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AN ELLIPTIC COLLOCATION-FINITE ELEMENT METHOD WITH INTERIOR PENALTIES*

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Abstract. A discontinuous collocation-finite element method with interior penalties is proposed and analyzed for elliptic equations. The integral orthogonalities are motivated by the interior penalty L^2-Galerkin procedure of Douglas and Dupont.

1. Introduction. Continuous L^2 and discontinuous H^{-1} finite element-collocation schemes have been defined and analyzed for the two-point boundary value problem [5], [8], and [11]. In this paper we generalize the one dimensional L^2 scheme to two dimensions, although the arguments presented here can be extended to R^n. We shall use discontinuous polynomial spaces.

Our primary interest here is in showing that one can define a collocation-finite element method on triangles and quadrilaterals that yields optimal L^2 estimates. Implementation of this scheme will be treated in a later paper.

Consider the boundary value problem

$$\begin{aligned} Lu &= f(x), \qquad x \in \Omega, \\ u(x) &= g(x), \qquad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$Lu = -\nabla \cdot a(x)\nabla u + b(x) \cdot \nabla u + c(x)u, \tag{1.2}$$

$b = (b_1, b_2)$, and Ω is a bounded domain in R^2 with piecewise smooth boundary $\partial\Omega$. Let $a(x)$, $b_i(x)$ and $c(x) \in C^\infty(\bar{\Omega})$ and let a_0 and a_1 be two positive constants such that

$$\begin{aligned} &a_0 \leqq a(x) \leqq a_1, && x \in \bar{\Omega}, \\ &|c(x)| \leqq a_1, && x \in \bar{\Omega}, \\ &|b_i(x)| \leqq a_1, && i = 1, 2, \quad x \in \bar{\Omega}. \end{aligned} \tag{1.3}$$

We further assume that given $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$ there exists a unique solution $u \in H^2(\Omega)$ to (1.1); moreover, we shall assume that the problem

$$\begin{aligned} L^*w &= q, \qquad x \in \Omega, \\ w &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

has 0-regularity; i.e.,

$$\|w\|_{H^2(\Omega)} \leqq C\|q\|_{L^2(\Omega)}.$$

Let $\mathscr{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a subdivision of Ω, where E_j is a quadrilateral or triangle. The boundary polygons can be curvilinear. If $h_j = \operatorname{diam}(E_j)$, we assume there exists a $\rho > 0$ such that each E_j contains a ball of radius ρh_j in its interior. We denote the edges of the polygons by $\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\}$ where $e_k \subset \Omega$, $1 \leqq k \leqq P_h$, and $e_k \subset \partial\Omega$, $P_h + 1 \leqq k \leqq M_h$. For $k \geqq 0$, let $H^k(\mathscr{E}_h) = \{v \in L^2(\Omega): v|_{E_j} \in H^k(E_j), j = 1, 2, \cdots, N_h\}$. The finite element subspace is taken to be

$$\mathscr{M}_h^r = \bigcap_{j=1}^{N_h} P_r(E_j),$$

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where $P_r(E_j)$ denotes the set of polynomials of degree less than $r+1$ on E_j. The subspace $\mathcal{M}_h^r$ satisfies the following approximation property [4], [9], [10]: given $w \in W_l^s(E_j)$, $0 \leqq s \leqq r+1$, $1 \leqq l \leqq \infty$, there exists a constant $C>0$, independent of w and h_j such that

$$\inf_{\chi \in \mathcal{M}_h^r} \|w-\chi\|_{W_l^k(E_j)} \leqq C\|w\|_{W_l^s(E_j)} h_j^{s-k}, \qquad 0 \leqq k \leqq s. \tag{1.4}$$

Here

$$\|\Phi\|_{W_l^k(E_j)} = \sum_{|\alpha| \leqq k} \left\| \frac{\partial^\alpha \Phi}{\partial x^\alpha} \right\|_{L^l(E_j)}.$$

In the L^2-finite element-collocation procedure we define $U_h \in \mathcal{M}_h^r$ by collocating at $K_r = \frac{1}{2}r(r-1)$ distinct points in each E_j and satisfying $(2r+1)N_h$ integral orthogonalities. These orthogonalities are taken with respect to the harmonic subset of $\mathcal{M}_h^r$. The definition of the integral equations was motivated by the interior penalty L^2-Galerkin procedure of Douglas and Dupont [6], [7], and G. Baker [3].

A somewhat similar procedure called the weak element method has been formulated by Babuška for Laplace's equation [2]. His procedure involves approximating the solution with harmonic polynomials that satisfy a jump condition.

