NONCONFORMING FE Methods for WAVE

PART II

PROPAGATION IN 2D-FRACTURED VISCOELASTIC MEDIA -

SPACE-FREQUENCY DOMAIN FORMULATION

1) CONTINUOUS PROBLEM, WEAK FORM, EXISTENCE & UNIQUENESS

2) GLOBAL NONCONFORMING Finite Element Method over a TRIANGULATION - UNIQUENESS -

3) MASSIVE DOMAIN DECOMPOSITION -

J. SANTOS, April 25, 2011

$\mathcal{T}^h(\Omega)$: non-overlapping partition of Ω into triangles of diameter bounded by h, so that $\Omega = \bigcup_{j=1}^{J} \Omega_j$

Set

$$NC_j = \left[P_1(\Omega_j)\right]^2 = \left[\{a + bx + cz\}\right]^2$$

with the DOF being the values of the polynomial at the mid points a_j of the sides of the triangle Ω_j –

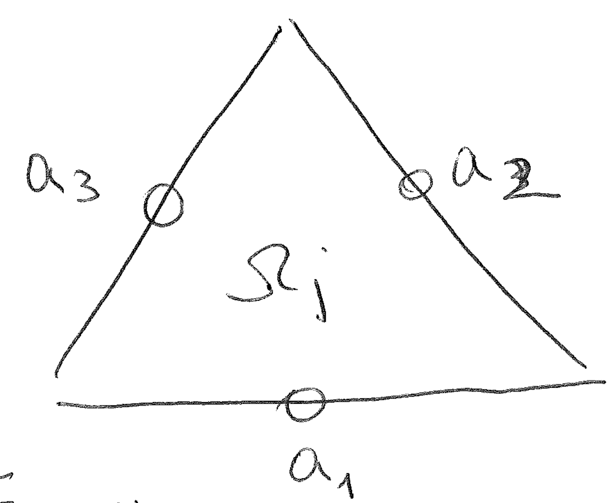

Fig 4

Denote by ξ_{jk} the mid points of

$\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k$.

Let $\Omega^f = \left\{ \begin{array}{l} \text{set of all triangles with a} \\ \text{side contained in the} \\ \text{fracture } \Gamma^f \end{array} \right\}$

$$= \bigcup_{j=1}^{J^f} \Omega_j = \bigcup_{j \in I_f} \Omega_j \quad I_f = \{1, \ldots, J^f\}$$

Then let $\Omega^R = \Omega \setminus \Omega^f = \bigcup_{j=J^f+1}^{J} \Omega_j$ be

the rest of the triangles $-$ $(I_R = \{J^f+1, \ldots, J\}$

Set

$$NC^{h,R} = \left\{ \begin{array}{l} v : v_j = v|_{\Omega_j} \in NC_j^h,\ v_j(\xi_{jk}) = v_k(\xi_{jk}) \\ \qquad \forall\ j,k \in I_R \end{array} \right\}$$

Next we need to define the elements related to Γ^f.

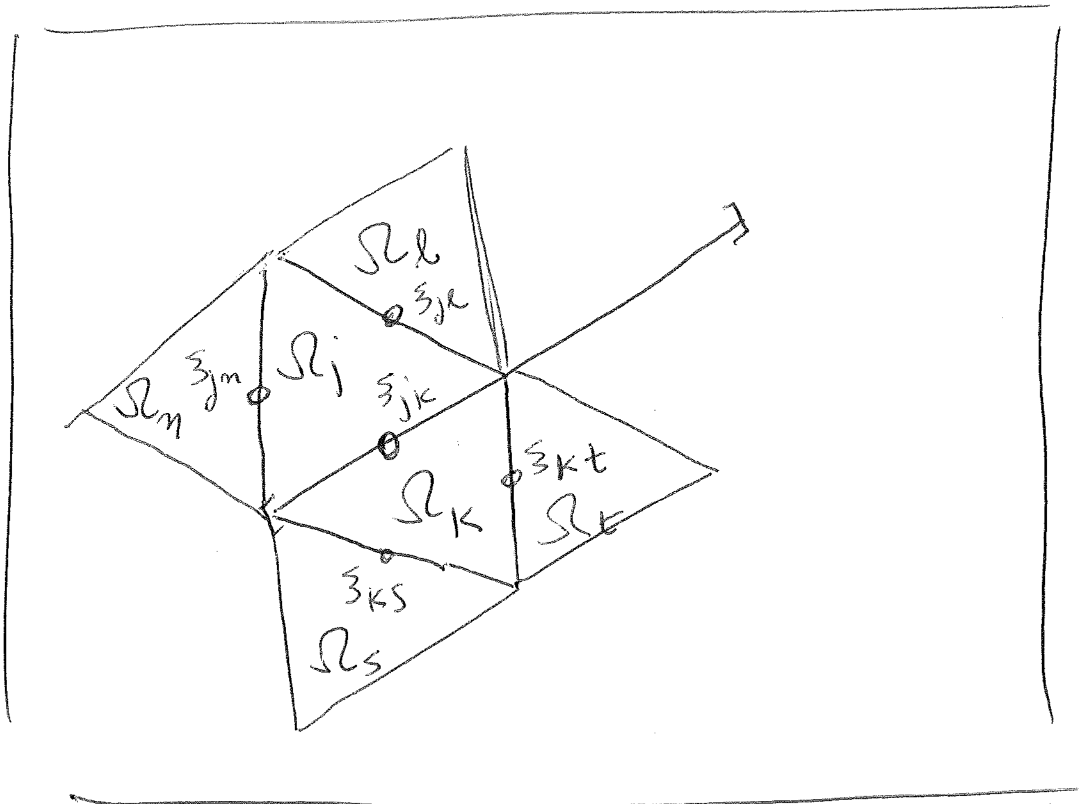

Fig 5

For $j \neq k$, $j, k \in I_f$, set

$$NC^{(j,k)} = \left\{ v^{(j,k)} = v|_{\Omega_j \cup \Omega_k} : \ v^{(j,k)}(\xi_{jl}) = v_l(\xi_{jl}), \ v^{(j,k)}(\xi_{jm}) = v_m(\xi_{jm}), \ v^{(j,k)}(\xi_{kt}) = v_t(\xi_{kt}), \ v^{(j,k)}(\xi_{ks}) = v_s(\xi_{ks}) \right\}$$

