1) A FEM FOR WAVE PROPAGATION IN 1D-FRACTURED VISCOELASTIC MEDIA -

2) A FEM FOR OSCILLATORY EXPERIMENTS in 1D-FRACTURED VISCOELASTIC MEDIA -

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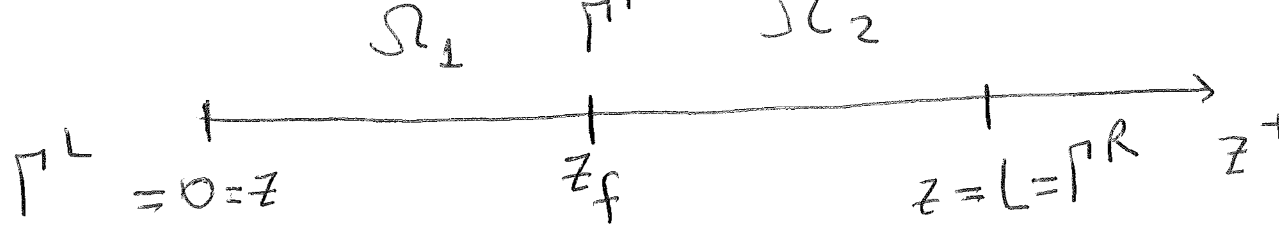
J. Santos, April 18, 2011

A FEM FOR WAVE PROPAGATION IN 1D FRACTURED VISCOELASTIC MEDIA

(1) $$-\omega^2 \rho u_z - \frac{\partial}{\partial z}\left(M \frac{\partial u_z}{\partial z}\right) = f, \quad \Omega_1 \cup \Omega_2$$

(2) $$M = \lambda + 2\mu = M_R + i M_I, \quad M_R > 0, \quad M_I > 0$$

(3) $$\tau_{zz} = M \frac{\partial u_z}{\partial z}, \quad \Omega_1 \cup \Omega_2$$



$$V^F = \{ v \in L^2(\Omega) : v|_{\Omega_j} \in H^1(\Omega_j), j = 1, 2 \}$$

Rewrite (1) in the form

(4) $$-\omega^2 \rho u_z - \nabla \cdot \tau_{zz} = f, \quad \Omega_1 \cup \Omega_2.$$

Multiply (4) by $v \in V^F$, $v = \begin{cases} v^{(1)}, & \Omega_1 \\ v^{(2)}, & \Omega_2 \end{cases}$ to get

$$-\omega^2 (\rho u_z^{(1)}, v^{(1)})_{\Omega_1} - (\nabla \cdot \tau_{zz}^{(1)}, v^{(1)})_{\Omega_1}$$

(5) $$-\omega^2 (\rho u_z^{(2)}, v^{(2)})_{\Omega_2} - (\nabla \cdot \tau_{zz}^{(2)}, v^{(2)})_{\Omega_2} = (f, v)_{\Omega}$$

Using integration by parts on each Ω_j:

(6)
$$\left(\nabla\cdot\tau_{zz}^{(1)}, v^{(1)}\right)_{\Omega_1} + \left(\tau_{zz}^{(1)}, \nabla v^{(1)}\right)_{\Omega_1} = \left\langle \tau_{zz}^{(1)}\cdot\nu_{12}, v^{(1)}\right\rangle_{\Gamma^F} + \left\langle \tau_{zz}^{(1)}\cdot\nu^{L}, v^{(1)}\right\rangle_{\Gamma^L}$$

(7)
$$\left(\nabla\cdot\tau_{zz}^{(2)}, v^{(2)}\right)_{\Omega_2} + \left(\tau_{zz}^{(2)}, \nabla v^{(2)}\right)_{\Omega_2} = \left\langle \tau_{zz}^{(2)}\cdot\nu_{21}, v^{(2)}\right\rangle_{\Gamma^F} + \left\langle \tau_{zz}^{(2)}\cdot\nu^{R}, v^{(2)}\right\rangle_{\Gamma^R}$$

where $\nu_{12} = 1$, $\nu^L = -1$, $\nu_{21} = -1$, $\nu^R = 1$ are the (1D) unit outward normals.

Using (6) (7) in (5)

(8)
$$-\omega^2\left(\rho u_z, v\right)_{\Omega} + \left(\tau_{zz}^{(1)}, \nabla v^{(1)}\right)_{\Omega_1} + \left(\tau_{zz}^{(2)}, \nabla v^{(2)}\right)_{\Omega_2}$$
$$-\left\langle \tau_{zz}^{(1)}\cdot\nu_{12}, v^{(1)}\right\rangle_{\Gamma^F} - \left\langle \tau_{zz}^{(1)}\cdot\nu^{L}, v^{(1)}\right\rangle_{\Gamma^L}$$
$$-\left\langle \tau_{zz}^{(2)}\cdot\nu_{21}, v^{(2)}\right\rangle_{\Gamma^F} - \left\langle \tau_{zz}^{(2)}\cdot\nu^{R}, v^{(2)}\right\rangle_{\Gamma^R} = (f, v)$$

