A FEM FOR WAVE PROPAGATION IN 2D-FRACTURED VISCOELASTIC MEDIA

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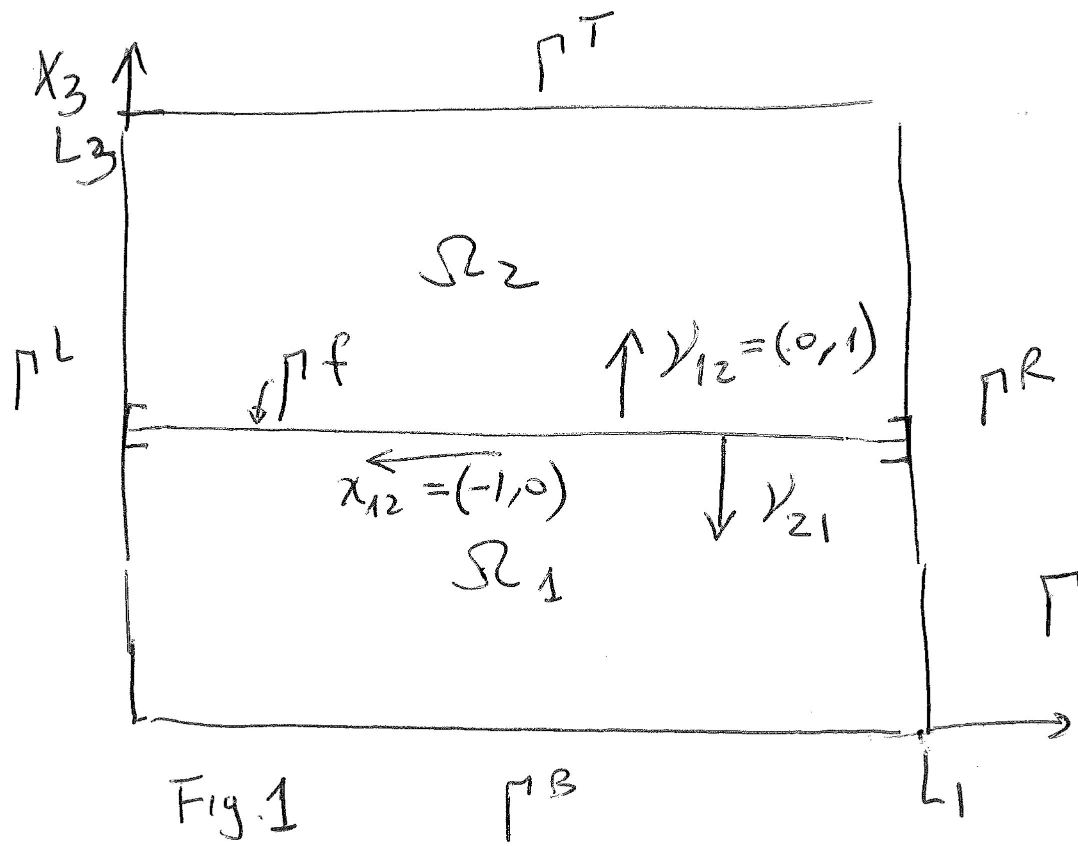
2D FRACTURED VE MEDIA

CASE 1: THE FRACTURE TOUCHES THE BOUNDARY —

$$(1)\quad -\omega^2 \rho u - \nabla\cdot\tau(u) = f \qquad \Omega = R \setminus \Gamma^f$$

$$(2)\quad \tau_{\ell m}(u) = 2\mu\,\varepsilon_{\ell m}(u) + \lambda\,\delta_{\ell m}\nabla\cdot u \quad ,\Omega$$



