# Fractional Taylor Series for Caputo Fractional Derivatives. Construction of Numerical Schemes

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**Abstract**

A fractional power series expansion is obtained for Caputo fractional derivative as a generalization of Taylor power series. The series obtained are independent from the point in which fractional derivative is defined. This is used to obtain Euler and Taylor numerical schemes to solve ordinary fractional differential equations. Finally the methods derived are applied to integrate numerically a first and a second order ordinary fractional differential equation using schemes of order $\alpha$, $2\alpha$ and $4\alpha$ for both equations.

*Key words:* Fractional derivative, Taylor series, numerical Euler method, numerical Taylor methods of general order.
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## 1 Introduction

Fractional Calculus is a tool of Mathematical Analysis applied to the study of integrals and derivatives of arbitrary order, not only fractional but also real. Commonly this fractional integrals and derivatives are not known for many scientist and up to recent years has been used only in a pure mathematical context. But during this last decades this integrals and derivatives have been applied in many context of sciences.

Nowadays it is impossible to describe a viscoelastic process without using a fractional derivative. Fractional derivatives have also been used in Anomalous Diffusion description, where this derivatives can explain sub-diffusive

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and super-diffusive phenomena observed in real systems. Other applications are Electromagnetic Theory, Circuit Theory, Biology, Atmospheric Physics, etc. A wide description of such application together with an exhaustive definition of all these integrals and derivatives can be found in the books [8] [7], [4].

The study of this fractional derivatives has two great difficulties, they do not have a clear significance since they can not be associated to a tangent direction as the usual first derivative is. This circumstance make impossible any intuitive *a priori* analysis of the problem. The second, but not less, difficulty is their complex integro-differential definition, which make a simple manipulation with standard integer operators, a complex operation that should be made carefully. A clear example of such situation is that summation rule is no fulfilled and so the $\alpha$ derivative of the $\beta$ derivative of a function is, in general, not equal to the $\alpha + \beta$ derivative of such function.

It is not necessary at all to comment how important is the well known Taylor power series of a given function in the history of mathematics, physics and many other sciences. Power series have become a fundamental tool in the study of elementary functions and also other not so elementary as can be checked in any book of analysis.

In Physics, Chemistry and many other Sciences this power expansions has allowed scientist to make an approximate study of many systems, neglecting higher order terms around the equilibrium point. This is a fundamental tool to linearize a problem which guaranties easy analysis.

In recent decades the power series expansion has been widely used in computational science obtaining an easy approximate of a function [1], numerical schemes to integrate a Cauchy problem [2], or gaining knowledge about the singularities of a function by comparing two different Taylor series expansions around different points [11] [12] [3].

In the context of the fractional derivatives, Taylor series has been developed for different definitions [5], but in general they consist on series in powers of $x^{n+\alpha}$, which in fact is not a purely fractional serie. Fractional Taylor serie has been developed for Riemann-Liouville derivative [9], and in the present work a similar study has been made for Caputo fractional derivative defined as

$$ {}_c\mathrm{D}_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\mathrm{D} f(t)}{(x-t)^\alpha} dt = \mathrm{I}_a^{1-\alpha} \mathrm{D} f(x) \tag{1} $$

where the fractional integral $\mathrm{I}_a^\alpha$ is defined as

$$ \mathrm{I}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt. \tag{2} $$

The use of this definition of the fractional derivative is justified since it has *"good physical properties"* [7] as, for example that the derivative of a constant is zero or that Cauchy problems requires initial conditions formulated in terms of integer order derivatives interpreted as initial position, initial velocity, etc.

As it has previously mentioned, elementary manipulations with entire order derivatives as the summation of index, Leibniz rule or the chain rule are not valid for Caputo fractional derivative as also happen with Riemann-Liouville definition. Taking this into account, and with no possibility of confusion it will be used the following convection for the sequential derivative in order to simplify notation:

$$ {}_c\mathrm{D}_a^{m\alpha} = \underbrace{{}_c\mathrm{D}_a^{\alpha}\,{}_c\mathrm{D}_a^{\alpha}\,{}_c\mathrm{D}_a^{\alpha} \cdots {}_c\mathrm{D}_a^{\alpha}}_{m \text{ times}}, \tag{3} $$

which is not equivalent to the derivative of order $m\alpha$. This is not the case for the integral operator $\mathrm{I}_a^{\alpha}$ for which $\mathrm{I}_a^{\alpha}\mathrm{I}_a^{\beta} = \mathrm{I}_a^{\alpha+\beta}$.

The present work is organized as follows. In section 2 a fractional McLaurin serie expansion is obtained for a general function in terms of the Caputo fractional derivative. The name of *fractional MacLaurin serie* is used since this is centered in the point $a$ used also to define the fractional integral and derivative. In section 3 this power serie is generalized and centered in any other point $a_1 > a$ within the radius of convergence of the power serie. In section 4 the general fractional Euler and Taylor method are developed, and those methods are applied to different Fractional Differential Equation (FDE) of First order in section 5.1 and second order in section 5.2. Finally conclusions and further lines are exposed.

## 2 Fractional MacLaurin power serie expansion

In the present section a fractional power serie for a function $f(x)$ in terms of its fractional derivatives ${}_c\mathrm{D}_a^{\alpha} f(a)$ is obtained and this is called MacLaurin power serie since the point in which the serie is constructed $x = a$ coincides with the point in which the Caputo derivative ${}_c\mathrm{D}_a^{\alpha}$ is defined. This dependence on the point in which the derivative is calculated is not useful at all since it make impossible, for example, to obtain directly a numerical Euler scheme for a Cauchy problem. It would be desirable using a serie in a point $a_1$ different to $a$ in order to express the solution at point $x_{n+1}$ as a function of the solution in $x_n$ as this is done in numerical schemes for Cauchy problems. But buildings can not be started by the roof, *i. e.* a MacLaurin serie is necessary to develop more sophisticated expressions.

