F.E. COSSERAT

WEAK FORM

UNIQUENESS

December 12, 2025

9-12-2005 (1)

J.S

$$\rho \ddot{u}_i = \sigma_{ji,j} + X_i \quad (1)$$

$$J \ddot{\varphi}_i = \varepsilon_{ijk}\sigma_{jk} + \tau_{ji,j} + Y_i \quad (2)$$

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\delta_{ij}\gamma_{kk} \quad (3)$$

$$\tau_{ji} = (\hat{\gamma}+\hat{\varepsilon})\kappa_{ji} + (\hat{\gamma}-\hat{\varepsilon})\kappa_{ij} + \beta\delta_{ij}\kappa_{kk} \quad (4)$$

$$\gamma_{ij} = u_{j,i} - \varepsilon_{ijk}\varphi_k \quad (5)$$

$$\kappa_{ij} = \varphi_{j,i} \quad (6)$$

<u>2D:</u> $u = (u_1, u_2, 0)$, $\varphi = (0, 0, \varphi_3)$, $X = (X_1, X_2, 0)$, $Y = (0, 0, Y_3)$

For $i=1$ (1) is

$$\rho \ddot{u}_1 = \underset{j=1}{\sigma_{11,1}} + \underset{j=2}{\sigma_{21,2}} + X_1 \quad (7)$$

$i=2$

$$\rho \ddot{u}_2 = \underset{j=1}{\sigma_{12,1}} + \sigma_{22,2} + X_2 \quad (8)$$

Levi Civita:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (2,1,3), (1,3,2) \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \end{cases} \quad (9)$$

$$\varepsilon_{ijk}\sigma_{jk} = \overset{=1}{\varepsilon_{123}}\,\overset{=0}{\sigma_{23}} + \overset{=1}{\varepsilon_{231}}\,\overset{=0}{\sigma_{31}} + \overset{=1}{\varepsilon_{312}}\,\sigma_{12} + \overset{=-1}{\varepsilon_{321}}\,\sigma_{21} + \overset{=-1}{\varepsilon_{132}}\,\overset{=0}{\sigma_{32}} + \overset{=-1}{\varepsilon_{213}}\,\overset{=0}{\sigma_{13}}$$

$$= \sigma_{12} - \sigma_{21} \qquad (10)$$

Then (τ exists only in z-direction)

$$J\ddot{\varphi} = \sigma_{12} - \sigma_{21} + \tau_{13,1} + \tau_{23,2} + Y_3 \qquad (11)$$

From (5)

$i=1, j=1$ $\quad \gamma_{11} = u_{1,1} - \overset{=0}{\varepsilon_{113}}\varphi = u_{1,1} \qquad (12)$

$i=1, j=2$ $\quad \gamma_{12} = u_{2,1} - \overset{=1}{\varepsilon_{123}}\varphi = u_{2,1} - \varphi \qquad (13)$

$i=2, j=1$ $\quad \gamma_{21} = u_{1,2} - \varepsilon_{213}\varphi = u_{1,2} + \varphi \qquad (14)$

$i=2, j=2$ $\quad \gamma_{22} = u_{2,2} - \overset{=0}{\varepsilon_{223}}\varphi = u_{2,2} \qquad (15)$

Next use (12)-(15) in the constitutive relation for σ:

$$\sigma_{11} = (\mu+\alpha)\gamma_{11} + (\mu-\alpha)\gamma_{11} + \lambda\delta_{11}\gamma_{kk}$$

sum over repeated index ← γ_{kk}

$$+ \lambda\delta_{11}(\gamma_{11} + \gamma_{22})$$

$$= (\mu+\alpha)\frac{\partial u_1}{\partial x_1} + (\mu-\alpha)\frac{\partial u_1}{\partial x_1} + \lambda\delta_{11}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)$$

$$= (\lambda+2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} \qquad (16)$$

$$\sigma_{12} = (\mu+\alpha)(u_{2,1} - \varphi) + (\mu-\alpha)(u_{1,2} + \varphi) \qquad (17)$$

$$= \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \alpha\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - 2\alpha\varphi$$

$$= 2\mu\, \varepsilon_{12}(\vec{u}) + \alpha \operatorname{curl} \vec{u} - 2\alpha\varphi$$

$$\sigma_{21} = (\mu+\alpha)\gamma_{21} + (\mu-\alpha)\gamma_{12}$$

$$= (\mu+\alpha)(u_{1,2} + \varphi) + (\mu-\alpha)(u_{2,1} - \varphi) \qquad (18)$$

$$= \mu(u_{2,1} + u_{1,2}) + \alpha\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) + 2\alpha\varphi$$

$$= 2\mu\, \varepsilon_{12}(\vec{u}) \cancel{+} - \alpha \operatorname{curl} \vec{u} + 2\alpha\varphi$$

$$\sigma_{22} = (\mu+\alpha)\gamma_{22} + (\mu-\alpha)\gamma_{22} + \lambda(\gamma_{11} + \gamma_{22})$$

$$= (\mu + \cancel{\alpha})\frac{\partial u_2}{\partial x_2} + (\mu - \cancel{\alpha})\frac{\partial u_2}{\partial x_2} + \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \qquad (19)$$

$$= (\lambda + 2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_1}{\partial x_1}$$

4-12-2025 (4)