$R = (0,L_1)\times(0,L_3)$

$\Gamma = \Gamma^L \cup \Gamma^R \cup \Gamma^B \cup \Gamma^T = \partial R$

Fig. 1

Let

$$V^f = \left\{ v \in [L^2(\Omega)]^2 : v^{(j)} = v|_{\Omega_j} \in [H^1(\Omega_j)]^2, j=1,2 \right\}$$

$$= [H^1(\Omega)]^2$$

Multiply (1) by $v \in V^f$ and use integration by parts on each Ω_j:

$$-\omega^2(\varrho u, v)_\Omega + \left(\tau_{\ell m}(u^{(1)}), \varepsilon_{\ell m}(v^{(1)})\right)_{\Omega_1}$$

$$+\left(\tau_{\ell m}(u^{(2)}), \varepsilon_{\ell m}(v^{(2)})\right)_{\Omega_2}$$

$$(3) \quad - \langle \tau(u^{(1)}) \cdot \nu_{12}, v^{(1)} \rangle_{\Gamma^f} - \langle \tau(u^{(2)}) \cdot \nu_{21}, v^{(2)} \rangle_{\Gamma^f}$$

$$- \langle \tau(u)\nu, v \rangle_\Gamma = (f, v), \quad v \in V^f -$$

Consider the bdy term on Γ^f in (3)

$$T_1 = - \langle \tau(u^{(1)})\nu_{12}, v^{(1)} \rangle_{\Gamma^f} - \langle \tau(u^{(2)}) \cdot \nu_{21}, v^{(2)} \rangle_{\Gamma^f}$$

$$= - \langle \tau(u^{(1)})\nu_{12}, v^{(1)} \rangle_{\Gamma^f} + \langle \tau(u^{(2)})\nu_{12}, v^{(2)} \rangle_{\Gamma^f}$$

Now we impose the B.C

(4) $\tau(u^{(1)})\nu_{12} = \tau(u^{(2)})\nu_{12}$ on Γ^f

Then $\equiv \tau^f(u)\nu_{12}$ [STRESS CONTINUITY AT THE FRACTURE]

$$T_1 = \langle \tau^f(u)\,\nu_{12},\, v^{(2)} - v^{(1)} \rangle_{\Gamma^f}$$

$$= \langle \left(\tau^f(u)\,\nu_{12}\,\nu_{12},\ \tau^f(u)\,\nu_{12}\,\chi_{12}\right),$$

(4-1) $\left(v^{(2)} - v^{(1)}\right)\cdot\nu_{12},\ \left(v^{(2)} - v^{(1)}\right)\cdot\chi_{12} \rangle_{\Gamma^f}$

where χ_{12} is the unit tangent on Γ^f oriented counterclockwise —

Next we impose the second B.C on Γ^f:

(5) $\left(\tau^f(u)\nu_{12}\,\nu_{12},\ \tau^f(u)\,\nu_{12}\,\chi_{12}\right) = D \begin{pmatrix} (u^{(2)} - u^{(1)})\cdot\nu_{12} \\ (u^{(2)} - u^{(1)})\cdot\chi_{12} \end{pmatrix}$ on Γ^f

where

$$D = \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \nu_1^{12} & \nu_3^{12} \\ \chi_1^{12} & \chi_3^{12} \end{bmatrix}$$

$$= \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} \alpha \nu_1^{12} & \alpha \nu_3^{12} \\ \beta \chi_1^{12} & \beta \chi_3^{12} \end{bmatrix}$$

(6)

$$= \begin{bmatrix} \alpha \nu_1^{12} \nu_1^{12} + \beta \chi_1^{12} \chi_1^{12} & \alpha \nu_1^{12} \nu_3^{12} + \beta \chi_1^{12} \chi_3^{12} \\ \alpha \nu_1^{12} \nu_3^{12} + \beta \chi_1^{12} \chi_3^{12} & \alpha \nu_3^{12} \nu_3^{12} + \beta \chi_3^{12} \chi_3^{12} \end{bmatrix}$$

$$= T^t \Lambda T ,$$

(7) $\alpha = K_1 + i\omega \xi_1$, $\beta = K_3 + i\omega \xi_3$.

$K_j \geq 0 \quad \xi_j > 0 \quad j = 1, 3$

Note that

(8) $D = D_R + i D_I$,

D_R is positive semidefinite and

D_I is positive definite –

Also, on Γ we impose the absorbing B.C.

$$(9) \quad -\tau(u)\nu = i\omega B u \quad , \quad B \text{ positive definite} -$$

Using (5) in (4-1)

$$(10) \quad T_1 = \left\langle D \begin{pmatrix} (u^{(2)}-u^{(1)})\cdot \nu_{12} \\ (u^{(2)}-u^{(1)})\cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)}-v^{(1)})\cdot \nu_{12} \\ (v^{(2)}-v^{(1)})\cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^F}$$