In order to obtain some interesting results first it is necessary to give some definitions:

**Definition 2.1** *Let $\alpha \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}$ an interval such that $a \in \Omega$, $a \leq x$ $\forall x \in \Omega$. Then the following set of functions are defined:*

$$ {}_a\mathcal{I}_\alpha = \{f \in \mathbf{C}(\Omega) : \mathrm{I}_a^\alpha f(x) \textit{ exist and is finite in } \Omega\} $$

$$ {}_a\mathcal{D}_\alpha = \{f \in \mathbf{C}(\Omega) : {}_c\mathrm{D}_a^\alpha f(x) \textit{ exist and is finite in } \Omega\} $$

Also, some interesting properties of the Caputo's definition of the fractional derivative would be necessary in order to develop a power expansion similar to the Taylor serie using fractional powers of the independent variable $x$. Similar properties can be described for the Riemann-Liouville derivative [9]. First property correspond to the application of the integral operator $\mathrm{I}_a^\alpha$ to a fractional derivative of a function.

**Proposition 2.1** *Let $\alpha \in (0, 1]$ and $f(x) \in {}_a\mathcal{D}_\alpha$ then*

$$ \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = f(x) - f(a). \tag{4} $$

This can be demonstrated applying equation (1) so $\mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = \mathrm{I}_a^1 \mathrm{D} f(x)$ since the integral operators holds the summation of index. Then property 2.1 is obtained directly.

The next proposition is a zero order approximation of a function in terms of its first $\alpha$ derivative.

**Proposition 2.2** *A function $f(x) \in {}_a\mathcal{I}_\alpha$ with the same conditions of proposition 2.1 can be written as*

$$ f(x) = f(a) + {}_c\mathrm{D}_a^\alpha f(\xi) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} \tag{5} $$

*for any $\xi \in (a, x)$, being ${}_c\mathrm{D}_a^\alpha f(x)$ continuous in $[a, x]$.*

In order to obtain expression (5) it is necessary to start from the same point of proposition 2.1

$$ \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_c\mathrm{D}_a^\alpha f(t) dt \tag{6} $$

and applying the Mean Value Theorem for the integral

$$ \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = \frac{{}_c\mathrm{D}_a^\alpha f(\xi)}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} dt = \frac{{}_c\mathrm{D}_a^\alpha f(\xi)}{\Gamma(\alpha+1)} (x-a)^\alpha \tag{7} $$

for any $\xi \in (a, x)$. Substituting expression (4) the desired proposition 2.2 is obtained.

More sophisticated properties with cumulative integrals and differential operators can be obtained and would be the starting point for a higher order approximations.

**Proposition 2.3** *Let $\alpha \in (0,1]$, $m, n \in \mathbb{N}$ and $f(x)$ an analytic function in $\Omega \subset \mathbb{R}$, $f(x) \in {}_a\mathcal{D}_{(m+1)\alpha}$ with $a, x \in \Omega$, $a < x$. Then*

$$\mathrm{I}_a^{m\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(x) - \mathrm{I}_a^{(m+1)\alpha}\,{}_c\mathrm{D}_a^{(m+1)\alpha} f(x) = {}_c\mathrm{D}_a^{m\alpha} f(a)\frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)} \tag{8}$$

Since ${}_c\mathrm{D}_a^{k\alpha} f(a)$ is a constant value it can be written

$$\mathrm{I}_a^{m\alpha}\,{}_c\mathrm{D}_a^{n\alpha} f(a) = {}_c\mathrm{D}_a^{n\alpha} f(a)\mathrm{I}_a^{m\alpha}1 = {}_c\mathrm{D}_a^{n\alpha} f(a)\frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{9}$$

On the other hand, applying proposition 2.1

$$\mathrm{I}_a^{(m+1)\alpha}\,{}_c\mathrm{D}_a^{(m+1)\alpha} f(x) = \mathrm{I}_a^{m\alpha}\mathrm{I}_a^{\alpha}\,{}_c\mathrm{D}_a^{\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(x) = \mathrm{I}_a^{m\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(x) - \mathrm{I}_a^{m\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(a), \tag{10}$$

and then equation (8) is obtained by direct substitution.

As it has been commented before, this proposition would be the initial point to construct the power serie of a sufficiently well behaved function $f(x)$. In order to obtain a proper expression for this serie another proposition is needed.

**Proposition 2.4** *In the same conditions of proposition 2.3 and being $m, k \in \mathbb{N}$, ${}_c\mathrm{D}_a^{m\alpha} f(x)$ continuous in $[a, x]$, and ${}_c\mathrm{D}_a^{m\alpha} f(x) \in {}_a\mathcal{I}_{k\alpha}$,*

$$\mathrm{I}_a^{k\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(x) = {}_c\mathrm{D}_a^{m\alpha} f(\xi)\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \tag{11}$$

*for any $\xi \in (a, x)$.*

Equation (11) is a generalization of corresponding equation from proposition 2.2. Using the definition of the integral operators

$$\mathrm{I}_a^{k\alpha}\,{}_c\mathrm{D}_a^{m\alpha} f(x) = \frac{1}{\Gamma(k\alpha)}\int_a^x (x-t)^{k\alpha-1}\,{}_c\mathrm{D}_a^{m\alpha} f(t)dt \tag{12}$$

and applying the Mean Value Theorem for ${}_c\mathrm{D}_a^{m\alpha} f(x)$ equation (11) is obtained directly.

Once demonstrated the last result, it is possible to obtain the desired expression for the fractional power serie for a function in terms of its Caputo fractional derivatives of order $\alpha \in (0,1]$.

