Effective viscoelastic medium from harmonic experiments

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Abstract

An effective viscoelastic medium is derived using harmonic FE numerical experiments in a poroviscoelastic fluid-saturated medium containing a dense set of horizontal fractures modeled as thin highly permeable layers.

Key words: , Poroviscoelasticity, Finite element methods, Effective anisotropic media

1 Introduction

Can Giorgi read this ???

La simulación numérica realística de propagación de ondas sísmicas en medios poroviscoelásticos fracturados y/o con diferentes tipos de anisotropía permite un mejor conocimiento de los fenómenos que tienen lugar en el subsuelo desde el punto de vista de la sísmica de exploración y el desarrollo de reservorios. Además, mediante procedimientos numéricos "upscaling" es posible representar heterogeneidades de las rocas, fluidos saturantes y fracturas en escala mesoscópica y conocer como ellos afectan las observaciones en la macroescala.

En este contexto, cobra importancia el estudio de los modelos diferenciales adecuados para la física numérica de rocas (mesoescala) y para los problemas globales en la macroescala, como así también las técnicas que deben aplicarse para su resolución. A estos efectos, luego de emplear el método de Elementos Finitos para la resolución de las ecuaciones gobernantes en el dominio espacio-tiempo y/o en el dominio espacio-frecuencia, los algoritmos deben ser implementados computacionalmente.

El plan de trabajo se orienta al desarrollo de rutinas de cálculo y la escritura eficiente de sus respectivos códigos computacionales para máquinas con arquitectura en paralelo.

2 A Biot's medium with fractures modeled as thin highly permeable layers

We consider a porous solid saturated by a single phase, compressible viscous fluid and assume that the whole aggregate is isotropic. Let $\mathbf{u}_s = (u_{s,i})$ and $\widetilde{\mathbf{u}}_f = (\widetilde{u}_{f,i})$, $i = 1, \cdots, E$ denote the averaged displacement vectors of the solid and fluid phases, respectively, where E denotes the Euclidean dimension. Also let

$$\mathbf{u}_f = \phi(\tilde{\mathbf{u}}_f - \mathbf{u}_s),$$

be the average relative fluid displacement per unit volume of bulk material, with ϕ denoting the effective porosity. Set $\mathbf{u} = (\mathbf{u}_s, \mathbf{u}_f)$ and note that

$$\xi = -\nabla \cdot \mathbf{u}_f,$$

represents the change in fluid content.

Let $\boldsymbol{\varepsilon} = \varepsilon_{ij}(u_s)$ be the strain tensor of the solid. Also, let $\boldsymbol{\sigma} = \sigma_{ij},\ i,j = 1,\cdots,E$, and p_f denote the stress tensor of the bulk material and the fluid pressure, respectively. Following [3], the stress-strain relations can be written in the form:

$$\sigma_{ij}(\mathbf{u}) = 2\mu\,\varepsilon_{ij}(\mathbf{u}_s) + \delta_{ij}(\lambda_c\,\nabla\cdot\mathbf{u}_s - B\,\xi), \tag{1a}$$
$$p_f(\mathbf{u}) = -B\,\nabla\cdot\mathbf{u}_s + M\xi. \tag{1b}$$

The coefficient μ is equal to the shear modulus of the bulk material, considered to be equal to the shear modulus of the dry matrix. Also

$$\lambda_c = K_c - \frac{2}{E}\mu, \tag{2}$$

with K_c being the bulk modulus of the saturated material. Following [21] [12] the coefficients in (1) can be obtained from the relations

$$\alpha = 1 - \frac{K_m}{K_s},\ M = \left(\frac{\alpha-\phi}{K_s} + \frac{\phi}{K_f}\right)^{-1} K_c = K_m + \alpha^2 K_{av},\ B = \alpha M \tag{3}$$

where K_s, K_m and K_f denote the bulk modulus of the solid grains composing the solid matrix, the dry matrix and the the saturant fluid, respectively. The coefficient α is known as the effective stress coefficient of the bulk material.

