NONCONFORMING FE Methods for WAVE
PART I

PROPAGATION IN 2D-FRACTURED VISCOELASTIC MEDIA -

SPACE-FREQUENCY DOMAIN FORMULATION

1) CONTINUOUS PROBLEM, WEAK FORM, EXISTENCE & UNIQUENESS

2) GLOBAL NONCONFORMING Finite Element Method over a TRIANGULATION - UNIQUENESS -

3) MASSIVE DOMAIN DECOMPOSITION -

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1

2D FRACTURED VE MEDIA

CASE 1: THE FRACTURE TOUCHES THE BOUNDARY –

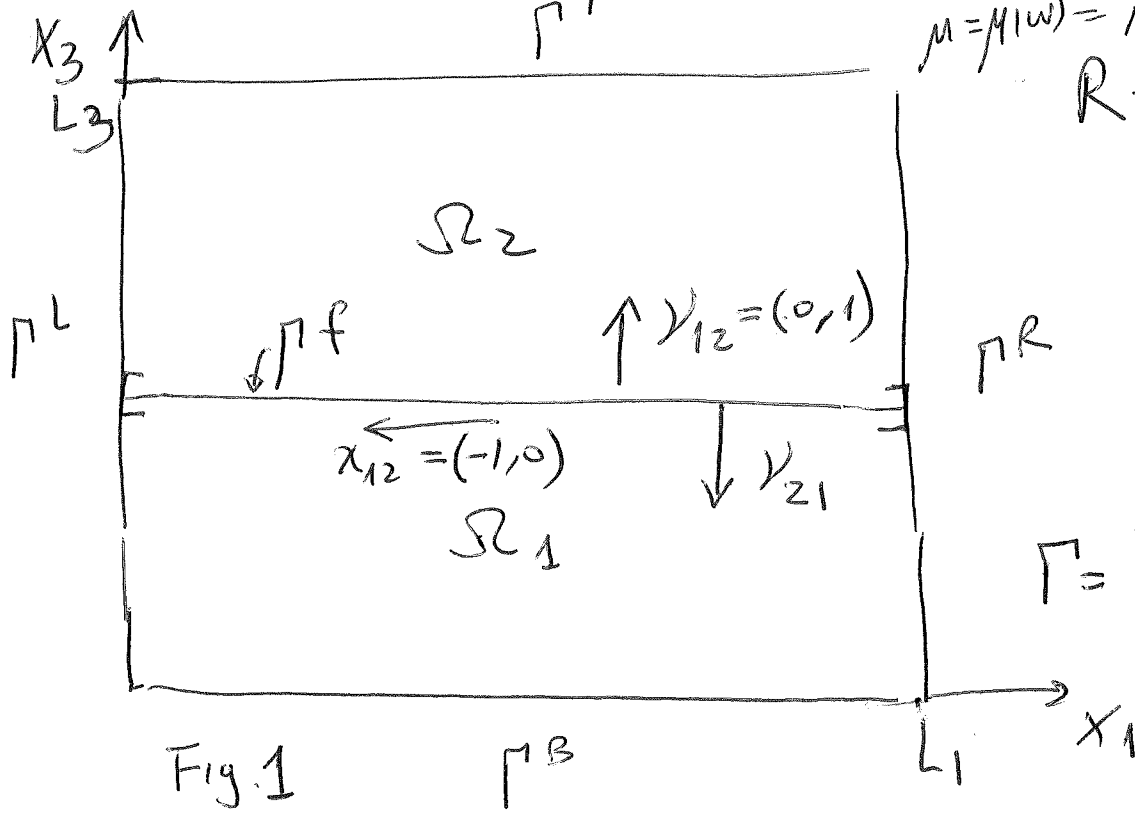
$$(1) \quad -\omega^2 \rho u - \nabla \cdot \tau(u) = f \quad \Omega = R \setminus \Gamma^f$$

$$(2) \quad \tau_{lm}(u) = 2\mu\, \varepsilon_{lm}(u) + \lambda\, \delta_{lm} \nabla \cdot u \quad , \Omega$$

$\lambda = \lambda(\omega) = \lambda_R + i\lambda_I$

$\mu = \mu(\omega) = \mu_R + i\mu_I$

$R = (0, L_1) \times (0, L_3)$



$\Gamma = \Gamma^L \cup \Gamma^R \cup \Gamma^B \cup \Gamma^T = \partial R$

Fig. 1

Let

$$V^f = \left\{ v \in [L^2(\Omega)]^2 : v^{(j)} = v|_{\Omega_j} \in [H^1(\Omega_j)]^2, j = 1, 2 \right\}$$

$$= [H^1(\Omega)]^2$$

Multiply (1) by $v \in V^f$ and use integration by parts on each Ω_j:

$$-\omega^2(\rho u, v)_\Omega + (\tau_{lm}(u^{(1)}), \varepsilon_{lm}(v^{(1)}))_{\Omega_1}$$

