

# Fracture-induced anisotropic attenuation

Short title: Fracture-induced anisotropy

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## Abstract

The triaxial nature of the tectonic stress in the earth's crust favours the appearance of vertical fractures. The resulting rheology is usually effective anisotropy with orthorhombic and monoclinic symmetries. In addition, the presence of fluids leads to azimuthally varying attenuation of seismic waves. A dense set of fractures embedded in a background medium enhances anisotropy and rock compliance. Fractures are modeled as boundary discontinuities in the displacement  $\mathbf{u}$  and particle velocity  $\mathbf{v}$  as  $[\boldsymbol{\kappa} \cdot \mathbf{u} + \boldsymbol{\eta} \cdot \mathbf{v}]$ , where the brackets denote discontinuities across the fracture surface,  $\boldsymbol{\kappa}$  is a fracture stiffness and  $\boldsymbol{\eta}$  is a viscosity related to the energy loss.

We consider a transversely isotropic background medium (e.g., thin horizontal plane layers), with sets of long vertical fractures. Schoenberg and Muir's theory combines the background medium and sets of vertical fractures, to provide the thirteen complex stiffnesses of the long-wavelength equivalent monoclinic and viscoelastic medium. The sym-

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metry plane is the horizontal plane. The equations for orthorhombic and transversely-isotropic media follow as particular cases. We compute the complex velocities of the medium as a function of frequency and propagation direction, which provide the phase velocities, energy velocities (wavefronts) and quality factors.

The effective medium ranges from monoclinic symmetry to hexagonal (transversely isotropic) symmetry from the low- to the high-frequency limits in the case of a particle-velocity discontinuity (lossy case) and the attenuation shows typical Zener relaxation peaks as a function of frequency. The attenuation of the coupled waves may show important differences when computed versus the ray or phase angles, with triplication appearing in the  $Q$  factor of the qS wave. We have performed a full-wave simulation to compute the field corresponding to the coupled qP-qS waves in the symmetry plane of an effective monoclinic medium. The simulations agree with the predictions of the plane-wave analysis.

**Keywords** Fractures · anisotropy · attenuation · Schoenberg-Muir theory · boundary conditions

## 1 Introduction

Wave propagation through fractures, faults and cracks is an important subject in seismology, exploration geophysics and mining. Faults in the Earth's crust constitute sources of earthquakes (Pyrak-Nolte et al., 1990), and hydrocarbon and geothermal reservoirs are mainly composed of fractured rocks (Nakagawa and Myer, 2009). Applications in geotechnical engineering, such as the analysis of dynamic stability of rock slopes and tunnels, involve the study of imperfect joints in rock masses (Perino et al., 2010; Fan et al., 2011). In geophysical prospecting, knowledge of reservoirs fracture orientations, densities and sizes is essential, since these factors control hydrocarbon pro-

duction (Hansen, 2002; Hall and Kendall, 2003; Grechka and Tsvankin, 2003; Barton, 2007). The analysis of the data exploits the fact that seismic-velocity and attenuation anisotropy due to the presence of fractures are sensitive to key properties of the reservoir, such as porosity, permeability and fluid type.

A few frequency-dependent models have been developed to describe anisotropy and attenuation. Carcione (1992) generalized Backus averaging to the anelastic case, obtaining the first model for  $Q$ -anisotropy (see Carcione, 2007). Analyses on sequences of sandstone-limestone and shale-limestone with different degrees of anisotropy indicate that the quality factors ( $Q$ ) of the shear modes are more anisotropic than the corresponding phase velocities, cusps of the qSV mode are more pronounced for low frequencies and midrange proportions, and in general, attenuation is higher in the direction perpendicular to layering or close to it, provided that the material with lower velocity is the more dissipative. This model has been further analyzed in Picotti et al. (2010), where it is shown how to obtain the medium properties with quasi-static numerical experiments. Other alternative models of  $Q$ -anisotropy were proposed by Carcione and Cavallini (1994) and Carcione et al. (1998). A brief description of all these phenomenological models can be found in Carcione (2007). Zhu and Tsvankin (2005) analyze in detail the attenuation in orthorhombic media, assuming homogeneous viscoelastic waves. They simplify the interpretation for processing purposes by introducing a set of attenuation-anisotropy parameters. A Backus type model to describe wave propagation in fractures has been introduced by Carcione (1996a), where plane layers are separated by thin continuous layers of viscous fluid. A similar model is considered in Liu et al. (2000), where the fracture is a very thin soft viscous layer.

On the other hand, a recently developed model (Chapman, 2003; Maultzsch, 2005) is explicitly describing the effects of cracks and fractures on wave propagation, since

the elastic constants are derived in terms of microstructural parameters and therefore the model is predictive. It describes attenuation and velocity dispersion at seismic frequencies and predicts how these effects are related to fluid type and size of the fractures. An approach to explicitly model cracks and fractures is proposed by Zhang and Gao (2009). The scheme treats the fractures as non-welded interfaces that satisfy the linear-slip displacement-discontinuity conditions instead of using equivalent medium theories. Hence, the algorithm can be used to characterize the seismic response of fractured media and to test equivalent medium theories. Other poroelasticity models describing anisotropic attenuation are given in Krzikalla & Müller (2011) and Carcione et al. (2011), who obtain the five complex and frequency-dependent stiffnesses of an equivalent medium corresponding to thin poroelastic layers.

Modeling fractures requires a suitable interface model for describing the dynamic response of the joint. Theories that consider imperfect contact were mainly based on the displacement discontinuity model at the interface. Pyrak-Nolte et al. (1990) proposed a non-welded interface model based on the discontinuity of the displacement and the particle velocity across the interface. The stress components are proportional to the displacement and velocity discontinuities through the specific stiffnesses and one specific viscosity, respectively. Displacement discontinuities conserve energy and yield frequency dependent reflection and transmission coefficients. On the other hand, velocity discontinuities generate energy loss at the interface. The specific viscosity accounts for the presence of a liquid under saturated conditions. The liquid introduces a viscous coupling between the two surfaces of the fracture (Schoenberg, 1980) and enhances energy transmission, but at the same time this is reduced by viscous losses. The model may account for slip and dilatancy effects. Chichinina et al. (2009a,b) describe anisotropic attenuation in a TI medium using Schoenberg's linear-slip model

with complex-valued normal and tangential fracture stiffnesses. The theory and laboratory experiments show that in the vicinity of the symmetry axis, P-wave attenuation is comparable to S-wave attenuation when the fracture is filled with a fluid. On the other hand, in the presence of dry fractures, P-wave attenuation is much greater than S-wave attenuation.

In this work, we generalize the orthorhombic model given in Schoenberg and Helbig (1997) to the anelastic monoclinic case, by introducing a particle-velocity discontinuity in the fracture surface, allowing us to describe  $Q$ -anisotropy. The medium consists of sets of vertical fractures embedded in a transversely isotropic (TI) background medium (generally horizontal fine layering) to form a long-wavelength equivalent monoclinic medium. Using the theory of Schoenberg and Muir (1989), we obtain the thirteen complex and frequency-dependent stiffnesses of this medium. We then obtain the quality factors and wave velocities as a function of frequency and propagation angle.

## 2 Interface model

Let us consider a planar fracture. The non-ideal characteristics of the interface are modeled by imposing suitable boundary conditions. The model proposed here is based on the discontinuity of the displacement and particle-velocity fields across the interface. Then, the boundary conditions at the interface are

$$\boldsymbol{\kappa} \cdot [\mathbf{u}] + \boldsymbol{\eta} \cdot [\mathbf{v}] = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad (1)$$

(Pyrak-Nolte et al., 1990; Carcione, 1996b), where  $\mathbf{u}$  and  $\mathbf{v}$  are the displacement and particle-velocity components, respectively,  $\boldsymbol{\sigma}$  is the  $3 \times 3$  stress tensor,  $\mathbf{n}$  is the unit normal to the fracture,  $\boldsymbol{\kappa}$  is the specific stiffness matrix, and  $\boldsymbol{\eta}$  is the specific viscosity matrix (both of dimension  $3 \times 3$ ). They have dimensions of stiffness and viscosity per

unit length, respectively. Moreover, the symbol “ $\cdot$ ” indicates scalar product and the brackets denote discontinuities across the interface, such that for a field variable  $\phi$ , it is  $[\phi] = \phi_2 - \phi_1$ , where 1 and 2 indicate the two sides of the fracture.

