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An alternative point of view to the theory of fractional Fourier transform

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The concept of the fractional Fourier transform is framed within the context of quantum evolution operators. This point of view yields an extension of the above concept and greatly simplifies the underlying operational algebra. It is also proved that a multidimensional extension can be performed by using a biorthogonal multiindex harmonic oscillator basis. It is finally shown that most of the proposed physical interpretations of the fractional Fourier transform are just trivial consequences of the analysis developed in this paper.

1. Introduction

The concept of Fourier transform of fractional order (F.O.F.T.) was introduced by Namias (1980), who established the main properties and provided the rules of the relevant operational calculus. The Namias ideas have been put on more rigorous basis by McBride & Kerr (1987), who eliminated, by a proper redefinition of the F.O.F.T. operator, the ambiguity contained in the first 'heuristic' formulation. Notwithstanding the weakness of the Namias approach from the purely mathematical point of view, the methods and the concepts developed in (Namias 1980) provide effective tools to deal with time-dependent Schrödinger equations or other types of non-homogenous parabolic equations. Furthermore they have stimulated interesting speculation in optics (Sashin *et al.* 1995) and in quantum optics (Aytür & Ozaktas 1995).

In this paper we propose a point of view different from the theory of F.O.F.T. by showing that it is a particular case of the evolution operator theory. The F.O.F.T. will be shown to be generated by an evolution operator belonging to a Hamiltonian admitting SU(1,1) as dynamical group. We will see how this point of view puts the F.O.F.T. in a wider context, allows the introduction of a generalized form of F.O.F.T., justifies the speculations of (Sashin *et al.* 1995, Aytür & Ozaktas 1995) and suggests further physical interpretations.

The paper is organized as follows. In Section 2 we recall the Namias definition of F.O.F.T. and show how it can be derived within the context of SU(1,1) evolution operator theory. In Section 3 we discuss the extension of the Namias operator to the multidimensional case and, as an important byproduct, we derive the Mehler formula for the Hermite functions with many variables and many indices. Final comments are contained in Section 4, which is also devoted to the physical interpretation of the concepts associated with F.O.F.T.

2. Fractional order Fourier transform and evolution operators

The F.O.F.T. operation, denoted by F_α, has been suggested by the following simple considerations.

(a) The harmonic oscillator functions are eigenfunctions of the ordinary F.T. denoted by $\mathsf{F}_{\frac{1}{2}\pi}$, namely

$$\mathsf{F}_{\frac{1}{2}\pi}\left[\mathrm{e}^{-\frac{1}{2}x^2}H_n(x)\right] = \mathrm{e}^{\frac{1}{2}\mathrm{i}n\pi}\left[\mathrm{e}^{-\frac{1}{2}x^2}H_n(x)\right]. \tag{1}$$

(b) The obvious generalization of the above identity can be written as

$$\mathsf{F}_\alpha\left[\mathrm{e}^{-\frac{1}{2}x^2}H_n(x)\right] = \mathrm{e}^{\mathrm{i}n\alpha}\left[\mathrm{e}^{-\frac{1}{2}x^2}H_n(x)\right]. \tag{2}$$

(c) By denoting the operator F_α by $\mathrm{e}^{\mathrm{i}\alpha\hat{A}}$ and by keeping the derivative of both sides of (2) with respect to α, we find the following differential realization for $\hat{A}$:

$$\hat{A} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 - \frac{1}{2}. \tag{3}$$

It is obvious that for α real

$$\mathsf{F}_\alpha = \mathrm{e}^{\mathrm{i}\alpha\hat{A}} \tag{4}$$

is a unitary operator, which follows from the hermiticity of $\hat{A}$. The action of F_α on a given function of x can be viewed as that of an evolution operator relevant to a Hamiltonian of a harmonic oscillator; with respect to this constant term $-\frac{1}{2}$ plays a minor role. We rewrite (4) as

$$\mathsf{F}_\alpha = \exp(-\tfrac{1}{2}\mathrm{i}\alpha)\exp(-\tfrac{1}{2}\mathrm{i}\alpha\mathrm{d}^2/\mathrm{d}x^2 + \tfrac{1}{2}\mathrm{i}\alpha x^2). \tag{5}$$