2. Notation and definition of method. We shall adopt the following notation. Let $1 \leqq k \leqq P_h$ and $e_k = \bar{E}_{j_1} \cap \bar{E}_{j_2}$. For $\Phi \in H^{1+\varepsilon}(\mathscr{E}_h)$, $\varepsilon > 0$, and $\tilde{x} \in e_k$, set

$$\{\Phi\}(\tilde{x}) = \tfrac{1}{2}\Big\{ \lim_{\substack{x \to \tilde{x} \\ x \in E_{j_1}}} \Phi(x) + \lim_{\substack{x \to \tilde{x} \\ x \in E_{j_2}}} \Phi(x) \Big\}.$$

On each e_k we select a normal direction $n = n_k$. A tangential direction τ is taken so that (n, τ) is a positively oriented basis at each point of e_k. We denote by E_{R} (E_{L}) the polygon whose inward (outward) normal is n. For $\Phi \in H^{1+\varepsilon}$, $\varepsilon > 0$, and $\tilde{x} \in e_k$, set

$$[\Phi](\tilde{x}) = \lim_{\substack{x \to \tilde{x} \\ x \in E_{\mathrm{L}}}} \Phi(x) - \lim_{\substack{x \to \tilde{x} \\ x \in E_{\mathrm{R}}}} \Phi(x).$$

One can easily verify that

$$-\int_{-e_k} a \frac{\partial u_{\mathrm{R}}}{\partial n} v_R \, ds - \int_{e_k} a \frac{\partial u_{\mathrm{L}}}{\partial n} v_L \, ds = -\int_{e_k} a \left\{ \frac{\partial u}{\partial n_k} \right\} [v] \, ds, \qquad v \in \mathcal{M}_h^r,$$

where $v_R = v|_{E_{\mathrm{R}}}$ and $v_L = v|_{E_{\mathrm{L}}}$.

For $P_h + 1 \leqq k \leqq M_h$ and $\tilde{x} \in e_k$, let

$$[\Phi](\tilde{x}) = \Phi(\tilde{x}) \quad \text{and} \quad \left\{ \frac{\partial \Phi}{\partial n_k} \right\}(\tilde{x}) = \frac{\partial \Phi}{\partial n_k}(\tilde{x}),$$

where n_k is the outer normal with respect to Ω.

For Φ, $\Psi \in H^2(\mathscr{E}_h)$, we set $(\Phi, \Psi) = \int_\Omega \Phi(x)\Psi(x)\,dx$, $\langle \Phi, \Psi \rangle = \int_{\partial\Omega} \Phi(s)\Psi(s)\,ds$, $(a\nabla\Phi, \nabla\Omega)_j = \int_{E_j} a\nabla\Phi \cdot \nabla\Psi \, dx$, $\langle \Phi, \Psi \rangle_j = \int_{\partial E_j} \Phi(s)\Psi(s)\,ds$, and $\langle\!\langle \Phi, \Psi \rangle\!\rangle_j = \int_{e_j} \varphi(s)\Psi(s)\,ds$. We also let $|\Phi|_j = \langle\!\langle \Phi, \Phi \rangle\!\rangle_j^{1/2}$, $\|\Phi\|_{\partial E_j} = \langle \Phi, \Phi \rangle_j^{1/2}$, and

$$\|\Phi\|_{l,E} = \Big(\sum_{|\alpha|=0}^{l} \|D^\alpha \Phi\|_{L^2(E)}^2 \Big)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geqq 0$, E is an open set in Ω, and $\Phi \in H^l(E)$.

We observe that since the solution u to (1.1) satisfies $u \in H^2(\Omega)$ and $[\partial u/\partial n] = 0$ on e_k,

$$\sum_{j=1}^{N_h} ((a\nabla u, \nabla v)_j + (b\cdot\nabla u, v)_j + (cu, v)_j) - \sum_{k=1}^{M_h} \left\langle\!\left\langle a\left\{\frac{\partial u}{\partial n_k}\right\}, [v]\right\rangle\!\right\rangle_k = (f, v), \qquad v \in \mathcal{M}_h^r. \tag{2.1}$$

Since $[u] = 0$ on any interior edge, we note that

$$-\sum_{k=1}^{M_h} \left\langle\!\left\langle a[u], \left\{\frac{\partial v}{\partial n_k}\right\}\right\rangle\!\right\rangle_k - \sum_{k=1}^{P_h} \langle\!\langle (b_1+b_2)[u], \{v\}\rangle\!\rangle = -\left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathcal{M}_h^r. \tag{2.2}$$

For $\Phi, \Psi \in H^2(\mathscr{E}_h)$, define

$$\begin{aligned} A(\Phi, \Psi) = &\sum_{j=1}^{N_h} ((a\nabla\Phi, \nabla\Psi)_j + (b\cdot\nabla\Phi, \Psi)_j + (c\Phi, \Psi)_j) \\ &- \sum_{k=1}^{M_h} \left(\left\langle\!\left\langle a\left\{\frac{\partial\Phi}{\partial n_k}\right\}, [\Psi]\right\rangle\!\right\rangle_k + \left\langle\!\left\langle a[\Phi], \left\{\frac{\partial\Psi}{\partial n_k}\right\}\right\rangle\!\right\rangle_k\right) \\ &- \sum_{k=1}^{P_h} \langle\!\langle (b_1+b_2)[\Phi], \{\Psi\}\rangle\!\rangle_k, \end{aligned} \tag{2.3}$$

$$J_0^\sigma(\Phi, \Psi) = \sum_{k=1}^{M_h} \sigma_k \frac{1}{l(e_k)} \langle\!\langle [\Phi], [\Psi]\rangle\!\rangle_k, \tag{2.4}$$

and

$$B_\sigma(\Phi, \Psi) = A(\Phi, \Psi) + J_0^\sigma(\Phi, \Psi), \tag{2.5}$$

where $l(e_k)$ is the length of e_k and σ_k is a nonnegative constant.