$NC^{(j,k)}$

[Continuity is <u>NOT</u> imposed at ξ_{jk}]

Now we set

$$NC^{h,f} = \left\{ v : v^{(j,k)} = v\Big|_{\Omega_j \cup \Omega_k} \in NC^{(j,k)} \;\middle|\; \forall \; j,k \in I_f \quad j \neq k \right\}$$

Next let us choose as our FE space

$$NC^h = NC^{h,R} \cup NC^{h,f} .$$

Set $\quad u^h = (u^{1,h}, u^{3,h})$.

For triangles Ω_j with $j \in I_R$, let

$\{ \varphi_{1,j}, \varphi_{2,j}, \varphi_{3,j} \}$ a local basis of

$P_1(\Omega_j) \qquad \left(\varphi_{1,j}(a_1) = 1, \;\; \varphi_{1,j}(a_2) = \varphi_{1,j}(a_3) = 0 \right)$

etc

Then set

$$(51) \quad u_j^{1,h} = u^1_{1,j} \varphi_{1,j} + u^1_{2,j} \varphi_{2,j} + u^1_{3,j} \varphi_{3,j} ,$$

$$(52) \quad u_j^{3,h} = u^3_{1,j} \varphi_{1,j} + u^3_{2,j} \varphi_{2,j} + u^3_{3,j} \varphi_{3,j} .$$

Then the local equations for

$u_j^h = u^h|_{\Omega_j}$ are (see (25))

(53)
$$-\omega^2 \left(\varrho u_j^h, (\varphi_{s,j},0)\right)_{\Omega_j} + \left(M \tilde{\varepsilon}(u_j^h), \tilde{\varepsilon}(\varphi_{sj},0)\right)_{\Omega_j}$$
$$+ \left\langle i\omega B \begin{pmatrix} u_j^h \cdot \nu \\ u_j^h \cdot \chi \end{pmatrix}, \begin{pmatrix} (\varphi_{sj},0)\cdot \nu \\ (\varphi_{sj},0)\cdot \chi \end{pmatrix} \right\rangle_{\Gamma_j}$$
$$= \left(f, (\varphi_{sj},0)\right)_{\Omega_j}, \quad s=1,2,3$$

and

(54)
$$-\omega^2 \left(\varrho u_j^h, (0,\varphi_{sj})\right)_{\Omega_j} + \left(M \tilde{\varepsilon}(u_j^h), \tilde{\varepsilon}(0,\varphi_{sj})\right)_{\Omega_j}$$
$$+ \left\langle i\omega B \begin{pmatrix} u_j^h \cdot \nu \\ u_j^h \cdot \chi \end{pmatrix}, \begin{pmatrix} (0,\varphi_{sj})\cdot \nu \\ (0,\varphi_{sj})\cdot \chi \end{pmatrix} \right\rangle_{\Gamma_j}$$
$$= \left(f, (0,\varphi_{sj})\right)_{\Omega_j}, \quad s=1,2,3 -$$

Next consider 2 elements Ω_j and Ω_K such that $j, K \in I_f$ and $\Gamma_{jK} \subset \Gamma^f$.