**Theorem 2.1** *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$ and $f(x)$ a continuous function in $[a,b]$ satisfying the following conditions:*

*(1) $\forall j = 1, ..., n, {}_c\mathrm{D}_a^{j\alpha} f \in \mathbf{C}([a,b])$ and ${}_c\mathrm{D}_a^{j\alpha} f \in {}_a\mathcal{I}_\alpha([a,b])$.*
*(2) ${}_c\mathrm{D}_a^{(n+1)\alpha} f(x)$ is continuous on $[a,b]$.*

*Then $\forall x \in [a,b]$,*

$$f(x) = \sum_{j=0}^{n} {}_c\mathrm{D}_a^{j\alpha} f(a) \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)} + R_n(x,a), \tag{13}$$

*being*

$$R_n(x,a) = {}_c\mathrm{D}_a^{(n+1)\alpha} f(\xi) \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}, \qquad a \le \xi \le x \tag{14}$$

*Proof:* Using proposition 2.3 for different values of $m$ it is possible to write

- Case $m = 0 \to f(x) - \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = f(a)$. Then $f(x) = f(a) + \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x)$.
- Case $m = 1 \to \mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) - \mathrm{I}_a^{2\alpha} {}_c\mathrm{D}_a^{2\alpha} f(x) = {}_c\mathrm{D}_a^\alpha f(a) \frac{(x-a)}{\Gamma(\alpha+1)}$ and then $\mathrm{I}_a^\alpha {}_c\mathrm{D}_a^\alpha f(x) = {}_c\mathrm{D}_a^\alpha f(a) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} + \mathrm{I}_a^{2\alpha} {}_c\mathrm{D}_a^{2\alpha} f(x)$
- Case $m = 2 \to \mathrm{I}_a^{2\alpha} {}_c\mathrm{D}_a^{2\alpha} f(x) - \mathrm{I}_a^{3\alpha} {}_c\mathrm{D}_a^{3\alpha} f(x) = {}_c\mathrm{D}_a^{2\alpha} f(a) \frac{(x-a)^{2\alpha}}{\Gamma(2\alpha+1)}$. Then $\mathrm{I}_a^{2\alpha} {}_c\mathrm{D}_a^{2\alpha} f(x) = {}_c\mathrm{D}_a^{2\alpha} f(a) \frac{(x-a)^{2\alpha}}{\Gamma(2\alpha+1)} + \mathrm{I}_a^{3\alpha} {}_c\mathrm{D}_a^{3\alpha} f(x)$
- General case $\to \mathrm{I}_a^{m\alpha} {}_c\mathrm{D}_a^{m\alpha} f(x) = {}_c\mathrm{D}_a^{m\alpha} f(a) \frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)} + \mathrm{I}_a^{(m+1)\alpha} {}_c\mathrm{D}_a^{(m+1)\alpha} f(x)$.

Substituting repetitively the resulting terms up to order $n$ the following serie is obtained

$$f(x) = \sum_{j=0}^{n} {}_c\mathrm{D}_a^{j\alpha} f(a) \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)} + \mathrm{I}_a^{(n+1)\alpha} {}_c\mathrm{D}_a^{(n+1)\alpha} f(x), \tag{15}$$

where the last term correspond to the integral form of the reminder term, so it is possible to write

$$R_n(x,a) = \mathrm{I}_a^{(n+1)\alpha} {}_c\mathrm{D}_a^{(n+1)\alpha} f(x). \tag{16}$$

Finally equation (14) is obtained applying proposition 2.4 with $m = k = n+1$ to equation (16).

## 3 Fractional Taylor power serie expansion

In the previous section theorem 2.1 allows to obtain a fractional power serie for a function in terms of its Caputo fractional derivatives evaluated at $a$, which is, in some sense, the initial point of the independent variable $x$. The ideal situation would be to obtain a similar serie with the derivatives evaluated in any other point $a_1 > a$, so the expansion can be constructed independently from the starting point $a$.

The next theorem makes real this possibility at least up to fourth order and gives an idea about how to obtain higher order approximations.

**Theorem 3.1** *Let $\alpha \in (0,1]$ and $f(x)$ a continuous function in $[a,b]$ satisfying the following conditions:*

*(1) $\forall j = 1, ..., 8, {}_c\mathrm{D}_a^{j\alpha} f \in \mathbf{C}([a,b])$ and ${}_c\mathrm{D}_a^{j\alpha} f \in {}_a\mathcal{I}_\alpha([a,b])$.*
*(2) ${}_c\mathrm{D}_a^{9\alpha} f(x)$ is continuous on $[a,b]$.*

*Let $a_1 \in (a,b]$. Then $\forall x \in [a,b]$,*

$$\begin{aligned} f(x) = f(a_1) + {}_c\mathrm{D}_a^{\alpha} f(a_1)\frac{\Delta_1}{\Gamma(\alpha+1)} + {}_c\mathrm{D}_a^{2\alpha} f(a_1)\frac{\Delta_2}{\Gamma(2\alpha+1)} + \\ + {}_c\mathrm{D}_a^{3\alpha} f(a_1)\frac{\Delta_3}{\Gamma(3\alpha+1)} + {}_c\mathrm{D}_a^{4\alpha} f(a_1)\frac{\Delta_4}{\Gamma(4\alpha+1)} + R_4(x,a_1,a), \end{aligned} \tag{17}$$

*being $\Delta_1$, $\Delta_2$, $\Delta_3$ and $\Delta_4$ the differences given by*

$$\begin{aligned} &\Delta_1 = [H^\alpha - L^\alpha] \\ &\Delta_2 = \left[H^{2\alpha} - L^{2\alpha} - \tfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} L^\alpha \Delta_1\right] \\ &\Delta_3 = \left[H^{3\alpha} - L^{3\alpha} - \tfrac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} L^\alpha \Delta_2 - \tfrac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} L^{2\alpha} \Delta_1\right] \\ &\Delta_4 = \left[H^{4\alpha} - L^{4\alpha} - \tfrac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} L^\alpha \Delta_3 - \tfrac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)} L^{2\alpha} \Delta_2 - \tfrac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} L^{3\alpha} \Delta_1\right] \end{aligned} \tag{18}$$

*and $H = (x-a)$, $L = (a_1 - a)$. $R_4(x,a_1,a)$ is the reminder term.*

*Proof:* Since $f(x)$ and ${}_c\mathrm{D}_a^{j\alpha} f(x)$ for $j = 1,2,3,4$ fulfill the conditions of theorem 2.1, there must be fractional power series expansions for $f(x)$, ${}_c\mathrm{D}_a^{\alpha} f(x)$,