J. S.

$$\sigma_{ji,j} = \frac{\partial \sigma_{ji}}{\partial x_j}$$

$j=1,\ i=1$ $\quad \sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial x_1}$

$$= \frac{\partial}{\partial x_1}\left[(\lambda+2\mu)\frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2}\right] \qquad (20)$$

$j=1,\ i=2$ $\quad \sigma_{12,1} = \frac{\partial \sigma_{12}}{\partial x_1}$

$$= \frac{\partial}{\partial x_1}\left[(\mu+\alpha)\frac{\partial u_2}{\partial x_1} + (\mu-\alpha)\frac{\partial u_1}{\partial x_2} - 2\alpha\varphi\right] \qquad (21)$$

$$= \frac{\partial}{\partial x_1}\left[\mu\underbrace{\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)}_{2\varepsilon_{12}(u)} + \alpha\underbrace{\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)}_{\mathrm{curl}\,\vec{u}} - 2\alpha\varphi\right]$$

$j=2,\ i=1$ $\quad \sigma_{21,2} = \frac{\partial \sigma_{21}}{\partial x_2}$

$$= \frac{\partial}{\partial x_2}\left[(\mu+\alpha)\frac{\partial u_1}{\partial x_2} + (\mu-\alpha)\frac{\partial u_2}{\partial x_1} + 2\alpha\varphi\right] \qquad (22)$$

$$= \frac{\partial}{\partial x_2}\left[\mu\underbrace{\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)}_{2\varepsilon_{12}(u)} + \alpha\underbrace{\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)}_{(-\mathrm{curl}\,\vec{u})} + 2\alpha\varphi\right]$$

$j=2\ \ i=2$ $\quad \sigma_{22,2} = \frac{\partial \sigma_{22}}{\partial x_2}$

$$= \frac{\partial}{\partial x_2}\left[(\lambda+2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_1}{\partial x_1}\right] \qquad (23)$$

test function

$$\left((\tau_{31}, \tau_{3,2}), \underbrace{(w_1, w_2)}_{\vec{q}}\right) = \left((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi, (w_1, w_2)\right)$$

$$= \left((\hat{\gamma}+\hat{\varepsilon})\varphi, \nabla\cdot w\right) + \langle(\hat{\gamma}+\hat{\varepsilon})\varphi, \nabla\cdot w\rangle$$

8-12-2025
J.S (5)

$$-\frac{\partial}{\partial x_1}\left[(\sigma_{11},\sigma_{21})^{(u)},(v_1,v_2)\right]$$

$$=(\sigma_{11},\sigma_{21})^{(u)},\left(\frac{\partial v_1}{\partial x_1},\frac{\partial v_2}{\partial x_1}\right)+\text{bry term}$$

$$=\sigma_{11}^{(u)}\frac{\partial v_1}{\partial x_1}+\sigma_{21}^{(u)}\frac{\partial v_2}{\partial x_1}+\text{bry term}$$

$$-\frac{\partial}{\partial x_2}\left((\sigma_{12}^{(u)},\sigma_{22}^{(u)}),(v_1,v_2)\right)$$

$$=\left((\sigma_{12}^{(u)},\sigma_{22}^{(u)}),\frac{\partial v_1}{\partial x_2},\frac{\partial v_2}{\partial x_2}\right)+\text{bry term}$$

$$=\sigma_{12}^{(u)}\frac{\partial v_1}{\partial x_2}+\sigma_{22}^{(u)}\frac{\partial v_2}{\partial x_2}+\text{bry term}$$

Then,

$$-\frac{\partial}{\partial x_1}\left[(\sigma_{11},\sigma_{21})^{(u)},(v_1,v_2)\right]-\frac{\partial}{\partial x_2}\left[(\sigma_{12},\sigma_{22})^{(u)}(v_1,v_2)\right]$$

$$=\left(\sigma_{11},\frac{\partial v_1}{\partial x_1}\right)+\left(\sigma_{21},\frac{\partial v_2}{\partial x_1}\right)+\left(\sigma_{12},\frac{\partial v_1}{\partial x_2}\right)+\left(\sigma_{22},\frac{\partial v_2}{\partial x_2}\right)$$

$$=\sigma_{11}(\vec{u})\,\varepsilon_{11}(\vec{v})+2\sigma_{12}(\vec{u})\,\varepsilon_{12}(\vec{v})+\sigma_{22}(\vec{u})\,\varepsilon_{22}(\vec{v})$$

8-12-2025

3.5 (6)

$$= -\left(\sigma_{11}(\vec{u}), \varepsilon_{11}(\vec{v})\right) + \left(\sigma_{21}(u), \frac{\partial v_2}{\partial x_1}\right) + \left(\sigma_{12}(u), \frac{\partial v_1}{\partial x_2}\right)$$

$$+ \left(\sigma_{22}(u), \varepsilon_{22}(\vec{v})\right) \quad + \text{bry term}$$

$$= \left((\lambda+2\mu)\,\varepsilon_{11}(\vec{u}) + \lambda\,\varepsilon_{22}(\vec{u}),\ \varepsilon_{11}(\vec{v})\right)$$

$$+ \left(\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) + \alpha \,\mathrm{curl}\,\vec{u} + 2\alpha\,\varphi,\ \frac{\partial v_2}{\partial x_1}\right)$$

$$+ \left(\mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) - \alpha\,\mathrm{curl}\,\vec{u} - 2\alpha\,\varphi,\ \frac{\partial v_1}{\partial x_2}\right)$$

$$+ \left((\lambda+2\mu)\,\varepsilon_{22}(\vec{u}) + \lambda\,\varepsilon_{11}(\vec{u}),\ \varepsilon_{22}(v)\right)$$