Using (10) and (9) in (3):

$$-\omega^2(\rho u, v)_\Omega + \sum_{j=1}^{2} \left(\tau_{\ell m}(u^{(j)}), \varepsilon_{\ell m}(v^{(j)})\right)_{\Omega_j}$$

$$(11) \quad + \left\langle D \begin{pmatrix} (u^{(2)}-u^{(1)})\cdot \nu_{12} \\ (u^{(2)}-u^{(1)})\cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)}-v^{(1)})\cdot \nu_{12} \\ (v^{(2)}-v^{(1)})\cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^F}$$

$$+ \langle i\omega B u, v \rangle_\Gamma = (f, v), \quad v \in V^f -$$

<u>Remark 1</u>: In the argument leading to the weak form (11) we <u>never</u> used the fact that Γ^{+} is an horizontal fracture, we just used that Ω_1 and Ω_2 are in contact at the fracture Γ^{f} and used the unit outer normal ν_{12} and the unit tangent χ_{12} oriented counterclockwise from Ω_1 into Ω_2.

<u>Remark 2</u> Changing $u^{(2)}-u^{(1)}$ and $v^{(2)}-v^{(1)}$ by $u^{(1)}-u^{(2)}$ and $v^{(1)}-v^{(2)}$ in the β-term Γ^{+} does not change that term. The same happens if we change ν_{12} and χ_{12} by ν_{21} and χ_{21}, unit outer normal and tangents from Ω_2 to Ω_1 -

Calculation of the B.C. (5) on Γ^f as in Fig 1:
(Horizontal fracture)

on Γ^f $\quad \nu^{12} = (0,1) = (\nu_1, \nu_3)$, $\quad \chi^{12} = (-1, 0) = (\chi_1, \chi_3)$

Then from (6)

$$D^{(1)} = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}$$

$$\tau^f(u)\,\nu = \left(\tau_{11}\nu_1 + \tau_{13}\nu_3 \,,\, \tau_{31}\nu_1 + \tau_{33}\nu_3\right)$$

Then on Γ^f as in Fig 1

$$\tau^f(u)\,\nu_{12} = (\tau_{13} \,,\, +\tau_{33})$$

$$\tau^f(u)\,\nu_{12}\,\nu_{12} = \tau_{33}\,, \quad \tau^f(u)\,\nu_{12}\,\chi_{12} = -\tau_{13}$$

$$\left(u^{(2)} - u^{(1)}\right)\cdot \nu_{12} = u_3^{(2)} - u_3^{(1)}$$

$$\left(u^{(2)} - u^{(1)}\right)\cdot \chi_{12} = (-1)\left(u_1^{(2)} - u_1^{(1)}\right)$$

Then on Γ^f the B.C. (5) is

$$(\tau_{33}, -\tau_{13}) = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} u_3^{(2)} - u_3^{(1)} \\ (-1)\,(u_1^{(2)} - u_1^{(1)}) \end{pmatrix}$$

