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Fractional order theory of thermo-viscoelasticity and application

Hany H. Sherief[1] · Mohammed A. El-Hagary[2]

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Abstract In this work, we derive a new fractional order theory for thermo-viscoelasticity. A uniqueness theorem for these equations is proved. A reciprocity theorem is also proved. A 1D problem for a viscoelastic half space is solved by using the Laplace transform technique. The solution in the transformed domain is obtained by a direct approach. The inverse transforms are obtained by using a numerical method. The temperature, displacement and stress distributions are computed and represented graphically.

Keywords Fractional calculus · Half Space · Reciprocity theorem · Thermo-viscoelasticity · Uniqueness theorem

1 Introduction

In recent years, viscoelastic materials became a very important study field. This is due to the massive use of polymers and composite materials in industry. The applications of these materials are various and numerous. For instance, they are used in the fabrication of medical diagnostic tools and also used in the NASA space programs. Also the investigation of seismic viscoelastic waves plays an important role for geophysical prospecting technology.

The mechanical model of linear viscoelasticity was represented by Gross (1953). Many authors worked to develop this model and discussed the behavior of viscoelastic materials like Gurtin and Sternberg (1962), Stratonova (1971), Malyi (1976), and Li (1978).

Pobedrya (1969), Il'yushin (1968), Kovalenko and Karnaukhov (1972) and Medri (1988) discussed the coupled theory of thermo-viscoelasticity and solved some problems in the context of this theory. The heat equation of this theory is a parabolic partial differential equation which predicts limited velocity of spread for heat waves contrary to physical observations. Due to this flaw, Sherief et al. (2011) introduced the generalized theory of thermo-viscoelasticity with one relaxation time. This theory is an extension to the generalized theory

✉ M.A. El-Hagary
melhagry@gmail.com

1 Department of Mathematics, University of Alexandria, Alexandria, Egypt

2 Department of Mathematics, Damiatta University, New Damiatta, Egypt

of thermoelasticity which was introduced by Lord and Shulman (1967) and Dhaliwal and Sherief (1980). Elhagary (2013) has solved a thermo-mechanical shock problem for generalized theory of thermo-viscoelasticity. Sherief et al. (2015) solved a 2D problem for a half space in the generalized theory of thermo-viscoelasticity.

Recently, fractional calculus has been used to describe an increasing numbers of physical processes, like, electromagnetism, astrophysics, quantum mechanics, and nuclear physics etc. Caputo and Mainardi (1971a, 1971b) and Caputo (1974) had got good experimental results when using fractional derivatives for description of viscoelastic materials and instituted the relationship between the theory of linear viscoelasticity and fractional order derivatives. Adolfsson and Enelund (2003), Adolfsson et al. (2004) constructed a newer fractional order model of visco-elasticity. Bagley and Torvik (1983, 1986) and Welch et al. (1999) introduced good contributions in this field. Ezzat et al. (2013, 2015), Ezzat and El-Bary (2017) constructed a model and solved a number of problems for fractional order thermo-viscoelasticity.

Povstenko (2005, 2009, 2011) investigated new thermoelasticity models that use fractional derivative. The fractional order theory of thermoelasticity was derived by Sherief et al. (2010). Sherief and AbdEl-Latief (2013, 2014) and Raslan (2015, 2016) solved some problems in the context of this theory.

In this work, the authors introduce a new fractional order theory of thermo-viscoelasticity. A modified law of heat conduction including both heat flux and its fractional time derivative is used to drive the equation of heat conduction. Uniqueness and reciprocity theorems for these equations are proved. The authors have solved a one-dimensional thermo-viscoelastic problem for a half space in order to illustrate the obtained results. In the following, a comma indicates a material derivative and a dot denotes differentiation with respect to time t.

1.1 Derivation of the fundamental equations

A continuous viscoelastic medium contained within a volume V and surrounded by a closed surface S is considered. It is subject to a body force F_i per unit mass and a heat source of strength Q per unit mass, Let the position vector of a point be denoted by $\mathbf{x}(x_i)$ where $(i = 1, 2, 3)$. The components of strain tensor e_{ij} are defined by (Fung 1965)

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.1}$$

where u_i are the components of the displacement vector.

Now, we define a linear thermo-viscoelastic material to be one for which the stress tensor components $\sigma_{ij}(x, t)$ are related to strain tensor components $e_{ij}(x, t)$ by a convolution integral as follows:

$$\sigma_{ij} = \int_0^t C_{ijkl}(t-\tau)\frac{\partial e_{kl}(x,\tau)}{\partial \tau}\, d\tau - \alpha_t \int_0^t m_{ij}(t-\tau)\frac{\partial T(x,\tau)}{\partial \tau}\, d\tau, \tag{1.2}$$

where C_{ijkl} and m_{ij} are tensor field called tensorial relaxation functions of the material, T is the absolute temperature and α_t is the coefficient of liner thermal expansion.

Substituting Eqs. (1.1) and (1.2) into the equation of motion which is given by (Fung 1965)

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{1.3}$$

where ρ is the density, we get

$$\rho \ddot{u}_i = \rho F_i + \frac{1}{2}\int_0^t C_{ijkl}(t-\tau)\frac{\partial}{\partial \tau}[u_{k,lj}+u_{l,kj}]\,d\tau - \alpha_t \int_0^t m_{ij}(t-\tau)\frac{\partial T_{,j}}{\partial \tau}\,d\tau. \tag{1.4}$$

The entropy equation for a thermally conducting viscoelastic solid subjected to small strain and small temperature changes is given by (Foutsitzi et al. 1996)

$$\rho T_0 \eta = \rho c_E (T - T_0) + \alpha_t T_0 \int_0^t m_{ij}(t-\tau)\frac{\partial e_{ij}(x,\tau)}{\partial \tau}\,d\tau, \tag{1.5}$$

where η is the entropy per unit mass and T_0 is reference temperature for which the medium is in equilibrium free of strain. Equation (1.5) can be written in the form

$$\rho T_0 \eta = \rho c_E (T - T_0) - \alpha_t T_0 \left[\int_0^t e_{ij}(x,t-\tau)\frac{\partial m_{ij}(\tau)}{\partial \tau}\,d\tau - m_{ij}(0)e_{ij}(x,t)\right]. \tag{1.6}$$

We shall use the linearized entropy balance equation, namely (Sherief et al. 2011)

$$\rho T_0 \dot{\eta} = -q_{i,i} + \rho Q, \tag{1.7}$$

where q_i is heat flux vector. Using Eq. (1.6) this reduces to

$$q_{i,i} = -\rho c_E \dot{T} + \alpha_t T_0 \left[\int_0^t \dot{e}_{ij}(x,t-\tau)\frac{\partial m_{ij}(\tau)}{\partial \tau}\,d\tau - m_{ij}(0)\dot{e}_{ij}(x,t)\right] + \rho Q. \tag{1.8}$$

Now, we shall use the definition of fractional derivatives of order $\alpha \in [0, 1]$ of the absolutely continuous function $f(t)$ given by (Miller and Ross 1993)

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f'(s)\,ds, \tag{1.9}$$

where $f(t)$ is a Lebesgue integrable function, $\alpha > 0$. If the function $f(t)$ is absolutely continuous, then

$$\operatorname*{Lim}_{\alpha \to 1} \frac{d^\alpha}{dt^\alpha} f(t) = f'(t).$$

We assume a generalized Fourier law of heat conduction of the form (Sherief et al. 2010)

$$q_i + \tau_0 \frac{\partial^\alpha q_i}{\partial t^\alpha} = -k_{ij} T_{,j}, \tag{1.10}$$

where k_{ij} is a thermal conductivity tensor, τ_0 is a constant with the dimension of time, called relaxation time, and α is a constant such that $0 < \alpha \le 1$.