$+$ bry term

$$= \left((\lambda+2\mu)\varepsilon_{11}(\vec{u}) + \lambda\,\varepsilon_{22}(\vec{u}),\ \varepsilon_{11}(\vec{v})\right)$$

$$+ \left(2\mu\,\varepsilon_{12}(u), \frac{\partial v_2}{\partial x_1}\right) + \left(2\mu\,\varepsilon_{21}(u), \frac{\partial v_1}{\partial x_2}\right)$$

$$+ \left(\alpha\,\mathrm{curl}\,\vec{u},\ \underbrace{\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}}_{\mathrm{curl}\,\vec{v}}\right)$$

$$+ \left(2\alpha\,\varphi,\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$$

$$+ \left((\lambda+2\mu)\,\varepsilon_{22}(u) + \lambda\,\varepsilon_{11}(\vec{u}),\ \varepsilon_{22}(\vec{v})\right)$$

$+$ bry term

8-12-2025
d.5

(7)

$$= \left((\lambda+2\mu)\,\varepsilon_{11}(\vec{u}) + \lambda\,\varepsilon_{22}(\vec{u}),\ \varepsilon_{11}(\vec{v}) \right)$$

$$+ \left((\lambda+2\mu)\,\varepsilon_{22}(\vec{u}) + \lambda\,\varepsilon_{11}(\vec{u}),\ \varepsilon_{22}(\vec{v}) \right)$$

$$+ \left(2\mu\,\varepsilon_{12}(\vec{u}),\ \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \right)$$

$$+ \left(\alpha\,\mathrm{curl}\,\vec{u},\ \underbrace{\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}}_{\mathrm{curl}\,\vec{v}} \right)$$

$$+ \left(2\alpha\varphi,\ \mathrm{curl}\,\vec{v} \right) \quad + \text{bry term}$$

$$= \left((\lambda+2\mu)\,\varepsilon_{11}(\vec{u}) + \lambda\,\varepsilon_{22}(\vec{u}),\ \varepsilon_{11}(\vec{v}) \right) \qquad (24)$$

$$+ \left((\lambda+2\mu)\,\varepsilon_{22}(\vec{u}) + \lambda\,\varepsilon_{11}(\vec{u}),\ \varepsilon_{22}(\vec{v}) \right)$$

$$+ \left(4\mu\,\varepsilon_{12}(\vec{u}),\ \varepsilon_{12}(\vec{v}) \right)$$

$$+ \left(2\alpha\,\mathrm{curl}\,\vec{u},\ \mathrm{curl}\,\vec{v} \right)$$

$$+ \left(2\alpha\varphi,\ \mathrm{curl}\,\vec{v} \right) \quad + \text{bry term}$$

<u>bry term</u>:

In space-freq. domain

$$-\omega^2 \rho \vec{u} - \nabla\cdot\tilde{\sigma} = \vec{X} \qquad (25)$$

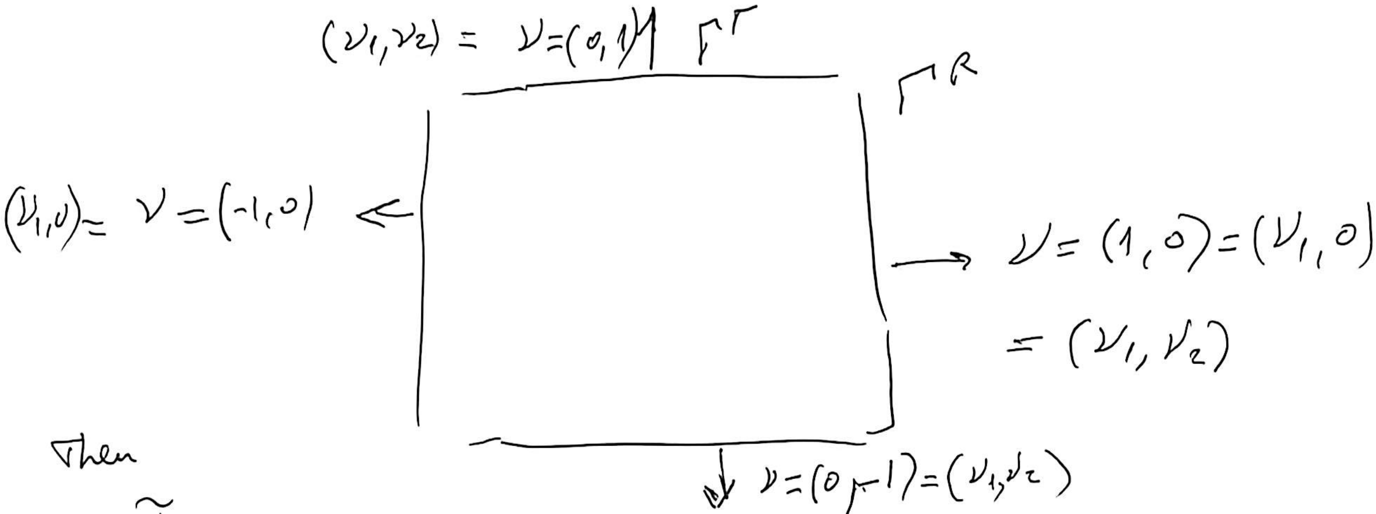
$$-\omega^2 (\rho\vec{u}, \vec{v}) + (\sigma_{ji}(\vec{u}), \varepsilon_{ji}(\vec{v})) - \langle \tilde{\sigma}\nu, \vec{v} \rangle_\Gamma$$