Let us analyze the boundary terms on Γ^F in (8):

$$T_1 = -\langle \tau_{zz}^{(1)} \cdot \nu_{12}, v^{(1)} \rangle_{\Gamma^F} - \langle \tau_{zz}^{(2)} \cdot \nu_{21}, v^{(2)} \rangle_{\Gamma^F}$$

$$= -\langle \tau_{zz}^{(1)}, v^{(1)} \cdot \nu_{12} \rangle_{\Gamma^F} - \langle \tau_{zz}^{(2)}, v^{(2)} \cdot \nu_{21} \rangle_{\Gamma^F}$$

Now we impose the first boundary condition (continuity of stress at Γ^F)

$$(9) \quad \tau_{zz}^{(1)} = \tau_{zz}^{(2)} = \tau_{zz}, \quad \Gamma^f$$

Then,

$$T_1 = -\langle \tau_{zz}^{(1)}, v^{(1)} \cdot \nu_{12} + v^{(2)} \cdot \nu_{21} \rangle_{\Gamma^F}$$

$$= -\langle \tau_{zz}^{(1)}, \left(v^{(2)} - v^{(1)}\right) \cdot \nu_{21} \rangle_{\Gamma^F}$$

Now we impose the second B.C., jump of displacement and velocity on Γ^F

$$(10) \quad -\tau_{zz} = \alpha\left(u_z^{(2)} - u_z^{(1)}\right) \cdot \nu_{21}, \quad \alpha = K_z + i\omega \xi_z$$

$$K_z \geq 0 \quad \xi_z > 0.$$

Since $\nu_{21} = -1$, (10) can be stated also as

$$\tau_{zz} = \alpha\,(u_z^{(2)} - u_z^{(1)}) = \alpha\,(u_z^{+} - u_z^{-})$$

(a 1D-version of the B.C. in Jose's paper)

Then,

$$T_1 = \langle \alpha\,(u_z^{(2)} - u_z^{(1)})\cdot\nu_{21}, (v^{(2)} - v^{(1)})\cdot\nu_{21}\rangle_{\Gamma^F}$$

$$= \langle \alpha\,(u_z^{(1)} - u_z^{(2)})\cdot\nu_{12}, (v^{(1)} - v^{(2)})\cdot\nu_{12}\rangle_{\Gamma^F}$$

$$(11) \quad = \langle \alpha\,(u_z^{(1)} - u_z^{(2)}), v^{(1)} - v^{(2)}\rangle_{\Gamma^F}$$

$$= \langle \alpha\,(u_z^{(2)} - u_z^{(1)}), v^{(2)} - v^{(1)}\rangle_{\Gamma^F}$$

We also impose the following B.C. on $\Gamma = \Gamma^L \cup \Gamma^R$

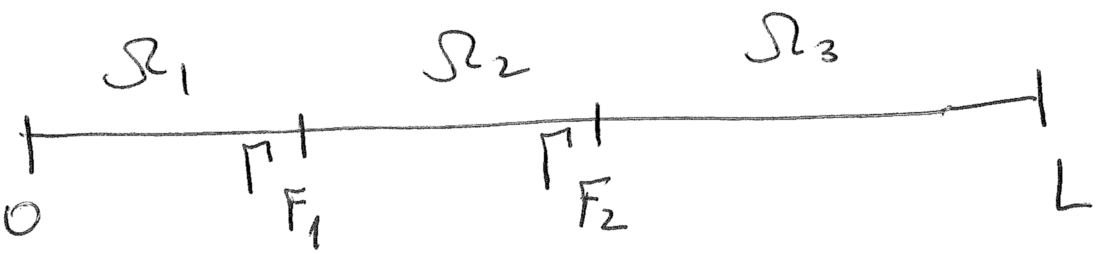
$$(12) \quad -\tau_{zz}\,\nu = i\omega D u_z \quad , D>0, \text{ on } \Gamma.$$

$$\left(D = \rho c, \; c = \sqrt{\frac{M}{\rho}}\right)$$

Thus, using (11) and (12) in (8) we get the WEAK FORM: Find $u_z \in V^F$ such that

$$
\begin{aligned}
(13)\quad & -\omega^2(\rho u_z, v) + \sum_{j=1}^{2} \left(\tau_{zz}(u_z^{(j)}), \nabla v^{(j)}\right)_{\Omega_j} \\
& + \left\langle (K_z + i\omega \xi_z)(u_z^{(2)} - u_z^{(1)}), v^{(2)} - v^{(1)} \right\rangle_{\Gamma^F} \\
& + \left\langle i\omega D\, u_z, v \right\rangle_{\Gamma} = (f, v), \quad v \in V^F.
\end{aligned}
$$