The particle velocity is given by

$$\mathbf{v} = \dot{\mathbf{u}}, \quad (2)$$

where a dot above a variable indicates time differentiation. In the Fourier domain,

$$\mathbf{v} = i\omega\mathbf{u}, \quad (3)$$

where  $\omega$  is the angular frequency and  $i = \sqrt{-1}$ . Equation (1) then becomes

$$[\mathbf{u}] = \mathbf{Z} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}), \quad (4)$$

where

$$\mathbf{Z} = (\boldsymbol{\kappa} + i\omega\boldsymbol{\eta})^{-1} \quad (5)$$

is a fracture compliance matrix, whose dimension is length/stress. This approach is equivalent to the linear-slip model introduced by Schoenberg (1980). In fact, equation (5), with  $\boldsymbol{\eta} = 0$ , is given in Coates and Schoenberg (1995) and Schoenberg and Helbig (1997). Three models have been studied by Liu et al. (2000) to obtain the expression of  $\mathbf{Z}$  for different fracture models. Two of the models only provide the real part of  $\mathbf{Z}$ . The third model describes the fracture as a thin and soft viscoelastic layer embedded in an isotropic elastic background medium, where  $\mathbf{Z}$  can be obtained in terms of the thickness, Lamé constants and viscosity of the soft material infill. The resulting equivalent medium has TI symmetry. Similarly, Carcione (1996a) described the infill by a viscous fluid.

The compliance matrix  $\mathbf{Z}$  of the set of fractures is the diagonal non-negative matrix

$$\mathbf{Z} = \begin{pmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{pmatrix} = \begin{pmatrix} (\kappa_1 + i\omega\eta_1)^{-1} & 0 & 0 \\ 0 & (\kappa_2 + i\omega\eta_2)^{-1} & 0 \\ 0 & 0 & (\kappa_3 + i\omega\eta_3)^{-1} \end{pmatrix}, \quad (6)$$

where  $Z_1$  is the normal compliance,  $Z_2$  is the horizontal tangential compliance, and  $Z_3$  is the vertical tangential compliance. The fact that  $Z_2 \neq Z_3$  means that the texture of the fracture surface has different roughnesses vertically and horizontally, while the fact that there are no off-diagonal components means that, across the fractures, the normal motion is uncoupled from the tangential motion.

A common situation in the earth's crust is to have a finely layered medium and vertical fractures. Figure 1 shows such a case, where  $q = 2$  sets of long vertical fractures are embedded in a TI medium with a vertical symmetry. This background medium is the long-wavelength equivalent of the finely layered medium, according to Backus (1962). The mechanical (viscoelastic) representation of the fracture boundary condition by a Kelvin-Voigt model is illustrated in the figure, according to the stress-displacement relation

$$\boldsymbol{\sigma} \cdot \mathbf{n} = (\boldsymbol{\kappa} + i\omega\boldsymbol{\eta}) \cdot [\mathbf{u}], \quad (7)$$

where the model simulates the fracture by a zero width layer of distributed spring-dashpots. The quantity  $\boldsymbol{\kappa} + i\omega\boldsymbol{\eta}$  is the complex modulus per unit length of the Kelvin-Voigt element (e.g., Carcione, 2007).

A displacement discontinuity yields compliance, while a discontinuity in the particle velocity implies an energy loss at the interface (Carcione, 1996b, 1998, 2007);  $\boldsymbol{\kappa} = 0$  gives the particle-velocity discontinuity model and  $\boldsymbol{\eta} = 0$  gives the displacement discontinuity model. On the other hand, if  $\boldsymbol{\kappa} \rightarrow \infty$  or  $\boldsymbol{\eta} \rightarrow \infty$ , the model gives a welded interface.

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## THE EQUIVALENT MONOCLINIC MEDIUM

Let us consider a background TI medium. The stress-strain relation is

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} \equiv \mathbf{C} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}, \quad (8)$$

where

$$2c_{66} = c_{11} - c_{12},$$

$\sigma_I$  denotes stress component and  $e_I$  denotes strain component in the Voigt notation (e.g., Carcione, 2007), such that

$$[e_1, e_2, e_3, e_4, e_5, e_6]^\top = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12}]^\top, \quad (9)$$

where  $\epsilon_{ij}$  are the strain components,  $c_{IJ}$  are the elasticity constants, and  $\mathbf{C}$  is the elasticity matrix.

The background medium can easily be generalized to a viscoelastic medium by using one of the three models of anisotropic anelasticity proposed in Chapter 4 of Carcione (2007). The generalization implies that the elasticity constants  $c_{IJ}$  become complex and frequency dependent. Here the purpose is to analyze the attenuation due to the fracture solely, however in the last example we consider an effective HTI medium to test the commonly used equation by which the total dissipation factor is equal to the dissipation factor of the background medium plus the dissipation factor due to the fractures.

Schoenberg and Muir (1989) showed how to combine arbitrary sets of elastic thin layers and find their long-wavelength-equivalent medium properties. The method is applicable also when a set of layers is infinitely thin and compliant, as it is the fracture



set shown in Figure 1. Nichols et al. (1989) and Hood (1991) simplified the procedure to include fractures by using compliance matrices instead of stiffness matrices. As shown below, the compliance matrix of the equivalent medium is found by addition of expanded  $6 \times 6$  versions of the fracture compliance matrices to the compliance matrix of the background medium.

Let us consider many sets of vertical fractures, so that the fracture normal of a given set makes an angle  $\beta$  with the  $x$ -axis (see Figure 1). We avoid the fracture index ( $q$ ) for simplicity. The expanded fracture compliance matrix of each set is

$$\mathbf{S}_f = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 & 0 & s_{16} \\ s_{12} & s_{22} & 0 & 0 & 0 & s_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & s_{45} & 0 \\ 0 & 0 & 0 & s_{45} & s_{55} & 0 \\ s_{16} & s_{26} & 0 & 0 & 0 & s_{66} \end{pmatrix}, \quad (10)$$

where

$$\begin{aligned} s_{11} &= \frac{3Z_N + Z_H}{8} + \frac{Z_N}{2} \cos 2\beta + \frac{Z_N - Z_H}{8} \cos 4\beta, \\ s_{12} &= \frac{Z_N - Z_H}{8} (1 - \cos 4\beta), \\ s_{16} &= \frac{Z_N \sin 2\beta}{2} + \frac{Z_N - Z_H}{4} \sin 4\beta, \\ s_{22} &= \frac{3Z_N + Z_H}{8} - \frac{Z_N}{2} \cos 2\beta + \frac{Z_N - Z_H}{8} \cos 4\beta, \\ s_{26} &= \frac{Z_N \sin 2\beta}{2} - \frac{Z_N - Z_H}{4} \sin 4\beta, \\ s_{44} &= \frac{Z_V (1 - \cos 2\beta)}{2}, \\ s_{45} &= \frac{Z_V \sin 2\beta}{2}, \\ s_{55} &= \frac{Z_V (1 + \cos 2\beta)}{2}, \\ s_{66} &= \frac{Z_N + Z_H}{2} - \frac{Z_N - Z_H}{2} \cos 4\beta \end{aligned} \quad (11)$$

(Schoenberg et al., 1999). We have introduced

$$Z_N = \frac{Z_1}{L}, \quad Z_H = \frac{Z_2}{L}, \quad Z_V = \frac{Z_3}{L}, \quad (12)$$

where  $L$  is a characteristic length, such that these quantities have dimensions of compliance, and

$$\begin{aligned} \kappa_N &= L\kappa_1, & \kappa_H &= L\kappa_2, & \kappa_V &= L\kappa_3, \\ \eta_N &= L\eta_1, & \eta_H &= L\eta_2, & \eta_V &= L\eta_3, \end{aligned} \quad (13)$$

such that these quantities have dimensions of stiffness and viscosity, respectively. It is

$$Z_N = \frac{1}{\kappa_N + i\omega\eta_N}, \quad Z_H = \frac{1}{\kappa_H + i\omega\eta_H}, \quad Z_V = \frac{1}{\kappa_V + i\omega\eta_V}. \quad (14)$$

Then, the complex and frequency-dependent stiffness matrix of the equivalent medium is given by

$$\mathbf{P}(\omega) = \left[ \mathbf{C}^{-1} + \sum_q \mathbf{S}_f^{(q)}(\omega) \right]^{-1}, \quad (15)$$

where the sum is over the  $q$  sets of fractures. The equivalent homogeneous anisotropic medium (in the long wavelength limit) is a monoclinic medium with a horizontal mirror plane of symmetry. Its complex and frequency dependent stiffness matrix has the form

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & p_{16} \\ p_{12} & p_{22} & p_{23} & 0 & 0 & p_{26} \\ p_{13} & p_{23} & p_{33} & 0 & 0 & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & 0 \\ 0 & 0 & 0 & p_{45} & p_{55} & 0 \\ p_{16} & p_{26} & p_{36} & 0 & 0 & p_{66} \end{pmatrix}. \quad (16)$$

At zero frequency, we obtain the lossless case, where  $Z_N = 1/\kappa_N$ ,  $Z_H = 1/\kappa_H$  and  $Z_V = 1/\kappa_V$ , and the medium remains monoclinic. At “infinite” frequency the fracture stiffnesses vanish ( $\mathbf{S}_f^{(q)} \rightarrow 0$ ) and we obtain the TI and lossless background medium. By “infinite” frequency we mean frequencies such that the wavelength is much larger than the distance between single fractures, i.e., the long-wavelength approximation. It can be shown that the wave velocities at zero frequency are smaller than the wave velocities at infinite frequency, i. e., the medium is more compliant.