$$= (\vec{X}, \vec{v})$$

$$\tilde{\tilde{\sigma}}\,\nu = \sigma_{ji}\nu_j = \left(\sigma_{11}\nu_1 + \sigma_{21}\nu_2,\ \sigma_{12}\nu_1 + \sigma_{22}\nu_2\right)$$

$$\tilde{\tilde{\sigma}}\,\nu\cdot\nu = \sigma_{ji}\nu_j\nu_i$$

$$= \sigma_{11}\nu_1\nu_1 + \sigma_{21}\nu_2\nu_1 + \sigma_{12}\nu_1\nu_2 + \sigma_{22}\nu_2\nu_2$$



Then

$$\tilde{\tilde{\sigma}}\,\nu\cdot\nu\big|_{\Gamma^R} = \sigma_{11}\big|_{\Gamma^R} \qquad \tilde{\tilde{\sigma}}\,\nu\cdot\nu\big|_{\Gamma^L} = \sigma_{11}\big|_{\Gamma^L}$$

$$\tilde{\tilde{\sigma}}\,\nu\cdot\nu\big|_{\Gamma^T} = \sigma_{22}\big|_{\Gamma^T} \qquad \tilde{\tilde{\sigma}}\,\nu\,\nu\big|_{\Gamma^B} = \sigma_{22}\big|_{\Gamma^L}$$

$$\sigma_{11}\Big|_{\Gamma^R} = (\lambda+2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_1}{\partial x_1}\Big|_{\Gamma^R}$$

$$\sigma_{22}\big|_{\Gamma} = (\lambda+2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_2}{\partial x_2}$$

Since σ_{12} σ_{21} do not show up in the BRYS we can use the B10's 1st order AB.C

$$-\langle\tilde{\tilde{\sigma}}\,\nu,\vec{v}\rangle = i\omega D(\vec{u}\cdot\nu,\vec{u}\cdot x) \quad (26)$$

$$D = A^{1/2}S^{1/2}A^{1/2} \quad A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

on Γ^R, $\vec{u}\cdot\nu = u_1$, on Γ^L, $\vec{u}\cdot\nu = -u_1$

on Γ^B $\vec{u}\cdot\nu = -u_2$, on Γ^T $\vec{u}\cdot\nu = u_2$

$$S = A^{-1/2} E A^{-1/2} \qquad E = \begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{pmatrix}$$

$$A^{1/2} = \begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{1/2} \end{pmatrix}$$

$$S = \begin{pmatrix} e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}\begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-1/2}(\lambda+2\mu) & 0 \\ 0 & e^{-1/2}\mu \end{pmatrix}\begin{pmatrix} e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda+2\mu)e^{-1} & 0 \\ 0 & e^{-1}\mu \end{pmatrix} = \begin{pmatrix} v_p^2 & 0 \\ 0 & v_s^2 \end{pmatrix}$$

$$S^{1/2} = \begin{pmatrix} v_p & 0 \\ 0 & v_s \end{pmatrix}$$

$$D = \begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{1/2} \end{pmatrix}\begin{pmatrix} v_p & 0 \\ 0 & v_s \end{pmatrix}\begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{1/2}v_p & 0 \\ 0 & e^{1/2}v_s \end{pmatrix}\begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{1/2} \end{pmatrix} = \begin{pmatrix} e\,v_p & 0 \\ 0 & e\,v_s \end{pmatrix}$$

Then using (24), 25) becomes

9-12-2025 J.S

(10)

$$-\omega^2(\rho\vec{u},\vec{v}) + \left((\lambda+2\mu)\varepsilon_{11}(\vec{u}) + \lambda\varepsilon_{22}(\vec{u}), \varepsilon_{11}(\vec{v})\right)$$

$$+\left((\lambda+2\mu)\varepsilon_{22}(\vec{u}) + \lambda\varepsilon_{11}(\vec{u}), \varepsilon_{22}(\vec{v})\right)$$

$$+(4\mu\,\varepsilon_{12}(\vec{u}), \varepsilon_{12}(\vec{v})) + (2\alpha\,\mathrm{curl}\,\vec{u}, \mathrm{curl}\,\vec{v})$$

$$+ (2\alpha\varphi, \mathrm{curl}\,\vec{v}) + i\omega\langle D(u\cdot\nu, u\cdot\chi), (v\cdot\nu, v\cdot\chi)\rangle_\Gamma$$

$$= (\vec{X}, \vec{v})$$

$$= -\omega^2(\rho\vec{u},\vec{v}) + \left[\underbrace{\begin{pmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & 4\mu\end{pmatrix}}_{E}\begin{pmatrix}\varepsilon_{11}(u)\\ \varepsilon_{22}(u)\\ \varepsilon_{12}(u)\end{pmatrix}, \begin{pmatrix}\varepsilon_{11}(\vec{v})\\ \varepsilon_{22}(\vec{v})\\ \varepsilon_{12}(\vec{v})\end{pmatrix}\right]$$

$$+(2\alpha\,\mathrm{curl}\,\vec{u}, \mathrm{curl}\,\vec{v}) + (2\alpha\varphi, \mathrm{curl}\,\vec{v}) \quad (27)$$

$$+ i\omega\langle D(u\cdot\nu, u\cdot\chi), (v\cdot\nu, v\cdot\chi)\rangle_\Gamma = (\vec{X},\vec{v})$$