$$+ (\tau_{lm}(u^{(2)}), \varepsilon_{lm}(v^{(2)}))_{\Omega_2}$$

(3)
$$- \langle \tau(u^{(1)}) \cdot \nu_{12}, v^{(1)} \rangle_{\Gamma^f} - \langle \tau(u^{(2)}) \cdot \nu_{21}, v^{(2)} \rangle_{\Gamma^f}$$

$$- \langle \tau(u) \nu, v \rangle_{\Gamma} = (f, v), \quad v \in V^f -$$

Consider the bdy term on Γ^f in (3)

$$T_1 = - \langle \tau(u^{(1)}) \nu_{12}, v^{(1)} \rangle_{\Gamma^f} - \langle \tau(u^{(2)}) \cdot \nu_{21}, v^{(2)} \rangle_{\Gamma^f}$$

$$= - \langle \tau(u^{(1)}) \nu_{12}, v^{(1)} \rangle_{\Gamma^f} + \langle \tau(u^{(2)}) \nu_{12}, v^{(2)} \rangle_{\Gamma^f}$$

Now we impose the B.C

(4) $\tau(u^{(1)})\nu_{12} = \tau(u^{(2)})\nu_{12}$ on Γ^f

Then $\equiv \tau^f(u)\nu_{12}$ [STRESS CONTINUITY AT THE FRACTURE]

$$T_1 = \langle \tau^f(u)\nu_{12}, v^{(2)} - v^{(1)} \rangle_{\Gamma^f}$$

$$= \langle \left(\tau^f(u)\nu_{12}\,\nu_{12}, \tau^f(u)\nu_{12}\chi_{12}\right),$$

(4-1) $$\left(v^{(2)} - v^{(1)}\right)\cdot\nu_{12}, \left(v^{(2)} - v^{(1)}\right)\cdot\chi_{12} \rangle_{\Gamma^f}$$

where χ_{12} is the unit tangent on Γ^f oriented counterclockwise —

Next we impose the second B.C. on Γ^f:

(5) $$\left(\tau^f(u)\nu_{12}\,\nu_{12}, \tau^f(u)\nu_{12}\chi_{12}\right) = D \begin{pmatrix} (u^{(2)} - u^{(1)})\cdot\nu_{12} \\ (u^{(2)} - u^{(1)})\cdot\chi_{12} \end{pmatrix}$$ on Γ^f

where

$$D = \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \nu_1^{12} & \nu_3^{12} \\ \chi_1^{12} & \chi_3^{12} \end{bmatrix}$$

$$= \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} \alpha \nu_1^{12} & \alpha \nu_3^{12} \\ \beta \chi_1^{12} & \beta \chi_3^{12} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \alpha \nu_1^{12} \nu_1^{12} + \beta \chi_1^{12} \chi_1^{12} & \alpha \nu_1^{12} \nu_3^{12} + \beta \chi_1^{12} \chi_3^{12} \\ \alpha \nu_1^{12} \nu_3^{12} + \beta \chi_1^{12} \chi_3^{12} & \alpha \nu_3^{12} \nu_3^{12} + \beta \chi_3^{12} \chi_3^{12} \end{bmatrix}$$

$$= T^t \Lambda T,$$

(7) $\alpha = K_1 + i\omega \xi_1, \quad \beta = K_3 + i\omega \xi_3.$

$K_j \geq 0 \quad \xi_j > 0 \quad j = 1, 3$

Note that

(8) $D = D_R + i D_I,$

D_R is positive semidefinite and

D_I is positive definite –

Also, on Γ we impose the absorbing B.C.

(9) $\quad -\tau(u)\nu = i\omega B u \quad$, B positive definite –

Using (5) in (4-1)

(10) $$T_1 = \left\langle D \begin{pmatrix} (u^{(2)} - u^{(1)}) \cdot \nu_{12} \\ (u^{(2)} - u^{(1)}) \cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)} - v^{(1)}) \cdot \nu_{12} \\ (v^{(2)} - v^{(1)}) \cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^F}$$

Using (10) and (9) in (3):

$$-\omega^2 (\rho u, v)_\Omega + \sum_{j=1}^{2} \left(\tau_{em}(u^{(j)}), \varepsilon_{em}(v^{(j)}) \right)_{\Omega_j}$$

(11) $$+ \left\langle D \begin{pmatrix} (u^{(2)} - u^{(1)}) \cdot \nu_{12} \\ (u^{(2)} - u^{(1)}) \cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)} - v^{(1)}) \cdot \nu_{12} \\ (v^{(2)} - v^{(1)}) \cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^F}$$

$$+ \langle i\omega B u, v \rangle_\Gamma = (f, v), \quad v \in V^f \text{ –}$$

Remark 1: In the argument leading to the weak form (11) we never used the fact that Γ^f is an horizontal fracture, we just used that Ω_1 and Ω_2 are in contact at the fracture Γ^f and used the unit outer normal ν_{12} and the unit tangent χ_{12} oriented counterclockwise from Ω_1 into Ω_2.