Now, taking divergence of both sides of (1.10) and using (1.8) and its time derivative, we arrive at

$$\begin{aligned}(k_{ij}T_{,j})_{,i} = \rho c_E \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right) T - \alpha_t T_0 \Bigg[\int_0^t \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right) e_{ij} \frac{\partial m_{ij}}{\partial \tau}\,d\tau \\ - m_{ij}(0)\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right) e_{ij}\Bigg] - \rho \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha}}{\partial t^{\alpha}}\right) Q.\end{aligned} \tag{1.11}$$

In a case of an isotropic body, the tensorial relaxation functions can be written as (Sherief et al. 2011)

$$C_{ijkl} = G_1\delta_{ij}\delta_{kl} + G_2[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}], \tag{1.12a}$$

$$m_{ij} = (3G_1 + 2G_2)\delta_{ij} = G\delta_{ij}, \qquad k_{ij} = k\delta_{ij}, \tag{1.12b}$$

where $G_1(t)$ and $G_2(t)$ are relaxation functions.

Substituting (1.12a), (1.12b) into (1.2) and (1.4) we get

$$\sigma_{ij} = 2\int_0^t G_2\frac{\partial e_{ij}}{\partial\tau}\,d\tau + \delta_{ij}\left[\int_0^t G_1\frac{\partial e}{\partial\tau}\,d\tau - \alpha_t\int_0^t G\frac{\partial T}{\partial\tau}\,d\tau\right]. \tag{1.13}$$

$$\rho\ddot{u}_i = \rho F_i + \int_0^t G_2\frac{\partial u_{i,jj}}{\partial\tau}\,d\tau + \int_0^t (G_1+G_2)\frac{\partial u_{j,ij}}{\partial\tau}\,d\tau - \alpha_t\int_0^t G\frac{\partial T_{,i}}{\partial\tau}\,d\tau \tag{1.14}$$

and the equation of heat conduction takes the form

$$\begin{aligned} kT_{,ii} = \rho c_E\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)T - \alpha_t T_0\Bigg[\int_0^t\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)e_{ij}\frac{\partial G}{\partial\tau}\,d\tau \\ - G(0)\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)e_{ij}\Bigg] - \rho\left(1 + \tau_0\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)Q, \end{aligned} \tag{1.15}$$

where e is the cubical dilatation given by $e = e_{kk}$.

2 Uniqueness theorem

As usual, to prove uniqueness we assume there exist two sets of functions $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$, etc., and let

$$\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}, \qquad e_{ij} = e_{ij}^{(1)} - e_{ij}^{(2)}.$$

Theorem *Given a regular region of space $V + S$ with boundary S then there exists at most one set of single-valued functions $\sigma_{ij}(x_k, t)$ and $e_{ij}(x_k, t)$ of class $C_{(1)}$, $u_i(x_k, t)$ and $T(x_k, t)$ of class $C_{(2)}$ in $V + S$, $t \geq 0$ which satisfy Eqs. (1.4) and (1.11) in V, $t > 0$ and the following equation on the boundary S, $t > 0$:*

$$T = f_1 \quad \textit{on } S, \tag{2.1}$$

$$u_i = u_{i1} \quad \textit{on } S. \tag{2.2}$$

The following equations in V, $t = 0$:

$$T = h_0, \quad \dot{T} = h_1, \qquad u_i = u_{i0}, \quad \dot{u}_i = \dot{u}_{i0}, \tag{2.3}$$

where we assume that the viscoelasticities and the conductivities satisfy the symmetry condition

$$C_{ijkl} = C_{klij}, \quad k_{ij} = k_{ji}. \tag{2.4}$$

We assume also that C_{ijkl} and k_{ij} satisfy the positive definiteness condition

$$C_{ijkl}(0)\xi_{ij}\xi_{kl} \geq c_1\xi_{ij}\xi_{ij}, \qquad k_{ij}\zeta_i\zeta_j \geq c_2\zeta_i\zeta_i \tag{2.5}$$

for some positive constants c_1 and c_2 and for all nonzero tensors ζ_i and ξ_{ij}.

Proof Equation (1.4) can be rewritten as the form

$$\rho\ddot{u}_i = \int_0^t C_{ijkl}(t-\tau)\frac{\partial e_{kl}}{\partial\tau}\,d\tau + \alpha_t\left[\int_0^t T_{,j}(x,t-\tau)\frac{\partial m_{ij}(\tau)}{\partial\tau}\,d\tau - m_{ij}(0)T_{,j}(x,t)\right]. \tag{2.6}$$

Applying the Laplace transform defined by the relation

$$\bar{f}(x,s) = \int_0^\infty e^{-st} f(x,t)\,dt,$$

to both sides of Eqs. (2.6) and (1.11) we obtain

$$\rho s^2\bar{u}_i = s\bar{C}_{ijkl}\bar{e}_{kl,j} + \alpha_t\big[s\bar{m}_{ij}\bar{T}_{,j} - 2m_{ij}(0)\bar{T}_{,j}\big], \tag{2.7}$$

$$(k_{ij}\bar{T}_{,j})_{,i} = \rho c_E s\big(1+\tau_0 s^a\big)\bar{T} - \alpha_t T_0 s\big(1+\tau_0 s^a\big)\big[s\bar{m}_{ij}\bar{e}_{ij} - 2m_{ij}(0)\bar{e}_{ij}\big]. \tag{2.8}$$

Multiplying Eq. (2.7) by $\bar{u}_i$ and (2.8) by $\bar{T}$, integrating over V and using the Green–Gauss theorem, we get

$$\int_V \rho s^2\bar{u}_i\bar{u}_i\,dV + \int_V s\bar{C}_{ijkl}\bar{e}_{kl}\bar{e}_{ij}\,dV + \alpha_t\left[\int_V s\bar{m}_{ij}\bar{T}\bar{e}_{ij}\,dV - \int_V 2m_{ij}(0)\bar{T}\bar{e}_{ij}\,dV\right] = 0, \tag{2.9}$$

$$\int_V k_{ij}\bar{T}_{,j}\bar{T}_{,i}\,dV + s\big(1+\tau_0 s^a\big)\int_V \rho c_E\bar{T}^2\,dV - \alpha_t T_0 s\big(1+\tau_0 s^a\big)\left[\int_V s\bar{m}_{ij}\bar{T}\bar{e}_{ij}\,dV - \int_V 2m_{ij}(0)\bar{T}\bar{e}_{ij}\,dV\right] = 0. \tag{2.10}$$