Then

$$-\omega^2(\rho\vec{u},\vec{v}) + \Lambda(\vec{u},\vec{v}) + 2(\alpha\,\mathrm{curl}\,\vec{u}, \mathrm{curl}\,\vec{v}) \quad (28)$$

$$+ 2(\alpha\varphi, \mathrm{curl}\,\vec{v}) + i\omega\langle D(u\cdot\nu, u\cdot\chi), v\cdot\nu, v\cdot\chi\rangle_\Gamma$$

$$= (\vec{X},\vec{v}) \qquad \vec{v}\in[H^1(\Omega)]^2$$

Note E is positive definite

Eq'N (4) 9-12-2025

$j=1,\ i=3$ $\qquad \tau_{13} = (\hat{\gamma}+\hat{\varepsilon})K_{13} + (\hat{\gamma}-\hat{\varepsilon})\overset{=0}{K_{31}} + \beta \overset{=0}{\delta_{31}} K_{kk} = (\hat{\gamma}+\hat{\varepsilon})K_{13}$ (29)

$j=3\ i=3$

$$\tau_{23} = (\hat{\gamma}+\hat{\varepsilon})K_{23} + (\hat{\gamma}-\hat{\varepsilon})\overset{=0}{K_{32}} = (\hat{\gamma}+\hat{\varepsilon})K_{23} \quad (30)$$

Compute K_{ij}:

$i=1,\ j=3$ $\qquad K_{13} = \varphi_{3,1} = \dfrac{\partial \varphi}{\partial x_1}$ (31)

$i=2\ j=3$ $\qquad K_{23} = \varphi_{3,2} = \dfrac{\partial \varphi}{\partial x_2}$ (32)

$$\tau_{23} = (\hat{\gamma}+\hat{\varepsilon})\frac{\partial \varphi}{\partial x_2} \quad (33)$$

$$\tau_{13} = (\hat{\gamma}+\hat{\varepsilon})\frac{\partial \varphi}{\partial x_1} \quad (34)$$

From (2)

$$-\omega^2 J \varphi = \sigma_{12} - \sigma_{21} + \tau_{13,1} + \tau_{23,2} + Y_3$$

$$= 2\mu\, \varepsilon_{12}(\vec{u}) + \alpha \operatorname{curl} \vec{u} - 2\alpha\varphi$$

$$-\left(2\mu\, \varepsilon_{12}(\vec{u}) - \alpha \operatorname{curl} \vec{u} + 2\alpha\varphi\right)$$

$$+ \frac{\partial}{\partial x_1}\left((\hat{\gamma}+\hat{\varepsilon})\frac{\partial \varphi}{\partial x_1}\right)$$

$$+ \frac{\partial}{\partial x_2}\left((\hat{\gamma}+\hat{\varepsilon})\frac{\partial \varphi}{\partial x_2}\right)$$

Then

$$-\omega^2 J\varphi - 2\alpha\, \mathrm{curl}\, \vec{u} + 4\alpha\varphi - \nabla\cdot\left((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi\right) = Y_3 \quad (35)$$

Test (35) against z

$$-\omega^2(J\varphi, z) - (2\alpha\, \mathrm{curl}\, \vec{u}, z) + 4(\alpha\varphi, z)$$

$$-\nabla\cdot((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi, z) = (Y_3, z)$$

Choose $z = \varphi$

$$-\omega^2(J\varphi,\varphi) - (2\alpha\, \mathrm{curl}\, \vec{u}, \varphi) + 4(\alpha\varphi,\varphi) \quad (36)$$

$$~~+\varphi((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi~~ - \nabla\cdot(\underbrace{(\hat{\gamma}+\hat{\varepsilon})\nabla\varphi}_{q}, \varphi) = (Y_3, \varphi)$$

Int. by parts in (36):

$$-\omega^2(J\varphi,\varphi) - (2\alpha\, \mathrm{curl}\, \vec{u},\varphi) + 4(\alpha\varphi,\varphi) \quad (37)$$

$$+((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi, \nabla\varphi) - \langle(\hat{\gamma}+\hat{\varepsilon})\nabla\varphi\cdot\nu, \varphi\rangle_{\Gamma} = (Y_3,\varphi)$$