From (2.1)–(2.5) we note that

$$B_\sigma(u, v) = (f, v) - \left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathcal{M}_h^r.$$

Set $h = \max_{1\leqq i\leqq N_h} h_i$. In the case $b \equiv 0$ and $\sigma_k(1/l(e_k)) = \sigma h^{-1}$, Douglas and Dupont have defined the following interior penalty L^2-Galerkin method [7]. Let $W_h \in \mathcal{M}_h^r$ satisfy

$$B_\sigma(W_h, v) = (f, v) - \left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathcal{M}_h^r. \tag{2.6}$$

Assuming $\mathscr{E}_h$ is quasi-uniform, they have shown that B_σ is positive definite on $\mathcal{M}_h^r$ if $\sigma \geqq \sigma_0$, where σ_0 is some appropriate positive constant. Moreover, they have proven that

$$B_{\sigma_0}(u - W_h, u - W_h)^{1/2} \leqq C_{\sigma_0}\|u\|_{l,\Omega} h^{l-1}, \qquad 2 \leqq l \leqq r+1,$$

and

$$\|u - W_h\|_{0,\Omega} \leqq C_{\sigma_0} B_{\sigma_0}(u - W_h, u - W_h)^{1/2} h,$$

where C_{σ_0} is a positive constant independent of u and h. Thus W_h is an optimal L^2 approximation of u. We shall later modify their analysis and extend it to include the general form (2.5) and equation (1.1).

We now observe that

$$\mathcal{M}_h^r = \mathcal{H}_h^r \oplus \mathcal{H}_h^{r\perp},$$

where

$$\mathcal{H}_h^r = \{v \in \mathcal{M}_h^r | \Delta v = 0 \text{ on } E_j,\ 1 \leqq j \leqq N_h\}$$

and

$$\mathcal{H}_h^{r\perp} = \{w \in \mathcal{M}_h^r | (w, v)_j = 0,\ v \in \mathcal{H}_h^r,\ 1 \leqq j \leqq N_h\}.$$

Since Δ is clearly an isomorphism from $\mathcal{H}_h^{r\perp}$ to $\mathcal{M}_h^{r-2}$ and the

$$\text{local dimension} \quad \mathcal{M}_h^r = \tfrac{1}{2}(r+1)(r+2) = K_{r+2},$$

it follows that the

$$\text{local dimension} \quad \mathcal{H}_h^r = K_{r+2} - K_r = 2r + 1.$$

Let $\{\alpha_{i,j}\}_{i=1}^{K_r}$ be a set of distinct points in the interior of E_j chosen so that if a polynomial of degree $r-2$ vanishes at all of them it vanishes identically.

We are now in position to define an interior penalty collocation-finite element procedure. We assume $f \in C(E_j)$, $1 \leqq j \leqq N_h$.

Let $U_h \in \mathcal{M}_h^r$ satisfy

$$(2.7) \qquad (LU_h)(\alpha_{i,j}) = f(\alpha_{i,j}), \qquad i = 1, 2, \cdots, K_r, \quad j = 1, 2, \cdots, N_h,$$

and

$$(2.8) \qquad B_\sigma(U_h, v) = (f, v) - \left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathcal{H}_h^r.$$

3. Global estimates. Because $\mathcal{E}_h$ is nondegenerate, the local inverse property holds.

Local inverse property. For $\Phi \in \mathcal{M}_h^r$, there exist constants $C_0 > 0$ and $\tilde{C}_0 > 0$ such that

$$(3.1) \qquad \|\Phi\|_{l,E_j} \leqq C_0 h_j^{-1} \|\Phi\|_{l-1,E_j},$$

$$(3.2) \qquad l(e_k)\left|\frac{\partial \Phi}{\partial n_k}\right|_k^2 \leqq C_0 l(e_k) h_j^{-1} \|\nabla\Phi\|_{0,E_j}^2 \leqq \tilde{C}_0 \|\nabla \Phi\|_{0,E_j}^2,$$

and

$$(3.3) \qquad \begin{aligned} l(e_k)|\Phi|_k^2 &\leqq l(e_k) C_0 (h_j^{-1}\|\Phi\|_{0,E_j}^2 + h_j \|\nabla\Phi\|_{0,E_j}^2) \\ &\leqq \tilde{C}_0(\|\Phi\|_{0,E_j}^2 + l(e_k) h_j \|\nabla \Phi\|_{0,E_j}^2), \end{aligned}$$

where e_k is an edge of E_j.