${}_c\mathrm{D}_a^{2\alpha} f(x)$, ${}_c\mathrm{D}_a^{3\alpha} f(x)$, ${}_c\mathrm{D}_a^{4\alpha} f(x)$, and using them at $x = a_1$ we have:

$$
\begin{aligned}
f(a) = f(a_1) - {}_c\mathrm{D}_a^{\alpha} f(a) \tfrac{L^{\alpha}}{\Gamma(\alpha+1)} - {}_c\mathrm{D}_a^{2\alpha} f(a) \tfrac{L^{2\alpha}}{\Gamma(2\alpha+1)} - \\
- {}_c\mathrm{D}_a^{3\alpha} f(a) \tfrac{L^{3\alpha}}{\Gamma(3\alpha+1)} - {}_c\mathrm{D}_a^{4\alpha} f(a) \tfrac{L^{4\alpha}}{\Gamma(4\alpha+1)} - R_{4,0}
\end{aligned}
$$

$$
\begin{aligned}
{}_c\mathrm{D}_a^{\alpha} f(a) = {}_c\mathrm{D}_a^{\alpha} f(a_1) - {}_c\mathrm{D}_a^{2\alpha} f(a) \tfrac{L^{\alpha}}{\Gamma(\alpha+1)} - {}_c\mathrm{D}_a^{3\alpha} f(a) \tfrac{L^{2\alpha}}{\Gamma(2\alpha+1)} - \\
- {}_c\mathrm{D}_a^{4\alpha} f(a) \tfrac{L^{3\alpha}}{\Gamma(3\alpha+1)} - {}_c\mathrm{D}_a^{5\alpha} f(a) \tfrac{L^{4\alpha}}{\Gamma(4\alpha+1)} - R_{4,1}
\end{aligned}
$$

$$
\begin{aligned}
{}_c\mathrm{D}_a^{2\alpha} f(a) = {}_c\mathrm{D}_a^{2\alpha} f(a_1) - {}_c\mathrm{D}_a^{3\alpha} f(a) \tfrac{L^{\alpha}}{\Gamma(\alpha+1)} - {}_c\mathrm{D}_a^{4\alpha} f(a) \tfrac{L^{2\alpha}}{\Gamma(2\alpha+1)} - \\
- {}_c\mathrm{D}_a^{5\alpha} f(a) \tfrac{L^{3\alpha}}{\Gamma(3\alpha+1)} - {}_c\mathrm{D}_a^{6\alpha} f(a) \tfrac{L^{4\alpha}}{\Gamma(4\alpha+1)} - R_{4,2}
\end{aligned}
\qquad (19)
$$

$$
\begin{aligned}
{}_c\mathrm{D}_a^{3\alpha} f(a) = {}_c\mathrm{D}_a^{3\alpha} f(a_1) - {}_c\mathrm{D}_a^{4\alpha} f(a) \tfrac{L^{\alpha}}{\Gamma(\alpha+1)} - {}_c\mathrm{D}_a^{5\alpha} f(a) \tfrac{L^{2\alpha}}{\Gamma(2\alpha+1)} - \\
- {}_c\mathrm{D}_a^{6\alpha} f(a) \tfrac{L^{3\alpha}}{\Gamma(3\alpha+1)} - {}_c\mathrm{D}_a^{7\alpha} f(a) \tfrac{L^{4\alpha}}{\Gamma(4\alpha+1)} - R_{4,3}
\end{aligned}
$$

$$
\begin{aligned}
{}_c\mathrm{D}_a^{4\alpha} f(a) = {}_c\mathrm{D}_a^{4\alpha} f(a_1) - {}_c\mathrm{D}_a^{5\alpha} f(a) \tfrac{L^{\alpha}}{\Gamma(\alpha+1)} - {}_c\mathrm{D}_a^{6\alpha} f(a) \tfrac{L^{2\alpha}}{\Gamma(2\alpha+1)} - \\
- {}_c\mathrm{D}_a^{7\alpha} f(a) \tfrac{L^{3\alpha}}{\Gamma(3\alpha+1)} - {}_c\mathrm{D}_a^{8\alpha} f(a) \tfrac{L^{4\alpha}}{\Gamma(4\alpha+1)} - R_{4,4},
\end{aligned}
$$

being $R_{4,i}$ the corresponding reminder terms of the fourth order serie of every ${}_c\mathrm{D}_a^{i\alpha} f(x)$ at $x = a_1$.

Then substituting every serie from equation (19) in the fourth order serie of $f(x)$ and grouping corresponding terms, equation (17) with differences (18) is obtained.

The reminder term $R_4(x, a_1, a)$ results as a combination of the reminder terms of expressions (19) and higher order terms with derivatives of order $5\alpha$, $6\alpha$, $7\alpha$ and $8\alpha$ evaluated at $x = a$. Explicit form is omitted because of its complex form, but can be directly computed following indications given above.

The method outlined above can be extended up to any order obtaining series similar to equation (17) for which the difference $\Delta_k$ would be given by

$$
\Delta_k = H^{k\alpha} - L^{k\alpha} - \Gamma(k\alpha+1) \sum_{j=1}^{k-1} \frac{L^{j\alpha} \Delta_{k-j}}{\Gamma(j\alpha+1)\Gamma\left((k-j)\alpha+1\right)}, \qquad (20)
$$

and the corresponding serie of order $n$ would be

$$
f(x) = \sum_{j=0}^{n} {}_c\mathrm{D}_a^{j\alpha} f(a_1) \frac{\Delta_j}{\Gamma(j\alpha+1)} + R_n(x, a_1, a) \qquad (21)
$$

with $R_n(x, a_1, a)$ being the reminder term of order $n\alpha$.

## 4 Application to the construction of numerical schemes

One application of Taylor power serie is the construction of numerical schemes to solve ODE Cauchy problems as is done in an elementary course of numerical analysis [2]. In the present section a similar approach is going to be followed in order to obtain numerical schemes to integrate the Cauchy problem for Fractional Differential Equation (FDE).

