FREDHOLM Alternative.

Theorem 14.2 (Friedman)

Let T a compact operator in a Hilbert space H and consider the equations

(1) $x - Tx = f$

(2) $y - T^*y = g$

Then the following alternative holds: either

i) $\exists!$ solution of (1) and (2) for any f, $g \in H$

ii) The equation

(3) $x - Tx = 0$.

has non-trivial solutions.

If ii) is true, then the dimension of the space of solutions of (3) is finite and equal to the dimension of the space N^* of solutions ~~y~~ y of $y - T^* y = 0$.

Also,

$$x - Tx = f$$

has solution, (not unique) if and only if

$$(f, x) = 0 \quad \forall \; x \in N^*.$$

<u>Application</u>: We have the problem

$$\Lambda(\hat{u}, v) =$$

$$-\omega^2(\hat{u}, v) + c^2(\nabla\hat{u}, \nabla v) + i\omega c\langle \hat{u}, v\rangle_\Gamma = (\hat{f}, v),$$

$v \in H^1(\Omega)$

$$|\Lambda(v,v)| \geq \frac{c^2}{\sqrt{2}} \|v\|_1^2 - \frac{(c^2+\omega^2)}{\sqrt{2}} \|v\|_0^2$$

$$= c_0 \|v\|_1^2 - \lambda_0 \|v\|_0^2$$

Let T_0 the operator associated with

$\Lambda(\hat{u}, v) + \lambda_0(\hat{u}, v)$, so that

$$T_0 : L^2(\Omega) \longrightarrow H^1(\Omega)$$

$$f \longrightarrow T_0 f = \hat{u}$$

NOTE 8

$$(\Lambda + \lambda_0)(\hat{u}, v) = (f, v)$$

$$(\Lambda + \lambda_0)(T_0 f, v) = (f, v)$$

Taking $v = T_0 f$,

$$\|T_0 f\|_1 \leq C_0^{-1} \|f\|_0$$

$\rightarrow$ T_0 is bounded. Consider T_0 as a map from $L^2(\Omega)$ in $L^2(\Omega)$

$$L^2(\Omega) \xrightarrow{T_0} H_1(\Omega) \xrightarrow[\text{Lemma Rellich}]{id} L^2(\Omega)$$

is compact because id is compact and T_0 is continuous.

Now recall we need to solve

$$\Lambda(\hat{u}, v) = (f, v) \quad \forall v \in H^1(\Omega) \iff$$

$$\Lambda(\hat{u}, v) + \lambda_0 (\hat{u}, v) = (f, v) + \lambda_0 (\hat{u}, v)$$

$$\iff (\Lambda + \lambda_0)(\hat{u}, v) = (f + \lambda_0 \hat{u}, v) \quad \forall v \in H^1$$

But we know that the solution operator satisfies

$$(\Lambda + \lambda_0)(T_0 f, v) = (f, v)$$

Then,

$$\hat{u} = T_0 (f + \lambda_0 \hat{u}) \iff$$

$$\hat{u} - \lambda_0 T_0 (\hat{u}) = T_0 f = f_1$$

Then,

$$\Lambda(\hat{u}, v) = (f, v) \iff$$

$$\hat{u} - \lambda_0 T_0 \hat{u} = T_0 f$$

This equation has a (~~unique~~) solution $\iff$

$$(T_0 f, v) = 0 \quad \forall\, v \in$$

$$N^* = \{ y : y - \lambda_0 T_0^* y = 0 \}$$

$$\cancel{\Lambda(\hat{u}, v) = (f, v)}$$

$$\cancel{\Lambda(T_0 f, v) = (f, v)}$$

$$\cancel{\Lambda(T_0 f, v) = (f, v)}$$

$$\cancel{= \Lambda^*(v, T_0 f) = (v, f)}$$

Consider the adjoint of Λ : NOTE 8 (5) (8)

$$\Lambda^*(u,v) = \overline{\Lambda(v,u)}$$

Then,

$$|\Lambda^*(u,u)| \geq C_0 \|u\|_1^2 - \lambda_0 \|u\|_0^2$$

Consider T_1 the solution operator for

$$(\Lambda+\lambda_0)^*(u,v) = (f,v) \quad , u = T_1 f \rightarrow$$

$$(\Lambda+\lambda_0)^*(T_1 f, v) = (f,v)$$

$$\overline{(\Lambda+\lambda_0)(v, T_1 f)} = (f,v) = \overline{(v,f)}$$

$$\Rightarrow \quad (\Lambda+\lambda_0)(v, T_1 f) = (v, f)$$

Now the choice $v = T_0 f$ in gives

$$(\Lambda+\lambda_0)(T_0 f, T_1 f) = (T_0 f, f)$$

and the choice $v = T_1 f$ in

$$(\Lambda+\lambda_0)(T_0 f, v) = (f, v) \quad \text{gives}$$

$$(\Lambda+\lambda_0)(T_0 f, T_1 f) = (f, T_1 f)$$

$$\longrightarrow \quad (T_0 f, f) = (f, T_1 f)$$

so that $T_1 = T_0^*$.

Now we observe that u is solution of $\boxed{\text{NOTE 8}}$ (6) 34

$$\Lambda^*(u,v) = (f,v) \iff \overline{\Lambda(v,u)} = \overline{(v,f)}$$

$$\cancel{\Lambda^*} \iff \Lambda(v,u) = (v,f) \iff$$

$$(\Lambda + \lambda_0)(v,u) = (v, f + \lambda_0 u) \iff$$

$$\overline{(\Lambda+\lambda_0)}^*(v,u) = (f + \lambda_0 u, v) = \overline{(v, f+\lambda_0 u)}$$

$$\|$$

$$(\Lambda+\lambda_0)^*(u,v) = (f+\lambda_0 u, v) \iff$$

$$\cancel{T_0^* u = T_1 u = T(f + \lambda_0 u)} \iff$$

$$u = T_0^*(f + \lambda_0 u) \iff$$

$$u - \lambda_0 T_0^* u = T_0^* f$$

Then, for $f = 0$,

$$u - \lambda_0 T_0^* u = 0 \iff \Lambda^*(u,v) = 0$$

$\iff u = 0$ as we proved before. Then,

$(T_0 f, v) = 0 \quad \forall \; v \in N^* = \{0\}$ is true and

$\hat{u} - \lambda_0 T_0^* u = T_0 f$ has ! solution.

(since $N^* = \{0\}$).

<u>Note</u>: $(f, v) = 0 \quad \forall\, v \in N^* \iff$

$(T_0 f, v) = 0 \quad \forall\, v \in N^*$

In fact:

$(T_0 f, v) = (f, T_0^* v)$. But since

$v \in N^* = \{ v : v - \lambda_0 T_0^* v = 0 \} \rightarrow$

$T_0^* v = \frac{1}{\lambda_0} v$ }

$\rightarrow$

$(T_0 f, v) = (f, T_0^* v) = (f, \frac{1}{\lambda_0} v)$

Then, $(T_0 f, v) = 0 \quad v \in N^* \iff (f, v) = 0 \quad v \in N^*$.