) and (6.3), the discrete Fourier transform of the sequence $\{\varphi(n\Delta t)\}_{n=0}^{L^*-1}$ is defined as a sequence $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$ with

$$\begin{aligned} \widehat{\varphi}(k\Delta\omega) &= \sum_{n=0}^{L^*-1} \varphi(n\Delta t) \left(e^{-i\frac{2\pi}{N}}\right)^{nk} \\ &= \sum_{n=0}^{L^*-1} \varphi(n\Delta t) \left(e^{-i\Delta\omega\Delta t}\right)^{nk}, \end{aligned} \tag{6.4}$$

where $\Delta\omega = 2\pi/\Delta t L^*$. In the same spirit, the inverse discrete Fourier transform of a sequence $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$ can be defined by

$$\begin{aligned} \varphi(n\Delta t) &= \frac{1}{L^*} \sum_{k=0}^{L^*-1} \varphi^*(k\Delta\omega) \left(e^{i\frac{2\pi}{N}}\right)^{kn} \\ &= \frac{1}{L^*} \sum_{k=0}^{L^*-1} \varphi^*(k\Delta\omega) \left(e^{-i\Delta\omega\Delta t}\right)^{kn}. \end{aligned} \tag{6.5}$$

A fast Fourier transform (FFT) is an algorithm to compute a sequence $\{\widehat{\varphi}(k\Delta\omega)\}_{k=0}^{L^*-1}$ in the frequency domain from a given sequence $\{\varphi(k\Delta t)\}_{k=0}^{L^*-1}$ in the time domain and reversely. Conventional Fourier transform algorithms require $0(L^{*2})$ flops, but FFTs require only $0(L^* \log L^*)$ flops ([.ct65.], [.gr69.]).

We modify the source function (6.1) so that (3.5) is satisfied and then normalize the resulting function:

$$f(x,t) = \frac{f_1(x,t)}{\max_{0\leq t\leq 2\ msec} f_1(x,t)},$$

where

$$f_1(x,t) = f_0(x,t) - \frac{1}{2\ msec}\int_0^{2\ msec} f_0(x,t)dt.$$

Using (6.4), (6.5), we solve Problem (2.1) both in the time domain and in the frequency domain. Various modifications of frequency spectrum and multiplication factor $m^*$ for the source function $f(x,t)$ are also attempted.

A simple band–pass filter with limit $[f_1, f_2, f_3, f_4](KHz)$ is used to modify the spectrum $\widehat{f}(x,\omega)$ such that

i) $f_1 = f_2 = 0$ and $f_3 = f_4 = f_{nyq}$, no modification is made for $\widehat{f}(x,\omega)$, i.e., $\widehat{f}(x,\omega) = \widehat{f}_0(x,\omega)$

ii) if $f_1 = f_2 = 0$ and $f_3 < f_4 \leq f_{nyq}$, $\widehat{f}(x,\omega)$ is unchanged for $\omega < f_3$,

$$\widehat{f}(x,\omega) = \widehat{f}_0(x,\omega) * \frac{f_4-\omega}{f_4-f_3}, \qquad f_3 < \omega < f_4,$$
$$\widehat{f}(x,\omega) = 0, \qquad f_4 < \omega \leq f_{nyq}.$$

In the case $0 \leq f_1 < f_2$, similar modifications by amplitude linearization for $\omega \in (f_1, f_2)$ are investigated.