Remark: If we have n interior fractures $\Gamma_{F_1} \ldots \Gamma_{F_n}$, $F_j = \partial\Omega_j \cap \partial\Omega_{j+1}$, $j = 1 \ldots n$



$\Omega = \cup \Omega_j$, we define

$$V^F = \{ v \in L^2(\Omega) : v|_{\Omega_j} \in H^1(\Omega_j) \}.$$

Then the weak form is: Find $u_z \in V^F$ such that

$$-\omega^2(\rho u_z, v) + \sum_{j=1}^{n+1}\left(\tau_{zz}(u^{(j)}), \nabla v^{(j)}\right)_{\Omega_j}$$

(14) $$+ \sum_{j=1}^{n}\left\langle (K_z^j + i\omega\xi_z^j)(u_z^{(j+1)} - u_z^{(j)}), v^{(j+1)} - v^{(j)}\right\rangle_{\Gamma_j^F}$$

$$+ \langle i\omega D u_z, v\rangle_{\Gamma} = (f, v), \quad v \in V^F.$$

<u>UNIQUENESS FOR (13)</u>: Take $v = u_z$ in (13) to get

$$-\omega^2(\rho u_z, u_z) + \sum_{j=1}^{2}\left([M_R^{(j)} + iM_I^{(j)}]\nabla u_z^{(j)}, \nabla u_z^{(j)}\right)_{\Omega_j}$$

(15) $$+ \left\langle (K_z + i\omega\xi_z)(u_z^{(2)} - u_z^{(1)}), u_z^{(2)} - u_z^{(1)}\right\rangle_{\Gamma^F}$$

$$+ \langle i\omega D u_z, u_z\rangle_{\Gamma} = 0.$$

Take imaginary part in (15) to get, since $M_I^{(j)} > 0$, $D > 0$, $\xi_z > 0$,

(16) $$\|\nabla u_z^{(1)}\| = \|\nabla u_z^{(2)}\| = 0$$

(17) $$u_z = 0 \quad , \text{ on } \Gamma$$

(18) $$u_z^{(2)} = u_z^{(1)} \quad \text{on } \Gamma^F$$

Now since $u_z^{(j)} \in H^1(\Omega_j)$, by Sobolev imbedding $u_z^{(j)} \in C_B^0(\Omega_j)$, and from (18) $u_z^{(1)}$ and $u_z^{(2)}$ are continuous at Γ^F

Then $u_z \in H^1(\Omega)$.

Also, from (18) and (16) and Poincare's inequality

$$\|u_z\|_0 \leq C \|\nabla u_z\|_0 = 0 ,$$

so that $u_z = 0$ and we have uniqueness — EXISTENCE ?

EXISTENCE FOR

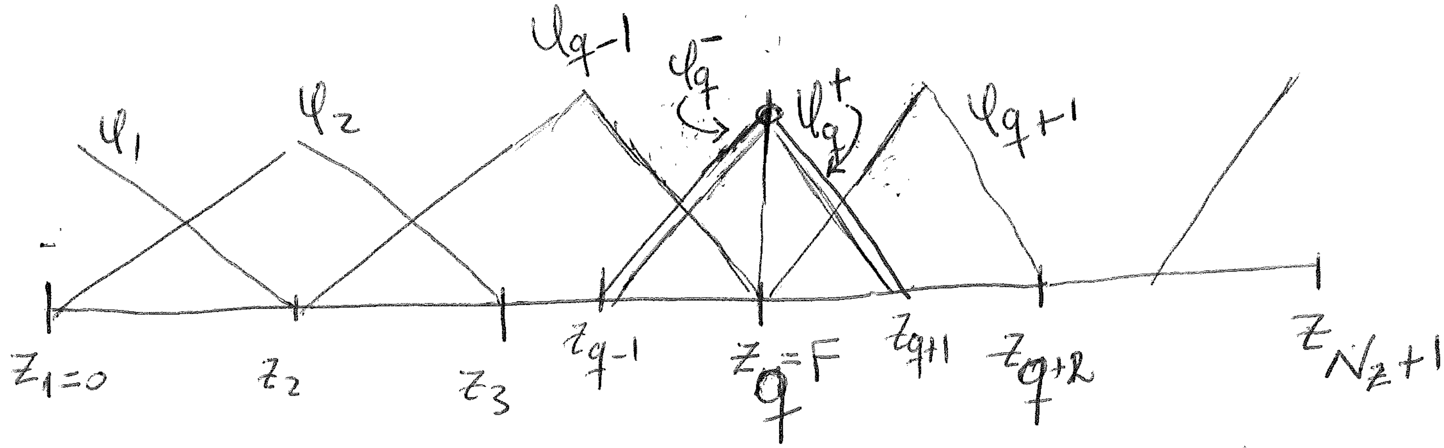
$$-\omega^2 \rho u_z - \nabla\cdot(\underbrace{\mu \nabla u_z}_{\tau_{zz}}) = f \quad , \Omega$$

