# Triangular and Tetrahedral Elements for Fractured Viscoelastic Media

Juan E. Santos[a,b,c,*], Fabio I. Zyserman[a,b], Patricia M. Gauzellino[b]

[a] *CONICET*
[b] *Departamento de Geofsica Aplicada, Fac. de Cs. Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, B1900FWA - La Plata, Argentina*
[c] *Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, Indiana, 47907-2067, USA*

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**Abstract**

We define the local matrices for triangular nonconforming elements

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## 1. Introduction

## 2. A continuous-time finite element method. Parallelepiped elements.

Let $\mathcal{T}^h$ be a quasiregular partition of $\Omega$ into parallelepipeds $\Omega_j$ of diameter bounded by $h$. Denote by $\xi_j$ and $\xi_{jk}$ the centroids of $\Gamma_j = \partial\Omega_j \cap \Gamma$ and $\Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k$, respectively.

To approximate the solid displacement vector we will employ the nonconforming finite element space $\mathcal{NC}^h$ presented in [24]. This choice is made based on the numerical dispersion analysis presented in [25], where it is shown that using this nonconforming space allows for using about half the number of points per wavelength as compared with standard bilinear elements to have a desired

*Corresponding author, fax: 0054-221-4236591
*Email addresses:* `santos@math.purdue.edu` (Juan E. Santos), `zyserman@fcaglp.unlp.edu.ar` (Fabio I. Zyserman), `gauze@fcaglp.unlp.edu.ar` (Patricia M. Gauzellino)

tolerance in numerical dispersion. See also [26] for the analysis of the numerical dispersion of waves in fluid-saturated poroelastic media when employing this non-conforming element to represent the solid displacement vector. The space $\mathcal{NC}^h$ is defined as follows. On the reference element $\widehat{R} = [-1,1]^3$ set

$$Q(\widehat{R}) = \mathrm{Span}\{1, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widetilde{\alpha}(\widehat{x}_1) - \widetilde{\alpha}(\widehat{x}_2), \widetilde{\alpha}(\widehat{x}_1) - \widetilde{\alpha}(\widehat{x}_3)\}, \quad \widetilde{\alpha}(z) = z^2 - \frac{5}{3}z^4.$$

Then let $\mathcal{NC}_j^h = [Q(\Omega_j)]^3$ and

$$\mathcal{NC}^h = \{v : v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h,\ v_j(\xi_{jk}) = v_k(\xi_{jk})\ \forall (j,k)\}.$$

The six local degrees of freedom are the values at the centroids $\xi_{jk}$ of the faces of $\Omega_j$.

To approximate the fluid displacement, we employ the vector part of the Raviart-Thomas-Nédélec space of zero order [18, 19], defined as follows.

$$\mathcal{M}^h = \{w \in H(\mathrm{div}, \Omega_p) : w|\Omega_j \in \mathcal{M}_j^h \equiv P_{1,0,0} \times P_{0,1,0} \times P_{0,0,1}\}.$$

The approximating properties of the finite element spaces defined above can be stated as follows. Let

$$[H_h^1(\Omega)]^3 = \{\psi : \psi|\Omega_j \in [H^1(\Omega_j)]^3\},$$

with $[H_h^1(\Omega_p)]^3$ defined in similar fashion. Also, if $\Gamma_{jk,p}$ denotes any inner interface $\Gamma_{jk}$ in $\Omega_p$ let

$$\widetilde{\Lambda}^h = \left\{\widetilde{\lambda}^h : \widetilde{\lambda}_{jk}^h = \mathrm{tr}\ _{\Gamma_{jk,p}}(\widetilde{\lambda}^h|_{\Omega_j}) \in [P_0(\Gamma_{jk,p})]^3 \equiv \widetilde{\Lambda}_{jk}^h, \quad \widetilde{\lambda}_{jk}^h + \widetilde{\lambda}_{kj}^h = 0\right\},$$

where $P_0(\Gamma_{jk,p})$ denotes the constant functions defined on $\Gamma_{jk,p}$.
*R*emark: Note that there are two copies of $[P_0(\Gamma_{jk,p})]^3$ assigned to each $\Gamma_{jk,p}$, one from $\Omega_j$ to $\Omega_k$ and another from $\Omega_k$ to $\Omega_j$.