Let start with a Cauchy problem for a general FDE given by:

$$\begin{cases} {}_c\mathrm{D}_0^\alpha y(t) = f(y) \\ y(0) = y_0 \end{cases} \tag{22}$$

with $\alpha \in (0, 1]$ and setting the origin at $t = a = 0$ for simplicity. It has been set $f(y)$ also for simplicity since in the problems studied below there is no explicit dependence on $t$ in the FDE but more general cases can be considered setting $f(t, y)$ instead.

To obtain different integration schemes for the problem (22) it is necessary to transform the continuous variable $t$ in a discrete analogue $t_j = jh$ with a fixed step $h$.

Using the fractional Taylor expansion given at theorem 3.1 up to the corresponding order it will make possible to obtain the schemes.

### *4.1 Fractional Euler Method*

The Euler method uses the first order Taylor serie to obtain the integration scheme. It is the easiest numerical integration method but the errors are usually very big so is not frequently used.

If we suppose that $y(t)$ is a solution of problem (22), then according to the fractional Taylor serie for $y(t)$ at $t_i$ it can be written

$$y(t_{i+1}) = y(t_i) + {}_c\mathrm{D}_0^\alpha y(t_i) \frac{\Delta_1}{\Gamma(\alpha + 1)} + R_1(t, t_i, 0), \tag{23}$$

with $\Delta_1 = h^\alpha \left((i+1)^\alpha - i^\alpha\right)$. Using the FDE of problem (22)

$$y(t_{i+1}) = y(t_i) + \frac{f(y(t_i))h^\alpha}{\Gamma(\alpha+1)} \left((i+1)^\alpha - i^\alpha\right) + R_1(t, t_i, 0) \tag{24}$$

Fractional Euler method consist in approximate the function $y(t)$ at $t_i$, $i = 0, ..., N$ for a set of values $w_i \approx y(t_i)$ neglecting the reminder term. Then

$$\begin{aligned} w_{i+1} &= w_i + h^\alpha \frac{f(w_i)}{\Gamma(\alpha+1)} \left((i+1)^\alpha - i^\alpha\right) \\ w_0 &= y_0, \end{aligned} \tag{25}$$

for every $i = 0, ..., N$.

Since the reminder term $R_1(t_{i+1}, t_i, 0)$ is of order $h^{2\alpha}$, the local truncation error is of order $h^\alpha$ and since $\alpha \in (0, 1]$ it would be $h^\alpha > h$. This method will present, in general, higher errors than the classical Euler scheme.

Given equation (23) and using equation (14) it is possible to obtain

$$R_1(t_{i+1}, t_i, 0) = h^{2\alpha} \left[ \frac{A\delta_1 i^\alpha}{\Gamma^2(\alpha+1)} - \frac{Ch^\alpha \delta_1 i^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{(D-B)i^{2\alpha}}{\Gamma(2\alpha+1)} \right], \tag{26}$$

where $t_i = hi$, $\delta_1 = (i+1)^\alpha - i^\alpha$ and $A = {}_c\mathrm{D}_0^{2\alpha\alpha} y(0)$, $B = {}_c\mathrm{D}_0^{2\alpha\alpha} y(\xi_1)$ is obtained applying the mean value theorem to the reminder term of $y(t_i)$, $C = {}_c\mathrm{D}_0^{3\alpha\alpha} y(\xi_2)$ is obtained applying the mean value theorem to the reminder term of ${}_c\mathrm{D}_0^{\alpha\alpha} y(t)$ and finally $D = {}_c\mathrm{D}_0^{2\alpha\alpha} y(\xi_3)$ is obtained applying the mean value theorem to the reminder term of $y(t_{i+1})$. All of this quantities $A$, $B$, $C$ and $D$ are unknown constants that can be bounded.

Using equation (26), local truncation error at step $i+1$ can be defined as:

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - y^{est}(t_{i+1})}{h^\alpha}, \tag{27}$$

where $y^{est}(t_{i+1})$ is the estimated value of $y(t)$ at $t = t_{i+1}$ using $y(t_i)$. It is clear from (24) that

$$\tau_{i+1}(h) = \frac{R_1}{h^\alpha}, \tag{28}$$

and then $\tau_{i+1}(h)$ neglects as $h^\alpha$.

### *4.2 Fractional Taylor Methods*

Similar to Euler method, there exist higher order methods based on Taylor series. They are commonly known as Taylor methods and they make possible to obtain an approximation or any desired order. All of them has the same difficulty: they require the evaluation of successive derivatives of the function of orders lower than the required approximation. This is no easy in general since many real problems make this an impossible task. In many other cases as described in [11] it can be computed with symbolic mathematical software obtaining series of hundred terms.

Depending on the order of the serie used for approximation it is possible to use second, third, fourth order methods, etc. The most commonly used are the second and fourth order methods and their fractional analogous would be described below.

Using the same description given above for Euler Method, and approximating the solution $y(t)$ at every $t_i$ for a set of values $w_i \approx y(t_i)$ the $2\alpha$ Taylor method for the FDE (22) is

$$\begin{aligned} w_{i+1} = & \; w_i + h^{\alpha} \frac{f(w_i)}{\Gamma(\alpha+1)} \left( (i+1)^{\alpha} - i^{\alpha} \right) + \\ & + h^{2\alpha} \frac{{}_c\mathrm{D}_0^{\alpha} f(w_i)}{\Gamma(2\alpha+1)} \left[ (i+1)^{2\alpha} - i^{2\alpha} - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} i^{\alpha} \left( (i+1)^{\alpha} - i^{\alpha} \right) \right] \\ w_0 = & \; y_0, \end{aligned} \tag{29}$$

for every $i = 0, ..., N$.