Choose $\vec{v} = \vec{u}$ in (28)

$$-\omega^2(\rho\vec{u},\vec{u}) + \Lambda(\vec{u},\vec{u}) + 2(\alpha\, \mathrm{curl}\, \vec{u}, \mathrm{curl}\, \vec{u})$$

$$+2(\alpha\varphi, \mathrm{curl}\, \vec{u}) + i\omega\langle D(u\cdot\nu, u\cdot\chi), u\cdot\nu, u\cdot\chi\rangle \quad (38)$$

$$= (\vec{X}, \vec{u})$$

Add (37) and (38): NO ~~Cancels~~ $(2\alpha\, \mathrm{curl}\, \vec{u}, \varphi)$ term

$$-\omega^2(\rho\vec{u},\vec{u}) + \Lambda(\vec{u},\vec{u}) + 2(\alpha\,\mathrm{curl}\,\vec{u},\mathrm{curl}\,\vec{u})$$

$$+\,4(\alpha\varphi,\varphi) + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi,\nabla\varphi) - \omega^2(J\varphi,\varphi) \qquad (39)$$

$$+\,i\omega\langle D(u\cdot\nu,u\cdot\chi),(u\cdot\nu,u\cdot\chi)\rangle_\Gamma$$

$$+$$

$$-\langle(\hat{\gamma}+\hat{\varepsilon})\nabla\varphi\cdot\nu,\varphi\rangle_\Gamma = (\vec{X},\vec{u}) + (Y_3,\varphi)$$

$$+\left[-\int 2\alpha\,\mathrm{curl}\,\vec{u}\,\bar{\varphi} + \int 2\alpha\varphi\,\overline{\mathrm{curl}\,\vec{u}}\,dg\right] \quad ?$$

Assume the B.C

$$-\langle(\hat{\gamma}+\hat{\varepsilon})\nabla\varphi\cdot\nu,\varphi\rangle_\Gamma = \langle i\omega C\varphi,\varphi\rangle_\Gamma \qquad (40)$$

C a positive constant. Then.

Then (39) is

$$-\omega^2(\rho\vec{u},\vec{u}) + \Lambda(\vec{u},\vec{u}) + (2\alpha\,\mathrm{curl}\,\vec{u},\mathrm{curl}\,\vec{u})$$

$$+\,4(\alpha\varphi,\varphi) + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi,\nabla\varphi) - \omega^2(J\varphi,\varphi) \qquad (40)$$

$$+\,i\omega\langle D(u\cdot\nu,u\cdot\chi),(u\cdot\nu,u\cdot\chi)\rangle_\Gamma$$

$$+\,i\omega\langle C\varphi,\varphi\rangle_\Gamma = (\vec{X},\vec{u}) + (Y_3,\varphi).$$

UNIQUENESS: ~~holds for ω small.~~

~~Maybe we need to solve~~ in space-time

+

Space-time formulation: from (2.8) 10-12-2025 J.S

$$(\rho \ddot{\vec{u}}, \vec{v}) + \Lambda(\vec{u}, \vec{v}) + (2\alpha \operatorname{curl} \vec{u}, \operatorname{curl} \vec{v})$$

$$+ 2(\alpha \varphi, \operatorname{curl} \vec{v}) + \langle D(\dot{\vec{u}}\cdot\vec{\nu}, \dot{\vec{u}}\cdot\vec{\chi}), \vec{v}\cdot\vec{\nu}, \vec{v}\cdot\vec{\chi} \rangle_\Gamma$$

$$= (\vec{X}, \vec{v}) \quad , \quad \vec{v} \in [H^1(\Omega)]^2 \quad (41),$$

FROM (37) and integration by parts and (40)

$$(J \ddot{\varphi}, z) - (2\alpha \operatorname{curl} \vec{u}, z) + (4\alpha \varphi, z)$$

$$+ ((\hat{\gamma} + \hat{\varepsilon}) \nabla \varphi, \nabla z) + \langle \mathbb{C} \dot{\varphi}, z \rangle_\Gamma \quad (42)$$

$$= (Y_3, \varphi) \quad , \quad \varphi \in H^1(\Omega) \, .$$

Add (41) and (42): Find $(\vec{u}, \varphi) \in [H^1(\Omega)]^2 \times H^1(\Omega)$:

$$(\rho \ddot{\vec{u}}, \vec{v}) + \Lambda(\vec{u}, \vec{v}) + 2(\alpha \operatorname{curl} \vec{u}, \operatorname{curl} \vec{v})$$

$$+ 2(\alpha \varphi, \operatorname{curl} \vec{v}) + \langle D(\dot{\vec{u}}\cdot\vec{\nu}, \dot{\vec{u}}\cdot\vec{\chi}), (\vec{v}\cdot\nu, \vec{v}\cdot\chi) \rangle_\Gamma$$

$$+ (J \ddot{\varphi}, z) - (2\alpha \operatorname{curl} \vec{u}, z) + (4\alpha \varphi, z)$$

$$+ ((\hat{\gamma} + \hat{\varepsilon}) \nabla \varphi, \nabla z) + \langle \mathbb{C} \dot{\varphi}, z \rangle_\Gamma \quad (43)$$

$$= (\vec{X}, \vec{v}) + \langle Y_3, \varphi) \quad , \quad \vec{v} \in [H^1(\Omega)]^2, \; z \in H^1(\Omega)$$

UNIQUENESS FOR (43)