Eliminating the last term in Eqs. (2.9) and (2.10), we obtain

$$\int_V \rho s^2\bar{u}_i\bar{u}_i\,dV + \int_V s\bar{C}_{ijkl}\bar{e}_{kl}\bar{e}_{ij}\,dV + \frac{k_{ij}}{T_0 s(1+\tau_0 s^\alpha)}\int_V \bar{T}_{,j}\bar{T}_{,i}\,dV + \int_V \frac{\rho c_E}{T_0}\bar{T}^2\,dV = 0. \tag{2.11}$$

By the initial value theorem of the Laplace transform (Churchill 1972), $s\bar{C}(s) \to C(0)$ as $s \to \infty$, then, for large s, we have

$$\int_V \left[\rho s^2\bar{u}_i\bar{u}_i + C_{ijkl}(0)\bar{e}_{kl}\bar{e}_{ij} + \frac{k_{ij}}{T_0 s(1+\tau_0 s^\alpha)}\bar{T}_{,j}\bar{T}_{,i} + \frac{\rho c_E}{T_0}\bar{T}^2\right] dV = 0. \tag{2.12}$$

By the hypothesis (2.5), we arrive at

$$\int_V \left[\rho s^2\bar{u}_i\bar{u}_i + c_1\bar{e}_{ij}\bar{e}_{ij} + \frac{c_2}{T_0 s(1+\tau_0 s^\alpha)}\bar{T}_{,i}\bar{T}_{,i} + \frac{\rho c_E}{T_0}\bar{T}^2\right] dV \leq 0. \tag{2.13}$$

The integrand in (2.13) is the sum of squares and cannot be negative. Therefore, we obtain

$$\int_V \left[\rho s^2\bar{u}_i\bar{u}_i + c_1\bar{e}_{ij}\bar{e}_{ij} + \frac{c_2}{T_0 s(1+\tau_0 s^\alpha)}\bar{T}_{,i}\bar{T}_{,i} + \frac{\rho c_E}{T_0}\bar{T}^2\right] dV = 0, \quad t \geq 0. \tag{2.14}$$

It follows from (2.14) that the difference functions are identically zero throughout the body for all time. According to Learch's theorem (Churchill 1972) the inverse Laplace transform of each is unique, and this completes the proof. □

3 Reciprocity theorem

We supplement Eqs. (1.4) and (1.11) with the boundary conditions

$$\sigma_{ij}n_j = p_i(x,t), \qquad T(x,t) = \upsilon(x,t), \quad x \in S,\ t > 0 \tag{3.1}$$

and the homogeneous initial conditions

$$u_i(x,0) = 0, \quad \dot{u}_i(x,0) = 0, \qquad T(x,0) = 0, \quad \dot{T}(x,0) = 0,\ x \in V. \tag{3.2}$$

Consider a bounded thermo-viscoelastic body subject to the action of a body forces F_i, surface tractions p_i, heat sources Q and heating of the surface to the temperature υ. We write these causes symbolically as

$$L = \{F_i, p_i, Q, \upsilon\}. \tag{3.3}$$

The causes C produce in the body the displacements u_i and the temperature increment θ. We write these results as

$$R = \{u_i, T\}. \tag{3.4}$$

Assume now that there exists another system of causes and effects, namely

$$L' = \left\{F_i', p_i', Q', \upsilon'\right\}, \qquad R' = \left\{u_i', T'\right\}. \tag{3.5}$$

If we perform over Eq. (1.2) the Laplace transform, we get the relation

$$\bar{\sigma}_{ij} = s\bar{C}_{ijkl}\bar{e}_{kl} - \bar{\beta}_{ij}\bar{T} \tag{3.6}$$

where $\bar{\beta}_{ij} = \alpha_t s\bar{m}_{ij}$

For system (3.7) the equivalent equation is

$$\bar{\sigma}_{ij}' = s\bar{C}_{ijkl}\bar{e}_{kl}' - \bar{\beta}_{ij}\bar{T}'. \tag{3.7}$$

Multiplying Eq. (3.6) by $\bar{e}_{kl}'\delta_{ik}\delta_{jl}$, Eq. (3.7) by $\bar{e}_{kl}\delta_{ik}\delta_{jl}$ and integrating the difference over V, we get

$$\int_V \left[\bar{\sigma}_{ij}\bar{e}_{ij}' - \bar{\sigma}_{ij}'\bar{e}_{ij}\right] dV = \alpha_t \int_V \bar{m}_{ij}\left[\bar{T}'\bar{e}_{ij} - \bar{T}\bar{e}_{ij}'\right] dV. \tag{3.8}$$

Now we have

$$\int_V \bar{\sigma}_{ij}\bar{e}_{ij}'\, dV = \int_V \bar{\sigma}_{ij}\bar{u}_{i,j}'\, dV = \int_S \bar{\sigma}_{ij}n_j\bar{u}_i'\, dS - \int_V \bar{\sigma}_{ij,j}\bar{u}_i'\, dV. \tag{3.9}$$

Performing the Laplace transform over Eq. (1.3) and using the homogeneous initial conditions (3.2) we obtain

$$\bar{\sigma}_{ij,j} + \rho\bar{F}_i = \rho s^2\bar{u}_i, \qquad x \in V. \tag{3.10}$$

Combining Eqs. (3.1), (3.9) and (3.10), we have

$$\int_V \bar{\sigma}_{ij}\bar{e}'_{ij}\,dV = \int_S \bar{p}_i\bar{u}'_i\,dS + \rho\int_V \bar{F}_i\bar{u}'_i\,dV - \rho\int_V s^2\bar{u}_i\bar{u}'_i\,dV. \tag{3.11}$$

Substituting from (3.11) and an analogous integral $\int_V \bar{\sigma}_{ij}\bar{e}'_{ij}\,dV$ into (3.8) we obtain the equation

$$\int_S \big[\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i\big]\,dS + \rho\int_V \big[\bar{F}_i\bar{u}'_i - \bar{F}'_i u_i\big]\,dV + \bar{\beta}_{ij}\int_V \big[\bar{T}'\bar{e}_{ij} - \bar{T}\bar{e}'_{ij}\big]\,dV = 0. \tag{3.12}$$

Equation (3.12) constitutes the first part of the reciprocity theorem since it contains only causes of a mechanical nature, namely, the mechanical forces and the surface tractions.