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## 2.1 Orthorhombic equivalent media

Assume two fracture sets with  $\beta = 0$  (fracture strike along the  $y$ -direction) and  $\beta = \pi/2$  (fracture strike points along the  $x$ -direction). The fracture stiffness matrices are given by

$$\mathbf{S}_f^{(1)} = \begin{pmatrix} Z_N^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_V^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_H^{(1)} \end{pmatrix}. \quad (17)$$

and

$$\mathbf{S}_f^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z_N^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_V^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_H^{(2)} \end{pmatrix}, \quad (18)$$

respectively.

The evaluation of equation (15) requires simple operations with  $3 \times 3$  matrices.

We obtain

$$\begin{aligned}
p_{11} &= \frac{c_{11} + Z_N^{(2)}(c_{11}^2 - c_{12}^2)}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{12} &= \frac{c_{12}}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{13} &= \frac{c_{13}[1 + Z_N^{(2)}(c_{11} - c_{12})]}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{22} &= \frac{c_{11} + Z_N^{(1)}(c_{11}^2 - c_{12}^2)}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{23} &= \frac{c_{13}[1 + Z_N^{(1)}(c_{11} - c_{12})]}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{33} &= \frac{c_{33} + (Z_N^{(1)} + Z_N^{(2)})(c_{11}c_{33} - c_{13}^2) + Z_N^{(1)}Z_N^{(2)}(c_{11} - c_{12})[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}{1 + Z_N^{(1)}Z_N^{(2)}(c_{11}^2 - c_{12}^2) + c_{11}(Z_N^{(1)} + Z_N^{(2)})}, \\
p_{44} &= \frac{c_{55}}{1 + c_{55}Z_V^{(2)}}, \\
p_{55} &= \frac{c_{55}}{1 + c_{55}Z_V^{(1)}}, \\
p_{66} &= \frac{c_{66}}{1 + c_{66}(Z_H^{(1)} + Z_H^{(2)})}.
\end{aligned} \tag{19}$$

Since the two sets are orthogonal, the equivalent medium has orthorhombic symmetry.

In the case of a single fracture set, e.g.,  $\beta = 0$ , equation (19) gives the stiffness matrix of Schoenberg and Helbig (1997):

$$\mathbf{P} = \begin{pmatrix} c_{11}(1 - \delta_N) & c_{12}(1 - \delta_N) & c_{13}(1 - \delta_N) & 0 & 0 & 0 \\ c_{12}(1 - \delta_N) & c_{11}(1 - \delta_N c_{12}^2/c_{11}^2) & c_{13}(1 - \delta_N c_{12}/c_{11}) & 0 & 0 & 0 \\ c_{13}(1 - \delta_N) & c_{13}(1 - \delta_N c_{12}/c_{11}) & c_{33}[1 - \delta_N c_{13}^2/(c_{11}c_{33})] & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}(1 - \delta_V) & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(1 - \delta_H) \end{pmatrix}, \tag{20}$$

where

$$\delta_N = [1 + 1/(Z_N c_{11})]^{-1}, \quad \delta_H = [1 + 1/(Z_H c_{66})]^{-1}, \quad \delta_V = [1 + 1/(Z_V c_{55})]^{-1}. \quad (21)$$

## 2.2 HTI equivalent media

If the background medium is isotropic ( $c_{11} = c_{12} + 2c_{55}$ ,  $c_{12} = c_{13}$ ,  $c_{55} = c_{66}$ ) and the fracture set is rotationally invariant, we have  $Z_H = Z_V \equiv Z_T$ , and the equivalent medium is TI with a horizontal symmetry axis (HTI), whose stiffness matrix is

$$\mathbf{P} = \begin{pmatrix} c_{11}c_N & c_{12}c_N & c_{12}c_N & 0 & 0 & 0 \\ c_{12}c_N & c_{11} - c_{12}^2 Z_N c_N & c_{12} - c_{12}^2 Z_N c_N & 0 & 0 & 0 \\ c_{12}c_N & c_{12} - c_{12}^2 Z_N c_N & c_{11} - c_{12}^2 Z_N c_N & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}c_T & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55}c_T \end{pmatrix}, \quad (22)$$

where

$$c_N = (1 + c_{11}Z_N)^{-1} \quad \text{and} \quad c_T = (1 + c_{55}Z_T)^{-1}. \quad (23)$$

The stiffness matrix in Coates and Schoenberg (1995) is equivalent to equation (22) with a rotation of  $\pi/2$  around the  $y$ -axis and considering the lossless case ( $\boldsymbol{\eta} = 0$ ). If  $\boldsymbol{\eta} \neq 0$ , equation (22) is equivalent to the medium studied by Chichinina et al. (2009b).

The fractured medium defined by equation (22) can be obtained from Backus averaging (Backus, 1962) of a periodic medium composed of two isotropic constituents with proportions  $p_i$ , and P-wave and S-wave moduli, respectively, given by  $E_i$  and  $\mu_i$ ,  $i = 1, 2$ . Backus's effective stiffness constants are given by (Carcione, 2007),

$$\begin{aligned} p_{11} &= [E_1 E_2 + 4p_1 p_2 (\mu_1 - \mu_2)(\lambda_1 + \mu_1 - \lambda_2 - \mu_2)]D, \\ p_{12} &= [\lambda_1 \lambda_2 + 2(\lambda_1 p_1 + \lambda_2 p_2)(\mu_2 p_1 + \mu_1 p_2)]D, \\ p_{13} &= (\lambda_1 p_1 E_2 + \lambda_2 p_2 E_1)D \\ p_{33} &= E_1 E_2 D, \\ p_{55} &= \mu_1 \mu_2 (p_1 \mu_2 + p_2 \mu_1)^{-1}, \\ p_{66} &= p_1 \mu_1 + p_2 \mu_2, \\ D &= (p_1 E_2 + p_2 E_1)^{-1}. \end{aligned} \quad (24)$$

Equation (22) is then obtained by taking the limits  $p_1 \rightarrow 1$  and  $p_2 \rightarrow 0$  (or  $p_2 \ll p_1$ ) and setting  $E_1 = c_{11}$  and  $\mu_1 = c_{55}$  (background medium) and  $E_2 = p_2/Z_N$  and  $\mu_2 = p_2/Z_H$  (fracture). This is shown in Schoenberg (1983) in the lossless case.

### 3 Properties of the effective medium

A general plane-wave solution for the displacement field  $\mathbf{u} = (u_x, u_y, u_z)^\top = (u_1, u_2, u_3)^\top$  is

$$\mathbf{u} = \mathbf{U} \exp[i\omega(t - s_1x - s_2y - s_3z)], \quad (25)$$

where  $s_i$  are the components of the slowness vector  $\mathbf{s} = (s_1, s_2, s_3)$ ,  $\mathbf{U}$  is a complex vector and  $t$  is the time variable. We consider homogeneous viscoelastic waves, such that  $\mathbf{s} = s(l_1, l_2, l_3)$ , where  $s = \sqrt{s_1^2 + s_2^2 + s_3^2}$ , and  $l_i$  are the direction cosines defining the propagation (and attenuation) directions. The complex velocity is

$$v = \frac{1}{s}. \quad (26)$$

#### 3.1 Symmetry plane of a monoclinic medium

In the symmetry plane of a monoclinic medium there is a pure shear wave and two coupled waves. The respective dispersion relations in the  $(x, y)$ -plane are

$$\begin{aligned} \Gamma_{33} - \rho v^2 &= 0, \\ (\Gamma_{11} - \rho v^2)(\Gamma_{22} - \rho v^2) - \Gamma_{12}^2 &= 0, \end{aligned} \quad (27)$$

where  $\rho$  is the density of the background medium,

$$\begin{aligned} \Gamma_{11} &= p_{11}l_1^2 + p_{66}l_2^2 + 2p_{16}l_1l_2, \\ \Gamma_{22} &= p_{66}l_1^2 + p_{22}l_2^2 + 2p_{26}l_1l_2, \\ \Gamma_{33} &= p_{55}l_1^2 + p_{44}l_2^2 + 2p_{45}l_1l_2, \\ \Gamma_{12} &= p_{16}l_1^2 + p_{26}l_2^2 + (p_{12} + p_{66})l_1l_2 \end{aligned} \quad (28)$$

(Carcione, 2007), where  $v$  is the complex velocity and  $l_1 = \sin \theta$  and  $l_2 = \cos \theta$ , with  $\theta$  the phase propagation angle.