2.1 The equations of motion

If $\omega = 2\pi f$ is the angular frequency, in the absence of body forces Biot's equations of motion in the diffusive range, stated in the space-frequency domain are [1] [2]

$$\nabla \cdot \sigma(\mathbf{u}) = 0, \tag{4}$$
$$i\omega \frac{\eta}{\kappa} \mathbf{u}_f(x,\omega) + \nabla p_f(\mathbf{u}) = 0, \tag{5}$$

where η is the fluid viscosity and κ the absolute permeability.

3 A variational formulation for c_{33}. 2D case.

In order to state a variational formulation we need to introduce some notation. For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle \cdot,\cdot \rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ and $|\cdot|_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_\Gamma$. Let us assume that $\Omega = (0,H)^2$ is partitioned into rectangles Ω_j:

$$\Omega = \cup_{j,l} \Omega_j$$

Let us introduce the following spaces:

$$\mathcal{W}_{33}(\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_{\Omega_j} \in [H^1(\Omega_j)]^2,\ \mathbf{v} \cdot \boldsymbol{\nu} = 0 \ \text{ on } \Gamma^L \cup \Gamma^R \cup \Gamma^B\},$$

Also, let

$$H_0(\mathrm{div};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 :\ \mathbf{v}|_{\Omega_j} \in H(\mathrm{div},\Omega_j), \mathbf{v} \cdot \boldsymbol{\nu} = 0 \ \text{ on } \Gamma\}.$$

and set

$$\mathcal{Z}_{33}(\Omega) = \mathcal{W}_{33}(\Omega) \times H_0(\mathrm{div};\Omega)\}$$

To obtain the variational formulation associated with c_{33}, multiply the first equation in (4) by $\mathbf{v}_s$ and the second by $\mathbf{v}_f$ with $(\mathbf{v}_s, \mathbf{v}_f) \in \mathcal{Z}_{33}(\Omega)$ and add the resulting equations. Use integration by parts and the following boundary conditions associted with c_{33}:

$$\boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu} \cdot \boldsymbol{\nu} = -\Delta P, \quad (x_1,x_3) \in \Gamma^T, \tag{6}$$
$$\boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu} \cdot \boldsymbol{\chi} = 0, \quad (x_1,x_3) \in \Gamma, \tag{7}$$
$$\mathbf{u} \cdot \boldsymbol{\nu} = 0, \quad (x_1,x_3) \in \Gamma^L \cup \Gamma^R \cup \Gamma^B. \tag{8}$$

where $\Gamma = \Gamma^L \cup \Gamma^B \cup \Gamma^R \cup \Gamma^T$,

$$\Gamma^L = \{(x_1,x_3) \in \Gamma : x_1 = 0\}, \quad \Gamma^R = \{(x_1,x_3) \in \Gamma : x_1 = H\},$$
$$\Gamma^B = \{(x_1,x_3) \in \Gamma : x_3 = 0\}, \quad \Gamma^T = \{(x_1,x_3) \in \Gamma : x_3 = H\}.$$

we see that our weak form is: find $\mathbf{u}^{(33)} = (\mathbf{u}_s^{(33)}, \mathbf{u}_f^{(33)}) \in \mathcal{Z}_{33}(\Omega)$ such that:

$$\begin{aligned}
&\Lambda(\mathbf{u}^{(33)}, \mathbf{v}) = i\omega \left(\frac{\eta}{\kappa} \mathbf{u}_f, \mathbf{v}_f \right) \\
&+ \sum_j \left(\sigma_{st}(\mathbf{u}), \varepsilon_{st}(\mathbf{v}_s) \right)_{\Omega_j} - \left(p_f(\mathbf{u}), \nabla \cdot \mathbf{v}_f \right)_{\Omega_j} \\
&= - \left\langle \Delta P, \mathbf{v}_s \cdot \boldsymbol{\nu} \right\rangle_{\Gamma^T}, \qquad \forall \mathbf{v} = (\mathbf{v}_s, \mathbf{v}_f) \in \mathcal{Z}_{33}(\Omega)
\end{aligned} \tag{9}$$