Remark 2, Changing $u^{(2)}-u^{(1)}$ and $v^{(2)}-v^{(1)}$ by $u^{(1)}-u^{(2)}$ and $v^{(1)}-v^{(2)}$ in the B-term Γ^f does not change that term. The same happens if we change ν_{12} and χ_{12} by ν_{21} and χ_{21}, unit outer normal and tangents from Ω_2 to Ω_1 –

Calculation of the B.C. (5) on Γ^f as in Fig 1:
(Horizontal fracture)

on Γ^f $\nu^{12} = (0,1)$, $\chi^{12} = (-1,0) = (\chi_1, \chi_3)$
$= (\nu_1, \nu_3)$

Then from (6)

$$D^{(1)} = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}$$

$$\tau^f(u)\nu = \left(\tau_{11}\nu_1 + \tau_{13}\nu_3 \, , \tau_{31}\nu_1 + \tau_{33}\nu_3\right)$$

Then on Γ^f as in Fig 1

$$\tau^f(u)\nu_{12} = (\tau_{13} \, , +\tau_{33})$$

$$\tau^f(u)\nu_{12}\nu_{12} = \tau_{33} \, , \quad \tau^f(u)\nu_{12}\chi_{12} = -\tau_{13}$$

$$\left(u^{(2)} - u^{(1)}\right)\cdot\nu_{12} = u_3^{(2)} - u_3^{(1)}$$

$$\left(u^{(2)} - u^{(1)}\right)\cdot\chi_{12} = (-1)\left(u_1^{(2)} - u_1^{(1)}\right)$$

Then on Γ^f the B.C. (5) is

$$(\tau_{33}, -\tau_{13}) = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} u_3^{(2)} - u_3^{(1)} \\ (-1)(u_1^{(2)} - u_1^{(1)}) \end{pmatrix}$$

or

$$(12) \quad \tau_{33} = (K_3 + i\omega \xi_3)\left(u_3^{(2)} - u_3^{(1)}\right),$$

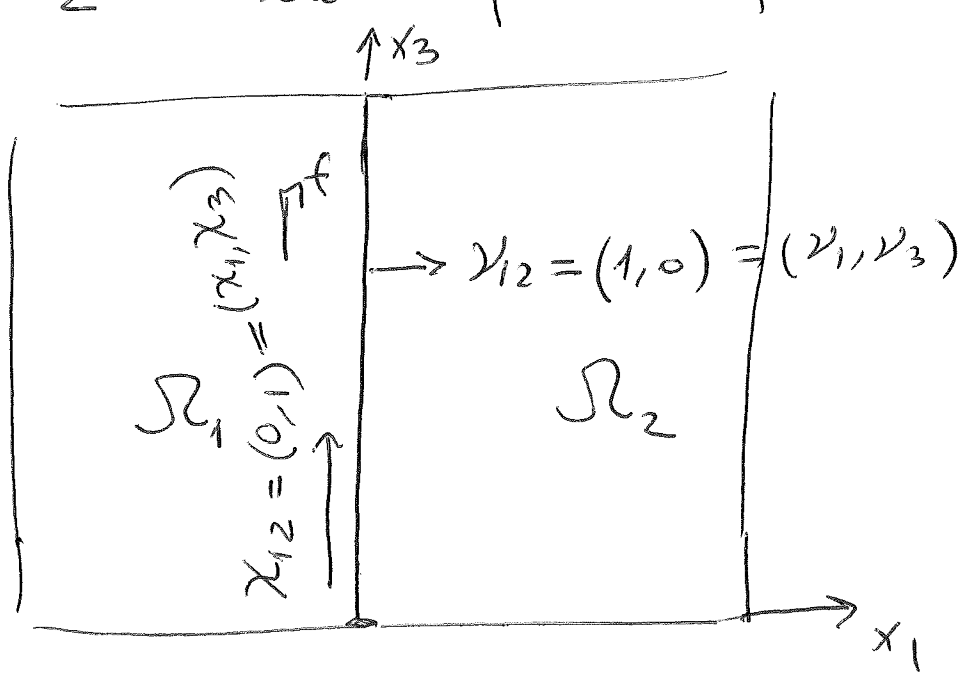
$$(13) \quad \tau_{13} = (K_1 + i\omega \xi_1)\left(u_1^{(2)} - u_1^{(1)}\right).$$

Now (12) and (13) are the B.C. in

((Carcione, J.G.R. (1996), V. 101 B12
p. 28177 – 28188) $[u_3^{(2)} = u_z^+,\ u_3^{(1)} = u_z^-]$

Next let us consider that Γ^f is as

in Fig 2 below (vertical fracture)



Then from (6)