If we label 1 the pure mode (the SH wave) and 2 and 3 the qS and qP waves, the corresponding complex velocities are (e.g., Carcione, 2007):

$$\begin{aligned} v_1 &= \sqrt{\rho^{-1}(p_{55}l_1^2 + p_{44}l_2^2 + 2p_{45}l_1l_2)}, \\ v_2 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{22}l_2^2 + p_{66} + 2l_1l_2(p_{16} + p_{26}) - A}, \\ v_3 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{22}l_2^2 + p_{66} + 2l_1l_2(p_{16} + p_{26}) + A}, \\ A &= \sqrt{(\Gamma_{11} - \Gamma_{22})^2 + 4\Gamma_{12}^2}. \end{aligned} \quad (29)$$

The phase velocity is given by

$$v_p = \left[ \operatorname{Re} \left( \frac{1}{v} \right) \right]^{-1} \quad (30)$$

and the quality factor is simply

$$Q = \frac{\operatorname{Re}(v^2)}{\operatorname{Im}(v^2)} \quad (31)$$

(e.g., Carcione, 2007). The values of the qP quality factor along orthogonal directions are

$$Q_P(\theta = \pi/2) = \frac{\operatorname{Re}(p_{11})}{\operatorname{Im}(p_{11})} \quad \text{and} \quad Q_P(\theta = 0) = \frac{\operatorname{Re}(p_{22})}{\operatorname{Im}(p_{22})}, \quad (32)$$

respectively, while those of the shear waves are

$$\begin{aligned} Q_{SV}(\theta = \pi/2) &= Q_{SV}(\theta = 0) = Q_{SH}(\theta = 0) = \frac{\operatorname{Re}(p_{44})}{\operatorname{Im}(p_{44})}, \\ \text{and } Q_{SH}(\theta = \pi/2) &= \frac{\operatorname{Re}(p_{55})}{\operatorname{Im}(p_{55})}. \end{aligned} \quad (33)$$

Next, we obtain the energy velocity at zero frequency (the lossless – elastic – limit)

times one unit of time. The SH-wave energy velocity is

$$\mathbf{v}_e = \frac{1}{\rho v_p} [(c_{55}^0 l_1 + c_{45}^0 l_2) \hat{\mathbf{e}}_1 + (c_{44}^0 l_2 + c_{45}^0 l_1) \hat{\mathbf{e}}_2]. \quad (34)$$

On the other hand, the qP and qS energy-velocity components  $v_{e1}$  and  $v_{e2}$  are

$$\begin{aligned} &\rho v_p (\Gamma_{11} + \Gamma_{22} - 2\rho v_p^2) v_{e1} \\ &= (\Gamma_{22} - \rho v_p^2) (c_{11}^0 l_1 + c_{16}^0 l_2) + (\Gamma_{11} - \rho v_p^2) (c_{66}^0 l_1 + c_{26}^0 l_2) - \Gamma_{12} [2c_{16}^0 l_1 + (c_{12}^0 + c_{66}^0) l_2] \end{aligned} \quad (35)$$

and

$$\begin{aligned} &\rho v_p (\Gamma_{11} + \Gamma_{22} - 2\rho v_p^2) v_{e2} \\ &= (\Gamma_{22} - \rho v_p^2) (c_{66}^0 l_2 + c_{16}^0 l_1) + (\Gamma_{11} - \rho v_p^2) (c_{22}^0 l_2 + c_{26}^0 l_1) - \Gamma_{12} [2c_{26}^0 l_2 + (c_{12}^0 + c_{66}^0) l_1] \end{aligned} \quad (36)$$

where  $c_{IJ}^0$  are the zero-frequency limit of the  $p_{IJ}$  (e.g., Carcione, 2007).

### 3.2 Orthorhombic media

The dispersion relation has the same form as in the lossless case, but replacing the (real) elasticity constants by the complex stiffnesses  $p_{IJ}$ , i.e., the components of matrix  $\mathbf{P}$ .

Following Schoenberg and Helbig (1997), we have

$$v^6 - a_2 v^4 + a_1 v^2 - a_0 = 0, \quad (37)$$

where

$$\begin{aligned} a_0 &= \Gamma_{11}\Gamma_{22}\Gamma_{33} - \Gamma_{12}^2\Gamma_{33} - \Gamma_{13}^2\Gamma_{22} - \Gamma_{23}^2\Gamma_{11} + \Gamma_{12}\Gamma_{23}\Gamma_{13} + \Gamma_{13}\Gamma_{12}\Gamma_{23}, \\ a_1 &= \Gamma_{22}\Gamma_{33} + \Gamma_{33}\Gamma_{11} + \Gamma_{11}\Gamma_{22} - \Gamma_{23}^2 - \Gamma_{13}^2 - \Gamma_{12}^2, \\ a_2 &= \Gamma_{11} + \Gamma_{22} + \Gamma_{33}, \end{aligned} \quad (38)$$

with

$$\begin{aligned} \Gamma_{11} &= p_{11}l_1^2 + p_{66}l_2^2 + p_{55}l_3^2, \\ \Gamma_{22} &= p_{66}l_1^2 + p_{22}l_2^2 + p_{44}l_3^2, \\ \Gamma_{33} &= p_{55}l_1^2 + p_{44}l_2^2 + p_{33}l_3^2, \\ \Gamma_{12} &= (p_{12} + p_{66})l_1l_2, \\ \Gamma_{13} &= (p_{13} + p_{55})l_3l_1, \\ \Gamma_{23} &= (p_{44} + p_{23})l_2l_3 \end{aligned} \quad (39)$$

the components of the Kelvin-Christoffel matrix (Carcione, 2007).

In the (three) symmetry planes of an orthorhombic medium, there is a pure shear wave (labeled 1 below) and two coupled waves. The corresponding complex velocities are given by a generalization of the lossless case (e.g., Carcione, 2007) to the lossy case:

$(x, y)$ -plane:

$$\begin{aligned} v_1 &= \sqrt{(\rho)^{-1}(p_{55}l_1^2 + p_{44}l_2^2)}, \\ v_2 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{22}l_2^2 + p_{66} - A}, \\ v_3 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{22}l_2^2 + p_{66} + A}, \\ A &= \sqrt{[(p_{22} - p_{66})l_2^2 - (p_{11} - p_{66})l_1^2]^2 + 4[(p_{12} + p_{66})l_1l_2]^2}; \end{aligned} \quad (40)$$

$(x, z)$ -plane:

$$\begin{aligned} v_1 &= \sqrt{(\rho)^{-1}(p_{66}l_1^2 + p_{44}l_3^2)}, \\ v_2 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{33}l_3^2 + p_{55} - A}, \\ v_3 &= (2\rho)^{-1/2} \sqrt{p_{11}l_1^2 + p_{33}l_3^2 + p_{55} + A}, \\ A &= \sqrt{[(p_{33} - p_{55})l_3^2 - (p_{11} - p_{55})l_1^2]^2 + 4[(p_{13} + p_{55})l_1l_3]^2}; \end{aligned} \quad (41)$$



$(y, z)$ -plane:

$$\begin{aligned} v_1 &= \sqrt{(\rho)^{-1}(p_{66}l_2^2 + p_{55}l_3^2)} \\ v_2 &= (2\rho)^{-1/2} \sqrt{p_{22}l_2^2 + p_{33}l_3^2 + p_{44} - A}, \\ v_3 &= (2\rho)^{-1/2} \sqrt{p_{22}l_2^2 + p_{33}l_3^2 + p_{44} + A}, \\ A &= \sqrt{[(p_{33} - p_{44})l_3^2 - (p_{22} - p_{44})l_2^2]^2 + 4[(p_{23} + p_{44})l_2l_3]^2}. \end{aligned} \quad (42)$$

In terms of angles,  $l_1 = \sin \theta$  and  $l_2 = \cos \theta$  in the  $(x, y)$ -plane,  $l_1 = \sin \theta$  and  $l_3 = \cos \theta$  in the  $(x, z)$ -plane, and  $l_2 = \sin \theta$  and  $l_3 = \cos \theta$  in the  $(y, z)$ -plane.

The complex velocities along the principal axes are:

$(x, y)$ -plane:

$$\begin{aligned} v_1(0^\circ) &= v_S(0^\circ) = \sqrt{p_{44}/\rho} \\ v_1(90^\circ) &= v_S(90^\circ) = \sqrt{p_{55}/\rho} \\ v_2(0^\circ) &= v_{qS}(0^\circ) = \sqrt{p_{66}/\rho} \\ v_2(90^\circ) &= v_{qS}(90^\circ) = \sqrt{p_{66}/\rho} \\ v_3(0^\circ) &= v_{qP}(0^\circ) = \sqrt{p_{22}/\rho} \\ v_3(90^\circ) &= v_{qP}(90^\circ) = \sqrt{p_{11}/\rho}; \end{aligned} \quad (43)$$

$(x, z)$ -plane:

$$\begin{aligned} v_1(0^\circ) &= v_S(0^\circ) = \sqrt{p_{44}/\rho} \\ v_1(90^\circ) &= v_S(90^\circ) = \sqrt{p_{66}/\rho} \\ v_2(0^\circ) &= v_{qS}(0^\circ) = \sqrt{p_{55}/\rho} \\ v_2(90^\circ) &= v_{qS}(90^\circ) = \sqrt{p_{55}/\rho} \\ v_3(0^\circ) &= v_{qP}(0^\circ) = \sqrt{p_{33}/\rho} \\ v_3(90^\circ) &= v_{qP}(90^\circ) = \sqrt{p_{11}/\rho}; \end{aligned} \quad (44)$$