From group theory, we introduce the following SU(1,1) generators (Dattoli *et al.* 1987)

$$\hat{K}_+ = \tfrac{1}{2}\mathrm{i}\mathrm{d}^2/\mathrm{d}x^2, \quad \hat{K}_- = -\tfrac{1}{2}\mathrm{i}x^2, \quad \hat{K}_0 = -\tfrac{1}{2}\left(x\mathrm{d}/\mathrm{d}x + \tfrac{1}{2}\right) \tag{6}$$

which satisfy the rules of commutation

$$\left[\hat{K}_+, \hat{K}_-\right] = -2\hat{K}_0, \quad \left[\hat{K}_0, \hat{K}_\pm\right] = \pm\hat{K}_\pm. \tag{7}$$

By using the ordinary ordering theorems, we can write F_α as the following product of exponential operators; see, for example, (Dattoli *et al.* 1987, Wei & Norman 1963):

$$\mathsf{F}_\alpha = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha}\mathrm{e}^{g(\alpha)\hat{K}_-}\mathrm{e}^{2h(\alpha)\hat{K}_0}\mathrm{e}^{f(\alpha)\hat{K}_+}, \tag{8}$$

where (see Dattoli *et al.* (1987) for the details of the derivation)

$$g(\alpha) = -\tan\alpha, \quad f(\alpha) = -\tan\alpha, \quad \mathrm{e}^{h(\alpha)} = \cos\alpha. \tag{9}$$

The action of F_α on a given function $F(x)$ can be evaluated by using the following two operational identities (Dattoli *et al.* 1990):

$$\mathrm{e}^{t\mathrm{d}^2/\mathrm{d}x^2}\phi(x) = \frac{1}{2\surd(\pi t)}\int_{-\infty}^{+\infty}\phi(x')\mathrm{e}^{-(x-x')^2/4t}\mathrm{d}x', \quad t > 0,$$

$$\mathrm{e}^{ax\mathrm{d}/\mathrm{d}x}\phi(x) = \phi(\mathrm{e}^a x)\,, \tag{10}$$

which, together with (8), after a little algebra yield

$$(\mathsf{F}_\alpha F)(x) = \frac{\mathrm{e}^{\mathrm{i}\left(\frac{1}{4}\pi\hat{\alpha}-\frac{1}{2}\alpha\right)}}{(2\pi)^{\frac{1}{2}}\,|\sin\alpha|^{\frac{1}{2}}}\mathrm{e}^{\frac{1}{2}\mathrm{i}x^2\tan\alpha}\int_{-\infty}^{+\infty}F(x')\exp\left\{-\mathrm{i}/2\tan\alpha\left(x/\cos\alpha - x'\right)^2\right\}\mathrm{d}x', \tag{11}$$

where $\hat{\alpha} = \mathrm{sign}(\alpha)$ and $-\pi < \alpha < \pi$ the values $\alpha = 0$ and $\alpha = \pm\pi$ reproducing the identity and reflection (with respect to $x = 0$) operators respectively.

Since we have clarified that F_α is just an evolution operator, most of the operational rules established in Namias (1980), McBride & Kerr (1987) follow from elementary quantum mechanics. To give an example we note that since F_α is unitary we have

$$\mathsf{F}_\alpha x = \left(\mathsf{F}_\alpha x \mathsf{F}_\alpha^\dagger\right)\mathsf{F}_\alpha\,, \tag{12}$$

and we also obtain

$$\mathsf{F}_\alpha x \mathsf{F}_\alpha^\dagger = \mathrm{e}^{\mathrm{i}\alpha\hat{A}}x\mathrm{e}^{-\mathrm{i}\alpha\hat{A}} = x\cos\alpha + \frac{1}{\mathrm{i}}\sin\alpha\frac{\mathrm{d}}{\mathrm{d}x}, \tag{13}$$

which is nothing but the Heisenberg evolution of x. The same procedure can be used to establish identities of the type

$$\mathsf{F}_\alpha x^m = \left(x\cos\alpha + \frac{1}{\mathrm{i}}\sin\alpha\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\mathsf{F}_\alpha\,. \tag{14}$$