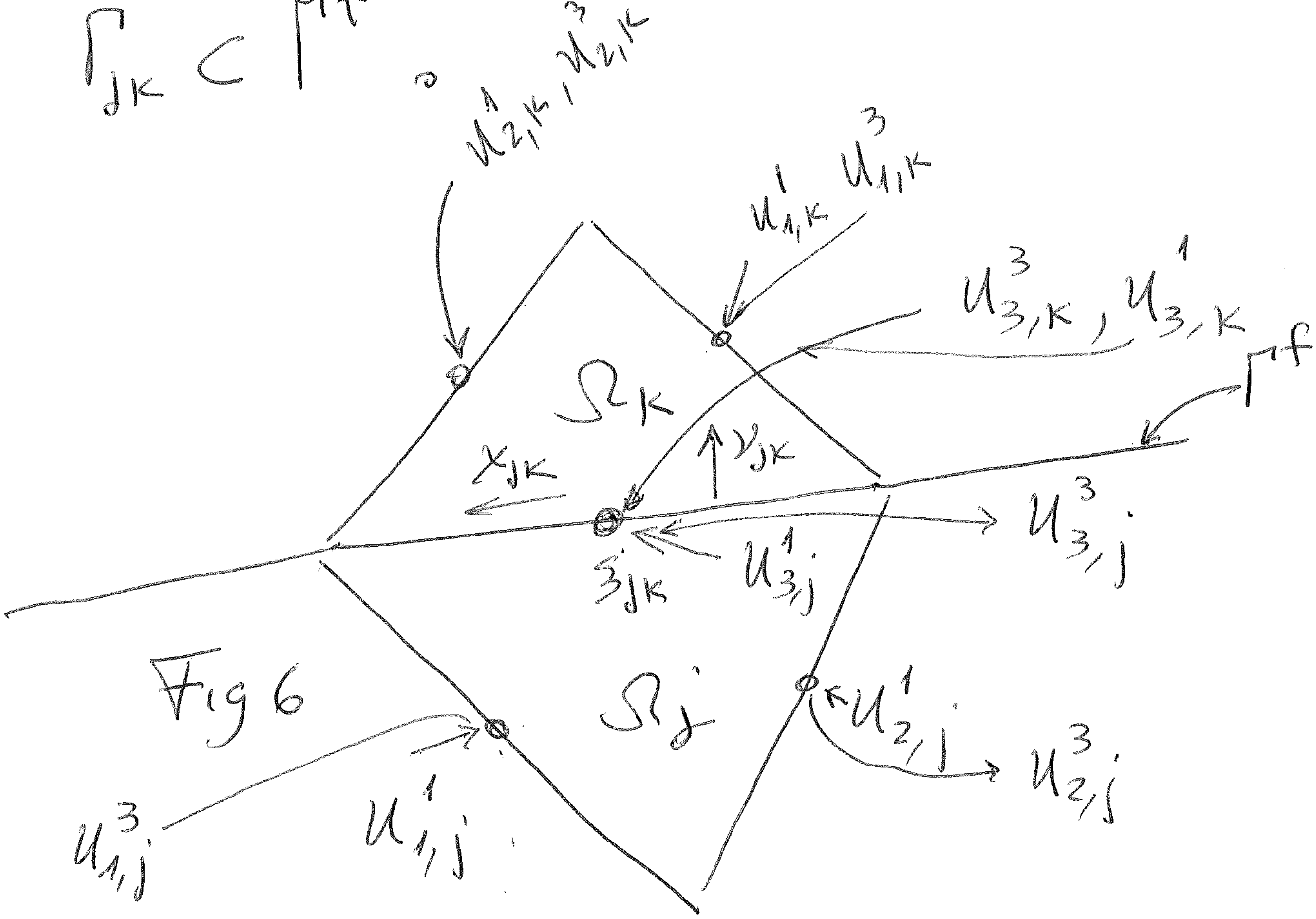

Fig 6

We write $u_j^{1,h}\ u_j^{3,h},\ u_K^{1,h},\ u_K^{3,h}$ as in (51) - (52) (change j by K for u_K) -

Taking the six test functions

$(\varphi_{sj}, 0), \ (0, \varphi_{sj})$ in Ω_j $\quad s=1,2,3$

and the six test functions

$(\varphi_{sk}, 0), \ (0, \varphi_{sk})$, in Ω_k, $s=1,2,3$

we get equations similar to (53) and (54) but adding the bry term on Γ^f:

$$
-\omega^2 \left(\rho u_j^h, (\varphi_{sj}, 0)\right)_j + \left(M \tilde{\varepsilon}(u_j^h), \tilde{\varepsilon}(\varphi_{sj}, 0)\right)_j
$$

$$
(55) \quad + \left\langle D \begin{pmatrix} (u_k^h - u_j^h)\cdot \nu_{jk} \\ (u_k^h - u_j^h)\cdot \chi_{jk} \end{pmatrix}, \begin{pmatrix} \left((0,0) - (\varphi_{sj}, 0)\right)\cdot \nu_{jk} \\ \left((0,0) - (\varphi_{js}, 0)\right)\cdot \chi_{jk} \end{pmatrix} \right\rangle_{\Gamma^f_{jk}}
$$

$s=1,2,3$.

$= 0$

(assuming the fracture Γ^f does not touch the bry Γ and no sources on the fracture) —

and

$$-\omega^2\left(\rho u_j^h, (0,\varphi_{sj})\right)_j + \left(M\tilde{\mathcal{E}}(u_j^h), \tilde{\mathcal{E}}(0,\varphi_{sj})\right)_j$$

$$+\left\langle D\begin{pmatrix}(u_k^h-u_j^h)\cdot\nu_{jk}\\ (u_k^h-u_j^h)\cdot\chi_{jk}\end{pmatrix}, \begin{pmatrix}\left((0,0)-(0,\varphi_{sj})\right)\cdot\nu_{jk}\\ \left((0,0)-(0,\varphi_{sj})\right)\cdot\chi_{jk}\end{pmatrix}\right\rangle_{\Gamma_{jk}^f} \tag{56}$$

$$= 0 \qquad s = 1, 2, 3$$

If the local unknowns (in Ω_j) are numbered as in Fig 6, only the test functions $(\varphi_{3j}, 0)$ and $(0, \varphi_{3j})$ will contribute with a non-zero term in (55) and (56). (the unknowns $u^1_{3,j}$, $u^3_{3,j}$, $u^1_{3,k}$, $u^3_{3,k}$ at ξ_{jk} where the jump condition is imposed will appear in this bry term)

Similarly,

$$-\omega^2\left(\varrho u_K^h,(\varphi_{sK},0)\right)_K+\left(M\tilde{\varepsilon}(u_K^h),\tilde{\varepsilon}(\varphi_{sK},0)\right)_K$$

$$+\left\langle D\begin{pmatrix}(u_K^h-u_j^h)\cdot\nu_{jK}\\(u_K^h-u_j^h)\cdot\chi_{jK}\end{pmatrix},\begin{pmatrix}((\varphi_{sK},0)-(0,0))\cdot\nu_{jK}\\((\varphi_{sK},0)-(0,0))\cdot\chi_{jK}\end{pmatrix}\right\rangle_{\Gamma_{jK}^f} \tag{57}$$

$$=0 \qquad s=1,2,3$$

$$-\omega^2\left(\varrho u_K^h,(0,\varphi_{sK})\right)_K+\left(M\tilde{\varepsilon}(u_K^h),\tilde{\varepsilon}(0,\varphi_{sK})\right)_K$$

$$+\left\langle D\begin{pmatrix}(u_K^h-u_j^h)\cdot\nu_{jK}\\(u_K^h-u_j^h)\cdot\chi_{jK}\end{pmatrix},\begin{pmatrix}((0,\varphi_{sK})-(0,0))\cdot\nu_{jK}\\((0,\varphi_{sK})-(0,0))\cdot\chi_{jK}\end{pmatrix}\right\rangle_{\Gamma_{jK}} \tag{58}$$

$$=0, \qquad s=1,2,3$$

Again, for the local numbering of the unknowns in Ω_K as in Fig 6, only $(\varphi_{3K},0)$, $(0,\varphi_{3K})$ will contribute with non-zero terms in $\langle\ \rangle_{\Gamma_{jK}}$ in (57) and (58)

In the $\langle\ \rangle_{\Gamma_{JK}}$ - term will

appear the local unknowns

$U^1_{3,J}$ $U^3_{3,J}$ $U^1_{3,K}$ $U^3_{3,K}$ (at ξ_{JK}) which are the

ones where the jump condition appears.