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\tau^f(u)\,\nu_{12} = (\tau_{11}, \tau_{31})$$

$$\tau^f(u)\,\nu_{12}\,\nu_{12} = \tau_{11}, \quad \tau^f(u)\,\nu_{12}\,\chi_{12} = \tau_{31}$$

$$(u^{(2)} - u^{(1)}) \cdot \nu_{12} = u_1^{(2)} - u_1^{(1)}$$

$$(u^{(2)} - u^{(1)}) \cdot \chi_{12} = u_3^{(2)} - u_3^{(1)}$$

so that on Γ^f as in Fig 2, the B.C. (5) is

$$(\tau_{11}, \tau_{31}) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} u_1^{(2)} - u_1^{(1)} \\ u_3^{(2)} - u_3^{(1)} \end{pmatrix}$$

or

$$(14) \quad \tau_{11} = (K_1 + i\omega\,\xi_1)\,(u_1^{(2)} - u_1^{(1)})$$

$$(15) \quad \tau_{31} = (K_3 + i\omega\,\xi_3)\,(u_3^{(2)} - u_3^{(1)}) -$$

Analysis of the term $\langle i\omega B u, v\rangle$ in (11):

We define

$$B = \begin{bmatrix} \nu_1^{12} & \chi_1^{12} \\ \nu_3^{12} & \chi_3^{12} \end{bmatrix} \begin{bmatrix} V_P & 0 \\ 0 & V_S \end{bmatrix} \begin{bmatrix} \nu_1^{12} & \nu_3^{12} \\ \chi_1^{12} & \chi_3^{12} \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} V_P \nu_1^{12}\nu_1^{12} + V_S \chi_1^{12}\chi_1^{12} & V_P \nu_1^{12}\nu_3^{12} + V_S \chi_1^{12}\chi_3^{12} \\ V_P \nu_1^{12}\nu_3^{12} + V_S \chi_1^{12}\chi_3^{12} & V_P \nu_3^{12}\nu_3^{12} + V_S \chi_3^{12}\chi_3^{12} \end{bmatrix}$$

V_P = P-wave phase velocity,

V_S = S-wave phase velocity.

Then, on Γ^T $\nu^{12} = (0,1)$, $\chi^{12} = (-1,0)$

and

$$B = \rho \begin{bmatrix} V_S & 0 \\ 0 & V_P \end{bmatrix}$$

Also, on Γ^L $\nu^{12} = (1,0)$, $\chi^{12} = (0,1)$

and

$$B = \rho \begin{bmatrix} V_P & 0 \\ 0 & V_S \end{bmatrix}$$

Then we rewrite (9) as

$$(17)\quad \left(-\tau(u)\nu\nu, -\tau(u)\nu\chi\right) = i\omega B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix}, \quad \Gamma$$

Then, on Γ^L we get

$$\left(-\tau_{11}, -\tau_{31}\right) = i\omega \rho \begin{pmatrix} V_p & 0 \\ 0 & V_s \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$

so that

$$(18\text{-}1)\quad \begin{cases} -\tau_{11} = i\omega\rho V_p u_1 \\ -\tau_{31} = i\omega\rho V_s u_3 \end{cases}$$

Also, on Γ^T,

$$-\left(\tau_{33}, -\tau_{13}\right) = i\omega\rho \begin{pmatrix} V_p & 0 \\ 0 & V_s \end{pmatrix} \begin{pmatrix} u_3 \\ -u_1 \end{pmatrix}$$

so that

$$(18\text{-}2)\quad \begin{aligned} -\tau_{33} &= i\omega\rho V_p u_3 \\ -\tau_{13} &= -i\omega\rho V_s u_1 \end{aligned}$$

Now (18-1) and (18-2) impose normal and tangential stresses proportional to normal and tangential velocities as desired —

Then we go back to equation (3) and write the big term on Γ as follows:

$$- \langle \tau(u)\nu , v \rangle_{\Gamma} = - \langle (\tau(u)\nu\nu , \tau(u)\nu\chi) ; \; v\cdot\nu , v\cdot\chi \rangle_{\Gamma}$$

using (17)

$$= i\omega \left\langle B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix} , \begin{pmatrix} v\cdot\nu \\ v\cdot\chi \end{pmatrix} \right\rangle_{\Gamma}$$

Then we rewrite the weak form (11) as follows: find $u \in V^f$ such that

$$-\omega^2 (\varrho u, v)_{\Omega} + (\tau_{em}(u), \varepsilon_{em}(v))_{\Omega}$$