Before concluding this section, let us note that we can derive from (11) the Mehler sum rule (Morse & Feschback 1953, p. 871), which is one of the starting points of the Namias procedure. Indeed by expanding $F(x)$ in harmonic oscillator eigenfunctions, we obtain

$$F(x) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{-\frac{1}{2}x^2}H_n(x), \tag{15}$$

where

$$a_n = \frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}H_n(x)\mathrm{e}^{-\frac{1}{2}x^2}F(x)\,\mathrm{d}x\,. \tag{16}$$

By noting that, according to (2) and (15)

$$(\mathsf{F}_\alpha F)(x) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\alpha}\mathrm{e}^{-\frac{1}{2}x^2}H_n(x) \tag{17}$$

by inserting (16) in (17), and using (11), after some manipulation we get

$$\sum_{n=0}^{\infty}\frac{\mathrm{e}^{in\alpha}}{2^n n!}H_n(x)H_n(x') = \frac{1}{(1-\mathrm{e}^{2\mathrm{i}\alpha})^{\frac{1}{2}}}\exp\left(\left\{2xx'\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{2\mathrm{i}\alpha}\left(x^2+x'^2\right)\right\}/\left\{1-\mathrm{e}^{2\mathrm{i}\alpha}\right\}\right) \tag{18}$$

which is just the Mehler sum rule. We will see that this result can be extended to generalized Hermite polynomials within the context of the multidimensional extension of the F.O.F.T. concept.

3. Generalized form of fractional Fourier transform

A first, almost trivial, example of generalization of F.O.F.T. is that of considering the following slightly generalized form of $\hat{A}$:

$$\hat{A} = -\frac{a}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{b}{2}x^2 + \frac{\mathrm{i}c}{2}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right), \tag{19}$$

where a, b, c are assumed real to ensure the hermiticity of $\hat{A}$. By using the operators (4) and (19) and by exploiting the same ordering procedure as before, we find the characteristic functions (Dattoli *et al.* 1987)

$$\mathcal{R}(\alpha) = \mathrm{e}^{h(\alpha)} = \cosh\left(\sqrt{\Delta}\alpha\right) + \frac{c}{2\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}\alpha\right),$$

$$\mathrm{e}^{h(\alpha)} f(\alpha) = -\frac{a}{\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}\alpha\right), \quad \mathrm{e}^{h(\alpha)} g(\alpha) = -\frac{b}{\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}\alpha\right)$$

$$\Delta = -ab + \tfrac{1}{4}c^2. \tag{20}$$

The use of the identities (10) finally yields

$$(\mathsf{F}_\alpha F)(x) = \frac{\exp\left(-\mathrm{sign}(\alpha)\frac{1}{2}\mathrm{i}\left[\frac{1}{2}\pi - x^2 g(-|\alpha|)\right]\right)}{(2\pi\left[-f(-|\alpha|)\right])^{\frac{1}{2}}}(\mathcal{R}(-|\alpha|))^{\frac{1}{2}}$$

$$\times \int_R \exp\left\{-\mathrm{sign}(\alpha)\frac{1}{2}\mathrm{i}\left[(\mathcal{R}(-|\alpha|)x - y)^2\right] / - f(-|\alpha|)\right\} F(y)\,\mathrm{d}y. \tag{21}$$

which holds for $\Delta > 0$, $a > 0$ and for every α.

The possibility of a more interesting generalization of F.O.F.T. is offered by the entangled harmonic oscillator eigenvalue equation (Dattoli & Torre 1995) satisfied by the functions $\mathcal{H}_{n,m}(x, y)$ and $\mathcal{G}_{n,m}(x, y)$, namely

$$\left(-\partial_z^T \hat{M}^{-1}\partial_z - 1 + \tfrac{1}{4}z^T \hat{M} z\right) T_{n,m}(x, y) = (n + m)T_{n,m}(x, y), \tag{22}$$

where $T_{n,m}$ is $\mathcal{H}_{n,m}$ or $\mathcal{G}_{n,m}$, with

$$\mathcal{H}_{n,m}(x, y) = \frac{\Delta_{\hat{M}}^{\frac{1}{4}}}{(2\pi)^{\frac{1}{2}}}\frac{1}{(n!m!)^{\frac{1}{2}}} H_{n,m}(x, y)\mathrm{e}^{-\frac{1}{4}z^T M z},$$