$(y, z)$ -plane:

$$\begin{aligned} v_1(0^\circ) &= v_S(0^\circ) = \sqrt{p_{66}/\rho} \\ v_1(90^\circ) &= v_S(90^\circ) = \sqrt{p_{55}/\rho} \\ v_2(0^\circ) &= v_{qS}(0^\circ) = \sqrt{p_{44}/\rho} \\ v_2(90^\circ) &= v_{qS}(90^\circ) = \sqrt{p_{44}/\rho} \\ v_3(0^\circ) &= v_{qP}(0^\circ) = \sqrt{p_{33}/\rho} \\ v_3(90^\circ) &= v_{qP}(90^\circ) = \sqrt{p_{22}/\rho}. \end{aligned} \quad (45)$$

The energy velocity can be computed for each frequency component, with the wavefront corresponding to the energy velocity at “infinite” frequency. Let us consider the  $(x, z)$ -plane of symmetry. The energy-velocity vector of the qP and qS waves is given by

$$\frac{\mathbf{v}_e}{v_p} = (l_1 + l_3 \cot \psi)^{-1} \hat{\mathbf{e}}_1 + (l_1 \tan \psi + l_3)^{-1} \hat{\mathbf{e}}_3. \quad (46)$$

(Carcione 2007; eq. 6.158), where

$$\tan \psi = \frac{\operatorname{Re}(\gamma^* X + \xi^* W)}{\operatorname{Re}(\gamma^* W + \xi^* Z)}, \quad (47)$$

defines the angle between the energy-velocity vector and the  $z$ -axis,

$$\begin{aligned} \gamma &= \sqrt{A \pm B}, \\ \xi &= \pm \operatorname{pv} \sqrt{A \mp B}, \\ B &= p_{11} l_1^2 - p_{33} l_3^2 + p_{55} \cos 2\theta, \end{aligned} \quad (48)$$

where the upper and lower signs correspond to the qP and qS waves, respectively.

Moreover,

$$\begin{aligned} W &= p_{55}(\xi l_1 + \gamma l_3), \\ X &= \gamma p_{11} l_1 + \xi p_{13} l_3, \\ Z &= \gamma p_{13} l_1 + \xi p_{33} l_3 \end{aligned} \quad (49)$$

(Carcione 2007; eqs. 6.121-6.123), where “pv” denotes the principal value, which has to be chosen according to established criteria.

On the other hand, the energy velocity of the SH wave is

$$\mathbf{v}_e = \frac{v_p}{\rho \operatorname{Re}(v)} \left[ l_1 \operatorname{Re} \left( \frac{p_{66}}{v} \right) \hat{\mathbf{e}}_1 + l_3 \operatorname{Re} \left( \frac{p_{44}}{v} \right) \hat{\mathbf{e}}_3 \right] \quad (50)$$

and

$$\tan \psi = \frac{\operatorname{Re}(p_{66}/v)}{\operatorname{Re}(p_{44}/v)} \tan \theta \quad (51)$$

(Carcione 2007; eq. 4.115).

In general, we have the property

$$v_p = v_e \cos(\psi - \theta), \quad (52)$$

where  $v_e = |\mathbf{v}_e|$ .

The quality factor expressions for each symmetry plane are similar to the equations obtained for the symmetry plane of the monoclinic medium. Along pure mode directions, we have

$$Q_{II} = \frac{\operatorname{Re}(p_{II})}{\operatorname{Im}(p_{II})}, \quad I = 1, \dots, 6. \quad (53)$$

The attenuation has a maximum for a given value of the specific viscosity. Let us consider, for instance, the P-wave and equation (19). A calculation yields

$$Q_{11}^{-1} = \frac{c_{11}\omega\eta_N}{k_N(k_N + c_{11}) + \omega^2\eta_N^2}. \quad (54)$$

This function has a maximum at

$$\omega\eta_N = \sqrt{k_N(k_N + c_{11})} \approx k_N \quad (55)$$

if  $k_N \gg c_{11}$ . On the other hand,

$$Q_{JJ}^{-1} = \frac{(c_{1J}^2/c_{JJ})\omega\eta_N}{(k_N + c_{11})(k_N + c_{11} - c_{1J}^2/c_{JJ}) + \omega^2\eta_N^2}, \quad J = 2, 3, \quad (56)$$

with a maximum at

$$\omega\eta_N = \sqrt{(k_N + c_{11})(k_N + c_{11} - c_{1J}^2/c_{JJ})}. \quad (57)$$

The attenuation of the shear waves (the 44, 55 and 66 components) have similar expressions. Two different values of the viscosity may give the same value of the quality factor, although the phase velocities differ.

### 3.3 Transversely isotropic media

The analysis for the HTI medium follows as a particular case of one of the symmetry planes of the orthorhombic medium. One can use the exact velocity expressions or the following approximations:

$$\begin{aligned} v_1^2 &= (c_{55}/\rho)(1 - \delta_H \sin^2 \theta), \\ v_2^2 &= (c_{55}/\rho)(1 - \delta_H \cos^2 2\theta - (c_{55}/c_{11})\delta_N \sin^2 2\theta), \\ v_3^2 &= (c_{11}/\rho)[1 - (c_{55}/c_{11})\delta_H \sin^2 2\theta - (1 - 2(c_{55}/c_{11})\delta_N \cos^2 \theta)^2], \end{aligned} \quad (58)$$

where  $\theta$  is the angle between the wavenumber vector and the vertical direction. Similar approximations have been obtained by Schoenberg and Douma (1988) (their equations (26)) for VTI media (horizontal fractures). Schoenberg and Douma (1988) have written

those expressions in terms of the coefficients  $\delta_N/(1 - \delta_N)$  and  $\delta_H/(1 - \delta_H)$ , instead of  $\delta_N$  and  $\delta_H$ , respectively, assuming  $\delta_N \ll 1$  and  $\delta_H \ll 1$ . Without imposing these conditions, more accurate expressions are obtained by using  $\delta_N$  and  $\delta_H$ , as in Chichinina et al. (2009a,b).

#### 4 Examples

We consider the TI background medium studied by Schoenberg and Helbig (1997), representing a typical shale,

$$\mathbf{C} = \rho \begin{pmatrix} 10 & 4 & 2.5 & 0 & 0 & 0 \\ 4 & 10 & 2.5 & 0 & 0 & 0 \\ 2.5 & 2.5 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad (59)$$

(in MPa), where  $\rho = 2300 \text{ kg/m}^3$ , e.g.,  $c_{11} = 23 \text{ GPa}$ . We assume a frequency  $f = \omega/(2\pi) = 50 \text{ Hz}$  and the low- and high-frequency limits for comparison.

First, we consider two orthogonal sets of fractures with  $\beta_1 = 0^\circ$  and  $\beta_2 = 90^\circ$ . The fracture stiffnesses of the first set are given by  $k_N^{(1)} = 9c_{11}$ ,  $k_H^{(1)} = \frac{8}{3}c_{66}$  and  $k_V^{(1)} = 4c_{55}$ . In the lossless case, we obtain  $\delta_N^{(1)} = 0.1$ ,  $\delta_H^{(1)} = 3/11$ , and  $\delta_V^{(1)} = 1/5$ , i.e., the values used by Schoenberg and Helbig (1997). Moreover, we set  $k_N^{(2)} = bk_N^{(1)}$ ,  $k_H^{(2)} = bk_H^{(1)}$  and  $k_V^{(2)} = bk_V^{(1)}$ , with  $b = 0.5$ . The fracture viscosities are assumed to be  $\eta_N^{(q)} = ak_N^{(q)}$ ,  $\eta_H^{(q)} = ak_H^{(q)}$  and  $\eta_V^{(q)} = ak_V^{(q)}$ , where  $a = 10^{-3} \text{ s}$ , for both fracture sets.

The stiffness-matrix components, given by equation (19), are

$$\begin{aligned} p_{11} &= (20.34, 0.70), \quad p_{12} = (6.93, 0.56), \quad p_{13} = (4.87, 0.22), \quad p_{22} = (18.83, 1.05), \\ p_{23} &= (4.60, 0.29), \quad p_{33} = (13.44, 0.09), \quad p_{44} = (3.13, 0.31), \quad p_{55} = (3.73, 0.22), \\ p_{66} &= (3.32, 0.53), \end{aligned} \quad (60)$$

in GPa. Figure 2 shows the dissipation factors as a function of frequency along pure mode directions. The attenuation behaves as relaxation peaks, similar to the Zener model. Figure 3 shows the energy velocity at the  $(x, z)$ -plane for 0 Hz (a), 50 Hz (b)

and “infinite” frequency (c). The high-frequency limit corresponds to the unfractured TI case. The dissipation factors versus phase and ray angle are represented in Figure 4, where the pure mode (the SH wave) shows more attenuation. The attenuation of the coupled waves have a different behaviour versus the ray angle, corresponding to the propagation of wave packets, compared to the representation versus the phase (propagation) angle, corresponding to the propagation of plane waves. In particular, the triPLICATION appears also in the  $Q$  factor of the qS wave. Experimental setups should consider these facts.