(19)

$$+ \left\langle D \begin{pmatrix} (u^{(2)}-u^{(1)})\cdot\nu_{12} \\ (u^{(2)}-u^{(1)})\cdot\chi_{12} \end{pmatrix} , \begin{pmatrix} (v^{(2)}-v^{(1)})\cdot\nu_{12} \\ (v^{2}-v^{(1)})\cdot\chi_{12} \end{pmatrix} \right\rangle_{\Gamma^f}$$

$$+ i\omega \left\langle B \begin{pmatrix} u\cdot\nu \\ u\cdot\chi \end{pmatrix} , \begin{pmatrix} v\cdot\nu \\ v\cdot\chi \end{pmatrix} \right\rangle_{\Gamma} = (f, v),$$

$$v \in V^f$$

CASE 2   AN INTERIOR FRACTURE

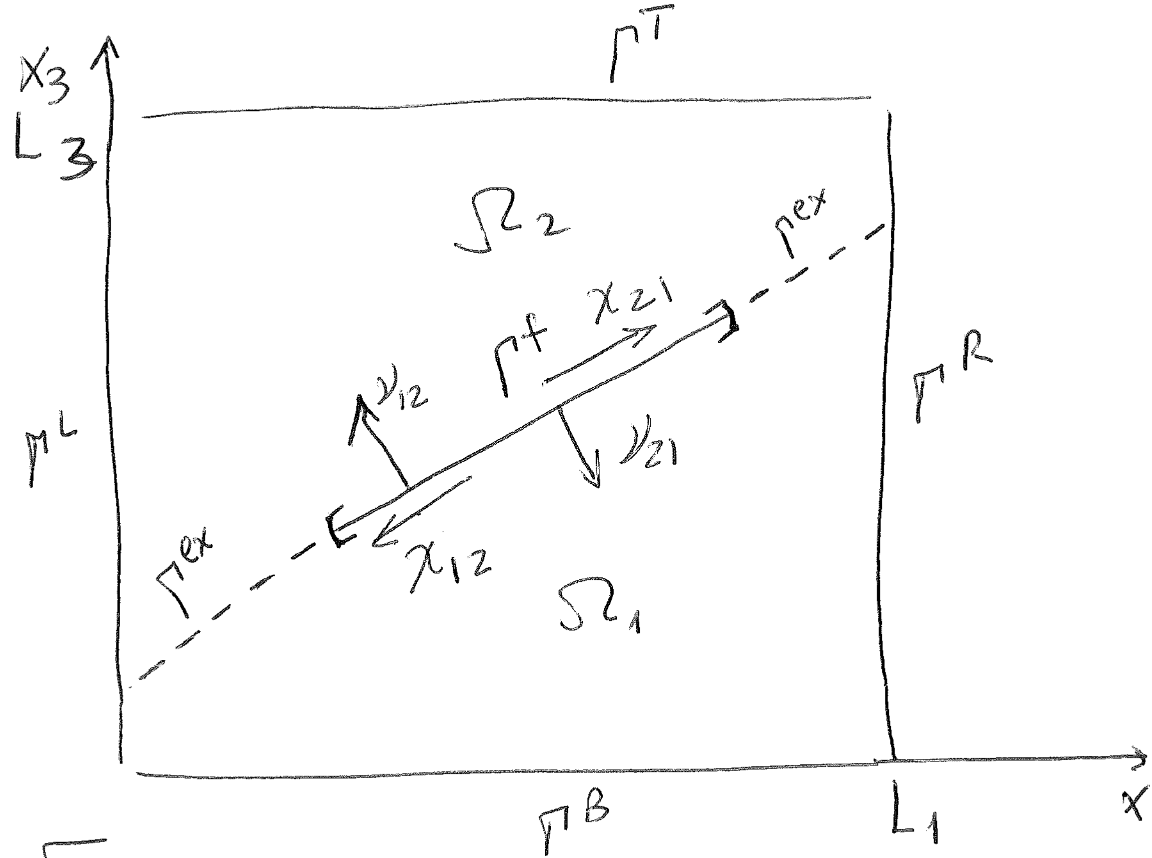


Fig. 3

Γ^f is a closed line inside $R=(0,L_1)\times(0,L_2)$ so that $\Omega = R\setminus\Gamma^f$ is an open set

We "continue" Γ^f until we get to $\Gamma=\partial R$.

(Γ^{ex}=dotted lines). so

we have $\hat{\Gamma}^f = \Gamma^f \cup \{\text{dotted lines set}\}$

$\equiv \Gamma^f \cup \Gamma^{ex}$

Now set

(13)

$$V^f = \left[H^1(\Omega)\right]^2$$

Let Ω_1 and Ω_2 the 2 subsets in Ω defined by the set $\hat{\Gamma}^f$.

Along the dotted line Γ^{ex} we have continuity of stress and displacements.