To derive the second part we take the Laplace transform of both sides of (1.11) and use the initial conditions (3.2) to obtain

$$(k_{ij}\bar{T}_{,j})_{,i} = \rho c_E\big(s + \tau_0 s^{1+\alpha}\big)\bar{T} - \alpha_t T_0\big(s + \tau_0 s^{1+\alpha}\big)\big[s\bar{m}_{ij} - m_{ij}(0)\big]\bar{e}_{ij} - \rho\big(1 + \tau_0 s^{\alpha}\big)Q. \tag{3.13}$$

The analogous equation for (3.13) is

$$\big(k_{ij}\bar{T}'_{,j}\big)_{,i} = \rho c_E\big(s + \tau_0 s^{1+\alpha}\big)\bar{T}' - \alpha_t T_0\big(s + \tau_0 s^{1+\alpha}\big)\big[s\bar{m}_{ij} - m_{ij}(0)\big]\bar{e}'_{ij} - \rho\big(1 + \tau_0 s^{\alpha}\big)Q'. \tag{3.14}$$

Multiplying Eq. (3.13) by $\bar{T}'$ and (3.14) by $\bar{T}$, subtracting the result and integrating over the volume V, we arrive at the identity

$$\begin{aligned}\int_V \big[(k_{ij}\bar{T}_{,j})_{,i}\bar{T}' - \big(k_{ij}\bar{T}'_{,j}\big)_{,i}\bar{T}\big]\,dV &= T_0[\bar{\beta}_{ij} - \alpha_{ij}]\big(s + \tau_0 s^{1+\alpha}\big)\int_V \big[\bar{e}'_{ij}\bar{T} - \bar{e}_{ij}\bar{T}'\big]\,dV \\ &\quad - \big(1 + \tau_0 s^{\alpha}\big)\int_V \rho\big[\bar{Q}\bar{T}' - \bar{Q}'\bar{T}\big]\,dV,\end{aligned} \tag{3.15}$$

where $\alpha_{ij} = \alpha_t m_{ij}(0)$.

Integrating by parts we find, after using the transformed boundary condition (3.1), that

$$\begin{aligned}\int_V (k_{ij}\bar{T}_{,j})_{,i}\bar{T}'\,dV &= \int_S k_{ij}\bar{T}_{,j}\bar{T}'n_i\,dS - \int_V k_{ij}\bar{T}_{,j}\bar{T}'_{,i}\,dV \\ &= \int_S k_{ij}\bar{\upsilon}'\bar{T}_{,j}n_i\,dS - \int_V k_{ij}\bar{T}_{,j}\bar{T}'_{,i}\,dV.\end{aligned} \tag{3.16}$$

Substituting (3.16) and an analogous expression for $\int_V (k_{ij}\bar{T}_{,j})_{,i}\bar{T}'\,dV$ into Eq. (3.15) we obtain

$$\begin{aligned}\int_S k_{ij}\big[\bar{\upsilon}'\bar{T}_{,j} - \bar{\upsilon}\bar{T}'_{,j}\big]n_i\,dS &= T_0[\bar{\beta}_{ij} - \alpha_{ij}]\big(s + \tau_0 s^{1+\alpha}\big)\int_V \big[\bar{e}'_{ij}\bar{T} - \bar{e}_{ij}\bar{T}'\big]\,dV \\ &\quad - \big(1 + \tau_0 s^{\alpha}\big)\int_V \rho\big[\bar{Q}\bar{T}' - \bar{Q}'\bar{T}\big]\,dV.\end{aligned} \tag{3.17}$$

Equation (3.17) is the second part of the reciprocity theorem. It contains thermal causes, namely: the heat sources and heating of the surface S.

Eliminating the integral $\int_V [\bar{T}'\bar{e}_{ij} - \bar{T}\bar{e}'_{ij}]\,dV$ from Eqs. (3.12) and (3.17) we arrive at a reciprocity theorem containing both systems of causes of L and L' and effects R and R':

$$\begin{aligned}
&\int_S k_{ij}\bar{\beta}_{ij}\big[\bar{\upsilon}'\bar{T}_{,j} - \bar{\upsilon}\bar{T}'_{,j}\big]n_i\,dS + \big(1+\tau_0 s^{\alpha}\big)\bar{\beta}_{ij}\int_V \rho\big[\bar{Q}\bar{T}' - \bar{Q}'\bar{T}\big]\,dV \\
&\quad = T_0\big[\bar{\beta}_{ij} - \beta_{ij}(0)\big]\big(s + \tau_o s^{1+\alpha}\big)\int_S \big[\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i\big]\,dS \\
&\qquad + T_0\big[\bar{\beta}_{ij} - \beta_{ij}(0)\big]\big(s + \tau_o s^{1+\alpha}\big)\int_V \big[\bar{F}_i\bar{u}'_i - \bar{F}'_i\bar{u}_i\big]\,dV.
\end{aligned} \tag{3.18}$$

To invert the Laplace transform in (3.18) we use the convolution theorem (Churchill 1972) and (Oberhettinger and Badii 1973) namely

$$L^{-1}\big[\bar{f}(s).\bar{g}(s)\big] = \int_0^t L^{-1}\big[\bar{f}(s)\big]_{t=t-z} L^{-1}\big[\bar{g}(s)\big]_{t=z}\,dz.$$