$$\mathcal{G}_{n,m}(x, y) = \frac{\Delta_{\hat{M}}^{\frac{1}{4}}}{(2\pi)^{\frac{1}{2}}}\frac{1}{(n!m!)^{\frac{1}{2}}} G_{n,m}(x, y)\mathrm{e}^{-\frac{1}{4}z^T M z},$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, c > 0, \quad \Delta_{\hat{M}} = ac - b^2 \neq 0 \tag{23}$$

and T denotes transpose. The functions $H_{n,m}$ and $G_{n,m}$ are generalized Hermite polynomials provided by the following Rodrigues type relation:

$$H_{n,m}(x,y) = (-1)^{n+m} e^{\frac{1}{2}z^T \hat{M} z} \frac{\partial^{n+m}}{\partial x^n \partial y^m} e^{-\frac{1}{2}z^T \hat{M} z},$$

$$G_{n,m}(x,y) = (-1)^{n+m} e^{\frac{1}{2}w^T \hat{M}^{-1} w} \frac{\partial^{n+m}}{\partial \xi^n \partial \eta^m} e^{-\frac{1}{2}w^T \hat{M}^{-1} w},$$

$$w = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \hat{M} z . \tag{24}$$

It is finally clear that the quadratic operator on the left-hand side of (22) is the generalization of the one-dimensional quantum harmonic oscillator Hamiltonian, in fact

$$\partial_z = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y, \end{pmatrix} \tag{25}$$

and

$$-\frac{1}{\Delta_{\hat{M}}} \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \tag{26}$$

is the entangled Laplace operator.

The bidimensional F.O.F.T. operator can be therefore defined as follows:

$$\mathsf{F}_\alpha = e^{-i\alpha} e^{i\alpha \hat{D}}, \quad \hat{D} = -\partial_x^T \hat{M}^{-1} \partial_x + \tfrac{1}{4} x^T \hat{M} x \tag{27}$$

and according to (22)

$$\mathsf{F}_\alpha T_{n,m} = e^{i(n+m)\alpha} T_{n,m} . \tag{28}$$

We can obtain the two-dimensional F.O.F.T. by employing the same procedure as before if we note that the SU(1,1) generators can be realized by the following operators:

$$K_+ = i\partial_z^T \hat{M}^{-1} \partial_z , \quad K_- = -i\tfrac{1}{4} z^T \hat{M} z , \quad K_0 = -\tfrac{1}{4}\left(z^T \partial_z + \partial_z^T z\right), \tag{29}$$

and further by noting that the bidimensional counterparts of equations (10) read

$$e^{\beta K_-} \psi(x,y) = e^{-i\frac{1}{4}z^T \hat{M} z} \psi(x,y),$$

$$e^{\beta K_0} \psi(x,y) = e^{-\frac{1}{2}\beta} \psi(e^{-\frac{1}{2}\beta} x, e^{-\frac{1}{2}\beta} y),$$

$$e^{\beta K_+} \psi(x,y) = \frac{\Delta_{\hat{M}}^{\frac{1}{2}}}{i4\pi\beta} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' e^{-1/4i\beta (z-z')^T \hat{M}(z-z')} \psi(z'), \quad \Im(\beta) > 0 \tag{30}$$

we can specify the action of the bidimensional F_α operator as follows:

$$(\mathsf{F}_\alpha F)(x,y) = -\frac{\Delta_{\hat{M}}^{\frac{1}{2}}}{4\pi\,|\sin\alpha|} e^{-i\alpha - \frac{1}{2}i\pi\hat{\alpha}} e^{\frac{1}{4}i \tan\alpha z^T \hat{M} z}$$

$$\times \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}y' \exp\left\{-\frac{i}{4\tan\alpha}\left((\cos\alpha)^{-1} z - z'\right)^T \hat{M}\left((\cos\alpha)^{-1} z - z'\right)\right\} F(x', y'),$$

$$-\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi\,, \alpha \neq 0, \tag{31}$$

or in a form closer to the Namias formula, as

$$(\mathsf{F}_\alpha F)(x, y) = -\frac{\mathrm{e}^{-\mathrm{i}\alpha - \frac{1}{2}\mathrm{i}\pi\hat{\alpha}} \Delta_{\hat{M}}^{\frac{1}{2}}}{4\pi\,|\sin\alpha|}$$

$$\times \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}y' \exp\left\{-\frac{1}{4}\mathrm{i}\cot\alpha\left(z^T\hat{M}z - z'^T\hat{M}z'\right) + \frac{\mathrm{i}}{2\sin\alpha} z^T\hat{M}z'\right\} F(x', y'),$$

$$-\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi\,, \alpha \neq 0. \tag{32}$$