For convenience we define the following norms on $H^2(\mathcal{E}_h)$:

$$(3.4) \qquad |||\Phi|||_\sigma^2 = \sum_{j=1}^{N_h} \|\varphi\|_{1,E_j}^2 + J_0^\sigma(\Phi, \Phi)$$

and

$$(3.5) \qquad |||\Phi|||_{1,\sigma}^2 = |||\Phi|||_\sigma^2 + \sum_{k=1}^{P_h} |\{\Phi\}|_k^2 \frac{l(e_k)}{\sigma_k} + \sum_{k=1}^{M_h} \left|\left\{\frac{\partial \Phi}{\partial n_k}\right\}\right|_k^2 \frac{l(e_k)}{\sigma_k}.$$

Using norm definitions, the local inverse property and the Cauchy-Schwarz inequality, we immediately have the following three lemmas.

LEMMA 1. *For $\Phi, \Psi \in H^2(\mathscr{E}_h)$, there exists a positive constant C such that*

$$|B_\sigma(\Phi, \Psi)| \leqq C|||\Phi|||_{1,\sigma}|||\Psi|||_{1,\sigma}.$$

LEMMA 2. *For $\varphi \in \mathscr{M}_h^r$, there exists a positive constant $C(\sigma)$ such that*

$$|||\Phi|||_{1,\sigma} \leqq C(\sigma)|||\Phi|||_\sigma.$$

We next establish a Garding's inequality for B_σ.

LEMMA 3. *For $\Phi \in \mathscr{M}_h^r$ there exist positive constants $\hat{a}$, $\hat{\sigma}$ and $\hat{\delta}(\hat{\sigma})$ such that if $\sigma_k \geqq \hat{\sigma}$*

$$B_\sigma(\Phi, \Phi) \geqq \hat{\delta}|||\Phi|||_\sigma^2 - \hat{a}\|\Phi\|_{0,\Omega}^2.$$

Proof. From (1.3) and (2.3) we see that

$$\begin{aligned} A(\Phi, \Phi) \geqq \sum_{j=1}^{N_h} (a_0\|\nabla\Phi\|_{0,E_j}^2 - a_1\|\nabla\Phi\|_{0,E_j}\|\Phi\|_{0,E_j} - a_1\|\Phi\|_{0,E_j}^2) \\ - \sum_{k=1}^{M_h} 2a_1 \left|\left\{\frac{\partial\Phi}{\partial n_k}\right\}\right|_k |[\Phi]|_k - \sum_{k=1}^{P_h} 2a_1|\{\Phi\}|_k|[\Phi]|_k. \end{aligned} \tag{3.6}$$

Using the local inverse property and (2.4), we obtain

$$\begin{aligned} &2a_1\left(\sum_{k=1}^{M_h} \left|\left\{\frac{\partial\Phi}{\partial n_k}\right\}\right|_k |[\Phi]|_k + \sum_{k=1}^{P_h} |\{\Phi\}|_k|[\Phi]|_k\right) \\ &\leqq C\left(\left(\sum_{k=1}^{M_h} \frac{1}{l(e_k)}|[\Phi]|_k^2\right)^{1/2}\left(\left(\sum_{j=1}^{N_h} \|\nabla\Phi\|_{0,E_j}^2\right)^{1/2}\right.\right. \\ &\qquad \left.\left. + \left(\sum_{j=1}^{N_h} (\|\Phi\|_{0,E_j}^2 + h_j l(e_k)\|\nabla\Phi\|_{0,E_j}^2)\right)^{1/2}\right)\right) \\ &\leqq C(J_0^1(\Phi, \Phi))^{1/2}\left(\sum_{j=1}^{N_h} \|\Phi\|_{1,E_j}^2\right)^{1/2} \\ &\leqq \frac{a_0}{4}\sum_{j=1}^{N_h} \|\Phi\|_{1,E_j}^2 + \bar{\sigma}J_0^1(\Phi, \Phi), \end{aligned} \tag{3.7}$$

where $\bar{\sigma} = \bar{\sigma}(a_0, a_1, C_0, \tilde{C}_0)$. Thus by (3.6), (3.7) and (2.5)

$$B_\sigma(\Phi, \Phi) \geqq \frac{a_0}{2}\sum_{j=1}^{N_h} \|\nabla\Phi\|_{0,E_j}^2 - \left(a_1 + \frac{4a_1^2}{a_0} + \frac{a_0}{4}\right)\|\Phi\|_{0,\Omega}^2 + \sum_{k=1}^{M_h} \frac{(\sigma_k - \bar{\sigma})}{l(e_k)}|[\Phi]|_k^2.$$

Letting $\hat{\sigma} = 2\bar{\sigma}$ and

$$\hat{\delta} = \min\left\{\frac{a_0}{2},, \min_k \left(\left|\frac{\sigma_k - \bar{\sigma}}{\sigma_k}\right|\right)\right\},$$

we obtain

$$B_\sigma(\Phi, \Phi) \geqq \tilde{\delta}|||\Phi|||^2 - \hat{a}\|\Phi\|_{0,\Omega}^2, \qquad \sigma_k \geqq \hat{\sigma}.$$

In our analysis we make use of the following trace theorem. A proof may be found in [1] for flat edges e_k. If e_k is a curvilinear edge, then a local flattening argument is used first.