Thus, we consider again equations

$$(20) \quad -\omega^2 \rho u - \nabla \cdot \tau(u) = f, \quad \Omega$$

$$(21) \quad \tau_{lm}(u) = 2\mu\, \varepsilon_{lm}(u) + \lambda\, \delta_{lm} \nabla \cdot u, \quad \Omega$$

with the B.C.

$$(22) \quad \tau(u^{(1)})\nu_{12} = \tau(u^{(2)})\nu_{12} \equiv \tau^f(u)\nu_{12}, \quad \hat{\Gamma}^f$$

$$(23) \quad u^{(1)} = u^{(2)}, \quad \Gamma^{ex},$$

$$(24) \quad \left(\tau^f(u)\nu_{12}\cdot\nu_{12},\ \tau^f(u)\nu_{12}\cdot\chi_{12}\right) = D\begin{pmatrix} (u^{(2)}-u^{(1)})\cdot\nu_{12} \\ (u^{(2)}-u^{(1)})\cdot\chi_{12}\end{pmatrix}, \quad \Gamma^f$$

$$(24\text{-}1) \quad \left(-\tau(u)\nu\cdot\nu,\ -\tau(u)\nu\cdot\chi\right) = i\omega B\begin{pmatrix} u\cdot\nu \\ u\cdot\chi\end{pmatrix}, \quad \Gamma,$$

where D is defined in (6) – (7) and (14) B by (16).

Then, we multiply (20) by $v \in V^f$ and use integration by parts on each Ω_j. Repeating the argument leading to (11) and using (22) and (23) on Γ^{ex} we recover the weak form (19).

So our weak formulation is: find $u \in V^f$ such that

$$-\omega^2 (\rho u, v)_\Omega + \left(\tau_{em}(u), \varepsilon_{em}(v)\right)_\Omega$$

$$+ \left\langle D \begin{pmatrix} (u^{(2)} - u^{(1)}) \cdot \nu_{12} \\ (u^{(2)} - u^{(1)}) \cdot \chi_{12} \end{pmatrix}, \begin{pmatrix} (v^{(2)} - v^{(1)}) \cdot \nu_{12} \\ (v^{(2)} - v^{(1)}) \cdot \chi_{12} \end{pmatrix} \right\rangle_{\Gamma^f} \tag{25}$$

$$+ \left\langle i\omega B \begin{pmatrix} u \cdot \nu \\ u \cdot \chi \end{pmatrix}, \begin{pmatrix} v \cdot \nu \\ v \cdot \chi \end{pmatrix} \right\rangle_\Gamma = (f, v), \quad v \in V^f.$$

UNIQUENESS FOR (25).

First, set

$$M(\omega) = \begin{pmatrix} \lambda(\omega)+2\mu(\omega) & \lambda(\omega) & 0 \\ \lambda(\omega) & \lambda(\omega)+2\mu(\omega) & 0 \\ 0 & 0 & 4\mu(\omega) \end{pmatrix} = M_R(\omega) + i M_I(\omega)$$

$$\tilde{\varepsilon}(u) = \begin{pmatrix} \varepsilon_{11}(u) \\ \varepsilon_{33}(u) \\ \varepsilon_{13}(u) \end{pmatrix} \qquad \varepsilon_{ij}(u) = \text{strain tensor}$$

Then, using (21), a calculation yields

$$\left(\tau_{\ell m}(u), \varepsilon_{\ell m}(v)\right) = \left(M(\omega)\, \tilde{\varepsilon}(u), \tilde{\varepsilon}(v)\right)$$

M_R and M_I are positive definite.

Then set $f = 0$ and take $v = u$ in (25) to get

$$-\omega^2(\rho u, u)_\Omega + (M_R(\omega)\tilde{\varepsilon}(u), \tilde{\varepsilon}(u))_\Omega$$

$$+ i\,(M_I(\omega)\tilde{\varepsilon}(u), \tilde{\varepsilon}(u))_\Omega$$

$$+ \left\langle (D_R + iD_I)\begin{pmatrix}(u^{(2)}-u^{(1)})\cdot\nu_{12}\\ (u^{(2)}-u^{(1)})\cdot\chi_{12}\end{pmatrix}, \begin{pmatrix}(u^{(2)}-u^{(1)})\cdot\nu_{12}\\ (u^{(2)}-u^{(1)})\cdot\chi_{12}\end{pmatrix}\right\rangle_{\Gamma^f}$$

(26)

$$+ i\omega\left\langle B\begin{pmatrix}u\cdot\nu\\ u\cdot\chi\end{pmatrix}, \begin{pmatrix}u\cdot\nu\\ u\cdot\chi\end{pmatrix}\right\rangle_\Gamma = 0$$

Then, take imaginary part in (26) to get

(27) $\varepsilon_{11}(u) = \varepsilon_{33}(u) = \varepsilon_{13}(u) = 0$, in $L^2(\Omega)$

(28) $u = 0$ in $L^2(\Gamma)$ (since B is pos. def.)

(29) $u^{(2)} = u^{(1)}$ in $L^2(\Gamma^f)$ (since D_I is pos. Def.).