Then we define the projections

$$P_h : [L^2(\Omega)]^3 \to \mathcal{W}^h : \qquad (P_h w - w, \varphi) = 0,\ \varphi \in \mathcal{W}^h, \tag{1}$$

$$R_h : [H^2(\Omega_p)]^3 \to \mathcal{NC}^h : \qquad (v_i^s - R_h v_i^s)(\xi) = 0,\ \xi = \xi_{jk} \text{ or } \xi_j, \tag{2}$$

$$\text{for} \quad v^s = (v_1^s, v_2^s, v_3^s),$$

$$Q_h : [H^1(\Omega_p)]^3 \to \mathcal{M}^h : \qquad \left\langle (v^f - Q_h v^f) \cdot \nu, 1 \right\rangle_B = 0;\ B = \Gamma_{jk,p} \text{ or } \Gamma_j, \tag{3}$$

$$S_h : [H^2(\Omega_p)]^3 \times H^1(\mathrm{div}; \Omega_p) \to \widetilde{\Lambda}^h : \qquad \langle \tau(v)\nu - S_h(v), 1 \rangle_B = 0, \tag{4}$$

$$v = (v^s, v^f), B = \Gamma_{jk,p} \text{ or } \Gamma_j.$$

Let us define the broken norms

$$\|v\|_{s,h,\Omega_p}^2 = \sum_{\Omega_j \subset \Omega_p} \|v\|_{s,\Omega_j}^2 .$$

The approximation properties of these operators can be stated as follows [19, 29]:

$$\|\nabla \times (\psi - \Pi_h \psi)\|_0 \le Ch \|\nabla \times \psi\|_1,\ \psi \in [H^1(\Omega)]^3,\ \nabla \times \psi \in H^1(\Omega), \tag{5}$$

$$\|P_h \varphi - \varphi\|_0 \le Ch \|\varphi\|_1, \qquad \forall \varphi \in H^1(\Omega), \tag{6}$$

$$\begin{aligned} &\Big[ \|v^s - R_h v^s\|_{\Omega_p} + h \|v^s - R_h v^s\|_{1,h,\Omega_p}^2 + h^2 \|v^s - R_h v^s\|_{2,h,\Omega_p}^2 \\ &\qquad + h^{\frac{1}{2}} \left( \sum_{\Omega_j \subset \Omega_p} \|v^s - R_h v^s\|_{0,\partial\Omega_j}^2 \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left( \sum_{\Omega_j \subset \Omega_p} \|\tau(v_j)\nu_j - S_h v_j\|_{0,\partial\Omega_j}^2 \right)^{1/2} \\ &\quad \le Ch^2 \left( \|v^s\|_{2,\Omega_p} + \|\nabla \cdot v^f\|_{1,\Omega_p} \right),\ v = \left(v^s, v^f\right) \in [H^2(\Omega_p)]^3 \times H^1(\mathrm{div}, \Omega_p), \end{aligned} \tag{7}$$

$$\|Q_h v^f - v^f\|_{0,\Omega_p} \le Ch \|\mathbf{v}^f\|_{1,\Omega_p},\ v^f \in [H^1(\Omega_p)]^3, \tag{8}$$

$$\|\nabla \cdot (v^f - Q_h v^f)\|_{0,\Omega_p} \le Ch \|\nabla \cdot v^f\|_{1,\Omega_p},\ v^f \in H^1(\mathrm{div}, \Omega_p). \tag{9}$$

Also note the orthogonality property for functions on $\mathcal{NC}^h$:

$$\left\langle v_j^s - v_k^s, 1 \right\rangle_{\Gamma_{jk}} = 0 \text{ for all interior interfaces } \Gamma_{jk}, \quad v^s \in \mathcal{NC}^h. \tag{10}$$

## 3. The case of tetrahedral elements

Let $\mathcal{T}^h(\Omega)$ be a nonoverlapping quasiregular partition of $\Omega = \Omega_p \cup \Omega_a$ into tetrahedral elements $\Omega_j$ of diameter bounded by $h$ and let $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$.

The nonconforming finite element space to approximate the solid displacement vector is defined as in [24] as follows: Let $\mathcal{N}C_j^h = [P_1(\Omega_j)]^3$ and

$$\mathcal{N}C^h = \{v\, : v_j = v|_{\Omega_j} \, \in \mathcal{N}C_j^h,\ v_j(\xi_{jk}) = v_k(\xi_{jk})\ \forall (j,k)\}.$$

The four local degres of freedom are the values at the centers $\xi_{jk}$ of the faces of $\Omega_j$. Next, let

$$S_1 = \{U : U = \alpha + \lambda \mathbf{x},\, \alpha \in [P_0]^3, \lambda \in P_0\} \tag{11}$$

and let

$$\mathcal{M}^h = \{v^f \in H(\mathrm{div}, \Omega_p)\, : v_j = v|_{\Omega_j}\, \in S_1\ \forall j\}. \tag{12}$$

The projections

$$\begin{aligned}
&R_h : [H^2(\Omega_p)]^3 \to [\mathcal{N}\mathcal{C}^h]^3,\\
&Q_h : [H^1(\Omega_p)]^3 \cup (H_h^1(\Omega_p))^3 \to \mathcal{M}^h,\\
&S_h : [H^2(\Omega_p)]^3 \times H^1(\mathrm{div}; \Omega_p) \to \widetilde{\Lambda}^h,
\end{aligned}$$

are defined identically than in (2), (3) and (4). The degrees of freedom

$$N_j(v^f) = \left\{ \left\langle v^f \cdot \nu, 1 \right\rangle_B,\, B \text{ any of the four faces of } \Omega_j \right\} \tag{13}$$

are $S_1$-solvent and conforming in $H(\mathrm{div}; \Omega_p)$ (see [20]), so that (3) uniquely defines $Q_h$.