Take $\vec{v} = \dot{\vec{u}}$, $z = \dot{\varphi}$; $\vec{X} = 0$, $Y_3 = 0$ in (43):

$$(\rho \ddot{\vec{u}}, \dot{\vec{u}}) + A(u, \dot{u}) + (2\alpha \operatorname{curl} \vec{u}, \operatorname{curl} \dot{\vec{u}})$$

$$+ (2\alpha \varphi, \operatorname{curl} \dot{\vec{u}}) + \langle D(\dot{\vec{u}}\cdot\vec{\nu}, \dot{\vec{u}}\cdot\vec{\chi}), (\dot{\vec{u}}\cdot\vec{\nu}, \dot{\vec{u}}\cdot\vec{\chi}) \rangle_\Gamma$$

$$+ (J \ddot{\varphi}, \dot{\varphi}) - (2\alpha \operatorname{curl} \vec{u}, \dot{\varphi}) + 4(\alpha \varphi, \dot{\varphi}) \quad (44)$$

$$+ ((\hat{\gamma} + \hat{\varepsilon}) \nabla \varphi, \nabla \dot{\varphi}) + \langle C \dot{\varphi}, \varphi \rangle_\Gamma = 0$$

Next,

$$\frac{d}{dt}(2\alpha\varphi, \operatorname{curl} \vec{u}) = (2\alpha\dot{\varphi}, \operatorname{curl} \vec{u}) + (2\alpha\varphi, \operatorname{curl} \dot{\vec{u}})$$

Then

$$(2\alpha\dot{\varphi}, \operatorname{curl} \vec{u}) - \frac{d}{dt}(2\alpha\varphi, \operatorname{curl} \vec{u}) - (2\alpha\varphi, \operatorname{curl} \dot{\vec{u}}) = 0$$

or

$$-(2\alpha\dot{\varphi}, \operatorname{curl} \vec{u}) + (2\alpha\varphi, \operatorname{curl} \dot{\vec{u}}) + \frac{d}{dt}(2\alpha\varphi, \operatorname{curl} \vec{u}) = 0 \quad (45)$$

$$(\rho \ddot{\vec{u}}, \dot{\vec{u}}) + \Lambda(u, \dot{u}) + (2\alpha \, \mathrm{curl}\, \dot{\vec{u}}, \mathrm{curl}\, \dot{u}) \quad (16 \text{[illegible]})$$

$$+ \langle D\dot{u}, \dot{u} \rangle$$

$$+ (J \ddot{\varphi}, \dot{u}) + 4(\alpha \dot{\varphi}, \dot{\varphi}) + ((\gamma + \varepsilon) \nabla \varphi, \nabla \dot{\varphi})$$

$$+ \langle C \dot{u}, \dot{\varphi} \rangle \quad (46)$$

$$+ (2\alpha \varphi, \mathrm{curl}\, \dot{\vec{u}}) - (2\alpha \, \mathrm{curl}\, \dot{\vec{u}}, \dot{\varphi}) = 0$$

Next

$$\int_0^t (2\alpha \varphi \, \mathrm{curl}\, \dot{u})(s) \, ds = 2\alpha \varphi \, \mathrm{curl}\, u \Big|_0^t - \int_0^t 2\alpha \dot{\varphi} \, \mathrm{curl}\, u \, ds$$

Then

$$\int_0^t (2\alpha \varphi, \mathrm{curl}\, \dot{u})(s) \, ds - \int_0 (2\alpha \, \mathrm{curl}\, \vec{u}, \dot{\varphi})$$

$$= 2(\alpha \varphi, \mathrm{curl}\, u)\Big|_0^t - \int_0^t (2\alpha, \mathrm{curl}\, u, \dot{\varphi}) \, ds - \int_0^t (2\alpha \dot{u}, \mathrm{curl}\, \vec{u}) \, ds$$

$$= 2(\alpha \varphi(t), \mathrm{curl}\, u(t)) - (2\alpha \varphi(0) \, \mathrm{curl}\, u(0)) - 4 \int_0^t \alpha \, \mathrm{curl}\, u(s) \, \dot{\varphi}(s) \, ds \quad (47)$$

$$\frac{1}{2}\frac{d}{dt}\Big[(\rho\dot{u},\dot{u}) + \Lambda(u,u) + (2\alpha\,\mathrm{curl}\,u, \mathrm{curl}\,u)$$ (17)

$$+ (J\dot{\varphi},\dot{\varphi}) + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi, \nabla\varphi)\Big]$$ (45)

$$+ \langle D\dot{u},\dot{u}\rangle + \langle c\,\dot{\varphi},\dot{\varphi}\rangle + 4(\alpha\varphi,\varphi)$$

$$+ (2\alpha\varphi, \mathrm{curl}\,\dot{u}) - (2\alpha\varphi\,\mathrm{curl}\,\bar{u}, \dot{\varphi}) = 0$$