$$\tau_{zz} \equiv \tau_{zz}^{(1)} = \tau_{zz}^{(2)} \quad \text{at} \quad \Gamma^F$$

$$\tau_{zz} = \alpha\left(u_z^{(2)} - u_z^{(1)}\right) \quad , \Gamma^F$$

$$-\tau_{zz}\nu = i\omega D u_z \quad , \Gamma .$$

GLOBAL Finite Element Method



$$u^h = \sum_{j=1}^{q-1} u_j^h \varphi_j + u_q^{h,-} \varphi_q^- + u_q^{h,+} \varphi_q^+$$

$$(19) \quad + \sum_{j=q+1}^{N_z+1} u_j^h \varphi_j$$

[So we have 2 DOF at $z = z_q = F$]

$$\varphi_q^- = \begin{cases} \dfrac{z - z_{q-1}}{h}, & z_{q-1} \le z \le z_q \\ 0 & \text{otherwise} \end{cases}$$

[Note that $(\varphi_q^-)^-\big|_{z=z_q} = 1$, $(\varphi_q^-)^+\big|_{z=z_q} = 0$]

$$\varphi_q^+ = \begin{cases} \dfrac{z_{q+1} - z}{h}, & z_q \le z \le z_{q+1} \\ 0 & \text{otherwise} \end{cases}$$

[Note that $(\varphi_q^+)^-\big|_{z=z_q} = 0$, $(\varphi_q^+)^+\big|_{z=z_q} = 1$.]

$$V^{F,h} = \operatorname{Span}\{\varphi_1 \ldots \varphi_{q-1}, \varphi_q^-, \varphi_q^+, \varphi_{q+1}, \ldots, \varphi_N\} \cap \left(C^0(\Omega_1) \cup C^0(\Omega_2)\right)$$

$$\dim V^{F,h} = N+1$$

The FEM is: find $u^h \in V^{F,h}$ such that

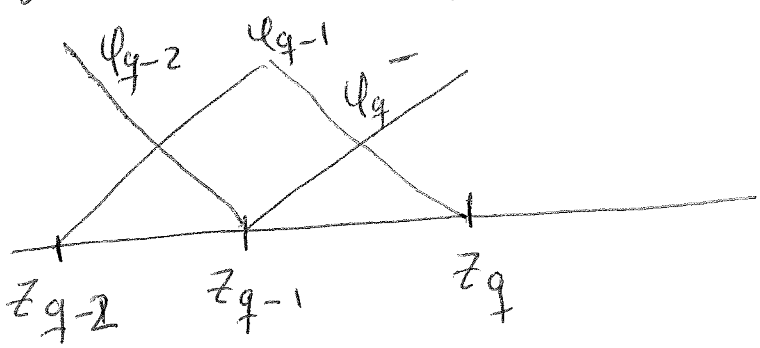
$$-\omega^2(\rho u^h, v) + \sum_{j=1}^{2}\left(M \frac{\partial u^h}{\partial z}, \frac{\partial v}{\partial z}\right)_{\Omega_j} + \langle i\omega D\, u^h, v\rangle_{\Gamma} + \langle (\kappa + i\omega \xi)(u^{h,+} - u^{h,-}), v^+ - v^- \rangle_{z=F} = (f, v), \quad v \in V^{F,h} \tag{20}$$

The algebraic problem associated with (20) is identical to that for the case without fractures, except for test functions that "see" φ_q^-, φ_q^+.

Then, we only compute for

$v = \varphi_{q-1}, \varphi_q^-, \varphi_q^+, \varphi_{q+1}$

Taking $v = \varphi_{q-1}$ in (20):



$$-\omega^2\left(\rho\left[u^h_{q-2}\varphi_{q-2}+u^h_{q-1}\varphi_{q-1}+u^{h,-}_q\varphi_q^-\right],\varphi_{q-1}\right)$$

$$+\left(M\frac{\partial}{\partial z}\left[u^h_{q-2}\varphi_{q-2}+u^h_{q-1}\varphi_{q-1}+u^{h,-}_q\varphi_q^-\right],\frac{\partial\varphi_{q-1}}{\partial z}\right)$$

(21)

$$+\left\langle(\kappa+i\omega\xi)(u^{h,+}_q-u^{h,-}_q),\underbrace{\varphi^+_{q-1}-\varphi^-_{q-1}}_{=0}\right\rangle_{z=z_q}$$

$$=(f,\varphi_{q-1})$$

Taking $v=\varphi_q^-$, φ_q^- only "sees" φ_{q-1}.