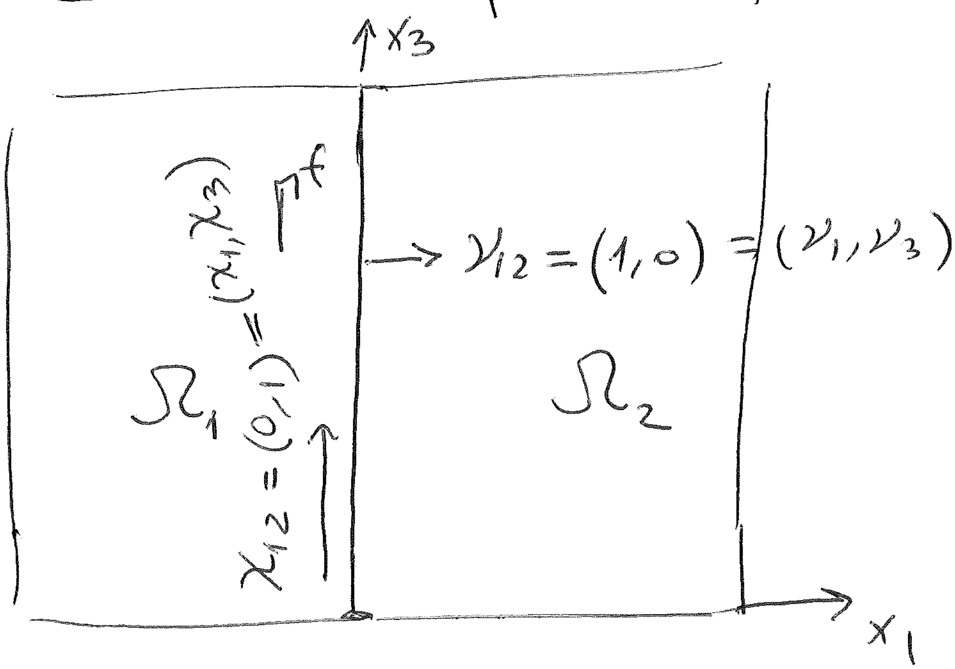
or

$$(12) \quad \tau_{33} = (K_3 + i\omega\, \xi_3)\,(u_3^{(2)} - u_3^{(1)}) \,,$$

$$(13) \quad \tau_{13} = (K_1 + i\omega\, \xi_1)\,(u_1^{(2)} - u_1^{(1)}) \,.$$

Now (12) and (13) are the B.C. in ((Carcione, J.G.R. (1996), v. 101 B12 p. 28177 – 28188) $[u_3^{(2)} = u_z^+,\ u_3^{(1)} = u_z^-]$

Next let us consider that Γ^f is as in Fig 2 below (vertical fracture)



Then from (6)

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\tau^f(u)\,\nu_{12} = (\tau_{11}, \tau_{31})$$

$$\tau^f(u)\,\nu_{12}\,\nu_{12} = \tau_{11}, \qquad \tau^f(u)\,\nu_{12}\,\chi_{12} = \tau_{31}$$

$$(u^{(2)} - u^{(1)}) \cdot \nu_{12} = u_1^{(2)} - u_1^{(1)}$$

$$(u^{(2)} - u^{(1)}) \cdot \chi_{12} = u_3^{(2)} - u_3^{(1)}$$

so that on Γ^f as in Fig 2, the B.C. (5) is

$$(\tau_{11}, \tau_{31}) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} u_1^{(2)} - u_1^{(1)} \\ u_3^{(2)} - u_3^{(1)} \end{pmatrix}$$

or

(14) $$\tau_{11} = (k_1 + i\omega\,\xi_1)(u_1^{(2)} - u_1^{(1)})$$

(15) $$\tau_{31} = (k_3 + i\omega\,\xi_3)(u_3^{(2)} - u_3^{(1)}) -$$

Analysis of the term $\langle i\omega B u, v\rangle$ in (11):

We define

$$B = \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} V_p & 0 \\ 0 & V_s \end{bmatrix} \begin{bmatrix} \nu_1^{12} & \nu_3^{12} \\ \chi_1^{12} & \chi_3^{12} \end{bmatrix}$$

(16)

$$= \begin{bmatrix} V_p \nu_1^{12}\nu_1^{12} + V_s \chi_1^{12}\chi_1^{12} & V_p \nu_1^{12}\nu_3^{12} + V_s \chi_1^{12}\chi_3^{12} \\ V_p \nu_1^{12}\nu_3^{12} + V_s \chi_1^{12}\chi_3^{12} & V_p \nu_3^{12}\nu_3^{12} + V_s \chi_3^{12}\chi_3^{12} \end{bmatrix}$$

V_p = P-wave phase velocity,

V_s = S-wave phase velocity —

Then, on Γ^T $\nu^{12} = (0,1)$, $\chi^{12} = (-1,0)$

and

$$B = \rho \begin{bmatrix} V_s & 0 \\ 0 & V_p \end{bmatrix}$$

Also, on Γ^L $\nu^{12} = (1,0)$, $\chi^{12} = (0,1)$

and

$$B = \rho \begin{bmatrix} V_p & 0 \\ 0 & V_s \end{bmatrix}$$

Then we rewrite (9) as

$$(17)\quad \left(-\tau(u)\nu\nu,\ -\tau(u)\nu\chi\right) = i\omega B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix},\ \Gamma$$