## 4. 2D case. Rectangular and Triangular elements

In this section we will assume that all physical quantities describing our domains $\Omega_a$ and $\Omega_p$ are independent of the $x_2$-direction (i.e., $x_2$ is the symmetry axis)

First consider the case that $\mathcal{T}^h(\Omega)$ is a nonoverlapping quasiregular partition of $\Omega = \Omega_p \cup \Omega_a$ into rectangles $\Omega_j$ of diameter bounded by $h$ such that $\overline{\Omega} = \cup_{j=1}^J \overline{\Omega}_j$.

To approximate each component of the solid displacement vector we employ the nonconforming finite element space $\mathcal{NC}^h$ as in [24], while to approximate the fluid displacement vector we choose $\mathcal{M}^h$, the vector part of the Raviart-Thomas-Nédélec space [18, 19] of zero order. Thus,

$$\widehat{R} = [-1,1]^2, \qquad Q(\widehat{R}) = \mathrm{Span}\{1, \widehat{x}_1, \widehat{x}_3, \widetilde{\alpha}(\widehat{x}_1) - \widetilde{\alpha}(\widehat{x}_3)\}, \quad \widetilde{\alpha}(z) = z^2 - \frac{5}{3}z^4,$$

with the degrees of freedom being the values at the midpoint of each edge of $\widehat{R}$. Next let $\mathcal{NC}_j^h = [Q(\Omega_j)]^2$ and

$$\mathcal{NC}^h = \{v \, : v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h, \ v_j(\xi_{jk}) = v_k(\xi_{jk}) \ \forall (j,k)\}, \tag{14}$$

$$\mathcal{M}^h = \{w \in H(\mathrm{div}, \Omega_p) : w|\Omega_j \in \mathcal{M}_j^h \equiv P_{1,0}(\Omega_j) \times P_{0,1}(\Omega_j)\}. \tag{15}$$

Next consider the case in which $\mathcal{T}^h$ is a quasiregular partition of $\Omega$ into triangles $\Omega_j$ of diameter bounded by $h$.

Next, if

$$\mathcal{NC}_j^h = [P_1(\Omega_j)]^2,$$

the space $\mathcal{NC}^h$ to approximate each component of the solid displacement in $\Omega_p$ is defined as in the rectangular case in (14). with the local degrees of freedom being the values at the mid-point of the edges of each $\Omega_j$.

Finally if

$$S_1 = \{U : u = (a + b \ x_1, c + d \ x_3), \ a, b, c \text{ constants}\},$$

the space to approximate the fluid displacement vector is

$$\mathcal{M}^h = \{v^f \in H(\mathrm{div}, \Omega_p) : v^f|_{\Omega_j} \in S_1(\Omega_j) \ \forall j\}.$$

The local degrees of fredom for $\mathcal{M}^h$ are the values of normal components at the mid-points of each edge of the triangle $\Omega_j$.

The approximating properties of the 2D finite element spaces defined above are the same than those stated in the 3D case.

Next we consider the SHTE-mode. In this case we will employ the spaces $\mathcal{V}^h$ and $\mathcal{W}^h$ to approximate the magnetic vector field $H$ and the scalar field $E_2$.

Also, to approximate the solid displacement $u_2^s$ in $\Omega_p$ we employ the (scalar) nonconforming finite element space $\mathcal{NC}^h$ (*i.e.* in (14) change the definition of $\mathcal{NC}_j^h$ to $\mathcal{NC}_j^h = Q(\Omega_j)$ (resp. $\mathcal{NC}_j^h = P_1(\Omega_j)$, depending on whether we have rectangular of triangular elements). To approximate the fluid displacement $u_2^s$ we choose the space of piecewise constants over the restriction of the partition $\mathcal{T}^h(\Omega)$ to $\Omega_p$. Thus, for the fluid displacement, the space $\widehat{\mathcal{M}}_j^h$ is defined as

$$\widehat{\mathcal{M}}_j^h = \{w : w|_{\Omega_j} \in P_0(\Omega_j)\}, \qquad \widehat{\mathcal{M}}^h = \{w \in L^2(\Omega) : w_j = w|_{\Omega_j} \in \widehat{\mathcal{M}}_j^h\}$$

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Table 1: Parameters characterizing the model used in Example 1.

Figure 1: Scheme of the domain and boundaries used in this work.

Figure 2: Trace of solid displacements measured on the surface. "A", "B" and "C" are direct arrivals of waves originated at the 500 mts, 700 m and 800 m depth interfaces respectively. "D" is a reflection on the lower boundary of the 200 mts width layer of a downwards travelling wave originated on the top boundary of the same layer.

Table 2: Material parameters for the second model

Table 3: Numerical estimate of the order of approximation of the studied finite element method, 1D case