Then,

$$-\omega^2\left(\rho\left[u^h_{q-1}\varphi_{q-1}+u^{h,-}_q\varphi_q^-\right],\varphi_q^-\right)$$

(22)

$$+M\left(\frac{\partial}{\partial z}\left[u^h_{q-1}\varphi_{q-1}+u^{h,-}_q\varphi_q^-\right],\frac{\partial\varphi_q^-}{\partial z}\right)$$

$$+\left\langle(\kappa+i\omega\xi)\cdot(u^{h,+}_q-u^{h,-}_q),\underbrace{(\varphi_q^-)^+}_{0}-\underbrace{(\varphi_q^-)^-}_{1}\right\rangle_{z=z_q}=(f,\varphi_q^-)$$

Taking $v=\varphi_q^+$, φ_q^+ only "sees" φ_{q+1}

Then,

$$-\omega^2\left(\varrho\,[u_q^{h,+}\varphi_q^+ + u_{q+1}^h\varphi_{q+1}],\varphi_q^+\right)$$

$$+\left(M\frac{\partial}{\partial z}\left[u_q^{h,+}\varphi_q^+ + u_{q+1}^h\varphi_{q+1}\right],\frac{\partial\varphi_q^+}{\partial z}\right)$$

(23)
$$+\left\langle (K+i\omega\xi)(u_q^{h,+}-u_q^{h,-}),\underbrace{(\varphi_q^+)^+}_{=1}-\underbrace{(\varphi_q^+)^-}_{=0}\right\rangle_{z=z_q}=(f,\varphi_q^+)$$

Finally, take $V=\varphi_{q+1}$ in (11) to get

$$-\omega^2\left(\varrho\,[u_q^{h,+}\varphi_q^+ + u_{q+1}^h\varphi_{q+1}],\varphi_{q+1}\right)$$

(24)
$$+\left(M\frac{\partial}{\partial z}\left[u_q^{h,+}\varphi_q^+ + u_{q+1}^h\varphi_{q+1}\right],\frac{\partial\varphi_{q+1}}{\partial z}\right)$$

$$+\left\langle (K+i\omega\xi)(u_q^{h,+}-u_q^{h,-}),\underbrace{\varphi_{q+1}^+-\varphi_{q+1}^-}_{=0}\right\rangle_{z=z_q}=(f,\varphi_{q+1})$$

Then we see that the jump at $z=z_q=F$ is taken care of in equations (22) and (23)

Next comes the algebraic problem.

Algebraic Problem for (20)

As usual,

$$\varphi_k = \begin{cases} (z - z_{k-1})/h_{k-1}, & z_{k-1} \le z \le z_k \\ 1 - \dfrac{z - z_k}{h_k}, & z_k \le z \le z_{k+1} \\ 0, & \text{otherwise} \end{cases}, \quad h_k = z_{k+1} - z_k$$

and recall that away from $z = z_q$, $z = 0$, $z = z_{N_z+1}$,

$$(\rho\, \varphi_{k-1}, \varphi_k) = \rho_{k-1} \frac{h_{k-1}}{6}$$

$$(\rho\, \varphi_k, \varphi_k) = \rho_{k-1} \frac{h_{k-1}}{3} + \rho_k \frac{h_k}{3}$$

(25)

$$(\rho\, \varphi_{k+1}, \varphi_k) = \rho_k \frac{h_k}{6}$$

and

$$\left(M \frac{\partial \varphi_{k-1}}{\partial z}, \frac{\partial \varphi_k}{\partial z}\right) = -\frac{1}{h_{k-1}} M_{k-1}$$

$$\left(M \frac{\partial \varphi_k}{\partial z}, \frac{\partial \varphi_k}{\partial z}\right) = \frac{1}{h_{k-1}} M_{k-1} + \frac{1}{h_k} M_k$$

(26)

$$\left(M \frac{\partial \varphi_{k+1}}{\partial z}, \frac{\partial \varphi_k}{\partial z}\right) = -\frac{1}{h_k} M_k$$

$$\langle i\omega D u^h, \varphi_k \rangle_{z=0} = i\omega\, \rho_k c_k u^h_k\, \delta_{k1}$$

Then, the choice $v = \varphi_K$ in (20), for $1 \le K \le q-1$ (see (21) for $K = q-1$) gives

$$
\begin{aligned}
&\left[-\omega^2 \frac{h_{K-1}}{6} e_{K-1} - \frac{1}{h_{K-1}} M_{K-1}\right](1-\delta_{K1})\, u^h_{K-1} \\
&+ \left[-\omega^2 \left[e_{K-1} \frac{h_{K-1}}{3}(1-\delta_{K1}) + e_K \frac{h_K}{3}\right] \right. \\
&\left. + \frac{M_{K-1}}{h_{K-1}}(1-\delta_{K1}) + \frac{M_K}{h_K}\right] u^h_K \\
&+ \left[-\omega^2 e_K \frac{h_K}{6} - \frac{M_K}{h_K}\right] u^h_{K+1} \\
&+ i\omega\, e_K C_K \delta_{K1}\, u^h_K = (f, \varphi_K)
\end{aligned}
\tag{27}
$$