In other words, these are the DOF

where we DO NOT IMPOSE continuity

at the mid points —

The 12 equations defined in

(55) (56) (57) and (58) define

a LOCAL system in the "ROMBOID"

element $\Omega_J \cup \Omega_K$ (our "superelement") —

THE DOF NOT related to the nodal

point $\xi_{J,K}$ are GLOBAL DOF :

that need to contribute to the assembly of the Global matrix if a Global procedure is employed.

ONE CHOICE IS TO ELIMINATE THE 4 DOF associated with the node ξ_{JK} in terms of the 8 GLOBAL DOF in $\Omega_J \cup \Omega_K$.

and consider $\Omega_J \cup \Omega_K$ as a "romboid" element with 8 DOF as when using rectangular Nonconforming elements.

MATHEMATICA COULD BE USED TO ELIMINATE SYMBOLICALLY the 4 DOF where the JUMP IS imposed.

to get the form of the local 8×8 system of equations associated with $\Omega_J \cup \Omega_K$ —

The same idea could be used to define an hybridized Domain Decomposition Procedure, where we DO NOT define Lagrange multipliers associated with $\tau(u)\nu_{JK}$ at the NODES ξ_{JK} where we have the jump condition imposed —

— · —

Finally, we will use the mid-point quadrature rule to compute the boundary terms:

$\langle\langle \alpha, \beta \rangle\rangle_{\Gamma_{JK}} = (\alpha \bar{\beta})(\xi_{JK}) \, |\Gamma_{JK}|$, where $|\Gamma_{JK}|$ is the measure of Γ_{JK} —

TO SUMMARIZE, the GLOBAL NON-CONFORMING Finite Element Method is:

Find $u^h \in NC^h$ such that

$$-\omega^2(\rho u^h, v) + \sum_j \left(M\, \tilde{\varepsilon}(u^h), \tilde{\varepsilon}(v) \right)_{\Omega_j}$$

$$+ \sum_{(j,k)\in I_f} \left\langle\!\left\langle D \begin{pmatrix} (u_k^h - u_j^h)\cdot \nu_{jk} \\ (u_k^h - u_j^h)\cdot \chi_{jk} \end{pmatrix}, \begin{pmatrix} (v_k - v_j)\cdot \nu_{jk} \\ (v_k - v_j)\cdot \chi_{jk} \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma_{jk}}$$

(59)

$$+ i\omega \left\langle\!\left\langle B \begin{pmatrix} u^h\cdot\nu \\ u^h\cdot\chi \end{pmatrix}, \begin{pmatrix} v\cdot\nu \\ v\cdot\chi \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma} = (f, v),$$

$v \in NC^h$

$\langle\langle \cdot,\cdot \rangle\rangle_\Gamma$ = mid point rule for $\langle \cdot,\cdot \rangle_\Gamma$.

UNIQUENESS FOR (59):

Set $f = 0$ in (59) and take $v = u^h$.

Taking imaginary part in the resulting equation

and using that $D_I > 0$, $M_I > 0$, $B > 0$

$$\sum_j \left(M_I \tilde{\mathcal{E}}(u^h), \tilde{\mathcal{E}}(u^h) \right)_{\Omega_j} + \sum_{(j,k)\in I_f} \langle\langle D_I, \rangle\rangle_{\Gamma_{jk}}$$

$$(60) \quad + \omega \left\langle\!\left\langle B \begin{pmatrix} u^h \cdot \nu \\ u^h \cdot x \end{pmatrix}, \begin{pmatrix} u^h \cdot \nu \\ u^h \cdot x \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma} = 0$$

Thus, since M_I and B are positive definite,

$$(61) \quad \mathcal{E}_{11}(u^h) = \mathcal{E}_{33}(u^h) = \mathcal{E}_{13}(u^h) = 0, \quad \Omega$$

$$(62) \quad u_1^h = u_3^h = 0, \quad \text{on } \Gamma.$$

We wish to show that $u_1^h = u_2^h = 0$ in Ω — [the argument below is in the paper by Ha, Santos Sheen in CMAME, 191, (2002) p. 5647–5670)

Take an element Ω_1, so that 2 sides of Ω_1 are contained in Γ. (a corner element) —

[There are typos in eq's (6.2) but the argument works]

Without loss of generality we may take Ω_1 with vertices $(-1,1)$, $(-1,-1)$, $(1,-1)$ and Ω_2 with vertices $(-1,1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$ be one of the corners of Ω (Fig 7)