Since $u \in [H^1(\Omega)]^2 = V^f$, and $u = 0$ on Γ

(30) $$|||v||| = \left(\sum_{l,m=1,3} \|\varepsilon_{lm}(v)\|_0^2\right)^{1/2}$$

defines a norm equivalent to the H^1- norm [Ciarlet, THE FEM for Elliptic Problems, North Holland, 1980]

so that

$$(31) \qquad C_1 \|V\|_1 \leq |||V||| \leq C_2 \|V\|_1$$

and from (27) we see that

$$(32) \quad \|u\|_1 = 0 \quad \text{in } \Omega -$$

and we have uniqueness —

Here (29) was not used for uniqueness —

If $M_I = 0$, $D_I = 0$, then taking imaginary part in (26) we get

$$(33) \qquad u = 0 \quad \text{in } L^2(\Gamma)$$

Then, from (24), (24-1)

$$(34) \quad \tau(u)\nu\nu = 0, \qquad \tau(u)\nu\chi = 0, \quad \Gamma \cup \Gamma^t$$

For example, on $\Gamma^R \doteq \{(x_1, x_3) : x_1 = L_1\}$

(35) $$\tau(u)\nu\nu = \tau_{11}(u) = (\lambda+2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_3}{\partial x_3} = 0\Big|_{x=L_1}$$

(36) $$\tau(u)\nu\chi = \tau_{31}(u) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\Big|_{x=L_1} = 0$$

Since

(37) $$u_1(L_1, x_3) = u_3(L_1, x_3) = 0,$$

(38) $$\frac{\partial u_1}{\partial x_3}\Big|_{x=L_1} = \frac{\partial u_3}{\partial x_3}\Big|_{x=L_1} = 0$$

From (35) – (38),

$$\frac{\partial u_1}{\partial x_1}\Big|_{x=L_1} = 0 \qquad \frac{\partial u_3}{\partial x_1} = 0$$

Thus, u_1, u_3 and its 1st derivatives vanish on Γ^R. The same argument holds on the other bries $\Gamma^L, \Gamma^B, \Gamma^T$ —

Then by the Cauchy - Kowaleski theorem

(19)

u_1 and u_3 vanish in a neighborhood of Γ except possible at the corners (and on Γ^f ?) -

Then by the unique continuation principle u vanishes on Ω -

(get a reference for unique continuation Principle -)

<u>Existence</u>: Consider the dual problem of (20) – (24-1):

(39) $-\omega^2 \rho u - \nabla\cdot\tau^*(u) = f$

(40) $\tau^*_{lm}(u) = 2\mu^*\varepsilon_{lm}(u) + \lambda^*\delta_{lm}\ \nabla\cdot u$

(41) $\tau^*(u^{(1)})\nu_{12} = \tau^*(u^{(2)})\nu_{12} = \tau^{*f}(u)\nu_{12}$, $\hat{\Gamma}^f$

(42) $u^{(1)} = u^{(2)}$, Γ^{ex}

(43) $\left(\tau^{f,*}(u)\nu_{12}\nu_{12}, \tau^{f,*}(u)\nu_{12}\chi_{12}\right) = D^*\begin{pmatrix}(u^{(2)}-u^{(1)})\cdot\nu_{12}\\ (u^{(2)}-u^{(1)})\cdot\chi_{12}\end{pmatrix}$, Γ^f

(44) $\left(-\tau^*(u)\nu\nu, -\tau^*(u)\nu\chi\right) = -i\omega B\begin{pmatrix}u\cdot\nu\\ u\cdot\chi\end{pmatrix}$

μ^*, λ^* are the conjugates of λ, μ.

$D^* = D_R - iD_I$.

Using again integration by parts we get the weak form

$$-\omega^2(\rho u, v) + \left(\left(M_R - iM_i\right)\tilde{\varepsilon}(u), \tilde{\varepsilon}(v)\right)_\Omega$$

$$+ \left\langle \left(D_R - i\,D_I\right)\begin{pmatrix}(u^2-u^1)\cdot\nu_{12}\\(u^2-u^1)\cdot\chi_{12}\end{pmatrix}, \begin{pmatrix}(v^2-v^1)\cdot\nu_{12}\\(v^2-v^1)\cdot\chi_{12}\end{pmatrix}\right\rangle_{\Gamma^f} \tag{45}$$

$$- i\omega\left\langle B\begin{pmatrix}u\cdot\nu\\u\cdot\chi\end{pmatrix}, \begin{pmatrix}v\cdot\chi\\v\cdot\chi\end{pmatrix}\right\rangle_\Gamma = (f, v)$$

TO SHOW UNIQUENESS FOR (45), set $f = 0$ in (45).
Then, taking imaginary parts in (45) $(D_I > 0,\ M_i > 0)$ we conclude that, (with the same argument than before)

$$u = 0.$$

and uniqueness holds for the adjoint problem.