$1 \le K \le q-1$

Next, from (22) the choice $v = \varphi_q^-$ in (20) gives

$$\left[-\omega^2 \frac{h_{q-1}}{6} - \frac{M_{q-1}}{h_{q-1}}\right] u^h_{q-1}$$

$$(28) \quad + \left[-\omega^2 \frac{h_{q-1}}{3} + \frac{M_{q-1}}{h_{q-1}}\right] u_q^{h,-}$$

$$+ \left(K_q + i\omega \xi_q\right)\left(u_q^{h,+} - u_q^{h,-}\right)(-1) = (f, \varphi_q^{-1}) = 0$$

<u>Here I assume $f \equiv 0$ near the fracture.</u>

From (28) we get

$$\left[-\omega^2 \frac{h_{q-1}}{6} - \frac{M_{q-1}}{h_{q-1}}\right] u^h_{q-1}$$

$$(29) \quad + \left[-\omega^2 \frac{h_{q-1}}{3} + \frac{M_{q-1}}{h_{q-1}} + (K_q + i\omega \xi_q)\right] u_q^{h,-}$$

$$+ \left[(-1)(K_q + i\omega \xi_q)\right] u_q^{h,+} = 0$$

Next, from (23), the choice $v = \varphi_q^+$ in (20) gives

$$\left[-\omega^2 \frac{h_q}{3} + \frac{M_q}{h_q}\right] u_q^{h,+}$$

$$+\left[-\omega^2 \frac{h_q}{6} - \frac{M_q}{h_q}\right] u_{q+1}^{h}$$

$$+ \left(k_q + i\omega \xi_q\right)\left(u_q^{h,+} - u_q^{h,-}\right)(1) = 0$$

so that

$$\left[(-1)\left(k_q + i\omega \xi_q\right)\right] u_q^{h,-}$$

(30) $$+ \left[-\omega^2 \frac{h_q}{3} + \frac{M_q}{h_q} + \left(k_q + i\omega \xi_q\right)\right] u_q^{h,+}$$

$$+ \left[-\omega^2 \frac{h_q}{6} - \frac{M_q}{h_q}\right] u_{q+1}^{h} = 0$$

Next, choose $v = \varphi_K$, $K = q+1, \ldots, N_z+1$

in (20) to get

(use that

$$\langle i\omega D u^h, \varphi_K \rangle_{z = z_{N_z+1}} = i\omega\, e_K\, c_K\, u_K^K\, \delta_{K, N_z+1}$$

(e_{N_z+1}, c_{N_z+1} : values of e, c in (z_{N_z}, z_{N_z+1})))

$$\left[-\omega^2 \frac{h_{k-1}}{6} e_{k-1} - \frac{1}{h_{k-1}} M_{k-1}\right] u^h_{k-1}$$

$$+ \left[-\omega^2 \left(e_{k-1} \frac{h_{k-1}}{3} + e_k \frac{h_k}{3} (1-\delta_{k,N_z+1}) \right) \right.$$

(31)

$$\left. + \frac{1}{h_{k-1}} M_{k-1} + \frac{M_k}{h_k} (1-\delta_{k,N_z+1}) \right] u^h_k$$

$$+ \left[-\omega^2 e_k \frac{h_k}{6} - \frac{M_k}{h_k} \right] (1-\delta_{k,N_z+1}) \, u^h_{k+1}$$

$$= (f, \varphi_k)$$

$$k = q+1, \dots, N_z+1$$

Equations (27), (29) (30) and (31) give a tridiagonal system for the $\underline{\underline{N_z+2}}$ unknowns

$$u^h_1, \dots, u^h_{q-1}, u^{h,-}_q, u^{h,+}_q, u^h_{q+1}, \dots, u^h_{N_z+1}.$$

The coefficients in the 3-digonal system are

$$a_{k-1} = \left(-\omega^2 \frac{h_{k-1}}{6} - \frac{M_{k-1}}{h_{k-1}}\right)(1-\delta_{k1})$$

$$a_k = -\omega^2\left[\rho_{k-1}\frac{h_{k-1}}{3}(1-\delta_{k1}) + \rho_k \frac{h_k}{3}\right]$$

(31-1) $$+ \frac{M_{k-1}}{h_{k-1}}(1-\delta_{k1}) + \frac{M_k}{h_k} + i\omega \rho_k c_k \delta_{k1}$$

$$a_{k+1} = -\omega^2 \rho_k \frac{h_k}{6} - \frac{M_k}{h_k},$$

$k = 1 \ldots q-1$

Now we remember shifting in 1 the unknowns, so that

$$u_q^{h,-} \Rightarrow u_q^h \qquad u_q^{h,+} \to u_{q+1}^h,$$

$$u_k^h \to u_{k+1}^h, \quad q = q+1 \ldots N_2+1.$$

Then

$$a_{q-1} = -\omega^2 \frac{h_{q-1}}{6} - \frac{M_{q-1}}{h_{q-1}}$$

(31-2) $$a_q = -\omega^2 \frac{h_{q-1}}{3} + \frac{M_{q-1}}{h_{q-1}} + (\kappa_q + i\omega \xi_q)$$