It is important to note that, in the limit $\alpha = \frac{1}{2}\pi$ we find that

$$\left(\mathsf{F}_{\frac{1}{2}\pi} F\right)(x, y) = \frac{\Delta_{\hat{M}}^{\frac{1}{2}}}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{\frac{1}{2}\mathrm{i}z^T\hat{M}z'} F(x', y')\,\mathrm{d}x'\,\mathrm{d}y',$$

$$\left(\mathsf{F}_{\frac{\pi}{2}} T_{n,m}\right)(x, y) = \mathrm{e}^{\mathrm{i}(n+m)\frac{\pi}{2}} T_{n,m}(x, y), \tag{33}$$

that is, either $\mathcal{H}_{n,m}$ or $\mathcal{G}_{n,m}$ is an eigenfunction of the bidimensional F.O.F.T. operator. Other definitions of the bidimensional Fourier transform are possible but do not satisfy the property (28). For a more general treatment of the two-dimensional Fourier transform the reader is addressed to Dattoli & Torre (1995) and Dattoli *et al.* (1997).

An important byproduct of the above results is the derivation of a generalized form of the Mehler sum rule for the entangled Hermite polynomials. Remember that the functions $\mathcal{H}_{n,m}$ and $\mathcal{G}_{n,m}$ provide a biorthogonal system in the sense that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{H}_{n,m}(x, y)\mathcal{G}_{n',m'}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \delta_{n,n'}\delta_{m,m'}\,. \tag{34}$$

We can therefore use this property to expand a two-variable function as follows:

$$F(x, y) = \sum_{n,m=0}^{\infty} a_{n,m}\mathcal{H}_{n,m}(x, y),$$

$$a_{n,m} = \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}y' F(x', y')\mathcal{G}_{n,m}(x', y')\,. \tag{35}$$

According to (28) we also find

$$(\mathsf{F}_\alpha F)(x, y) = \sum_{n,m=0}^{\infty} a_{n,m}\mathrm{e}^{\mathrm{i}(n+m)\alpha}\mathcal{H}_{n,m}(x, y)\,. \tag{36}$$

By means of $(35)_2$, equation (36) can be cast in the form

$$(\mathsf{F}_\alpha F)(x,y) = \sum_{n,m=0}^{\infty} \mathrm{e}^{\mathrm{i}(n+m)\alpha} \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}y' F(x',y') \mathcal{G}_{n,m}(x',y') \mathcal{H}_{n,m}(x,y) \quad (37)$$

which, using (3), yields (for $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi, \alpha \neq 0$)

$$\sum_{n,m=0}^{\infty} \mathrm{e}^{\mathrm{i}(n+m)\alpha} \frac{1}{n!m!} G_{n,m}(x',y') H_{n,m}(x,y) = \frac{1}{\left(1-\mathrm{e}^{2\mathrm{i}\alpha}\right)^{\frac{1}{2}}} \times \exp\left\{ \frac{2z^T \hat{M} z' \mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{2\mathrm{i}\alpha}\left(z^T \hat{M} z + z'^T \hat{M} z'\right)}{2\left(1-\mathrm{e}^{2\mathrm{i}\alpha}\right)} \right\}. \quad (38)$$

This last identity is fairly important, being the extension of the Mehler formula to the multivariable multiindex Hermite functions.