LEMMA 4. *If $w \in H^2(\mathscr{E}_h)$, then there exists a constant $C > 0$ such that*

$$|w|_k^2 \leqq C(h_j\|\nabla w\|_{0,E_j}^2 + h_j^{-1}\|w\|_{0,E_j}^2)$$

and

$$\left|\frac{\partial w}{\partial n}\right|_k^2 \leqq C(h_j\|w\|_{2,E_j}^2 + h_j^{-1}\|\nabla w\|_{0,E_j}^2),$$

where k is such that e_k is an edge belonging to E_j.

An immediate consequence of the trace theorem and (1.4) is the following approximation result.

Approximation property. If $w \in H^l(\mathscr{E}_h)$, $2 \leqq l \leqq \hat{r}+1$, then there exists a $\chi \in \mathscr{M}_h^r$ such that

$$\|w-\chi\|_{0,\Omega} \leqq C\left(\sum_{j=1}^{N_h} (\|w\|_{l,E_j} h_j^l)^2\right)^{1/2} \tag{3.8}$$

and

$$|||w-\chi|||_{1,\sigma} \leqq C\left(\sum_{j=1}^{N_h} (\|w\|_{l,E_j} h_j^{l-1})^2\right)^{1/2}, \tag{3.9}$$

where $C = C(\sigma)$ is a constant independent of w and h_j. Inequalities (3.8) and (3.9) also hold if $\mathscr{M}_h^r$ is replaced by $\mathscr{M}_h^r \cap C^0(\Omega)$ [10].

LEMMA 5. *Let $\Psi \in H^2(\mathscr{E}_h)$ and assume that*

$$B_\sigma(\Psi, v) = 0, \qquad v \in \mathscr{M}_h^1.$$

Then there exists a positive constant C such that

$$\|\Psi\|_{0,\Omega} \leqq C|||\Psi|||_{1,\sigma} h, \tag{3.10}$$

where $h = \max_{1\leqq j \leqq N_h} h_j$.

Proof. Consider the auxiliary problem

$$L^*\Gamma = \Psi, \qquad x \in \Omega,$$
$$\Gamma = 0, \qquad x \in \partial\Omega.$$

Since $\Gamma \in H^2(\Omega)$, we have

$$\|\Psi\|_{0,\Omega}^2 = (\Psi, L^*\Gamma) = B_\sigma(\Psi, \Gamma) = B_\sigma(\Psi, \Gamma - \Gamma^*), \qquad \Gamma^* \in \mathscr{M}_h^1.$$

Using (3.9) with $l = 2$ and $w = \Gamma$ we let $\Gamma^* = \chi$.

By Lemma 1 and the 0-regularity of L^* we have

$$\|\Psi\|_{0,\Omega}^2 \leqq C|||\Psi|||_{1,\sigma}|||\Gamma-\chi|||_{1,\sigma} \leqq C|||\Psi|||_{1,\sigma}\|\Gamma\|_{2,\Omega} h \leqq C|||\Psi|||_{1,\sigma}\|\Psi\|_{0,\Omega} h.$$

We now derive an optimal L^2 estimate for the interior penalty L^2-Galerkin procedure.

THEOREM 1. *Let $W_h \in \mathscr{M}_h^r$ satisfy*

$$B_\sigma(W_h, v) = (f, v) - \left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathscr{M}_h^r, \tag{3.11}$$

where $\sigma_k \geqq \hat{\sigma}$. There exists a constant $C > 0$ such that if h is sufficiently small

$$|||W_h - u|||_{1,\sigma} \leqq C\left(\sum_{j=1}^{N_h} (\|u\|_{l,E_j} h_j^{l-1})^2\right)^{1/2}, \qquad 2 \leqq l \leqq r+1, \tag{3.12}$$

and

$$\|W_h - u\|_{0,\Omega} \leqq C|||W_h - u|||_{1,\sigma} h. \tag{3.13}$$

Proof. It follows from Lemmas 3 and 5 that W_h is uniquely determined by (3.11). Let $\chi \in \mathcal{M}_h^r$ satisfy (3.8) and (3.9) with $w = u$. We set $\xi = u - W_h$, $\zeta = u - \chi$ and $\eta = \chi - W_h$. By Lemmas 2 and 5

$$\|\xi\|_{0,\Omega} \leqq C|||\xi|||_{1,\sigma} h \leqq C(|||\zeta|||_{1,\sigma} + |||\eta|||_\sigma)h. \tag{3.14}$$

Thus

$$\|\eta\|_{0,\Omega} \leqq \|\xi\|_{0,\Omega} + \|\zeta\|_{0,\Omega} \leqq C(|||\zeta|||_{1,\sigma} + |||\eta|||_\sigma)h + \|\zeta\|_{0,\Omega}. \tag{3.15}$$