Equation (3.18) finally reduces to

$$\begin{aligned}
&\int_S k_{ij}n_i\,dS\int_0^t\bigg[T_{,j}(x,t-\tau)\int_0^{\tau}\beta_{ij}(\tau-\varsigma)\upsilon'(x,\varsigma)d\varsigma - T'_{,j}(x,t-\tau) \\
&\quad\times\int_0^{\tau}\beta_{ij}(\tau-\varsigma)\upsilon(x,\varsigma)\,d\varsigma\bigg]d\tau \\
&\quad+\int_V \rho\,dV\int_0^t\bigg[T'(x,t-\tau)\int_0^{\tau}\beta_{ij}(\tau-\varsigma)Q(x,\varsigma)\,d\varsigma - T(x,t-\tau) \\
&\quad\times\int_0^{\tau}\beta_{ij}(\tau-\varsigma)Q'(x,\varsigma)\,d\varsigma\bigg]d\tau \\
&\quad+\tau_0\int_V \rho\,dV\int_0^t\bigg[\frac{\partial T'(x,\tau)}{\partial\tau}\int_0^{t-\tau}\beta_{ij}(\varsigma)Q(x,t-\tau-\varsigma)\,d\varsigma \\
&\quad-\frac{\partial T(x,\tau)}{\partial\tau}\int_0^{t-\tau}\beta_{ij}(\varsigma)Q'(x,t-\tau-\varsigma)\,d\varsigma\bigg]d\tau \\
&= T_0\int_S dS\int_0^t\bigg[\frac{\partial u'_i(x,\tau)}{\partial\tau}\int_0^{t-\tau}\beta_{ij}(\varsigma)p_i(x,t-\tau-\varsigma)\,d\varsigma \\
&\quad-\frac{\partial u_i(x,\tau)}{\partial\tau}\int_0^{t-\tau}\beta_{ij}(\varsigma)p'_i(x,t-\tau-\varsigma)\,d\varsigma\bigg]d\tau \\
&\quad+T_0\tau_0\int_S dS\int_0^t\bigg[\frac{\partial^{1+\alpha}u'_i(x,\tau)}{\partial\tau^{1+\alpha}}\int_0^{t-\tau}\beta_{ij}(\varsigma)p_i(x,t-\tau-\varsigma)\,d\varsigma \\
&\quad-\frac{\partial^{1+\alpha}u_i(x,\tau)}{\partial\tau^{1+\alpha}}\int_0^{t-\tau}\beta_{ij}(\varsigma)p'_i(x,t-\tau-\varsigma)\,d\varsigma\bigg]d\tau \\
&\quad-T_0\alpha_{ij}\int_S dS\int_0^t\bigg[\frac{\partial u'_i(x,\tau)}{\partial\tau}p_i(x,t-\tau) - \frac{\partial u_i(x,\tau)}{\partial\tau}p'_i(x,t-\tau)\bigg]d\tau \\
&\quad-T_0\alpha_{ij}\int_S dS\int_0^t\bigg[\frac{\partial^{1+\alpha}u'_i(x,\tau)}{\partial\tau^{1+\alpha}}p_i(x,t-\tau) - \frac{\partial^{1+\alpha}u_i(x,\tau)}{\partial\tau^{1+\alpha}}p'_i(x,t-\tau)\bigg]d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ T_0 \int_V dV \int_0^t \left[\frac{\partial u_i'(x,\tau)}{\partial \tau} \int_0^{t-\tau} \beta_{ij}(\varsigma) F_i(x, t-\tau-\varsigma)\, d\varsigma \right. \\
&\left. - \frac{\partial u_i(x,\tau)}{\partial \tau} \int_0^{t-\tau} \beta_{ij}(\varsigma) F_i'(x, t-\tau-\varsigma)\, d\varsigma \right] d\tau \\
&+ T_0 \tau_0 \int_V dV \int_0^t \left[\frac{\partial^{1+\alpha} u_i'(x,\tau)}{\partial \tau^{1+\alpha}} \int_0^{t-\tau} \beta_{ij}(\varsigma) F_i(x, t-\tau-\varsigma)\, d\varsigma \right. \\
&\left. - \frac{\partial^{1+\alpha} u_i(x,\tau)}{\partial \tau^{1+\alpha}} \int_0^{t-\tau} \beta_{ij}(\varsigma) F_i'(x, t-\tau-\varsigma)\, d\varsigma \right] d\tau \\
&- T_0 \alpha_{ij} \int_V dV \int_0^t \left[\frac{\partial u_i'(x,\tau)}{\partial \tau} F_i(x, t-\tau) - \frac{\partial u_i(x,\tau)}{\partial \tau} F_i'(x, t-\tau) \right] d\tau \\
&- T_0 \alpha_{ij} \int_V dV \int_0^t \left[\frac{\partial^{1+\alpha} u_i'(x,\tau)}{\partial \tau^{1+\alpha}} F_i(x, t-\tau) - \frac{\partial^{1+\alpha} u_i(x,\tau)}{\partial \tau^{1+\alpha}} F_i'(x, t-\tau) \right] d\tau.
\end{aligned}
\tag{3.19}
$$

4 A half space problem

Consider a homogeneous isotropic thermos-viscoelastic solid occupying the region $0 \leq x \leq \infty$, and assumed that the body is initially quiescent. For zero body forces and the absence of a heat source, the equation of motion is given by

$$
\begin{aligned}
\rho \ddot{u}_i = &\int_0^t G_2(t-\tau) \frac{\partial u_{i,jj}(x,\tau)}{\partial \tau} d\tau + \int_0^t (G_1 + G_2)(t-\tau) \frac{\partial u_{i,ij}(x,\tau)}{\partial \tau} d\tau \\
&- \alpha_t \int_0^t G(t-\tau) \frac{\partial T_{,i}(x,\tau)}{\partial \tau} d\tau,
\end{aligned}
\tag{4.1}
$$

and the equation of heat conduction has the form

$$
\begin{aligned}
k \nabla^2 T = &\rho c_E \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) T - \alpha_t T_0 \left[\int_0^t \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) e_{ij} \frac{\partial G}{\partial \tau} d\tau \right. \\
&\left. - G(0) \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) e_{ij} \right],
\end{aligned}
\tag{4.2}
$$

where the relaxation functions can be taken in the form (Sherief et al. 2011)

$$
\begin{aligned}
G_1(t) &= \lambda \left(1 - A_1 e^{-\beta t}\right), \\
G_2(t) &= \mu \left(1 - A_2 e^{-\beta t}\right), \\
G(t) &= 3G_1(t) + 2G_2(t),
\end{aligned}
\tag{4.3}
$$

where A_1, A_2, and, β are non-dimensional empirical constants such that $\beta \geq 0$.

The boundary conditions are assumed to be

$$
\sigma_{xx}(0,t) = 0 \quad \text{and} \quad T(0,t) - T_0 = C H(t), \quad t > 0, \tag{4.4}
$$

where C is a constant and $H(\cdot)$ is the Heaviside unit step function.

Since the problem is solved in one dimension, all considered functions will depend only on the space variables x and time t. The displacement vector has the components $(u(x,t),0,0)$ and the cubical dilatation has the form $e=\frac{\partial u}{\partial x}$, and Eqs. (4.1), (4.2), and (1.13) then reduce to

$$\rho\frac{\partial^2 u}{\partial t^2}=\int_0^t (G_1+2G_2)\frac{\partial}{\partial\tau}\left(\frac{\partial^2 u}{\partial x^2}\right)d\tau-\alpha_t\int_0^t G\frac{\partial}{\partial\tau}\left(\frac{\partial T}{\partial x}\right)d\tau, \tag{4.5}$$

$$\begin{aligned}k\frac{\partial^2 T}{\partial x^2}=\rho c_E\left(\frac{\partial T}{\partial t}+\tau_0\frac{\partial^{\alpha+1}T}{\partial t^{\alpha+1}}\right)-\alpha_t T_0\Bigg[\int_0^t\left(\frac{\partial e}{\partial t}+\tau_0\frac{\partial^{\alpha+1}e}{\partial t^{\alpha+1}}\right)\frac{\partial G}{\partial\tau}\,d\tau\\ -G(0)\left(\frac{\partial e}{\partial t}+\tau_0\frac{\partial^{\alpha+1}e}{\partial t^{\alpha+1}}\right)\Bigg],\end{aligned} \tag{4.6}$$

$$\sigma_{xx}=\int_0^t (G_1+2G_2)\frac{\partial e}{\partial\tau}\,d\tau-\alpha_t\int_0^t G\frac{\partial T}{\partial\tau}\,d\tau. \tag{4.7}$$