Note that in (9), we can write

$$\begin{aligned}
&\sum_j \left(\sigma_{st}(\mathbf{u}), \varepsilon_{st}(\mathbf{v}_s) \right)_{\Omega_j} - \left(p_f(\mathbf{u}), \nabla \cdot \mathbf{v}_f \right)_{\Omega_j} \\
&\qquad = \sum_j \left(\mathbf{M}\, \widetilde{\boldsymbol{\varepsilon}}(\mathbf{u})), \widetilde{\boldsymbol{\varepsilon}}(\mathbf{v}) \right)_{\Omega_j}.
\end{aligned} \tag{10}$$

In (10), the complex matrix $\mathbf{M}(\omega) = \mathbf{M}_R(\omega) + i\mathbf{M}_I(\omega)$ and the column vector $\widetilde{\boldsymbol{\varepsilon}}(\mathbf{u})$ are defined by

$$\mathbf{M} = \begin{pmatrix} \lambda_c + 2\mu & \lambda_c & B & 0 \\ \lambda_c & \lambda_c + 2\mu & B & 0 \\ B & B & M & 0 \\ 0 & 0 & 0 & 4\mu \end{pmatrix}, \ \boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} e_{11}(\mathbf{u}_s) \\ e_{33}(\mathbf{u}_s) \\ \nabla \cdot \mathbf{u}_f \\ e_{13}(\mathbf{u}^s) \end{pmatrix}.$$

It will be assumed that the real part $\mathbf{M}_R(\omega)$ is positive definite since in the elastic limit it is associated with the strain energy density. Furthermore, the imaginary parts $\mathbf{M}_I(\omega)$ are assumed to be positive definite because of the restriction imposed on our system by the first and second laws of thermodynamics.

Also, since $\mathbf{F}_R(\omega), \mathbf{F}_I(\omega)$ are nonnegative, $\mathbf{M}_R(\omega)$ and $\mathbf{M}_I(\omega)$ are positive definite and the real part of Π is positive and the imaginary part of Π is negative, for the choice $v = u$ in (9) all imaginary terms (dissipative energy terms) are nonnegative and all real terms are positive (strain energy terms). This in turn will imply uniqueness.

4 The Finite Element Method

Let

$$\mathcal{V}_{33}^h(\Omega) = \{\mathbf{v}_s \in [L^2(\Omega)]^2 : \mathbf{v}|_{\Omega_j} \in [P_{1,1})(\Omega_j)]^2,\ \mathbf{v}_s \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma^L \cup \Gamma^R \cup \Gamma^B\},$$

$$\mathcal{W}_{33}^h(\Omega) = \{\mathbf{v}_f \in [L^2(\Omega)]^2 : \mathbf{v}|_{\Omega_j} \in (P_{1,0} \times P_{0,1})(\Omega_j),\ \mathbf{v}_f \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\},$$

$$\mathcal{Z}_{33}^h(\Omega) = \times \mathcal{V}_{33}^h(\Omega) \times \mathcal{W}_{33}^h(\Omega).$$

Now repeatting the argument leading to (9), using integration by parts we see that the Finite Element Method can be formulated as follows: find $\mathbf{u}^h = (\mathbf{u}_s^h, \mathbf{u}_f^h) \in \mathcal{Z}_{33}^h(\Omega)$ such that

$$\begin{aligned}
&\Lambda(\mathbf{u}^h, \mathbf{v}) = i\omega \left(\frac{\eta}{\kappa}\mathbf{u}_f^h, \mathbf{v}\right) \\
&+(\lambda_c \nabla \cdot \mathbf{u}_s^h, \nabla \cdot \mathbf{v}_s) + 2(\mu e_{11}(\mathbf{u}_s^h), e_{11}(\mathbf{v}_s)) \\
&\qquad +2(\mu e_{33}(\mathbf{u}_s^h), e_{33}(\mathbf{v}_s)) + 4(\mu e_{13}(\mathbf{u}_s^h), e_{13}(\mathbf{v}_s)) \\
&\qquad +(B\nabla \cdot \mathbf{u}_f^h, \nabla \cdot \mathbf{v}^s) + (B\nabla \cdot \mathbf{u}_s^h, \nabla \cdot \mathbf{v}^f) + (M\nabla \cdot \mathbf{u}_f^h, \nabla \cdot \mathbf{v}^f) \\
&= -\left\langle \Delta P, \mathbf{v}_s \cdot \nu \right\rangle_{\Gamma^T}, \qquad \forall \mathbf{v} = (\mathbf{v}_s, \mathbf{v}_f) \in \mathcal{Z}_{33}^h(\Omega)
\end{aligned} \tag{11}$$