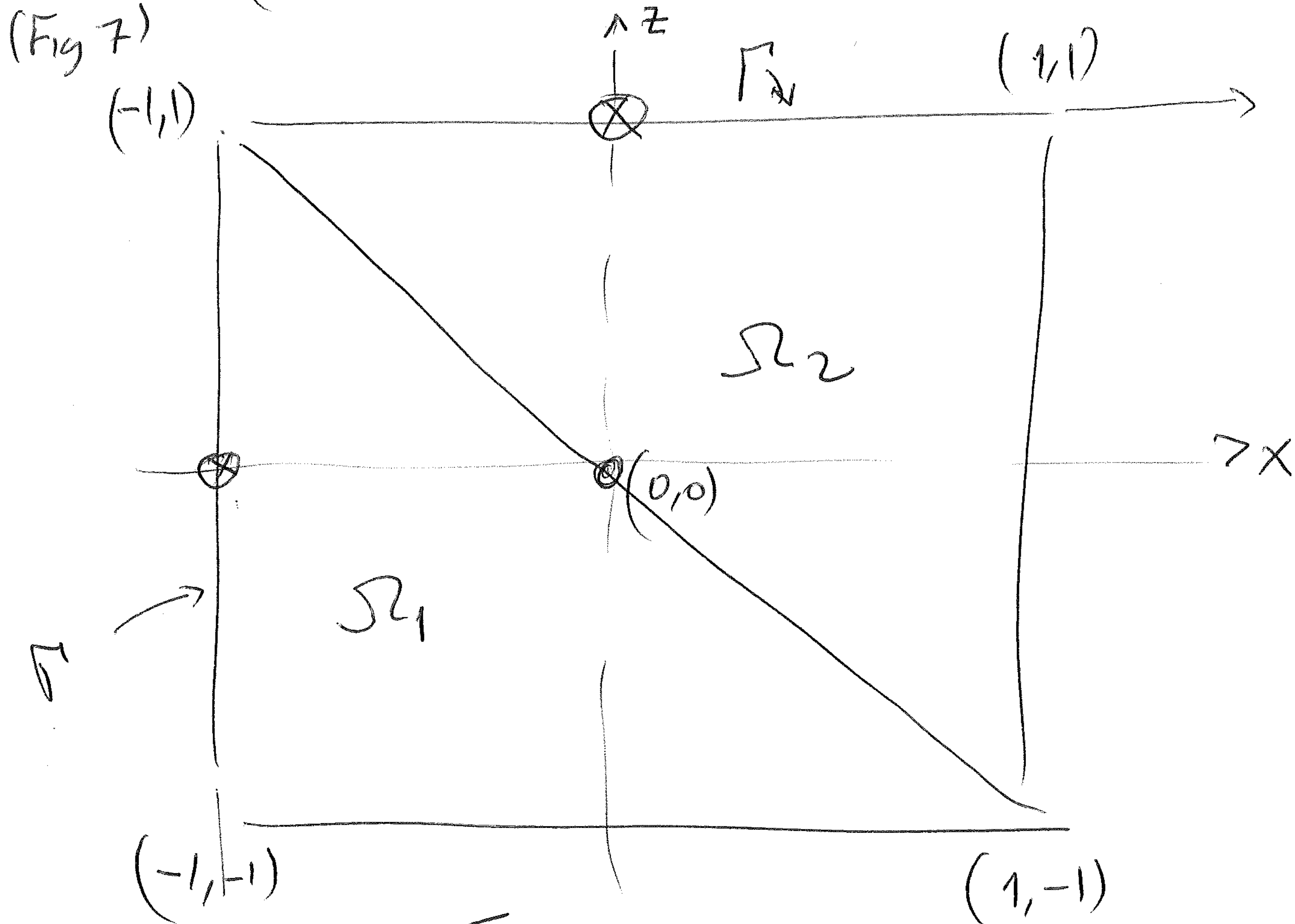

Fig 7

(63) $u_1^h|_{\Omega_1} = a_1 + b_1 x + c_1 \cdot z \qquad u_3^h|_{\Omega_1} = a_2 + b_2 x + c_2 z$

(64) $u_1^h|_{\Omega_2} = p_1 + q_1 x + r_1 z \qquad u_3^h|_{\Omega_2} = p_2 + q_2 x + r_2 z$

From (60),

(65) $u_1^h\big|_{\Omega_1}(-1,0) = a_1 - b_1 = 0$

(66) $u_3^h\big|_{\Omega_1}(-1,0) = a_2 - b_2 = 0$

(67) $u_1^h\big|_{\Omega_2}\ (0,1) = p_1 + r_1 = 0$

(68) $u_3^h\big|_{\Omega_2}(0,1) = p_2 + r_2 = 0$

and

(69) $\varepsilon_{11}\big|_{\Omega_1} = \dfrac{\partial u_1^h}{\partial x} = b_1 = 0$

(70) $\varepsilon_{33}\big|_{\Omega_1} = \dfrac{\partial u_3^h}{\partial z} = c_2 = 0$

(71) $\varepsilon_{13}\big|_{\Omega_1} = \dfrac{1}{2}\left(\dfrac{\partial u_1^h}{\partial z} + \dfrac{\partial u_3^h}{\partial x}\right) = \dfrac{1}{2}\left(c_1 + b_2\right) = 0$

(72) $\varepsilon_{11}\big|_{\Omega_2} = \dfrac{\partial u_1^h}{\partial x} = q_1 = 0$

(73) $\varepsilon_{33}\big|_{\Omega_2} = \dfrac{\partial u_3^h}{\partial z} = r_2 = 0$

(74) $\varepsilon_{13}\big|_{\Omega_2} = \dfrac{1}{2}\left(\dfrac{\partial u_1^h}{\partial z} + \dfrac{\partial u_3^h}{\partial x}\right)\Big|_{\Omega_2} = \dfrac{1}{2}\left(r_1 + q_2\right) = 0$

From (69) and (65)

$$a_1 = 0 \qquad (\text{and } b_1 = 0)$$

so that (from 71)

$$(75) \qquad u_1^h|_{\Omega_1} = c_1 z = -b_2 z$$

From (70) and (66)

$$(76) \qquad u_3^h|_{\Omega_1} = a_2(1+x) = b_2(1+x)$$

From (72) and (67)

$$u_1^h|_{\Omega_2} = p_1 + r_1 z =$$

$$(77) \qquad = -r_1 + r_1 z = -r_1(1-z)$$

From (73) and (68) $(r_2 = p_2 = 0)$ (and (74))

$$(78) \qquad u_3^h|_{\Omega_2} = q_2 x = -r_1 x$$

Now at $(0,0)$ $\quad u^h|_{\Omega_1}(0,0) = u^h|_{\Omega_2}(0,0)$

(since $u^h \in NC^h$)