$$a_{q+1} = -(\kappa_q + i\omega \xi_q)$$

$$a_q = -(k_q + i\omega \xi_q)$$

$$a_{q+1} = -\omega^2 \frac{h_q}{3} + \frac{M_q}{h_q} + (k_q + i\omega \xi_q)$$

(31-3)

$$a_{q+2} = -\omega^2 \frac{h_q}{6} - \frac{M_q}{h_q}$$

$$a_k = -\omega^2 \frac{h_{k-1}}{6} - \frac{M_{k-1}}{h_{k-1}}$$

$$a_{k+1} = -\omega^2\left(e_{k-1}\frac{h_{k-1}}{3} + e_k \frac{h_k}{3}(1-\delta_{k,N_z+1})\right)$$

(31-4) $$+ \frac{M_{k-1}}{h_{k-1}} + \frac{M_k}{h_k}(1-\delta_{k,N_z+1})$$

$$a_{k+2} = \left(-\omega^2 e_k \frac{h_k}{6} - \frac{M_k}{h_k}\right)(1-\delta_{k,N_z+1})$$

$$k = q+1, \ldots, N_z$$

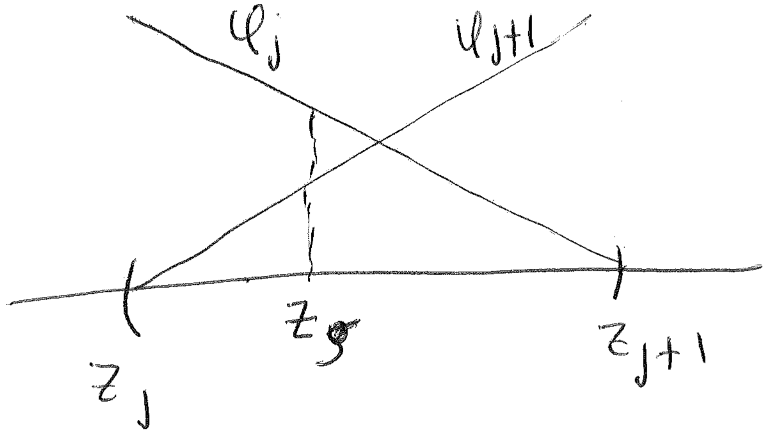
Source function f to compute the RHS.

Take

$$f(z,t) = \frac{\partial}{\partial z}\delta(z - z_s)\, g(\omega)$$ 17

Then, choose z_s an interior point in an interval

$$(z_j,\ z_{j+1})$$



$$(f(z,t), \varphi_k) = -\left(\delta(z - z_s) g(\omega),\ \frac{\partial \varphi_k(z)}{\partial z}\right)$$

$\neq 0$ only for $k = j,\ k+1$

$$(f, \varphi_j) = -g(\omega) \left.\frac{\partial \varphi_j}{\partial z}\right|_{z = z_s} = + g(\omega)\frac{1}{h}$$

$$(f, \varphi_{j+1}) = -g(\omega)\frac{\partial \varphi_{j+1}}{\partial z} = -g(\omega)\frac{1}{h}$$

Choose $g(\omega)$: Fourier transform of

$$g(t) = -2\xi(t - t_0)\, e^{-\xi(t - t_0)^2}$$

OSCILLATORY EXPERIMENTS

THE DIFFERENTIAL MODEL IS

$$(32)\quad -\omega^2 \rho\, u_z - \nabla\cdot \tau_{zz}(u_z) = 0, \quad \Omega_1 \cup \Omega_2$$

$$(33)\quad \tau_{zz}(u) = M \frac{\partial u_z}{\partial z}, \quad M = \lambda + 2\mu = M_R + i M_I \quad \Omega_1 \cup \Omega_2$$

with the boundary conditions

$$(34)\quad \tau_{zz}(u_z)\nu\nu = -\Delta P, \quad \Gamma^T = \{z = L\}$$

$$(35)\quad u_z = 0, \quad \Gamma^B = \{z = 0\}$$

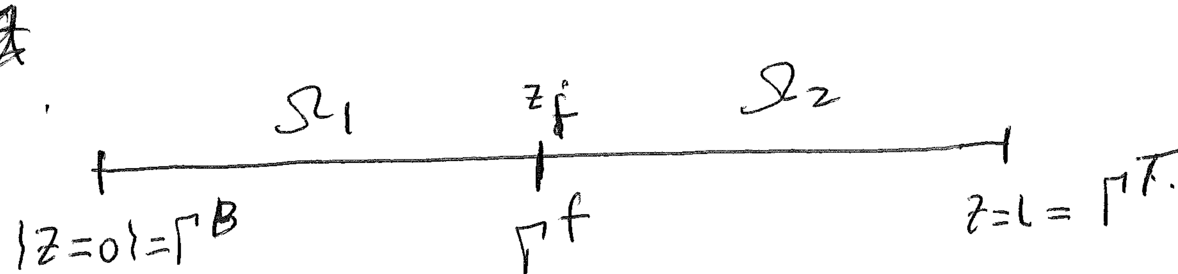
and the "jump" condition at the fracture Γ^f:

$$(36)\quad \tau_{zz}^{(1)} = \tau_{zz}^{(2)} = \tau_{zz}, \quad \Gamma^f$$

$$(37)\quad -\tau_{zz} = \alpha\left(u_z^{(2)} - u_z^{(1)}\right)\cdot\nu_{21}, \quad \Gamma^f$$

$$\alpha = K_z + i\omega\,\xi_z \qquad K_z \geq 0, \; \xi_z > 0 \; -$$

~~Let~~



$$\Omega_1 = (0, z_f), \quad \Omega_2 = (z_f, L),$$

Let

$$V_0^F = \{ v \in L^2(\Omega) : v|_{\Omega_1} \in H^1_{0,\Gamma^L}(\Omega_1),\ v|_{\Omega_2} \in H^1(\Omega_2) \}$$

where

$$H^1_{0,\Gamma^L}(\Omega_1) = \{ v \in H^1(\Omega_1) : v = 0 \text{ at } z = 0 \}$$

Proceeding as in the argument leading to (13) we get the weak form: Find $u_z \in V_0^F$:

$$-\omega^2(\rho u_z, v)_\Omega + \sum_{j=1}^{2} (\tau_{zz}(u_z^{(j)}), \nabla v^{(j)})_{\Omega_j}$$

(38)

$$+ \langle (\kappa_z + i\omega \xi_z)(u_z^{(2)} - u_z^{(1)}), v^{(2)} - v^{(1)} \rangle_{\Gamma^f}$$

$$= -\langle \Delta P, v^{(2)} \cdot \nu \rangle_\Gamma, \quad v \in V_0^F.$$

Uniqueness for (38) follows with the same argument than that used for the wave propagation problem —

Finite Element Method

$$V_0^{F,h} = \{ v : v|_{\Omega_j} \in P_1,\ v = 0 \text{ at } z = 0 \} \cap (C^0(\Omega_1) \cup C^0(\Omega_2))$$

(if $v \in V_0^{F,h}$ then v is C^0-p.w. linear in Ω_1 and Ω_2 and we do not impose continuity at ~~z=z_f~~

$z = z_q = z_f$.)

Then, let

$$u^h = \sum_{j=2}^{q-1} u_j^h \varphi_j + u_q^{h,-} \varphi_q^- + u_q^{h,+} \varphi_q^+$$

(39)
$$+ \sum_{j=q+1}^{N_z+1} u_j^h \varphi_j$$

(same notation as in (19)) –

The FEM is: Find $u^h \in V_0^{F,h}$ such that

$$-\omega^2 (\rho u^h, v)_\Omega + \sum_{j=1}^{2} \left(M_j \frac{\partial u^h}{\partial z}, \frac{\partial v}{\partial z} \right)_{\Omega_j}$$

(40)
$$+ \left\langle (K_z + i\omega \xi_z)(u_z^{(2)} - u_z^{(1)}),\ v^{(2)} - v^{(1)} \right\rangle_{\Gamma^f}$$

$$= -\left\langle \Delta P,\ v^{(2)} \cdot \nu \right\rangle_{\Gamma^T}, \quad v \in V_0^{F,h}$$

ALGEBRAIC PROBLEM ASSOCIATED WITH (40) -

The algebraic problem is almost identical to that for the wave propagation problem, changing as follows:

1) Equation (27) holds for $2 \leq k \leq q-1$ (so that the $(1-\delta_{k1})$-terms may be deleted), ~~and~~ the boundary term of coefficient

$$i\omega e_k c_k \delta_{k1}$$

~~vanishes~~ is deleted

and the right-hand side is equal to zero.

2) Equations (29) and (30) remain identical

3) Equation (31) remains identical except ~~a~~ that the right-hand side changes to

$$-\Delta P\, \delta_{k, N_z+1}$$

The definition of the coefficients in (31-1) remains unchanged except that $k = 2 - q-1$ (so that the factors $(1-\delta_{k1})$ may be deleted and the bry term $i\omega e_k c_k \delta_{k1} = 0$.

The other definitions of the coefficients in (31-2) (31-3) and (31-4) remain identical.

Thus we solve a TRIDIAGONAL SYSTEM for the UNKNOWNS

$$u_2^n, \ldots, u_{q-1}^n, u_q^{n,-}, u_q^{n,+}, u_{q+1}^n \ldots, u_{Nz+1}^n$$

($N_z + 1$ - unknowns) —