Let us introduce the following non-dimensional variables:

$$x^*=c_1\xi x,\qquad u^*=c_1\xi u,\qquad t^*=c_1^2\xi t,\qquad \tau_0^*=c_1^2\xi\tau_0,$$
$$G_i^*=\frac{G_i}{\lambda+2\mu},\quad i=1,2,\qquad \theta=\alpha_t(T-T_0),\qquad \sigma_{xx}^*=\frac{\sigma_{xx}}{(\lambda+2\mu)},$$

where $c_1=\sqrt{(\lambda+2\mu)/\rho}$, $\xi=\rho c_E/k$

In terms of these non-dimensional variables Eqs. (4.5)–(4.7), and (4.3) become

$$\frac{\partial^2 u}{\partial t^2}=\int_0^t (G_1+2G_2)\frac{\partial}{\partial\tau}\left(\frac{\partial^2 u}{\partial x^2}\right)d\tau-\int_0^t G\frac{\partial}{\partial\tau}\left(\frac{\partial\theta}{\partial x}\right)d\tau, \tag{4.8}$$

$$\begin{aligned}\frac{\partial^2\theta}{\partial x^2}=\left(\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^{\alpha+1}\theta}{\partial t^{\alpha+1}}\right)-\varepsilon_0\Bigg[\int_0^t\left(\frac{\partial e}{\partial t}+\tau_0\frac{\partial^{\alpha+1}e}{\partial t^{\alpha+1}}\right)\frac{\partial G}{\partial\tau}\,d\tau\\ -G(0)\left(\frac{\partial e}{\partial t}+\tau_0\frac{\partial^{\alpha+1}e}{\partial t^{\alpha+1}}\right)\Bigg],\end{aligned} \tag{4.9}$$

$$\sigma_{xx}=\int_0^t (G_1+2G_2)\frac{\partial e}{\partial\tau}\,d\tau-\int_0^t G\frac{\partial\theta}{\partial\tau}\,d\tau, \tag{4.10}$$

$$\begin{aligned}G_1(t)&=\alpha_1\left(1-A_1e^{-\beta t}\right),\\ G_2(t)&=\alpha_2\left(1-A_2e^{-\beta t}\right),\end{aligned} \tag{4.11}$$

where

$$\varepsilon=\frac{\alpha_t^2T_0(\lambda+2\mu)}{\rho c_E},\qquad \alpha_1=\frac{\lambda}{\lambda+2\mu},\qquad \alpha_2=\frac{\mu}{\lambda+2\mu},$$
$$G(0)=3\alpha_1(1-A_1)+2\alpha_2(1-A_2).$$

The boundary conditions (4.4) become

$$\sigma_{xx}(0,t)=0\quad\text{and}\quad\theta(0,t)=\theta_0H(t) \tag{4.12}$$

where $\theta_0=\alpha_tC$.

4.1 Solution in the Laplace transform domain

Applying the Laplace transform (denoted by an over-bar) to both sides of Eqs. (4.8)–(4.11), and using the initial conditions, we arrive at

$$\big(\bar{\gamma}_1 D^2 - s^2\big)\bar{u} = \bar{\gamma}_2 \frac{\partial \bar{\theta}}{\partial x}, \tag{4.13}$$

$$\big(D^2 - s\big(1+\tau_0 s^\alpha\big)\big)\bar{\theta} = \varepsilon s\big(1+\tau_0 s^\alpha\big)\big(2G(0) - \bar{\gamma}_2\big)\bar{e}, \tag{4.14}$$

$$\bar{\sigma}_{xx} = \bar{\gamma}_1 \bar{e} - \bar{\gamma}_2 \bar{\theta}, \tag{4.15}$$

where

$$\bar{\gamma}_1 = s(\bar{G}_1 + 2\bar{G}_2), \qquad \bar{\gamma}_2 = s\bar{G}, \quad \text{and} \quad D = \frac{\partial}{\partial x}. \tag{4.16}$$

The transformed boundary condition (4.12) becomes

$$\bar{\sigma}_{xx}(0,s) = 0 \quad \text{and} \quad \bar{\theta}(0,s) = \frac{\theta_0}{s}. \tag{4.17}$$

Differentiating both sides of Eq. (4.13) with respect to x, we obtain

$$\big(\bar{\gamma}_1 D^2 - s^2\big)\bar{e} = \bar{\gamma}_2 D^2 \bar{\theta}. \tag{4.18}$$

Eliminating $\bar{e}$ between Eqs. (4.15) and (4.18), we get

$$\big\{\bar{\gamma}_1 D^4 - D^2\big[s^2 + s\big(1+\tau_0 s^\alpha\big)\big(\bar{\gamma}_1 + \varepsilon\bar{\gamma}_2\big(2G(0) - \bar{\gamma}_2\big)\big)\big] + s^3\big(1+\tau_0 s^\alpha\big)\big\}\bar{\theta} = 0. \tag{4.19}$$

The above equation can be factorized as

$$\big(D^2 - k_1^2\big)\big(D^2 - k_2^2\big)\bar{\theta} = 0, \tag{4.20}$$

where k_1^2 and k_2^2 are the roots with positive real parts of the characteristic equation

$$\bar{\gamma}_1 k^4 - k^2\big[s^2 + s\big(1+\tau_0 s^\alpha\big)\big(\bar{\gamma}_1 + \varepsilon\bar{\gamma}_2\big(2G(0) - \bar{\gamma}_2\big)\big)\big] + s^3\big(1+\tau_0 s^\alpha\big) = 0. \tag{4.21}$$

Since $\bar{\theta}$ must remain bounded as x $\to \infty$, the solution of Eq. (4.20) is given by

$$\bar{\theta} = \sum_{i=1}^{2}\big(\bar{\gamma}_1 k_i^2 - s^2\big)C_i e^{-k_i x}, \tag{4.22}$$

where C_i, $i = 1, 2$ are parameters depending on s.

Similarly, eliminating $\bar{\theta}$ between Eqs. (4.15) and (4.18), we find that $\bar{e}$ satisfies an equation identical to Eq. (4.19). Thus, we obtain the solution compatible with Eq. (4.18):

$$\bar{e} = \sum_{i=1}^{2} \bar{\gamma}_2 k_i^2 C_i e^{-k_i x}. \tag{4.23}$$

Integrating both sides of (4.23) with respect to x, we obtain

$$\bar{u} = -\sum_{i=1}^{2} \bar{\gamma}_2 k_i C_i e^{-k_i x}. \tag{4.24}$$

Substituting form Eqs. (4.22) and (4.23) into (4.16), we get

$$\bar{\sigma}_{xx} = \sum_{i=1}^{2} \bar{\gamma}_2 s^2 C_i e^{-k_i x}. \tag{4.25}$$

Applying the boundary conditions (4.17), we arrive at the linear system of equations

$$\sum_{i=1}^{2} \left(\bar{\gamma}_1 k_i^2 - s^2\right) C_i = \frac{\theta_0}{s}, \tag{4.26}$$

$$\sum_{i=1}^{2} s^2 \bar{\gamma}_2 C_i = 0. \tag{4.27}$$

The solution of the above system of equations is given by

$$C_1 = \frac{\theta_0}{s\bar{\gamma}_1 (k_1^2 - k_2^2)}, \tag{4.28}$$

$$C_2 = \frac{-\theta_0}{s\bar{\gamma}_1 (k_1^2 - k_2^2)}. \tag{4.29}$$

This completes the solution of the problem in the Laplace transform domain.