Then from (76)

$$u_3^h|_{\Omega_1}(0,0) = b_2 = u_3^h|_{\Omega_2}(0,0) = -r_1 \cdot 0 = 0$$

$$\rightarrow b_2 = 0$$

From (77):

$$u_1^h|_{\Omega_1}(0,0) = 0 = u_1^h|_{\Omega_2}(0,0) = -r_1$$

$$\rightarrow r_1 = 0$$

Then $b_2 = r_1 = 0$ and from

(75) – (78) $u^h|_{\Omega_1} = u^h|_{\Omega_2} = 0$

Next, take ~~3~~ 2 triangles Ω_3, Ω_4

Ω_3 adjacent to Ω_2 and Ω_4 with a side contained in Γ

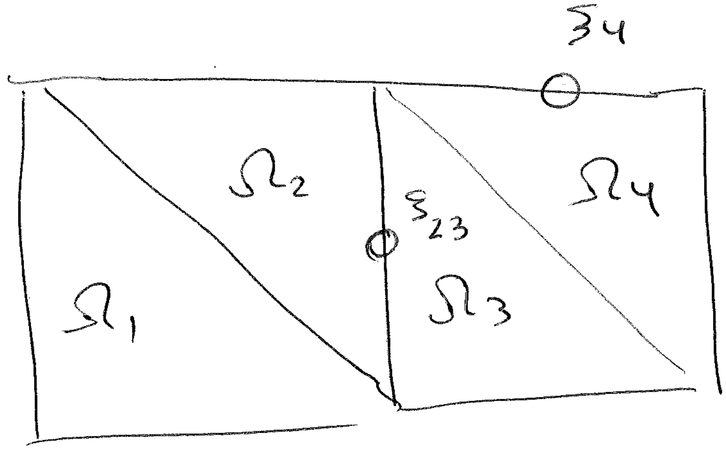

Fig 8

Now $u^h(\xi_{23}) = u^h(\xi_4) = 0$ and we can repeat the argument to show that

$u^h|_{\Omega_3} = u^h|_{\Omega_4} = 0$ –

We repeat the argument until we show that u^h vanishes for all triangles with 1 side contained in Ω – Now we start over inside Ω until the domain is exausted –

<u>Remark</u>: when we follow this argument inside the domain and we get to a triangle Ω_j with one side Γ_{jk} contained in the fracture Γ^f, Ω_j will have a neighbor Ω_k so that $(j,k) \in I_f$ and from (60)

$$0 = \left\langle\!\left\langle D_I \begin{pmatrix} (u_k^h - u_j^h)\cdot \nu_{jk} \\ (u_k^h - u_j^h)\cdot \chi_{jk} \end{pmatrix}, \begin{pmatrix} (u_k^h - u_j^h)\cdot \nu_{jk} \\ (u_k^h - u_j^h)\cdot \chi_{jk} \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma_{jk}}$$

so that (since we are using the mid point rule)

$$(u_k^h - u_j^h)\cdot \nu_{jk}\Big|_{\xi_{jk}} = 0$$

$$(u_k^h - u_j^h)\cdot \chi_{jk}\Big|_{\xi_{jk}} = 0$$

i.e.

$$u_k^h(\xi_{jk}) = u_j^h(\xi_{jk}) \tag{79}$$

Then we are in the situation as in Fig 9

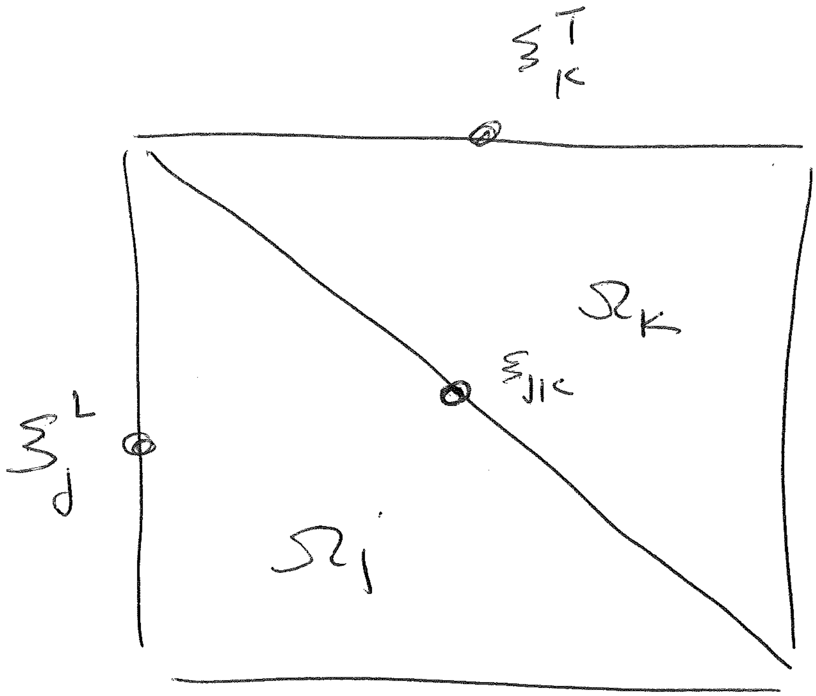

Fig 9

We know that $u^h|_{\Omega_j}(\xi_j^L) = u^h|_{\Omega_k}(\xi_k^T) = 0$

and from (79)

$$u_j^h(\xi_{jk}) = u_k^h(\xi_{jk})$$

so we are in the same position then when the argument started (Fig 7) and we conclude that

$$u^h|_{\Omega_j} = u^h|_{\Omega_k} = 0 \quad -$$

Then this argument still holds for domains with fractures — EXISTENCE FOLLOWS FROM FINITE DIMENSIONALITY —

A DOMAIN DECOMPOSITION ITERATION

Introduce the Lagrange multipliers

$$\Lambda_{-1}^h = \{ \eta^h : \eta^h\big|_{\Gamma_{JK}} = \eta^h_{JK} \in [P_0(\Gamma_{JK})]^2$$

$$\forall\ J,K \in I_R \} .$$

Thus we <u>do not</u> have Lagrange multipliers on the interelement boundaries $\Gamma_{JK} \subset \Gamma^f$.