Next, we consider two sets of fractures with  $\beta_1 = 20^\circ$  and  $\beta_2 = 65^\circ$  and perform the analysis in the symmetry axis of the effective monoclinic medium. The fracture parameters corresponding to the first example are considered. The stiffness-matrix components, given by equation (15), are

$$\begin{aligned} p_{11} &= (18.05, 1.1), & p_{12} &= (8.98, 0.29), & p_{13} &= (4.83, 0.23), & p_{16} &= (-1.07, 0.13), \\ p_{22} &= (17.27, 1.26), & p_{23} &= (4.69, 0.26), & p_{26} &= (-0.15, 0.05), & p_{33} &= (13.44, 0.09), \\ p_{36} &= (-0.22, 0.03), & p_{44} &= (3.37, 0.25), & p_{45} &= (-0.67, 0.11), & p_{55} &= (3.70, 0.19), \\ p_{66} &= (4.53, 0.46), \end{aligned} \tag{61}$$

in GPa. Figure 5 shows the low-frequency limit energy velocities (a) and dissipation factors (b) in the  $(x, y)$  symmetry plane as a function of the phase (propagation) angle. The frequency in Figure 5b is  $f = 50$  Hz. In this case, the qS wave shows the maximum attenuation.

In order to verify the shape of the wave fronts shown in Figure 5, we perform a 2D full-wave numerical simulation of qP-qS propagation in the symmetry plane, where the effective medium is defined by the low-frequency elasticity constants:  $c_{11}^0 = 17.8$  GPa,  $c_{22}^0 = 17$  GPa,  $c_{12}^0 = 8.9$  GPa,  $c_{16}^0 = -1.08$  GPa,  $c_{26}^0 = -0.16$  GPa, and  $c_{66}^0 = 4.44$  GPa. The density is  $\rho = 2300$  kg/m<sup>3</sup>. The algorithm solves the particle-velocity/stress formulation based on the Fourier pseudospectral method for computing the spatial

derivatives and a 4th-order Runge-Kutta technique for calculating the wavefield recursively in time (e.g., Carcione, 2007). The source is a vertical force with a Ricker time history, located at the center of the mesh. The simulation use a  $455 \times 455$  mesh with 1 m grid spacing and the central frequency of the source is 80 Hz. The algorithm has a time step of 0.1 ms and a snapshot of the vertical-particle velocity is computed at 80 ms (see Figure 6). It is verified that the results of the modeling algorithm and plane-wave analysis are in agreement.

Finally, we test the approximation

$$Q^{-1}(\theta) \approx Q_b^{-1}(\theta) + Q_f^{-1}(\theta), \quad (62)$$

where  $Q_b$  and  $Q_f$  are the quality factors of the background medium and fracture set in the same lossless background medium. Equation (62) is commonly used in the literature to obtain the total quality factor due to different attenuation mechanisms (e.g., Chichinina et al. 2009a,b). We assume for simplicity an isotropic background medium, and therefore  $Q_b$  is independent of  $\theta$ . The effective medium has HTI symmetry and the stiffness matrix is given by equation (22), with  $c_{IJ}$  complex. Let us consider the simplest model, i.e.,  $c_{11} \rightarrow c_{11}(Q_P + i)$  and  $c_{55} \rightarrow c_{55}(Q_S + i)$ , where  $Q_P$  and  $Q_S$  are the P-wave and S-wave quality factors of the background medium. (Note that  $c_{12} = c_{11} - 2c_{55}$ .) Let us assume  $Q_P = 20$  and  $Q_S = 15$ . Figure 7 compares the exact dissipation factors to the approximate dissipation factors, corresponding to the qP and qS waves, respectively. As can be appreciated, equation (62) is a rough approximation and should be used with caution. The approximation improves if  $Q$  increases.

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## 5 Conclusions

We have presented a theory to obtain the wave velocities and attenuation of many sets of vertical fractures embedded in a transversely isotropic medium. The anisotropic effective medium has monoclinic symmetry. Fractures are modeled as boundary discontinuities in the displacement and particle velocity fields. The theory generalizes an existing model describing the acoustic properties of a single set of fractures embedded in an isotropic background medium. The expressions of the complex and frequency-dependent stiffness constants corresponding to two orthogonal sets of fractures are obtained explicitly, where the effective medium has orthorhombic symmetry. The phase, energy and quality factors as a function of the propagation and ray (energy) angle are obtained for homogeneous viscoelastic plane waves (wavenumber and attenuation directions coincide). We consider the symmetry plane of a monoclinic medium and the three symmetry planes of an orthorhombic medium. The examples show that the effective media have high anisotropy and show relaxation attenuation peaks, similar to Zener viscoelastic models. Finally, we tested the commonly used equation stating that the dissipation factor of the effective medium is equal to the sum of the dissipation factors of the background medium and fractured background (lossless) medium. The results indicated that this equation is not a good approximation for realistic  $Q$  values of the background medium.

The novel model can be important in determining the orientation of fractures in the reservoir and the overlying cap rock. This plays an important role during production, and other applications, such as CO<sub>2</sub> injection and monitoring.

## 6 References

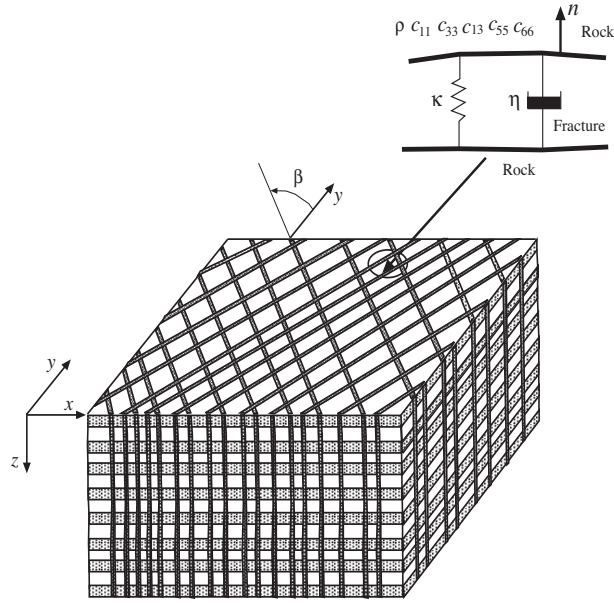
- Backus GE (1962) Long-wave elastic anisotropy produced by horizontal layering. *J. Geophys. Res.* 7: 4427-4440.
- Barton N (2007) Fracture-induced seismic anisotropy when shearing is involved in production from fractured reservoirs. *Journal of Seismic Exploration* 16: 115-143.
- Carcione JM (1992) Anisotropic Q and velocity dispersion of finely layered media. *Geophys. Prospect.* 40: 761-783.
- Carcione JM (1996a) Plane-layered models for the analysis of wave propagation in reservoir environments. *Geophys. Prospect.* 44: 3-26.
- Carcione JM (1996b) Elastodynamics of a non-ideal interface: Application to crack and fracture scattering. *J. Geophys. Res.* 101: 28177-28188.
- Carcione JM (2007) Wave fields in real media: Wave propagation in anisotropic, anelastic, porous and electromagnetic media, *Handbook of Geophysical Exploration*, vol. 38, Elsevier (2nd edition, revised and extended).
- Carcione JM and Cavallini F (1994) A rheological model for anelastic anisotropic media with applications to seismic wave propagation. *Geophys. J. Int.* 119: 338-348.
- Carcione JM, Cavallini F and Helbig K (1998) Anisotropic attenuation and material symmetry. *Acustica* 84: 495-502.
- Carcione JM, Santos JE and Picotti P (2011) Anisotropic poroelasticity and wave induced fluid flow. Harmonic finite-element simulations. *Geophys. J. Internat.*, in press.