Hence by Lemmas 1 and 3, (3.14) and (3.15)

$$\begin{aligned}\hat{\delta}|||\eta|||_\sigma^2 &\leqq B_\sigma(\eta, \eta) + \hat{a}\|\eta\|_{0,\Omega}^2 = B_\sigma(-\zeta, \eta) + \hat{a}\|\eta\|_{0,\Omega}^2 \\ &\leqq C_\sigma(|||\zeta|||_{1,\sigma}|||\eta|||_\sigma + \|\eta\|_{0,\Omega}^2) \leqq C_\sigma|||\zeta|||_{1,\sigma}^2 + \left(\frac{\hat{\delta}}{2} + C^* h^2\right)|||\eta|||_\sigma^2,\end{aligned} \tag{3.16}$$

or for h sufficiently small,

$$|||\eta|||_\sigma \leqq C_\sigma|||\zeta|||_{1,\sigma}. \tag{3.17}$$

Using (3.15), (3.17), (3.8) and (3.9) we have (3.12) and (3.13).

For the remainder of this paper we assume $u \in H^{r+1}(\mathscr{E}_h)$, $\hat{r} > 2$.

We now define a local projection $Z_h = P_h(u)$ of u onto $\mathcal{M}_h^r$ by

$$(\Delta Z_h)(\alpha_{i,j}) = (\Delta u)(\alpha_{i,j}), \qquad i = 1, 2, \cdots, K_r, \quad j = 1, 2, \cdots, N_h, \tag{3.18}$$

and

$$\int_{E_j} (Z_h - u)v\, dx = 0, \qquad v \in \mathscr{H}_h^r, \quad j = 1, 2, \cdots, N_h. \tag{3.19}$$

We have the following error estimate for $Z_h - u$.

LEMMA 6. *Let Z_h be defined by* (3.18) *and* (3.19). *There exists a constant $C > 0$ such that*

$$\|u - Z_h\|_{k,E_j} \leqq C\|u\|_{l,E_j} h_j^{l-k}, \qquad 3 < l \leqq r+1, \quad 0 \leqq k \leqq l, \tag{3.20a}$$

$$\|u - Z_h\|_{k,E_j} \leqq C\|u\|_{W_\infty^l(E_j)} h_j^{l-k+1}, \qquad l = 2, 3, \quad r = 2, \quad 0 \leqq k \leqq l, \tag{3.20b}$$

$$|||u - Z_h|||_{1,\sigma} \leqq \begin{cases} C\left(\sum_{j=1}^{N_h} \|u\|_{l,E_j}^2 h_j^{2(l-1)}\right)^{1/2}, & 3 < l \leqq r+1, \\ C \sum_{j=1}^{N_h} \|u\|_{W_\infty^l(E_j)} h_j^l, & l = 2, 3, \quad r = 2. \end{cases} \tag{3.21}$$

Proof. Since existence and uniqueness of Z_h are equivalent we deduce from (3.18) and (3.19) that if $u \equiv 0$ then $Z_h \equiv 0$. $Z_h = P_h u$ is a local projection and $P_h\chi = \chi$, $\chi \in \bigcap_{j=1}^{N_h} P_r(E_j)$. Inequalities (3.20a) and (3.20b) follow directly from the approximation property (1.4) ([4] and [9]). Inequality (3.21) follows from Lemma 4 and (3.20a) and (3.20b).

We next define $\tilde{U}_h \in \mathcal{M}_h^r$ by

$$\Delta\tilde{U}_h(\alpha_{i,j}) = (\Delta u)(\alpha_{i,j}), \qquad i = 1, 2, \cdots, K_r, \quad j = 1, 2, \cdots, N_h, \tag{3.22}$$

and

$$B_\sigma(\tilde{U}_h, v) = (f, v) - \left\langle g, a\frac{\partial v}{\partial n}\right\rangle, \qquad v \in \mathscr{H}_h^r, \tag{3.23}$$

where $\sigma_k \geqq \hat{\sigma}$. For h sufficiently small $\tilde{U}_h$ exists and is unique. This is easily seen by setting the data $(\Delta u)(\alpha_{ij}), f$ and g equal to zero in (3.22) and (3.23) and using Lemmas 3 and 5.

Let $\nu = W_h - \tilde{U}_h$ where W_h is defined by (3.11). We note that

$$B_\sigma(\nu, \nu) = B_\sigma(\nu, \nu - v), \qquad v \in \mathcal{H}_h^r.$$

Since $Z_h - \tilde{U}_h \in \mathcal{H}_h^r$ we have by Lemmas 1 and 2 that

$$B_\sigma(\nu, \nu) = B_\sigma(\nu, W_h - Z_h) \leqq C|||\nu|||_\sigma |||W_h - Z_h|||_\sigma.$$

By Lemma 3

$$\hat{\delta}|||\nu|||_\sigma^2 - \hat{a}\|\nu\|_{0,\Omega}^2 \leqq C|||\nu|||_\sigma |||W_h - Z_h|||_\sigma,$$

and since $\mathcal{M}_h^1 \subset \mathcal{H}_h^r$, by Lemma 5,

$$\|\nu\|_{0,\Omega} \leqq C|||\nu|||_\sigma h. \tag{3.24}$$

Thus for h sufficiently small

$$|||\nu|||_\sigma \leqq C(\sigma)|||W_h - Z_h|||_\sigma. \tag{3.25}$$

Using (3.24), (3.25), Lemma 6 and Theorem 1, we note the following result.