Integrate in time and use (44 B) (48)

$$\frac{1}{2}\Big[(\rho\dot{u},\dot{u})(t) + \Lambda(u(t),u(t)) + \underbrace{(2\alpha\,\mathrm{curl}\,\overset{\times}{u}, \mathrm{curl}\,u)(t)}_{2\|\alpha^{1/2}\|\mathrm{curl}\,u\|_0^2(t)}$$

$$+ (J\dot{\varphi}(t),\dot{\varphi}(t)) + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi(t), \nabla\varphi(t))$$

$$+ \boxed{4(\alpha\varphi(t), \overset{\times}{\varphi}(t))}\Big] \quad 4\|\alpha^{1/2}\alpha(t)\|_0^2$$

$$= \frac{1}{2}\Big[(\rho\dot{u},\dot{u})(0) + \Lambda(u,u)(0) + (2\alpha\,\mathrm{curl}\,u, \mathrm{curl}\,u)(0)$$

$$\to Q(0) \quad + (J\dot{\varphi}(0),\dot{\varphi}(0)) + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi(0), \nabla\varphi(0))$$ (49)

$$+4(\alpha\varphi(0),\varphi(0))\Big]$$

$$+ \boxed{2\alpha\varphi(t)\,\overset{\times}{\mathrm{curl}}\,u(t)} - (2\alpha\varphi(0), \mathrm{curl}\,u(0))$$

$$- 4\int_0^t \alpha\,\mathrm{curl}\,u(s)\,\dot{\varphi}(s)\,ds = 0$$

$$\frac{1}{2}\Big[(\rho\dot{u},\dot{u})(t)+\Lambda(u(t),u(t))$$

$$+\,(2\alpha\,\mathrm{curl}\,\vec{u},\mathrm{curl}\,\vec{u})(t)$$

$$+\,(J\dot{\varphi},\dot{\varphi})(t)+((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi,\nabla\varphi)(t)+4(\alpha\varphi,\varphi)(t)\Big]$$

$$-\,Q(0)+(2\alpha\,\varphi(t),\mathrm{curl}\,u(t))$$

$$-\,(2\alpha\,\varphi(0),\mathrm{curl}\,\vec{u}(0)) \qquad (50)$$

$$-\,4\int_0^t\alpha\,\mathrm{curl}\,u(s)\,\dot{\varphi}(s)\,ds=0$$

$$\left|4\int_0^t\alpha\,\mathrm{curl}\,u(s)\,\dot{\varphi}(s)\,ds\right|$$

$$\le C\int_0^t\|\alpha^{1/2}\,\mathrm{curl}\,\vec{u}(s)\|_0\,\|\dot{\varphi}(s)\|_0\,ds \qquad (51)$$

$$\le C\int_0^t\left(\|\mathrm{curl}\,\vec{u}(s)\|_0^2+\|\dot{\varphi}(s)\|_0^2\right)ds$$

~~[illegible]~~

$$|(\alpha\varphi(t),\mathrm{curl}\,u(t))|=|(\alpha^{1/2}\varphi(t),\alpha^{1/2}\,\mathrm{curl}\,\vec{u}(t))|$$

$$\le\|\alpha^{1/2}\varphi(t)\|_0\,\|\alpha^{1/2}\,\mathrm{curl}\,u(t)\|_0$$

$$\leq \frac{1}{2}\left(\|\alpha^{1/2}\varphi(t)\|_0^2 + \|\alpha^{1/2}\operatorname{curl} u(t)\|_0^2\right) \quad (52)$$

$$= \frac{1}{2}\left[(\alpha\varphi,\varphi)(t) + (\alpha \operatorname{curl} u, \operatorname{curl} u)(t)\right]$$

From (50)-(52)

$$\frac{1}{2}\Big[\|\rho^{1/2}\dot{\vec{u}}(t)\|_0^2 + \Lambda(\vec{u}(t),\vec{u}(t)) + 2\|\alpha^{1/2}\operatorname{curl} u\|_0^2(t)$$

$$+ \|J^{1/2}\varphi(t)\|_0^2 + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi,\nabla\varphi)(t) \quad (53)$$

$$+ 4\|\alpha^{1/2}\varphi(t)\|_0^2\Big]$$

$$= Q(0) + (2\alpha\varphi(0), \operatorname{curl} u(0))$$

$$+ \left[\frac{1}{2}\left[\|\alpha^{1/2}\varphi\|_0^2(t) + \|\alpha^{1/2}\operatorname{curl}\vec{u}\|_0^2(t)\right]\right]$$

$$+ C\int_0^t \left(\|\operatorname{curl}\vec{u}(s)\|_0^2 + \|\dot{\varphi}(s)\|_0^2\right) ds$$

Absorbed in LHS of (53).

$$\frac{1}{2}\|\rho^{1/2}\dot{\vec{u}}(t)\|_0^2 + \Lambda(\vec{u}(t),\vec{u}(t)) + \frac{3}{2}\|\alpha^{1/2}\operatorname{curl}\vec{u}(t)\|_0^2$$

$$+\frac{1}{2}\|J^{1/2}\dot{\varphi}(t)\|_0^2 + ((\hat{\gamma}+\hat{\varepsilon})\nabla\varphi,\nabla\varphi)(t)$$

$$~~+ 4(\alpha\varphi(t),\varphi(t))~~ + 4\|\alpha^{1/2}\varphi(t)\|_0^2$$

(5 by)

$$\leq Q(0) + (2\alpha\varphi(0);\operatorname{curl}\vec{u}(0))$$

$$+ C\int_0^t\left(\|\operatorname{curl}\vec{u}(s)\|_0^2 + \|\dot{\varphi}(s)\|_0^2\right)ds$$

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→ (54) says for zero initial conditions $\vec{u}(t)\equiv 0$, $\varphi(t)\equiv 0$ and we have uniqueness