To complete this section we note that the rules of the operational calculus of the bidimensional operator F_α can be established fairly straightforwardly and in fact we can infer that

$$\mathsf{F}_\alpha z \mathsf{F}_\alpha^\dagger = \mathrm{e}^{\mathrm{i}\alpha\hat{D}} z \mathrm{e}^{-\mathrm{i}\alpha\hat{D}} = z\cos\alpha - 2\mathrm{i}\sin\alpha M^{-1} \partial_z \quad (39)$$

which facilitates the following alternative definition of F.O.F.T. operations:

$$(\mathsf{F}_\alpha F)(x,y) = F\left(x\cos\alpha - 2i\sin\alpha\left(\frac{c}{\Delta}\partial_x - \frac{b}{\Delta}\partial_y\right), y\cos\alpha - 2i\sin\alpha\left(-\frac{b}{\Delta}\partial_x + \frac{a}{\Delta}\partial_y\right)\right). \quad (40)$$

The multidimensional generalization (more than two indices and two variables) of the F.O.F.T. is straightforward and obtained from (31) by replacing $\hat{M}$ with an $n \times n$ symmetric matrix, z with an n-component vector, the integral with an n-dimensional integral and F with an n-variable function.

4. Concluding remarks

The fact that we have viewed the operator F_α as an ordinary evolution operator clarifies in a fairly transparent way its physical meaning. It is therefore obvious that any evolution problem treated by means of a Schrödinger equation involving a quadratic potential may be considered a F.O.F.T.

In particular the paraxial propagation of an optical beam through a lens-like medium is a relevant genuine example.

It is indeed well known that the propagation of an electromagnetic wave in a non-homogeneous medium with a quadratic refractive index profile is governed by an equation of the type (Yariv 1975)

$$\left[\frac{1}{2k_0(z)}\frac{\partial^2}{\partial\eta^2} - \frac{k_2(z)}{2}\eta^2\right]\psi(\eta,z) = \mathrm{i}\frac{\partial}{\partial z}\psi(\eta,z), \quad \phi(\eta) = \psi(\eta,0), \quad (41)$$

where z is the longitudinal coordinate of propagation and η is one of the transverse coordinates (x or y). Furthermore $k_0(z)$ is the wave number on the axis of propagation and $k_2(z)$ is associated to the refractive index. In the case in which $k_0(z)$ and $k_2(z)$ are not explicitly z-dependent‡ the solution of (41) is just the F.O.F.T. of $\phi(\eta)$. We must also emphasize that in the general time-dependent case the solution can be found in a F.O.F.T. like form but in that case the characteristic functions (f, g, h) are obtained as the solution of a nonlinear system of first-order differential equations, namely (Dattoli *et al.* 1987)

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \mathrm{e}^{h} & -\mathrm{e}^{h}f \\ \mathrm{e}^{h}g & -\mathrm{e}^{h}fg+\mathrm{e}^{-h} \end{pmatrix} = \begin{pmatrix} 0 & 1/k_0(z) \\ -k_2(z) & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{h} & -\mathrm{e}^{h}f \\ \mathrm{e}^{h}g & -\mathrm{e}^{h}fg+\mathrm{e}^{-h} \end{pmatrix}. \tag{42}$$

It is also clear that the same method can be applied to other problems involving, for example, the evolution of squeezed states.

The concepts developed in this paper indicate that the idea of F.O.F.T. can be extended to other types of solvable Hamiltonians admitting SU(1,1) as dynamical group. This aspect of the problems will be considered in a forcoming investigation.

In the previous sections we have stressed the hermiticity of the operators $\hat{A}$ and $\hat{D}$ which ensures the unitarity of F_α and thus the fact that $\mathsf{F}_\alpha^\dagger$ produces a F.O.F.T. which is the complex conjugate of that associated with F_α. We must emphasize that the unitarity of the operator-generating transform of the fractional type is not crucial. Namias has in fact introduced the fractional Hankel transform defined by the operator (Namias 1980)

$$\hat{H}_\alpha = \mathrm{e}^{\mathrm{i}\alpha\hat{B}},$$

$$\hat{B} = -\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{4x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4} + \frac{v^2-1}{4x^2} - \frac{(v+1)}{2}; \tag{43}$$

$\hat{H}_\alpha$ can be viewed as a non-unitary evolution operator since

$$\hat{B} \neq \hat{B}^\dagger = -\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{4x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4} + \frac{v^2}{4x^2} - \frac{(v+1)}{2}. \tag{44}$$

However, the techniques developed in the previous sections can be exploited in this case too, the only difference being that two different Hankel transforms should be considered, one associated with $\hat{B}$ and the other with $\hat{B}^\dagger$.