Let

$$\eta^h_{JK} \sim -\tau(u_J^h)\nu_{JK}$$

at the mid points ξ_{JK} of Γ_{JK}

$\forall\ J,K \in I_R$

If the triangle Ω_J does not touch the fracture, testing against

$$v \in NC_J^h = P_1(\Omega_J),$$

$$-\omega^2(\rho u_j^h, v)_j + (\tau_{lm}(u_j^h), \varepsilon_{lm}(v))_j$$

$$-\sum_K \langle \tau(u_j^h)\nu_{jK}, v\rangle_{\Gamma_{jK}} - \langle \tau(u_j^h)\nu_j, v\rangle_{\Gamma_j} = (f, v)$$

or

$$-\omega^2(\rho u_j^h, v) + (\tau_{lm}(u_j^h), \varepsilon_{lm}(v))_j$$

(80)
$$+\sum_K \langle\langle \eta_{jK}^h, v\rangle\rangle_{\Gamma_{jK}} + i\omega\left\langle\left\langle B\begin{pmatrix} u_j^h\cdot\nu_j \\ u_j^h\cdot\chi_j\end{pmatrix}, \begin{pmatrix} v\cdot\nu_j \\ v\cdot\chi_j\end{pmatrix}\right\rangle\right\rangle_{\Gamma_j}$$

$$= (f, v) \qquad v \in NC_j^h,$$

and we impose the usual Robin condition

(81) $$\tau(u_j^h)\nu_{jK} + \beta_{jK} u_j^h = -\tau(u_K^h)\nu_{Kj} + \beta_{jK} u_K^h, \quad \Gamma_{jK}, \quad (j,K)\in I_R$$

(82) $$\tau(u_K^h)\nu_{Kj} + \beta_{jK} u_K^h = -\tau(u_j^h)\nu_{jK} + \beta_{jK} u_j^h$$

Now we write (80) – (82) in iterative form, using the Lagrange multipliers in (81) – (82):

$$-\omega^2 (\varrho u_j^{h,n}, v)_j + (\tau_{\ell m}(u_j^{h,n}), \varepsilon_{\ell m}(v))_j$$

$$(83) \quad + i\omega \left\langle\left\langle B \begin{pmatrix} u_j^{h,n}\cdot\nu_j \\ u_j^{h,n}\cdot\chi_j \end{pmatrix}, \begin{pmatrix} v\cdot\nu_j \\ v\cdot\chi_j \end{pmatrix} \right\rangle\right\rangle_{\Gamma_j}$$

$$+ \sum_K \langle\langle \eta_{jK}^{h,n}, v \rangle\rangle_{\Gamma_{jK}} = (f, v)_j, \quad v \in NC_j^h,$$

$$(84) \quad \eta_{jK}^{h,n} = -\eta_{Kj}^{h,n-1} + \beta_{jK}\left[u_j^{h,n}(\xi_{jK}) - u_K^{h,n-1}(\xi_{jK})\right],$$

$$\forall (j,K) \in I_R -$$

Next, consider a triangle Ω_j with a common face with Ω_K and $\Gamma_{jK} \subset \Gamma^f$, so that $j,K \in I_f$ –

Define a new local space supported on $\Omega_j \cup \Omega_K$

$$NC_{jK} = \left\{ v \in L^2_{(\Omega_j \cup \Omega_K)} : v_j \equiv v|_{\Omega_j} \in \left[P_1(\Omega_j)\right]^2 \right. \\ \left. v_K \equiv v|_{\Omega_K} \in \left[P_1(\Omega_K)\right]^2 \right\}$$