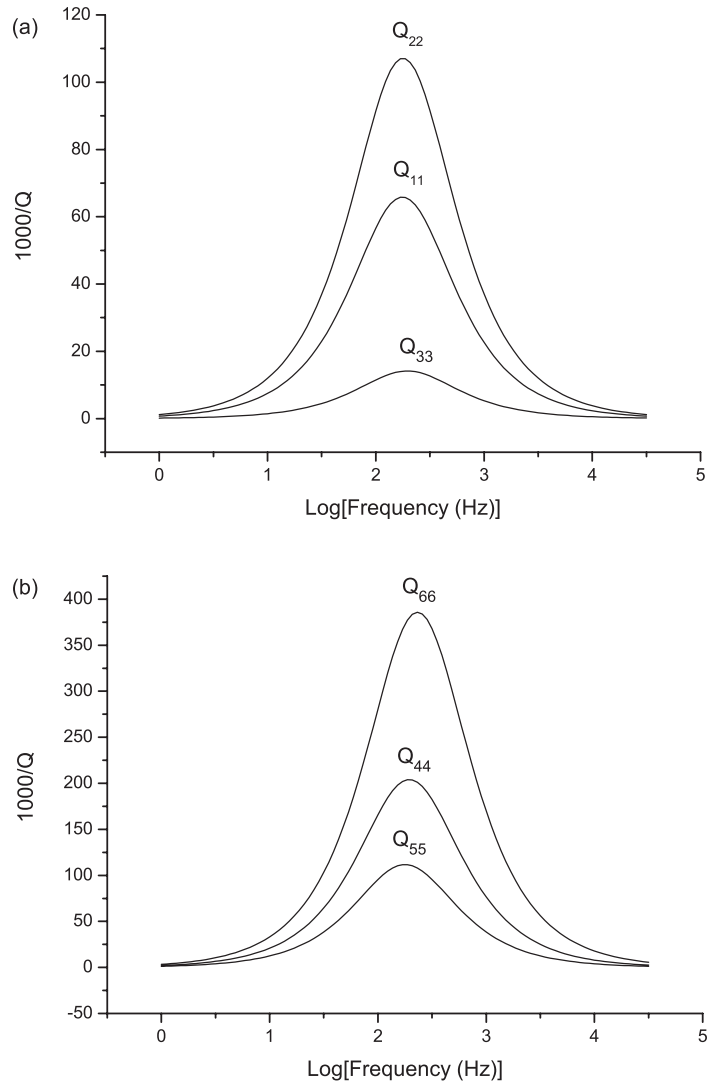
- 
- Chapman M (2003) Frequency dependent anisotropy due to meso-scale fractures in the presence of equant porosity. *Geophys. Prospect.* 51: 369-379.
- Chichinina TI, Obolentseva, IR and Ronquillo-Jarillo G (2009a) Anisotropy of seismic attenuation in fractured media: Theory and ultrasonic experiment. *Transport in Porous Media* 79: 1-14.
- Chichinina TI, Obolentseva, IR, Gik L, Bobrov B. and Ronquillo-Jarillo G. (2009b) Attenuation anisotropy in the linear-slip model: Interpretation of physical modeling data. *Geophysics* 74: WB165-WB176.
- Coates RT and Schoenberg M (1995) Finite-difference modelling of faults and fractures: *Geophysics* 60: 1514-1526.
- Fan LF, Ren F., and Ma GW (2011) An extended displacement discontinuity method for analysis of stress wave propagation in viscoelastic rock mass. *Journal of Rock Mechanics and Geotechnical Engineering* 3: 73-81.
- Grechka V and Tsvankin I (2003) Feasibility of seismic characterization of multiple fracture sets. *Geophysics* 68: 1399-1407.
- Hall SA and Kendall JM (2003) Fracture characterization at Valhall: Application of P-wave amplitude variation with off set and azimuth AVOA analysis to a 3D ocean-bottom data set. *Geophysics* 68: 1150-1160.
- Hansen BRH (2002) Evaluating the impact of fracture-induced anisotropy on reservoir rock property estimates made from seismic data. Report No.: GPH 7/02, Curtin University of Technology.
- Hood JA (1991) A simple method for decomposing fracture-induced anisotropy. *Geophysics* 56: 1275-1279.

- Krzikalla F and Müller T (2011) Anisotropic P-SV-wave dispersion and attenuation due to inter-layer flow in thinly layered porous rocks. *Geophysics* 76: WA135; doi:10.1190/1.3555077
- Liu E, Hudson JA and Pointer T (2000) Equivalent medium representation of fractured rock. *J. Geophys. Res.* 105: 2981-3000.
- Maultzsch S (2005) Analysis of frequency-dependent anisotropy in VSP Data. PhD thesis, University of Edinburgh.
- Nakagawa S and Myer, LR (2009) Fracture permeability and seismic wave scattering – Poroelastic linear-slip interface model for heterogeneous fractures, *SEG Expanded Abstracts* 28, 3461; doi:10.1190/1.3255581.
- Nichols D, Muir F and Schoenberg M (1989) Elastic properties of rocks with multiple sets of fractures: 63rd Ann. Internal. Mtg., Soc. Expl. Geophys., Extended Abstracts, 471-474.
- Perino A, Zhu, JB, Li JC, Barla G. and Zhao J (2010) Theoretical methods for wave propagation across jointed rock masses. *Rock Mech. Rock Eng.* 43: 799-809.
- Picotti S, Carcione JM, Santos JE and Gei D (2010) Q-anisotropy in finely layered media. *Geophys. Res. Lett.* 37: L06302, doi:10.1029/2009GL042046.
- Pyrak-Nolte LJ, Myer LR and Cook NGW (1990) Transmission of seismic waves across single natural fractures. *J. Geophys. Res.* 95: 8617-8638.
- Schoenberg M (1980) Elastic wave behavior across linear slip interfaces. *J. Acous. Soc. Am.* 68: 1516-1521.
- Schoenberg M (1983) Reflection of elastic waves from periodically stratified media with interfacial slip. *Geophys. Prosp.* 31: 265-292.

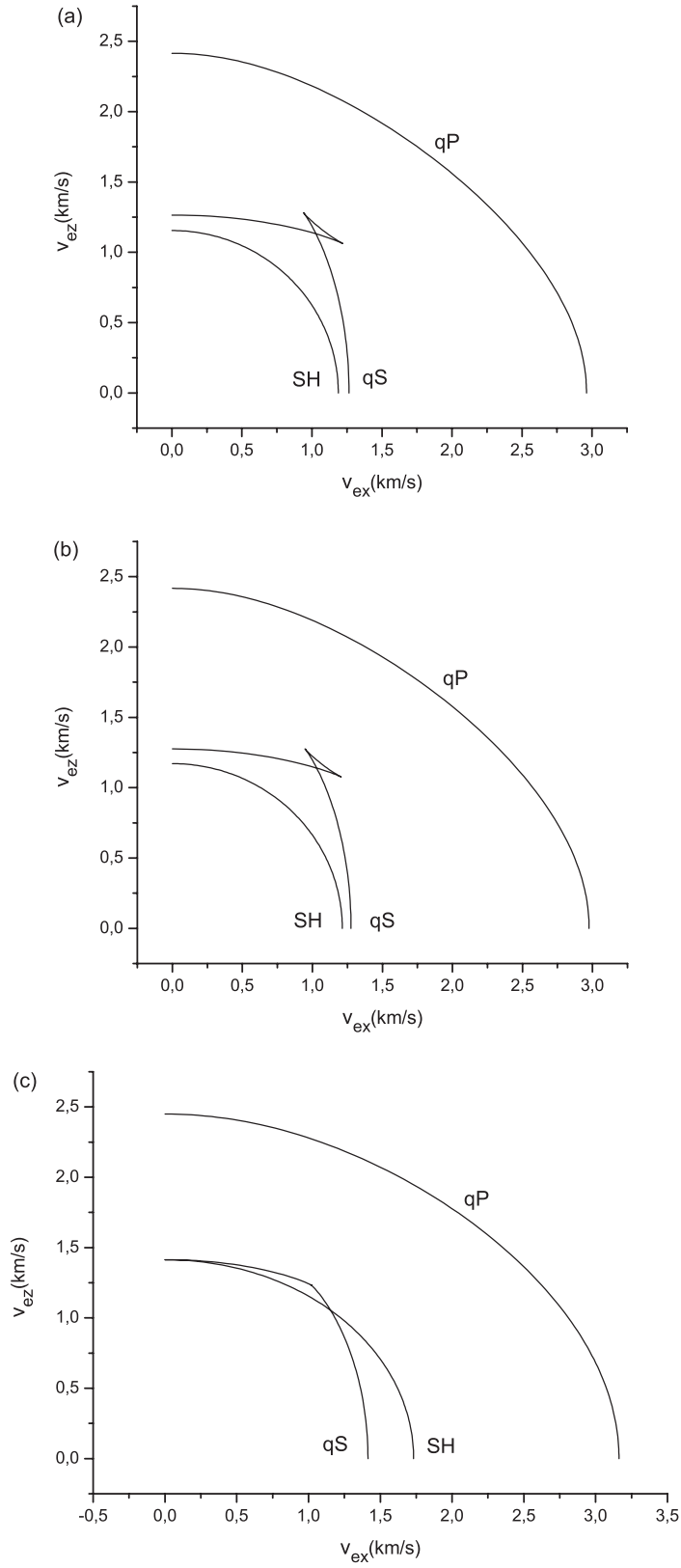
- Schoenberg M and Douma J (1988) Elastic wave propagation in media with parallel fractures and aligned cracks. *Geophys. Prosp.* 36: 571-590.
- Schoenberg M and Helbig K (1997) Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth. *Geophysics* 62: 1954-1974.
- Schoenberg M and Muir F (1989) A calculus for finely layered media. *Geophysics* 54: 581-589.
- Schoenberg M, Dean S and Sayers C (1999) Azimuth-dependent tuning of seismic waves reflected from fractured reservoirs. *Geophysics* 64: 1160-1171.
- Zhang J and Gao H (2009) Elastic wave modelling in 3-D fractured media: An explicit approach. *Geophys. J. Internat.* 177: 1233-1241.
- Zhu Y and Tsvankin I (2005). Plane-wave attenuation anisotropy in orthorhombic media, *Geophysics* 72: D9-D19.