Then, on Γ^L we get

$$\left(-\tau_{11},\ -\tau_{31}\right) = i\omega\rho \begin{pmatrix} V_P & 0 \\ 0 & V_S \end{pmatrix}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$

so that

$$(18\text{-}1)\quad \begin{cases} -\tau_{11} = i\omega\rho V_P\, u_1 \\ -\tau_{31} = i\omega\rho V_S\, u_3 \end{cases}$$

Also, on Γ^T,

$$-\left(\tau_{33},\ -\tau_{13}\right) = i\omega\rho \begin{pmatrix} V_P & 0 \\ 0 & V_S \end{pmatrix}\begin{pmatrix} u_3 \\ -u_1 \end{pmatrix}$$

so that

$$(18\text{-}2)\quad \begin{aligned} -\tau_{33} &= i\omega\rho V_P\, u_3 \\ -\tau_{13} &= -i\omega\rho\, V_S\, u_1 \end{aligned}$$

Now (18-1) and (18-2) impose normal and tangential stresses proportional to normal and tangential velocities as desired —

Then we go back to equation (3) and write the big term on Γ as follows:

$$- \langle \tau(u)\nu, v \rangle_{\Gamma} = - \langle (\tau(u)\nu\nu, \tau(u)\nu\chi), v\cdot\nu, v\cdot\chi \rangle_{\Gamma}$$

using (17)

$$= i\omega \left\langle B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix}, \begin{pmatrix} v\cdot\nu \\ v\cdot\chi \end{pmatrix} \right\rangle_{\Gamma}$$

Then we rewrite the weak form (11) as follows: find $u \in V^f$ such that

$$- \omega^2 (\rho u, v)_{\Omega} + (\tau_{em}(u), \varepsilon_{em}(v))_{\Omega}$$

(19)
$$+ \left\langle D \begin{pmatrix} (u^{(2)} - u^{(1)})\cdot\nu_{12} \\ (u^{(2)} - u^{(1)})\cdot\chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)} - v^{(1)})\cdot\nu_{12} \\ (v^{(2)} - v^{(1)})\cdot\chi_{12} \end{pmatrix} \right\rangle_{\Gamma^f}$$

$$+ i\omega \left\langle B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix}, \begin{pmatrix} v\cdot\nu \\ v\cdot\chi \end{pmatrix} \right\rangle_{\Gamma} = (f, v),$$