The fractional Hankel transform can be shown to be contained in the F.O.F.T. This fact can be understood as follows. The function

$$L_p(x^2+y^2) = \frac{1}{2\pi^{\frac{1}{2}}}L_p(x^2+y^2)\mathrm{e}^{-\frac{1}{4}(x^2+y^2)}, \tag{45}$$

where L_p denotes Laguerre polynomials (Morse & Feschback 1953), satisfies the differential equation

$$\left[-\partial_x^2 - \partial_y^2 + \tfrac{1}{4}\left(x^2+y^2\right)\right]L_p(x^2+y^2) = 2pL_p(x^2+y^2). \tag{46}$$

‡ Note that within the present context z plays the role of time and thus of α.

We can therefore define the following bidimensional operators:

$$\mathsf{F}_\alpha = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha Q_x}\mathrm{e}^{\mathrm{i}\alpha Q_y}, \qquad Q_\eta = -\partial_\eta^2 + \tfrac{1}{4}\eta^2 \tag{47}$$

which yield

$$(\mathsf{F}_\alpha\psi)(x, y) = -\frac{\mathrm{e}^{-\mathrm{i}\alpha - \frac{1}{2}\mathrm{i}\pi\hat{\alpha}}}{4\pi\,|\sin\alpha|}\mathrm{e}^{-\frac{\mathrm{i}}{4}\cot\alpha(x^2+y^2)}$$

$$\times\int_{-\infty}^{+\infty}\mathrm{d}x'\int_{-\infty}^{+\infty}\mathrm{d}y'\exp\left\{\frac{1}{4}\mathrm{i}\cot\alpha\left(x'^2+y'^2\right)+\frac{\mathrm{i}}{2\sin\alpha}\left(xx'+yy'\right)\right\}\psi(x', y'),$$

$$-\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi, \tag{48}$$

the value $\alpha = 0$ reproducing the unit operator.

By using polar coordinates, by assuming that ψ has a cylindrical symmetry and by exploiting the following integral representation for the J_0-Bessel function:

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}x\cos\theta}\,\mathrm{d}\theta, \tag{49}$$

$$(\mathsf{F}_\alpha\psi)(\rho) = -\frac{\mathrm{e}^{-\mathrm{i}\alpha-\frac{1}{2}\mathrm{i}\pi\hat{\alpha}}}{4\pi\,|\sin\alpha|}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\cot\alpha\,\rho^2}\int_0^{+\infty}\rho' d\rho' \mathrm{e}^{-\frac{1}{4}\mathrm{i}\cot\alpha\,\rho'^2}\psi(\rho')J_0\left(\frac{\rho\rho'}{2\sin\alpha}\right), \tag{50}$$

which is a particular case of the fractional Hankel transform introduced in Namias (1980). By using the same procedure leading to the Mehler sum rule for the Hermite polynomials, we can exploit the fact that L_p is an orthogonal basis (Morse & Feschback 1953), to get the following Mehler addition formula for Laguerre polynomials:

$$\sum_{p=0}^{\infty}L_p(\rho)L_p(\rho')\frac{\mathrm{e}^{2\mathrm{i}p\alpha}}{2\pi} = \frac{1}{1-\mathrm{e}^{2\mathrm{i}\alpha}}\exp\left\{-\frac{\left(\rho^2+\rho'^2\right)\mathrm{e}^{2\mathrm{i}\alpha}}{2\left(1-\mathrm{e}^{2\mathrm{i}\alpha}\right)}\right\}J_0\left(\frac{\rho\rho'}{2\sin\alpha}\right). \tag{51}$$

The more general case of fractional-order Hankel transform, involving the generalized Laguerre polynomials, will be discussed elsewhere.

Before closing we want to further emphasize the usefulness of the F.O.F.T. concept from the mathematical point of view; we have seen that by inverting the procedure we can derive the Mehler sum rule as a consequence of the fractional-order transform. This fact can be exploited to obtain Mehler type sum rules for other types of special functions.

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