LEMMA 7. *Let $\tilde{U}_h \in \mathcal{M}_h^r$ satisfy* (3.22) *and* (3.23) *with $\sigma_k \geqq \hat{\sigma}$. There exist constants $C_1(\sigma) > 0$ and $C_2 > 0$ such that for h sufficiently small,*

$$|||u - \tilde{U}_h|||_{1,\sigma} \leqq C_1(\sigma)\left(\sum_{j=1}^{N_h} (\|u\|_{l,E_j} h_j^{l-1})^2\right)^{1/2}, \qquad 3 < l \leqq r+1, \tag{3.26}$$

$$|||u - \tilde{U}_h|||_{1,\sigma} \leqq C_1(\sigma) \sum_{j=1}^{N_h} \|u\|_{W_\infty^l(E_j)} h_j^l, \qquad l = 2, 3, \quad r = 2, \tag{3.27}$$

and

$$\|u - \tilde{U}_h\|_{0,\Omega} \leqq C_2\{|||u - W_h|||_{1,\sigma} + |||u - \tilde{U}_h|||_{1,\sigma}\}h. \tag{3.28}$$

It should be noted that Lemma 7 gives an optimal L^2 rate of convergence for the L^2-finite element-collocation method when $L = \Delta$. If $r \geqq 3$ the norm on the solution is also optimal, optimal in the sense that it can not be weakened.

We now consider the general second order differential operator given by (1.2). Let $\theta_h = U_h - \tilde{U}_h$ where U_h and $\tilde{U}_h$ are defined by (2.7)–(2.8) and (3.22)–(3.23) respectively. Since

$$B_\sigma(\theta_h, v) = 0, \qquad v \in \mathcal{H}_h^r, \tag{3.29}$$

we have

$$B_\sigma(\theta_h, \theta_h) = B_\sigma(\theta_h, \theta_h - \chi), \qquad \chi \in \mathcal{H}_h^r. \tag{3.30}$$

Let $W_h^* \in \mathcal{M}_h^r$ satisfy $W_h^* \in \mathcal{H}_h^{r\perp}$ and

$$(\Delta W_h^*)(\alpha_{i,j}) = \frac{1}{a}(\nabla a \cdot \nabla(u - U_h) - b \cdot \nabla(u - U_h) - c(u - U_h))(\alpha_{i,j}),$$
$$i = 1, 2, \cdots, K_r, \quad j = 1, 2, \cdots, N_h.$$

We deduce that W_h^* is uniquely defined since if $\Delta W_h^*(\alpha_{ij}) = 0$, $i = 1, 2, \cdots, K_r$, $j = 1, 2, \cdots, N_h$ then $W_h^* \in H_h^r \cap H_h^{r\perp} = \{0\}$. One can easily verify that

$$\Delta\theta_h = \Delta W_h^*. \tag{3.31}$$

From (3.30), (3.31), Lemma 1 and Lemma 2 we see that

$$B_\sigma(\theta_h, \theta_h) \leqq C(\sigma)|||\theta_h|||_\sigma |||W_h^*|||_\sigma. \tag{3.32}$$

Taking $\sigma_k \geqq \hat{\sigma}$, we obtain by Lemma 3 and (3.32)

$$\frac{\hat{\delta}}{2}|||\theta_h|||_\sigma^2 - \hat{a}\|\theta_h\|_{0,\Omega}^2 \leqq C(\sigma)|||W_h^*|||_\sigma^2. \tag{3.33}$$

Using (3.29) and Lemma 5, we note

$$\|\theta_h\|_{0,\Omega} \leqq C|||\theta_h|||_\sigma h. \tag{3.34}$$

Substituting (3.34) into (3.33) we see that for h sufficiently small,

$$|||\theta_h|||_\sigma \leqq C(\sigma)|||W_h^*|||_\sigma.$$

To bound $|||W_h^*|||_\sigma$ it suffices to obtain an estimate of $\|W_h^*\|_{0,E_j}$ and then use the local inverse property of $\mathcal{M}_h^r$.