$$v \in V^f$$

CASE 2 AN INTERIOR FRACTURE

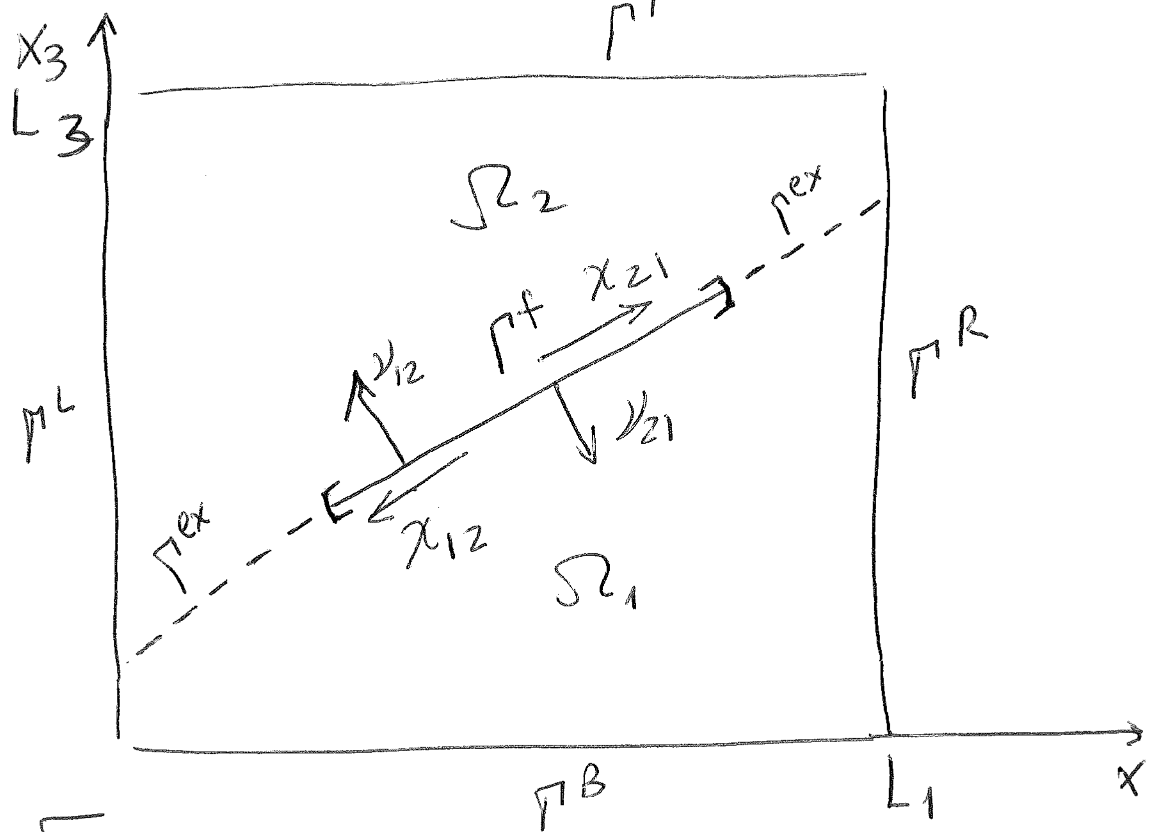


Fig. 3

Γ^f is a closed line inside $R=(0,L_1)\times(0,L_2)$ so that $\Omega = R\setminus\Gamma^f$ is an open set

We "continue" Γ^f until we get to $\Gamma=\partial R$. (Γ^{ex}=dotted lines). so

we have $\hat{\Gamma}^f = \Gamma^f \cup \{\text{dotted lines set}\}$
$\equiv \Gamma^f \cup \Gamma^{ex}$

Now set

$$V^f = \left[H^1(\Omega)\right]^2$$

Let Ω_1 and Ω_2 the 2 subsets in Ω defined by the set Γ^f.

Along the dotted line Γ^{ex} we here continuity of stress and displacements.

Thus, we consider again equations

$$(20)\quad -\omega^2 \rho\, u - \nabla \cdot \tau(u) = f, \quad \Omega$$

$$(21)\quad \tau_{em}(u) = 2\mu\, \varepsilon_{em}(u) + \lambda\, \delta_{em} \nabla \cdot u, \quad \Omega$$

with the B.C.

$$(22)\quad \tau(u^{(1)})\,\nu_{12} = \tau(u^{(2)})\,\nu_{12} \equiv \tau^f(u)\,\nu_{12}, \quad \hat{\Gamma}^f$$

$$(23)\quad u^{(1)} = u^{(2)}, \quad \Gamma^{ex},$$

$$(24)\quad \left(\tau^f(u)\nu_{12}\,\nu_{12},\ \tau^f(u)\nu_{12}\,\chi_{12}\right) = D \begin{pmatrix} (u^{(2)} - u^{(1)})\cdot\nu_{12} \\ (u^{(2)} - u^{(1)})\cdot\chi_{12} \end{pmatrix}, \quad \Gamma^f$$

where D is defined in (6) – (7) and (14)
B by (16).
Then, we multiply (20) by $v \in V^f$
and use integration by parts on each
Ω_j. Repeating the argument leading
to (11) and using (22) and (23) on Γ^{ex}
we recover the weak form
So our weak formulation is: find
$u \in V^f$ such that

$$
\begin{aligned}
&-\omega^2 (\rho u, v)_\Omega + (\tau_{lm}(u), \varepsilon_{lm}(v))_\Omega \\
&+ \left\langle D \begin{pmatrix} (u^{(2)} - u^{(1)}) \cdot \nu_{12} \\ (u^{(2)} - u^{(1)}) \cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)} - v^{(1)}) \cdot \nu_{12} \\ (v^{(2)} - v^{(1)}) \cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^f} \qquad (25) \\
&+ \left\langle i\omega B \begin{pmatrix} u \cdot \nu \\ u \cdot \chi \end{pmatrix}, \begin{pmatrix} v \cdot \nu \\ v \cdot \chi \end{pmatrix} \right\rangle_{\Gamma} = (f, v), \quad v \in V^f
\end{aligned}
$$