5 The Algebraic Problem associated with (11)

The first step is to construct a global basis for $\mathcal{Z}_{33}^h(\Omega)$. Let $\varphi_j, j = 1, \cdots, N$, $\psi_j, j = N+1, \cdots, N+M$ be global basis for $\mathcal{V}_{33}^h(\Omega)$ and $\mathcal{W}_{33}^h(\Omega)$, respectively. Set

$$\mathbf{u}_s^h = \sum_{j=1}^{N} u_{s,j}\varphi_j, \quad \mathbf{u}_f^h = \sum_{j=N+1}^{N+M} u_{f,j}\psi_j$$

Choose $v = (\varphi_i, 0), i = 1, \cdots, N$ in (11) to get

$$\Lambda(\mathbf{u}^{(33)}, (\varphi_i, 0)) = \sum_k (\mathbf{M}\ \widetilde{\boldsymbol{\varepsilon}}(\mathbf{u})), \widetilde{\boldsymbol{\varepsilon}}(\varphi_i, 0))_{\Omega_k} = -\left\langle \Delta P, (\varphi_i, 0) \cdot \boldsymbol{\nu} \right\rangle_{\Gamma^T}. \tag{12}$$

Also, choose $v = (0, \psi_i), i = N+1, \cdots, N+M$ in (11) to get

$$\Lambda(\mathbf{u}^{(33)},(0,\psi_i)) = i\omega\left(\frac{\eta}{\kappa}\mathbf{u}_f,\psi_i\right)_{\Omega_k}\sum_k(\mathbf{M}\ \widetilde{\boldsymbol{\varepsilon}}(\mathbf{u})),\widetilde{\boldsymbol{\varepsilon}}(0,\psi_i))_{\Omega_k} = 0. \tag{13}$$

Set

$$\mathbf{A}_{ij} = \left(\frac{\eta}{\kappa}\psi_j,\psi_i\right)_{\Omega_k} \qquad i,j = 1,\cdots,N \tag{14}$$

$$\mathbf{B}_{ij} = (\mathbf{M}\ \widetilde{\boldsymbol{\varepsilon}}(\varphi_j,\psi_j)),\widetilde{\boldsymbol{\varepsilon}}(\varphi_i,0))_{\Omega_k}, \quad i = 1,\cdots,N \tag{15}$$

$$\mathbf{B}_{ij} = (\mathbf{M}\ \widetilde{\boldsymbol{\varepsilon}}(\varphi_j,\psi_j)),\widetilde{\boldsymbol{\varepsilon}}(0,\psi_i))_{\Omega_k}, \quad i = N+1,\cdots,M \tag{16}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A} \end{pmatrix}.$$

$$\mathbf{D} = \mathbf{B} + \mathbf{C}$$

Note that the real part of $\mathbf{B}$ is positive definite and the imag. part of $\mathbf{B}$ is nonnegative. Also, $\mathbf{C}$ is nonnegative. Set

$$F_i = -\left\langle \Delta P,(\varphi_i,0)\cdot\boldsymbol{\nu}\right\rangle_{\Gamma^T}, i = 1,\cdots,N, \quad F_i = 0,\ i = N+1,\cdots,n+M \tag{17}$$

$$X_i = u_{s,i}, \quad i = 1,\cdots,N, X_i = u_{f,i}, \quad i = N+1,\cdots,N+M. \tag{18}$$

Then in matrix form (12)-(12) can be stated as

$$\mathbf{D}X = F \tag{19}$$

where $\mathbf{D}$ is a complex matrix with positive definite real part and nonnegative imaginary part.

We need to solve (19) using either global or iterative parallelizable solvers. Here is where Giorgi can help us, we hope.

The 3D extension of this problem is immediate, the algebraic problem is huge and we need to define efficient parallel iterative solvers.

6 Numerical experiments

to be performed

7 Conclusions

bla bla bla

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