Numerical methods described in Honig and Hirdes (1984) are used to get the inversion solution of the problem in the normal domain.

4.2 Numerical results and discussion

For purposes of numerical evaluation, polymethylmethacrylate (PMMA) material was chosen. The constants of the problem are given by

$$k = 187\ \text{W/(mK)}, \qquad \alpha_t = 6.3(10)^{-5}\ \text{K}^{-1}, \qquad c_E = 1475\ \text{J/(kgK)},$$

$$\eta = 9149.73, \qquad \tau_0 = 0.02\ \text{s}, \qquad \mu = 0.19(10)^{10}\ \text{kg}/\left(\text{ms}^2\right),$$

$$\lambda = 0.4(10)^{10}\ \text{kg}/\left(\text{ms}^2\right), \qquad \rho = 1160\ \text{kg/m}^3, \qquad T_0 = 293, \qquad \beta = 0.5.$$

Firstly, the computations were carried out for three values of time, namely for $t = 0.025, 0.05$, and 0.075. The outcomes are shown graphically in Figs. 1, 2 and 3 for the temperature increment θ, displacement component u and stress component σ_{xx} distributions, respectively. Secondary, Figs. 4, 5 and 6 show the computations for two different values of α, namely for $\alpha = 1.0$ (the case of thermo-viscoelastic (TV)), and $\alpha = 0.5$ (the case of fractional thermo-viscoelastic (FTV)) materials. Thirdly, the influence of the constants A_1 and A_2 and $t = 0.08$ and $\alpha = 0.5$ for stress and displacement components are showed in Figs. 7, 8, 9 and 10, respectively.

There are two waves emanating from the surface of the half space ($x = 0$). The first wave is mainly mechanical, while the second wave is mainly thermal. As expected from the order of the partial differential equations, the finite velocities of propagation of the waves is apparent in all figures. The front of the first wave appears as a finite jump in the graph of σ_{xx} while it is not clear in the graph of the temperature θ because the values of these discontinuities are very small. Also, it appears as a discontinuous first derivative (sharp peak) in the graph of u.