With the expression given at Theorem 3.1 it is possible to obtain the $4\alpha$ order method given by:

$$\begin{aligned} w_{i+1} = & \; w_i + h^{\alpha} \frac{f(w_i)}{\Gamma(\alpha+1)} \delta_1 + h^{2\alpha} \frac{{}_c\mathrm{D}_0^{\alpha} f(w_i)}{\Gamma(2\alpha+1)} \delta_2 + h^{3\alpha} \frac{{}_c\mathrm{D}_0^{2\alpha} f(w_i)}{\Gamma(3\alpha+1)} \delta_3 + h^{4\alpha} \frac{{}_c\mathrm{D}_0^{3\alpha} f(w_i)}{\Gamma(4\alpha+1)} \delta_4 \\ w_0 = & \; y_0, \end{aligned} \tag{30}$$

for every $i = 0, ..., N$, and being $\delta_i$ the differences given by

$$\begin{aligned} \delta_1 &= \left[ (i+1)^{\alpha} - i^{\alpha} \right] \\ \delta_2 &= \left[ (i+1)^{2\alpha} - i^{2\alpha} - \tfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} i^{\alpha} \delta_1 \right] \\ \delta_3 &= \left[ (i+1)^{3\alpha} - i^{3\alpha} - \tfrac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} i^{\alpha} \delta_2 - \tfrac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} i^{2\alpha} \delta_1 \right] \\ \delta_4 &= \left[ (i+1)^{4\alpha} - i^{4\alpha} - \tfrac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} i^{\alpha} \delta_3 - \tfrac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2} i^{2\alpha} \delta_2 - \tfrac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} i^{3\alpha} \delta_1 \right] \end{aligned} \tag{31}$$

As it also happens with classical Taylor methods, the computation of the fractional derivatives of $f(y)$ will be, in general, a difficult task, increased this time with the difficulties of the fractional derivative. This will make Taylor method hard to be applied in many problems, since it will require huge calculations in order to estimate the remaining terms, being fractional Euler method sufficiently accurate.

It is possible to make a similar analysis of the truncation error as the one made for the Fractional Euler method. In this case it would be necessary to obtain the expression for many reminder terms and calculations will become very large to be outlined here. General details are given below.

In a general Taylor serie of order $N$ the reminder term $R_N$ is of order $h^{(N+1)\alpha}$. Then we will have to compute all the reminder terms of $y(t)$ and all their $N$ fractional derivatives which also will be of the same order. Once substituted, all the reminder terms obtained would be of such order or ever greater, so the general expression of $R_N$ for $y(t_{i+1})$ as a function of $y(t_i)$ and its derivatives will be of order $h^{(N+1)\alpha}$. Then the local truncation error defined in (27) will be of order $h^{N\alpha}$. This result generalizes for the fractional Taylor method the classical obtained for the truncation error of the Taylor method.

## 5 Numerical Study of Fractional Differential Equations

In the present section the three methods described above are going to be tested in two different cases for which analytical solution is known.

In the first part the schemes are tested with a first order FDE and in the last one with a second order FDE analogous to a fractional oscillator. In that former case the three schemes are compared also with a symplectic scheme obtained in [12] [10].

### 5.1 First Order FDE

The first problem studied is given by the first order Cauchy problem

$$\begin{cases} {}_c\mathrm{D}_0^\alpha y(t) = \lambda y \\ y(0) = y_0 \end{cases} \tag{32}$$

with $\alpha \in (0, 1]$, which solution is given by $y(t) = y_0 E_\alpha(\lambda t^\alpha)$, being $E_\alpha(z)$ the Mittag-Leffler function [4,8], that generalizes the elementary exponential function $e^z$.

This case is very simple and sequential fractional derivatives of the function gives ${}_c\mathrm{D}_0^{k\alpha} y = \lambda^k y$ so Taylor Method can be made up to any order. Numerical schemes for this simple case are:

**Fractional Euler**

$$w_{i+1}^{E} = w_{i}^{E}\left[1 + \frac{h^{\alpha}\lambda}{\Gamma(\alpha+1)}\delta_1\right] \tag{33}$$

**Fractional $2\alpha$ Taylor**

$$w_{i+1}^{T2} = w_{i}^{T2}\left[1 + \frac{h^{\alpha}\lambda}{\Gamma(\alpha+1)}\delta_1 + \frac{h^{2\alpha}\lambda^2}{\Gamma(2\alpha+1)}\delta_2\right] \tag{34}$$

**Fractional $4\alpha$ Taylor**

$$w_{i+1}^{T4} = w_{i}^{T4}\left[1 + \frac{h^{\alpha}\lambda}{\Gamma(\alpha+1)}\delta_1 + \frac{h^{2\alpha}\lambda^2}{\Gamma(2\alpha+1)}\delta_2 + \frac{h^{3\alpha}\lambda^3}{\Gamma(3\alpha+1)}\delta_3 + \frac{h^{4\alpha}\lambda^4}{\Gamma(4\alpha+1)}\delta_4\right] \tag{35}$$

where the differences $\delta_j$ are defined in equation (31) and $w_0^E = w_0^{T2} = w_0^{T4} = y_0$.

In figure 1 are shown the solutions for two different values of $\alpha$ and their corresponding absolute values. Initial condition is $x(0) = 1$ in all cases and parameter $\lambda = 1$. Calculations has been made with single precision and the exact solution has been computed using a MATLAB routine made by I. Podlubny [6]. As the order of approximation depends on a power of $\alpha$ it can be observed that increasing the value of $\alpha$ decreases the errors as can be expected since increasing $\alpha$ increases the order of the scheme and then it should be more accurate.

### *5.2 Second Order FDE*

The next problem studied correspond to a second order FDE given by

$$\begin{cases} {}_c\mathrm{D}_0^{2\alpha} y(t) = -\omega^2 y \\ y(0) = y_0 \\ {}_c\mathrm{D}_0^{\alpha} y(0) = p_0 \end{cases} \tag{36}$$

and in that case two initial conditions for the function $y(t)$ and its first fractional derivative is needed.