For $\Phi \in P_r(E_j)$ we define the discrete norm

$$|||\Phi||| = \|\Phi_1\|_{0,E_j} + h_j^3\left(\sum_{i=1}^{K_r} |(\Delta\Phi)(\alpha_{i,j})|\right), \tag{3.35}$$

where $\Phi = \Phi_1 + \Phi_2$, $\Phi_1 \in \mathcal{H}_h^r$ and $\Phi_2 \in \mathcal{H}_h^{r\perp}$. One can easily verify that $|||\cdot|||$ is a norm on $P_r(E_j)$. Since all norms are equivalent on a finite dimensional subspace, we have that there exists a constant $C > 0$ independent of h_j such that

$$\|\Phi\|_{0,E_j} \leqq C|||\Phi|||, \qquad \Phi \in P_r(E_j). \tag{3.36}$$

Now

$$(\Delta W_h^*)(\alpha_{i,j}) = \left[-\frac{1}{a}(\nabla a \cdot \nabla\theta_h - b \cdot \nabla\theta_h - c\theta_h) + \frac{1}{a}(\nabla a \cdot \nabla\beta_h - b \cdot \nabla\beta_h - c\beta_h)\right](\alpha_{i,j}),$$

where $\beta_h = u - \tilde{U}_h$. We observe that

$$\sum_{i=1}^{K_r} |(\Delta W_h^*)(\alpha_{i,j})| \leqq C(\|\theta_h\|_{W_\infty^1(E_j)} + \|\beta_h\|_{W_\infty^1(E_j)}). \tag{3.37}$$

Since $\theta_h \in P_r(E_j)$ we have by norm equivalence

$$\|\theta_h\|_{W_\infty^1(E_j)} \leqq C\|\theta_h\|_{1,E_j} h_j^{-1}. \tag{3.38}$$

By Sobolev's inequality [1]

$$\|\beta_h\|_{W_\infty^1(E_j)} \leqq C(\|\beta_h\|_{3,E_j} h_j + \|\beta_h\|_{0,E_j}, h_j^{-2}). \tag{3.39}$$

Substituting (3.38), (3.39) into (3.37) we have

$$|||W_h^*||| = h_j^3 \sum_{i=1}^{K_r} |(\Delta W_h^*)(\alpha_{i,j})| \leqq C(\|\theta_h\|_{1,E_j} h_j^2 + \|\beta_h\|_{3,E_j} h_j^4 + \|\beta_h\|_{0,E_j} h_j). \tag{3.40}$$

By the local inverse property,

$$\|\nabla W_h^*\|_{0,E_j} \leqq C\|W_h^*\|_{0,E_j} h_j^{-1} \tag{3.41}$$

and

$$|W_h^*|_k \leqq C(\|W_h^*\|_{0,E_j} h_j^{-1/2} + \|\nabla W_h^*\|_{0,E_j} h_j^{1/2}), \tag{3.42}$$

where e_k is an edge of E_j. Since $(1/l(e_k))h_j \leqq C(\rho)$, we conclude from (3.40)–(3.42) that

$$|||W_h^*|||_\sigma^2 \leqq C \sum_{j=1}^{N_h} (\|\theta_h\|_{1,E_j}^2 h_j^2 + \|\beta_h\|_{3,E_j}^2 h_j^6 + \|\beta_h\|_{0,E_j}^2).$$

Let u_I be an interpolate of u on E_j and set $\zeta_h = u_I - \tilde{U}_h$. By (3.1) and (1.4) we have

$$h_j^3\|\beta_h\|_{3,E_j} \leqq h_j^3(\|u-u_I\|_{3,E_j} + \|\zeta_h\|_{3,E_j}) \leqq C(h_j^3\|u-u_I\|_{3,E_j} + \|\zeta_h\|_{0,E_j}).$$

Thus

$$|||W_h^*|||_\sigma \leqq C\left(\left(\sum_{j=1}^{N_h} (\|\theta_h\|_{1,E_j}h_j)^2\right)^{1/2} + \left(\sum_{j=1}^{N_h} (\|u-u_I\|_{3,E_j}h_j^3)^2\right)^{1/2} + \|\zeta_h\|_{0,\Omega}\right). \tag{3.43}$$

From (3.33), (3.34) and (3.43), we see that for h sufficiently small,

$$|||\theta_h|||_\sigma \leqq C(\sigma)\left(\left(\sum_{j=1}^{N_h} (\|u-u_I\|_{3,E_j}h_j^3)^2\right)^{1/2} + \|\zeta_h\|_{0,\Omega}\right). \tag{3.44}$$

Using inequalities (1.4) and (3.44) and Lemma 7, we note the following result.

THEOREM 2. *Let $u \in H^{\hat{r}+1}(\mathscr{E}_h)$, $\hat{r} > 2$, and let U_h be defined by* (2.7)–(2.8) *with $\sigma_k \geqq \hat{\sigma}$. There exist constants $C_\sigma > 0$ and $C_2 > 0$ such that for h sufficiently small,*

$$|||U_h - u|||_\sigma \leqq C_\sigma\left(\sum_{j=1}^{N_h} (\|u\|_{l,E_j}h_j^{l-1})^2\right)^{1/2}, \qquad 3 < l \leqq \min(\hat{r}+1, r+1),$$

$$|||U_h - u|||_{1,\sigma} \leqq C_\sigma \sum_{j=1}^{N_h} \|u\|_{W_\infty^l(E_j)}h_j^l, \qquad l = 2, 3, \quad r = 2,$$

and

$$\|U_h - u\|_{0,\Omega} \leqq C_2|||U-u|||_{1,\sigma}h.$$

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