Fig. 1 Temperature distributions for different values of time

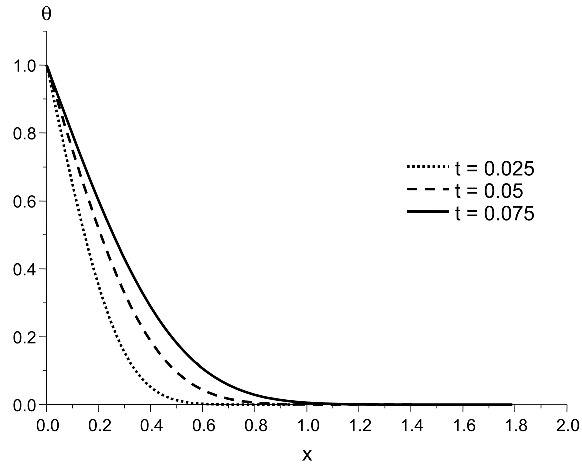


Fig. 2 Displacement distributions for different values of time

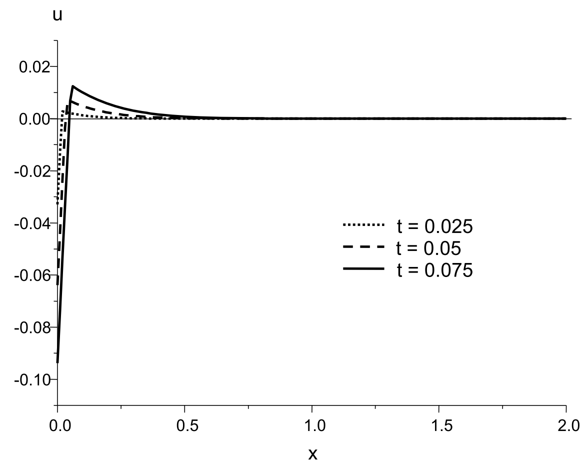


Fig. 3 Stress Distributions for different values of time

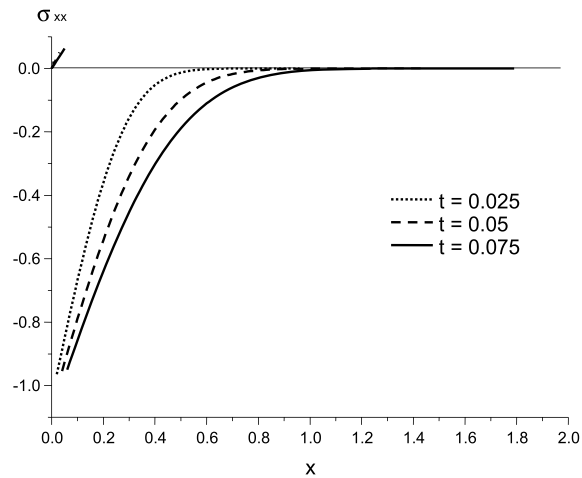


Fig. 4 Temperature distribution for $t = 0.1$

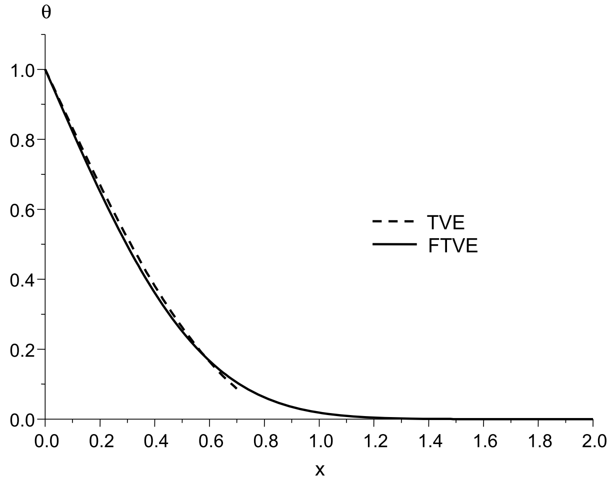


Fig. 5 Displacement distributions for $t = 0.1$

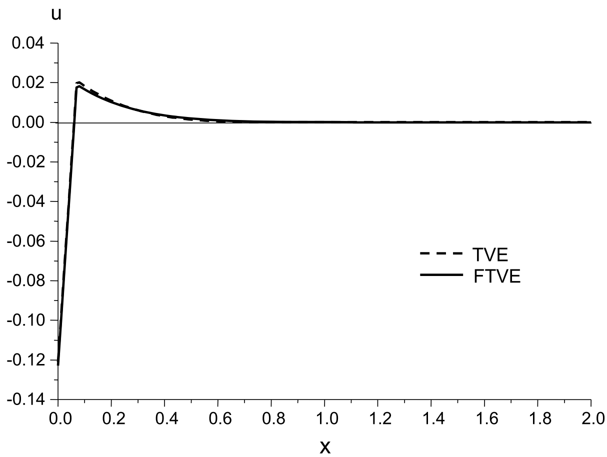
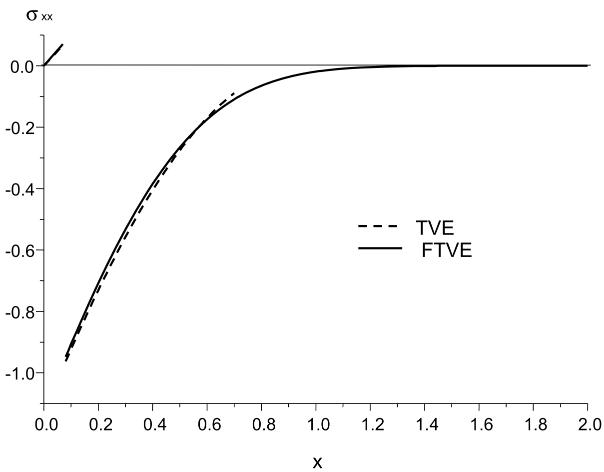


Fig. 6 Stress distribution for $t = 0.1$



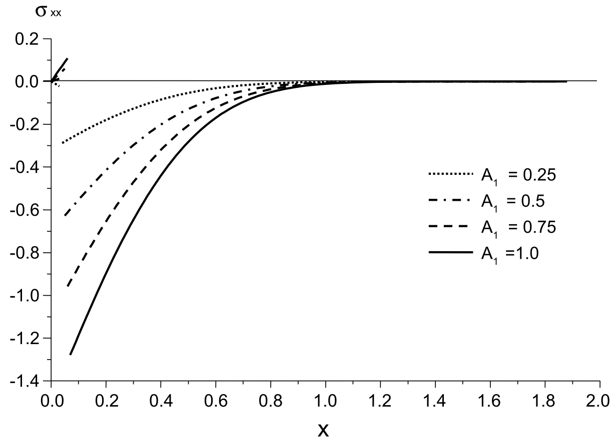


Fig. 7 Influence of A_1 on the stress component σ_{xx} for fixed $A_2 = 0.5$ at $t = 0.08$ and $\alpha = 0.5$

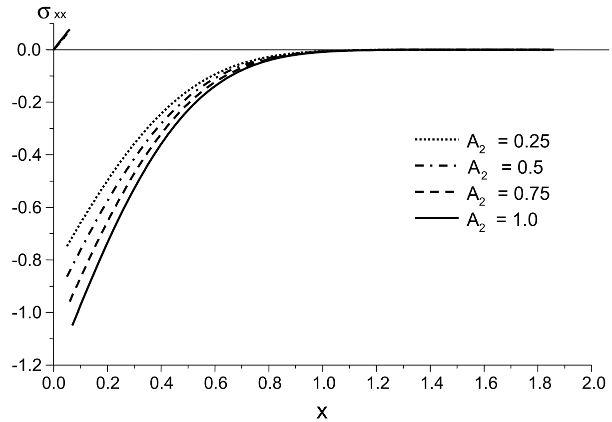


Fig. 8 Influence of A_2 on the stress component σ_{xx} for fixed $A_1 = 0.5$ at $t = 0.08$ and $\alpha = 0.5$

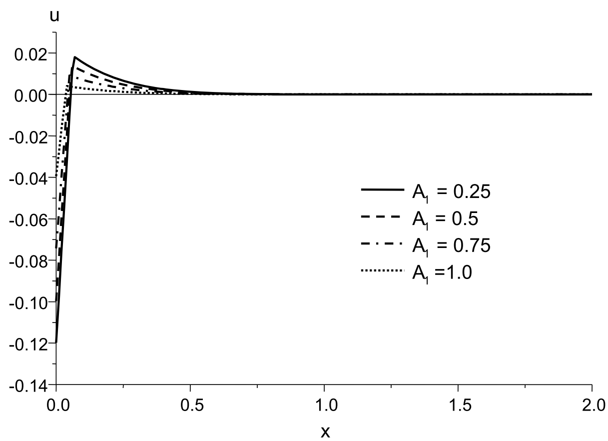


Fig. 9 Influence of A_1 on the displacement component u for fixed $A_2 = 0.5$ at $t = 0.08$ and $\alpha = 0.5$

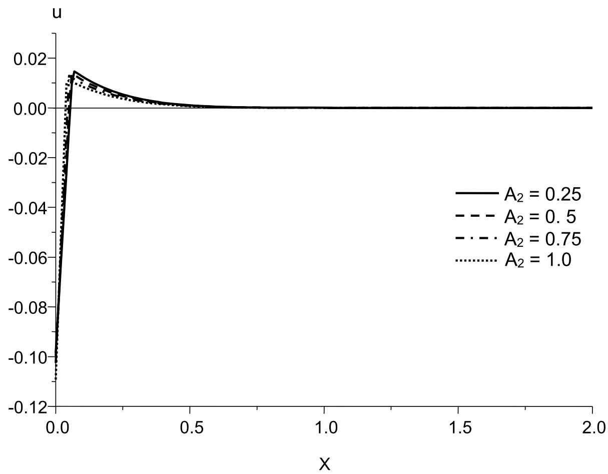


Fig. 10 Influence of A_2 on the displacement component u for fixed $A_1 = 0.5$ at $t = 0.08$ and $\alpha = 0.5$

In Figs. 1, 2 and 3, it is clear that all functions have infinite wave propagation speed. For $t = 0.025$, the two wave fronts are located at the positions $x = 0.0156$ and $x = 0.963$, approximately, while for $t = 0.075$ the waves arrive at the positions $x = 0.0469$ and $x = 1.730$, approximately.

In Figs. 4, 5, and 6, it is found that a change in α has a small effect on the magnitudes of the functions considered and on the speed of the first wave. A change in α, however, has a great effect on the location of the second front wave and hence on the speed of propagation of the second wave. For example, in the case of TVE the positions of the front waves are located at $x = 0.0493$ and 0.568, respectively, and the speed of the two waves $v_1 = 0.616$ and $v_2 = 7.1$. For the FTVE case, the front waves are located at $x = 0.0501$ and 1.846, respectively, and the speed of the two waves are $v_1 = 0.626$ and $v_2 = 23.075$.

Also, in Figs. 7, 8, 9 and 10, it was observed that viscoelastic effects decrease the absolute value of the stress for all values of the parameters A_1 and A_2. The increase of either of A_1 or A_2 tends also to decrease the absolute values of stress. Also, viscoelastic effects tend to decrease the absolute values of the displacement.

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