This problem can be transformed into a system of two first order FDE

$$\begin{cases} {}_c\mathrm{D}_0^{\alpha} y(t) = p(t) \\ {}_c\mathrm{D}_0^{\alpha} p(t) = -\omega^2 y(t) \end{cases} \tag{37}$$

with initial conditions $y(0) = y_0$ and $p(0) = p_0$. This system is known as Fractional Oscillator and has been studied in [12], [10,15] and in [12] [13,14] adding an external periodic forcing. In all cases cited numerical integration has been of maximum interest and a symplectic scheme has been used. This scheme, developed partially by the author is explained at [12] and [10] and correspond to

**Symplectic Scheme**

$$\begin{cases} p^S_{i+1} = p_0 - \dfrac{\omega^2 h^\alpha}{\Gamma(\alpha+1)} \displaystyle\sum_{k=0}^{i} w_k \left[(i+1-k)^\alpha - (i-k)^\alpha\right] \\ w^S_{i+1} = w_0 + \dfrac{h^\alpha}{\Gamma(\alpha+1)} \displaystyle\sum_{k=0}^{i} p_{k+1} \left[(i+1-k)^\alpha - (i-k)^\alpha\right] \end{cases} \tag{38}$$

The main problem to compute algorithm (38) is the high number of operations needed. In every step it calculates the a sum over all previous elements so operations increase as $N^2$ and also does the computation time. This will not happen with Euler and Taylor schemes because the number of operations increase linearly with $N$.

Exact solution of (37) is known and in terms of the complex variable $\Theta(t) =$

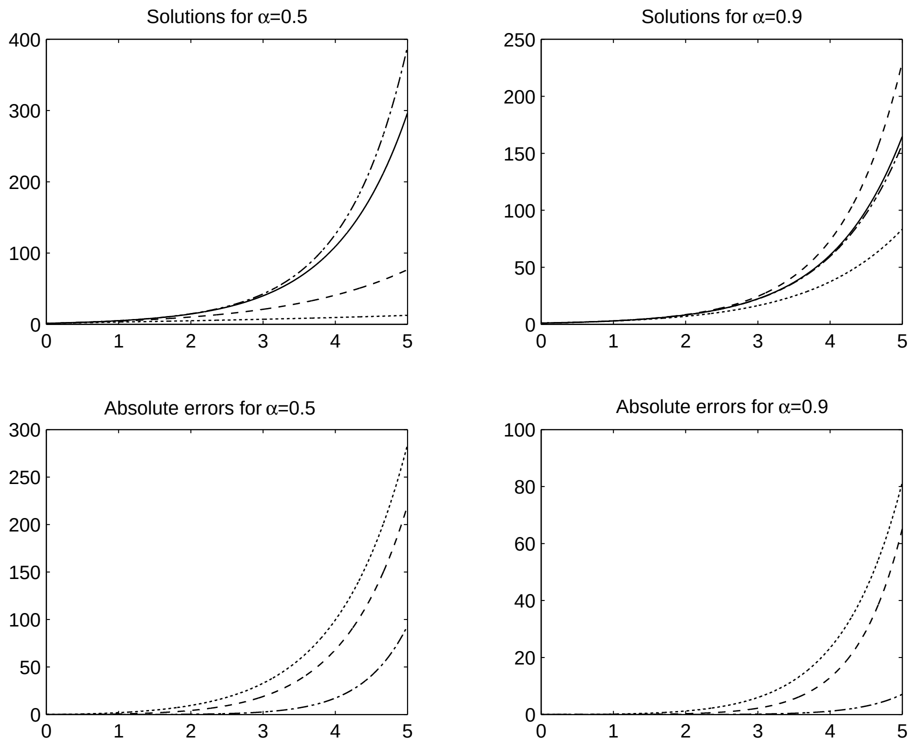


Fig. 1. Plots for the solutions of the FDE (32) and absolute errors for $h = 0.001$ and $N = 5000$. Continuous line correspond to exact solution, dotted line to computed Euler solution, dashed line to computed $2\alpha$ order Taylor solution and dashed-dotted line to $4\alpha$ order solution.

$\sqrt{\omega}y(t) + ip(t)/\sqrt{\omega}$ is $\Theta(t) = \Theta(0)E_\alpha(i\omega t^\alpha)$

The particular form of problem (37) makes possible to compute the derivatives of any order so the three methods described above can be obtained:

**Fractional Euler Method**

$$\begin{cases} w_{i+1}^E = w_i^E + \dfrac{p_i^E h^\alpha}{\Gamma(\alpha+1)}\delta_1 \\ p_{i+1}^E = p_i^E - \dfrac{w_i^E h^\alpha}{\Gamma(\alpha+1)}\delta_1 \end{cases} \tag{39}$$

**Fractional $2\alpha$ Taylor Method**

$$\begin{cases} w_{i+1}^{T2} = w_i^{T2} + \dfrac{p_i^{T2} h^\alpha}{\Gamma(\alpha+1)}\delta_1 - \dfrac{\omega^2 w_i^{T2} h^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2 \\ p_{i+1}^{T2} = p_i^{T2} - \dfrac{\omega^2 w_i^{T2} h^\alpha}{\Gamma(\alpha+1)}\delta_1 - \dfrac{\omega^2 p_i^{T2} h^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2 \end{cases} \tag{40}$$

**Fractional $4\alpha$ Taylor Method**

$$\begin{cases} w_{i+1}^{T4} = w_i^{T4} + \dfrac{p_i^{T4} h^\alpha}{\Gamma(\alpha+1)}\delta_1 - \dfrac{\omega^2 w_i^{T4} h^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2 - \dfrac{\omega^2 p_i^{T4} h^{3\alpha}}{\Gamma(3\alpha+1)}\delta_3 + \dfrac{\omega^4 w_i^{T4} h^{4\alpha}}{\Gamma(4\alpha+1)}\delta_4 \\ p_{i+1}^{T4} = p_i^{T4} - \dfrac{\omega^2 w_i^{T4} h^\alpha}{\Gamma(\alpha+1)}\delta_1 - \dfrac{\omega^2 p_i^{T4} h^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2 + \dfrac{\omega^4 w_i^{T4} h^{3\alpha}}{\Gamma(3\alpha+1)}\delta_3 + \dfrac{\omega^4 p_i^{T4} h^{4\alpha}}{\Gamma(4\alpha+1)}\delta_4 \end{cases} \tag{41}$$

where the differences $\delta_j$ are defined in equation (31) and $w_0^E = w_0^{T2} = w_0^{T4} = y_0$ and $p_0^E = p_0^{T2} = p_0^{T4} = p_0$.