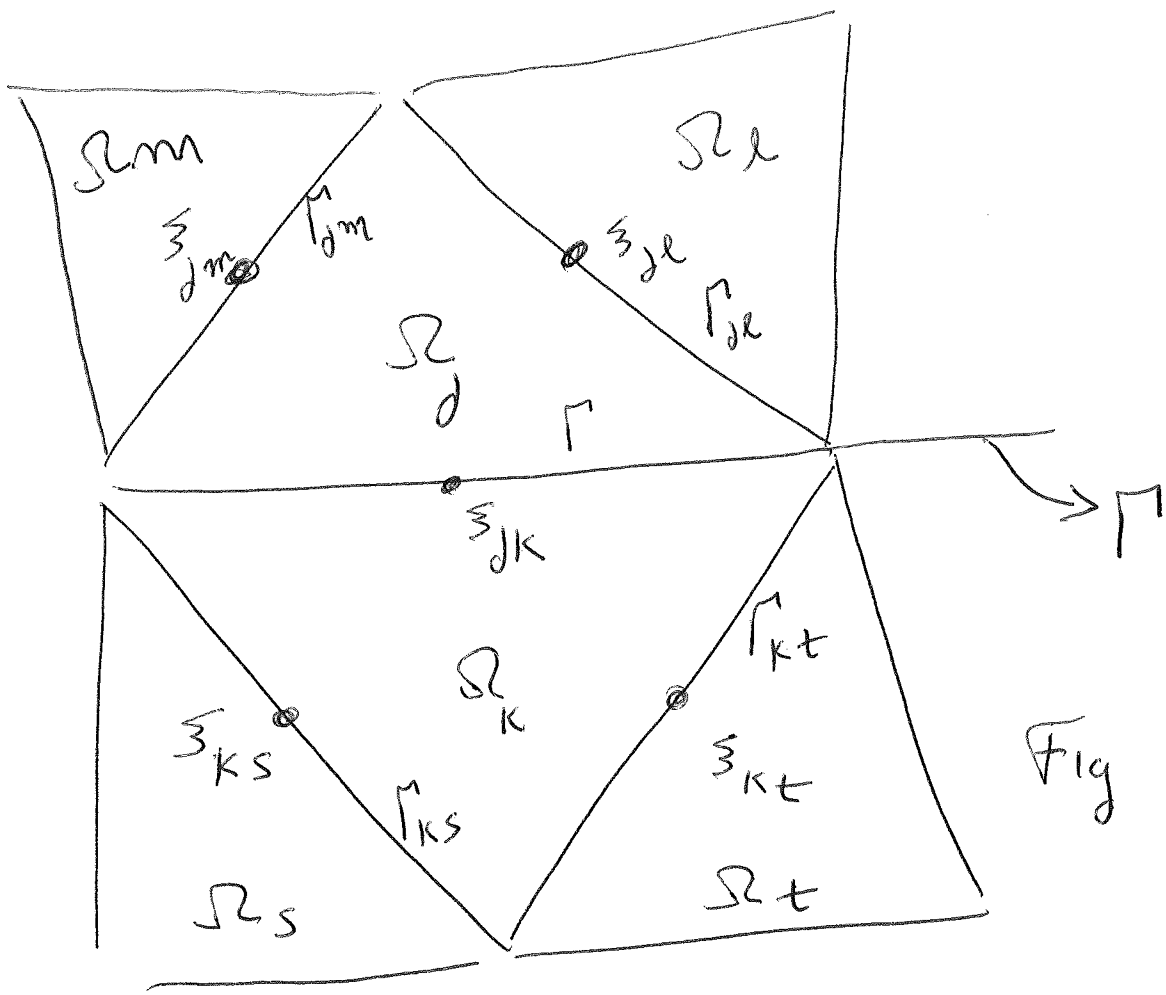

Fig 10

Let $\quad u_{jK} = u|_{\Omega_j \cup \Omega_K} \qquad (\, , \,)_{jK} = (\, , \,)_{\Omega_j \cup \Omega_K}$

Testing against $v \in NC_{jk}$ and using integration by parts in Ω_j and Ω_k, with the argument leading to (11) we get the iterative procedure:

$$-\omega^2(\rho\, u_{jk}^{h,n}, v)_{jk} + (\tau_{\ell m}(u_{jk}^{h,n}), \varepsilon_{\ell m}(v))_{jk}$$

$$+ \left\langle\!\left\langle D \begin{pmatrix} (u_k^{h,n} - u_j^{h,n}) \cdot \nu_{jk} \\ (u_k^{h,n} - u_j^{h,n}) \cdot \chi_{jk} \end{pmatrix}, \begin{pmatrix} (v_k - v_j) \cdot \nu_{jk} \\ (v_k - v_j) \cdot \chi_{jk} \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma_{jk}}$$

(85)

$$+ \langle\langle \eta_{j\ell}^{h,n}, v \rangle\rangle_{\Gamma_{j\ell}} + \langle\langle \eta_{jm}^{h,n}, v \rangle\rangle_{\Gamma_{jm}}$$

$$+ \langle\langle \eta_{ks}^{h,n}, v \rangle\rangle_{\Gamma_{ks}} + \langle\langle \eta_{kt}^{h,n}, v \rangle\rangle_{\Gamma_{kt}}$$

$$+ i\omega \left\langle\!\left\langle B \begin{pmatrix} u_{jk}^{h,n} \cdot \nu_j \\ u_{jk}^{h,n} \cdot \chi_j \end{pmatrix}, \begin{pmatrix} v \cdot \nu_j \\ v \cdot \chi_j \end{pmatrix} \right\rangle\!\right\rangle_{\Gamma_j} = (f, v)$$

$v \in NC_{jk}$, $j,k \in I_f$

$$(86)\quad \eta^{h,n}_{jl} = -\eta^{h,n-1}_{lj} + \beta_{jl}\left(u^{h,n}_{j}(\xi_{jl}) - u^{h,n-1}_{l}(\xi_{jl})\right)$$

$$(87)\quad \eta^{h,n}_{j,m} = -\eta^{h,n-1}_{m,j} + \beta_{jm}\left(u^{h,n}_{j}(\xi_{jm}) - u^{h,n-1}_{m}(\xi_{jm})\right)$$

$$(88)\quad \eta^{h,n}_{k,s} = -\eta^{h,n-1}_{s,k} + \beta_{ks}\left(u^{h,n}_{k}(\xi_{ks}) - u^{h,n-1}_{s}(\xi_{ks})\right)$$

$$(89)\quad \eta^{h,n}_{k,t} = -\eta^{h,n-1}_{t,k} + \beta_{kt}\left(u^{h,n}_{k}(\xi_{kt}) - u^{h,n-1}_{t}(\xi_{kt})\right),$$

$$\{(j,l),\ (j,m),\ (k,s),\ (k,t)\} \in I_R .$$

In (85),

$$u^{h,n}_{k} = u^{h,n}_{jk}\big|_{\Omega_k}, \quad u^{h,n}_{j} = u^{h,n}_{jk}\big|_{\Omega_j} .$$

Remark: In (85) we can take as test functions (all of them are in NC_{jk})

$(\varphi_{sj},0),\ (0,\varphi_{sj})$, in Ω_j $\quad s=1,2,3$

$(0,0),(0,0)$ in Ω_K

$(\varphi_{sK},0),(0,\varphi_{sK})$, in Ω_K, $s=1,2,3$

$(0,0)\ (0,0)$ in Ω_j

to obtain equations similar to (55) (56) (57) and (58) for the 12 DOF (unknowns) defining $u_j^{h,n}$ $u_K^{h,n}$ at the nodes ξ_{jK}, ξ_{Ks}, ξ_{Kt} ξ_{je} and ξ_{jm}.

The Lagrange multipliers can be eliminated from such equations using (86)–(89). After solving for $u_j^{h,n}$, $u_K^{h,n}$ (and for $u_j^{h,n}$ in (83)) we update the Lagrange multipliers using (84) or (86)–(89).

Then our iterative procedure is defined by (83) and (84) for $j,k \in I_R$ and by (85) – (89) for $j,k \in I_f$ –

<u>Remark</u> about implementation.

1) We need to define the fracture location and each triangle has to have each side labeled as if the side is contained or not in the fracture – (ABAQUS can do it?)

2) EACH TRIANGLE NEEDS TO KNOW which are its neighbors, this may be done by ABAQUS – ?

in order to perform the Domain Decomposition without IF statements to FIND OUT THIS AT EACH ITERATION –