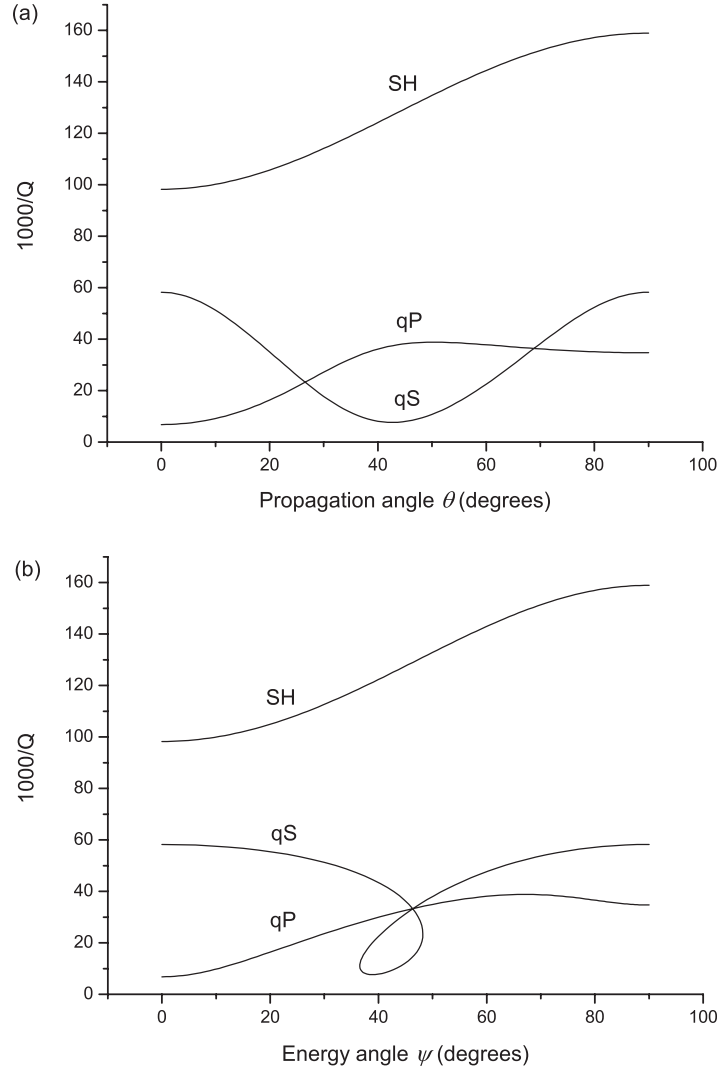
**Fig. 1** Two sets of vertical fractures embedded in a TI medium. In this case, transverse isotropy is due to fine layering. If the two sets are orthogonal, the equivalent medium is orthorhombic, otherwise the symmetry is monoclinic with a horizontal single plane of symmetry. The boundary condition at the fracture describes an imperfect bonding in terms of the specific stiffness  $\kappa$  and specific viscosity  $\eta$ . Angle  $\beta$  is measured from the  $y$ -axis towards the strike direction.



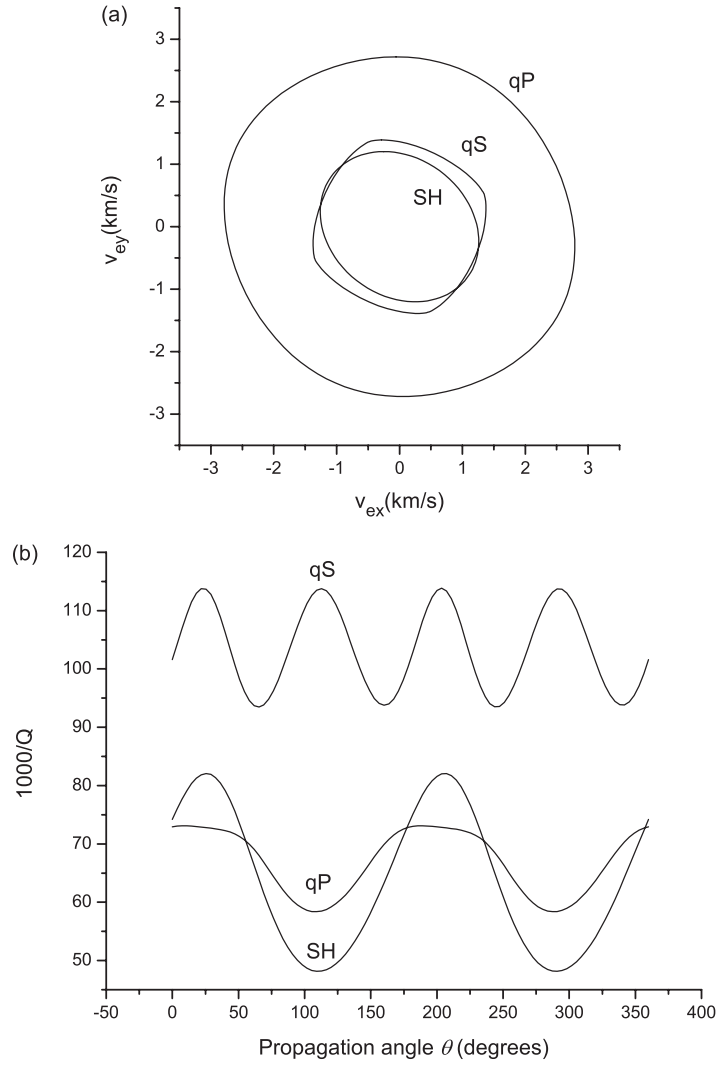
**Fig. 2** Dissipation factors (53) as a function of frequency, where (a) corresponds to the P wave and (b) corresponds to the S waves.



**Fig. 3** Energy velocities in the  $(x, z)$  symmetry plane of the equivalent orthorhombic medium, where (a)  $f = 0$  Hz, (b)  $f = 50$  Hz and (c)  $f = \infty$  Hz.

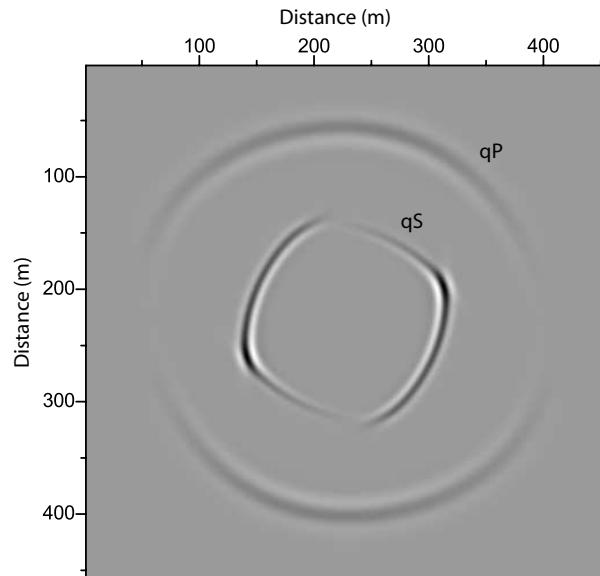


**Fig. 4** Quality factors in the  $(x, z)$  symmetry plane as a function of the phase (propagation) angle (a) and ray (energy) angle (b). The equivalent medium has orthorhombic symmetry and the frequency is  $f = 50$  Hz.

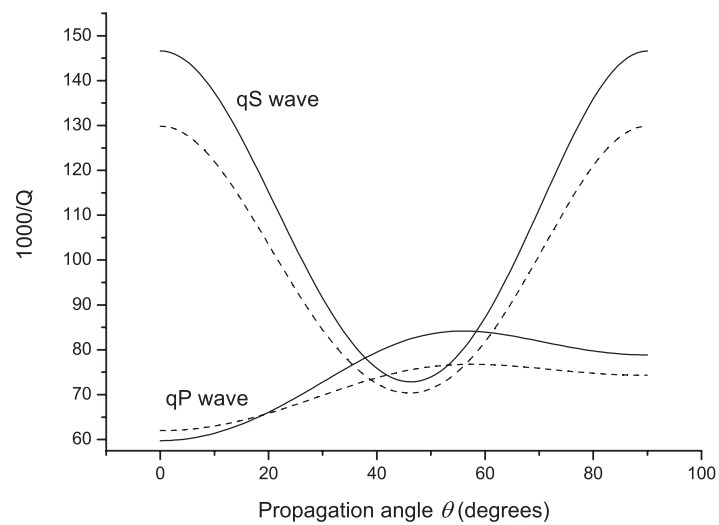


**Fig. 5** Low-frequency limit energy velocities (a) and dissipation factors (b) in the  $(x, y)$  symmetry plane as a function of the phase (propagation) angle. The equivalent medium has monoclinic symmetry and the frequency in (b) is  $f = 50$  Hz.





**Fig. 6** Snapshot in the  $(x, y)$  symmetry plane of the monoclinic effective medium. The elasticity constants are those of the low-frequency limit.



**Fig. 7** Exact (solid line) and approximate (dashed line) dissipation factors of the qP and qS waves as a function of the propagation angle. The frequency is 50 Hz.