In figure 2 are shown the solutions and their corresponding absolute errors computed for the fractional harmonic oscillator (37) with $\omega = 1$, initial conditions $x(0) = 1$, $p(0) = 0$ and two different values of $\alpha$. Exact solution has been computed using the same MATLAB routine made by Prof. I. Podlubny cited above with complex argument. An very small difference can be appreciated between this exact solution and the one calculated with the symplectic scheme, while a other Euler and Taylor methods has bigger errors as expected. As also happened before, errors increase when decreases $\alpha$.

## 6 Conclusions

Formally it is possible to obtain Fractional Taylor serie of general order $N\alpha$ with $\alpha \in (0, 1]$ is obtained in a general form and its reminder term as it is computed in the present work. This is done for a serie centered in the definition point $a$ called MacLaurin fractional serie and then generalized for any other point $a_1 > a$. This generalization is needed in order to define a numerical Taylor scheme to integrate Fractional Ordinary Differential Equations (FDE).

This serie is used to construct numerical schemes of order $4\alpha$ and also a general method is outlined to obtain a Taylor Scheme of a higher general order $N\alpha$. Truncation errors of a Fractional Taylor Method of order $N\alpha$ are of order $N\alpha$, as it would be expected since this fractional methods are a generalization of classical schemes.

This schemes have been applied to two different Fractional Differential equations, which general solutions are known, as a test for the method. In general, solutions obtained by numeric computations require small computation time compared with others, like the symplectic scheme (38). Most of the schemes developed up to nowadays are computed inverting the differential equation and evaluating the integral by any numerical method. In general this requiere high computational times since the number of operations for the solution in $t_n$ grows as $n^2$. This is not the case in Taylor methods since only the solution in $t_n$ is required in order to calculate the solution in $t_{n+1}$, and then the computational time grows linearly with the number of steps.

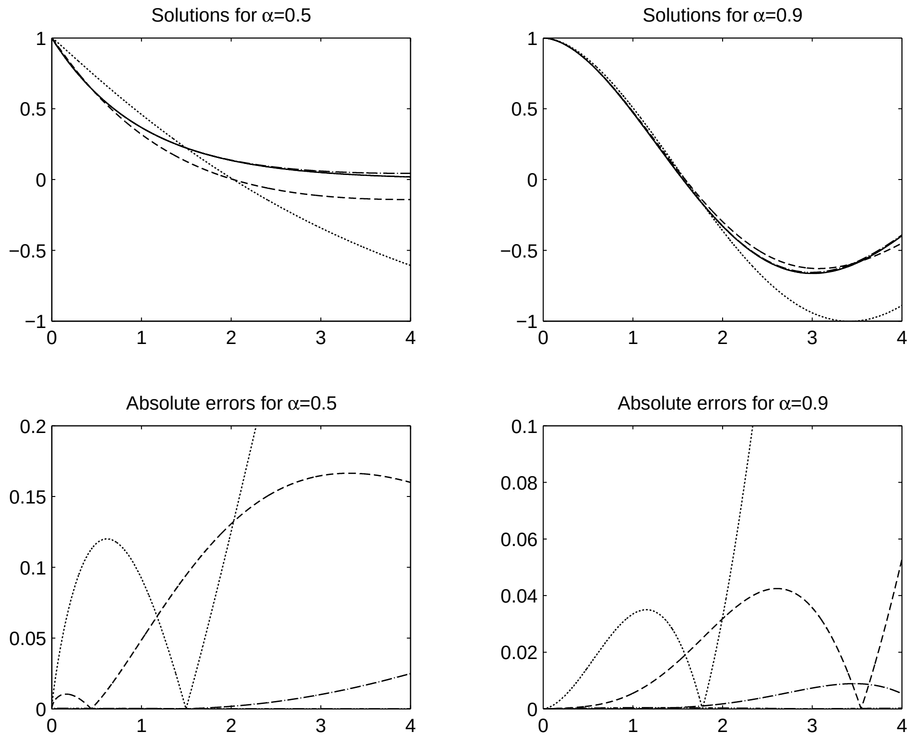


Fig. 2. Plots for the solutions and absolute errors for the FDE (37) with $h = 0.001$ and $N = 5000$. Black line correspond to exact solution, dotted line to computed Euler solution, dashed line to computed $2\alpha$ order Taylor solution, dashed-dotted line to $4\alpha$ order solution and long dashed line, that cannot be distinguished in this scale, to the solution computed using the symplectic scheme.

Errors of the computed solution grows as could be expected and $4\alpha$ order method is the most accurate of the three schemes tested as shown in figures 1 and 2. This is the same for any initial condition. Higher order methods will show best accuracy.

In general, increasing the number of terms will make possible, for a fixed time step $h$, a most accurate solution that will be valid for a greater number of iterations. But this will be true only within the radius of convergence of the serie. This radius of convergence will be the distance of the initial solution to the nearest singularity of the function.

This has been used with nonlinear ordinary differential equation of integer order to localice this singularity by extending the problem to the complex plane. Comparing two of this Taylor series of different orders it is possible to estimate its radius of convergence and even localize them with high accuracy. This has been done in [11] and applied to classical problems like the logistic equation, the Pendulum, the Van der Pol Oscillator or the Henon-Heiles system. There exist general routines that can be used to integrate classical problems using Taylor methods as the one written in Fortran described in [3] which also localize the singularity in a similar way. Nothing similar has been done with fractional derivative problems and future work will be done in this direction.

## 7 Acknowledgments

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