Set:

$$\Lambda(u,v) = -\omega^2(\rho u, v)_\Omega + (M\tilde{\varepsilon}(u), \tilde{\varepsilon}(v))_\Omega$$

$$+ \left\langle D\begin{pmatrix}(u^2-u^1)\cdot\nu_{12}\\(u^2-u^1)\cdot\chi_{12}\end{pmatrix}, \begin{pmatrix}(v^2-v^1)\cdot\nu_{12}\\(v^2-v^1)\cdot\chi_{12}\end{pmatrix}\right\rangle_{\Gamma^f}$$

(46)

$$+ \left\langle i\omega B\begin{pmatrix}u\cdot\nu\\u\cdot\chi\end{pmatrix}, \begin{pmatrix}v\cdot\nu\\v\cdot\chi\end{pmatrix}\right\rangle_{\Gamma}$$

Then

$$2|\Lambda(u,u)| \geq \left(|Re(\Lambda(u,u))| + |Im(\Lambda(u,u))|\right)$$

$$\geq |Re(\Lambda(u,u))|$$

$$= -\omega^2(\rho u, u)_\Omega + (M_R\tilde{\varepsilon}(u), \tilde{\varepsilon}(u))_\Omega$$

$$+ \left\langle D_R\begin{pmatrix}(u^2-u^1)\cdot\nu_{12}\\(u^2-u^1)\cdot\chi_{12}\end{pmatrix}, \begin{pmatrix}(u^2-u^1)\cdot\nu_{12}\\(u^2-u^1)\cdot\chi_{12}\end{pmatrix}\right\rangle_{\Gamma^f}$$

(47)

$$\geq C_1\|\tilde{\varepsilon}(u)\|_0^2 - \omega^2\rho_{MAX}\|u\|_0^2$$

$$= (C_1+\omega^2\rho_{MAX})\|\tilde{\varepsilon}(u)\|_0^2 - (\omega^2\rho_{MAX}+C_1)\|u\|_0^2$$

Next, Korn's 2nd inequality gives

$$(48) \quad \sum_{l,m=1,3} \| \varepsilon_{lm}(v)\|_0^2 + \|v\|_0^2 \geq c_1 \|v\|_1^2$$

Using (48) in (47)

$$(49) \quad |\operatorname{Re}(\Lambda(u,u)| \geq C_3 \| u\|_1^2 - C_4 \|u\|_0^2$$

which is a Gårding inequality for our problem.

Also, since our boundary is piecewise smooth, the imbedding

$$H^1(\Omega) \xrightarrow{id} L^2(\Omega) \text{ is compact}$$

(Rellich Lemma)

Next, since the operator

$$H^1(\Omega) \longrightarrow L^2(\Gamma \cup \Gamma^f) \text{ is}$$

continuous, $\left[|v|_{0,\Gamma} \leq C \|v\|_{1,\Omega}^{1/2} \|v\|_{0,\Omega}^{1/2} \right]$

we have

$$\left| \langle u^{(2)} - u^{(1)}, v^{(2)} - v^{(1)} \rangle_{\Gamma^f} \right|$$

$$\le \| u^{(2)} - u^{(1)} \|_{L^2(\Gamma^f)} \| v^{(2)} - v^{(1)} \|_{L^2(\Gamma^f)}$$

$$\le C \| u^{(2)} - u^{(1)} \|_{0,\Omega}^{1/2} \| \cancel{u}^{(2)} - u^{(1)} \|_{1,\Omega}^{1/2}$$

$$\| v^{(2)} - v^{(1)} \|_{0,\Omega}^{1/2} \| v^{(2)} - v^{(1)} \|_{1,\Omega}^{1/2}$$

$$\le C \| u^{(2)} - u^{(1)} \|_{1,\Omega} \| v^{(2)} - v^{(1)} \|_{1,\Omega}.$$

[since $u^{(2)}$ inside Ω is $u|_{\Omega_2}$, $u^{(1)}$ inside Ω is $u|_{\Omega_1}$ etc.]

$$\le C \| u|_{\Omega_2} - u|_{\Omega_1} \|_{1,\Omega} \| v|_{\Omega_2} - v|_{\Omega_1} \|_{1,\Omega}$$

$$\le C \left[\left(\| u|_{\Omega_2} \|_{1,\Omega} + \| u|_{\Omega_1} \|_{1,\Omega} \right) \left(\| v|_{\Omega_2} \|_{1,\Omega} + \| v|_{\Omega_1} \|_{1,\Omega} \right) \right]$$

$$\le C \left[2 \| u \|_{1,\Omega} \left(2 \| v \|_{1,\Omega} \right) \right]$$

$$\le C_5 \| u \|_{1,\Omega} \| v \|_{1,\Omega}.$$

Then,

$$(50)\quad |\Lambda(u,v)| \leq C \, \|u\|_1 \, \|v\|_1 .$$

Thus, Λ is H^1-continuous and satisfies the Garding inequality (49). Since we have uniqueness for the adjoint problem, existence follows from the Fredholm alternative —