THERMOPOROELASTICITY

EXISTENCE, UNIQUENESS

PART I

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12/8/19

THERMO POROELASTICITY EXISTENCE UNIQUENESS (1)

5-8-19

$$\varepsilon^m = \nabla\cdot\underline{u}^s, \quad \varepsilon^f = \nabla\cdot\underline{u}^f$$

$$\varepsilon = \alpha\,\varepsilon^m + \varepsilon^f = \alpha\,\nabla\cdot\underline{u}^s + \nabla\cdot\underline{u}^f$$

$$\tau_{ij}(u,\theta) = 2\mu\,\varepsilon_{ij}(u) + \delta_{ij}\Big(\lambda\nabla\cdot u^s + \alpha M(\alpha\,\nabla\cdot u^s + \nabla\cdot u^f) - \beta\,\theta\Big)\Big[-\beta(\theta + t_1\dot{\theta})\Big] \quad (1)$$

$$= 2\mu\,\varepsilon_{ij}(u) + \delta_{ij}\Big(\lambda\nabla\cdot u^s + \alpha^2 M\nabla\cdot u^s + \alpha M\,\nabla\cdot u^f - \beta\theta\Big) + f_{i,j}$$

$$\sigma_f = -\phi p_f = \phi M(\alpha\,\nabla\cdot u^s + \nabla\cdot u^f) - \beta_f\,\theta \Big[-\beta_f(\theta + t_2\dot{\theta})\Big] \quad (2)$$

$$= \alpha M\phi\,\nabla\cdot u^s + \phi M\,\nabla\cdot u^f - \beta_f\,\theta\Big[-\beta_f(\theta \ldots\Big]$$

$$= \sigma_f(u,\theta)$$

$$\rho\,\ddot{\underline{u}} + \rho_f\,\ddot{\underline{u}}_f - \nabla\cdot\underline{\sigma} = f^s \quad (3)$$

$$\rho_f\,\ddot{\underline{u}}_s + g\,\ddot{\underline{u}}_f + \frac{\eta}{K}\dot{\underline{u}}_f + \nabla p_f = f^f \quad (4)$$

2D CASE

$$e(u^s) = \varepsilon_{11}(u^s) + \varepsilon_{33}(u^s)$$

$$e(u^f) = \varepsilon_{11}(u^f) + \varepsilon_{33}(u^f)$$

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij}(u^s)$$

$$+\, \delta_{ij}\left(\underbrace{(\lambda+\alpha^2 M)}_{\lambda_\mu} e(u^s) + \alpha M\, e(u^f) - \beta\, \Theta\right)$$

$$+p_f = -\alpha M\, e(u^s) - M\, e(u^f) + \beta_f \Theta$$

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij}(u^s) + \delta_{ij}\left(\lambda_\mu e(u^s) + B\, e(u^f) - \beta\Theta\right) = \sigma_{ij}(u^s, u^f, \Theta)$$

$$+p_f = -B\, e(u^s) - M\, e(u^f) + \beta_f \Theta = p_f(u^s, u^f, \Theta)$$

$$B = \alpha M \quad - \quad \lambda_\mu = \lambda + \alpha^2 M \quad -$$

Testing against $v = v^s \in [H^1(\Omega)]^2$:

$$-\left(\frac{\partial}{\partial x_j}\sigma_{ij}, v\right) = \left(\frac{\partial}{\partial x_j}\left[2\mu\, \varepsilon_{ij}(u^s) + \delta_{ij}\left[\lambda_\mu e(u^s) + B\, e(u^f) - \beta\Theta\right]\right], v\right)$$

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$$= \left(2\mu\, \varepsilon_{ij}(u^s), \frac{\partial}{\partial x_j}(v_i)\right)$$

$$+ \left(\delta_{ij}\left(\lambda_\mu\, e(u^s) + B\, e(u^f) - \beta\theta\right), \frac{\partial v_i}{\partial x_j}\right)$$

$$- \langle \sigma_{ij}\nu_j, v_i \rangle$$

$$= \left(2\mu\, \varepsilon_{ij}(u^s), \varepsilon_{ij}(v)\right)$$

$$+ \left(\lambda_\mu\, e(u^s) + B\, e(u^f) - \beta\theta, \underbrace{\delta_{ij}\frac{\partial v_i}{\partial x_j}}_{\nabla\cdot v = e(v)}\right)$$

$$- \langle \sigma_{ij}\nu_j, v_i \rangle$$

$$= \left(2\mu\, \varepsilon_{ij}(u^s), \varepsilon_{ij}(v)\right)$$

$$+ \left(\lambda_\mu\, e(u^s) + B\, e(u^f), e(v)\right)$$

$$- (\beta\theta, e(v)) - \langle \sigma_{ij}\nu_j, v_i \rangle$$

$$\cancel{= \left(\left((\lambda_\mu + 2\mu)\,\varepsilon_{11}(u^s) + \lambda_\mu\, \varepsilon_{33}(u^s)\right), \varepsilon_{11}(v)\right)}$$

$$\cancel{+ \left(\left((\lambda_\mu + 2\mu)\,\varepsilon_{33}(u^s) + \lambda_\mu\, \varepsilon_{11}(u^s)\right), \varepsilon_{33}(v)\right)}$$

$$\cancel{+ 4\left(\mu\, \varepsilon_{13}(u), \varepsilon_{13}(v)\right) + \left(B\, e(u^s), e(v)\right)}$$

$$\cancel{- (\beta\theta}$$

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$$= (2\mu\, \varepsilon_{11}(u^s), \varepsilon_{11}(v)) + (2\mu\, \varepsilon_{33}(u^s), \varepsilon_{33}(v))$$

$$+ (2\mu\, \varepsilon_{13}(u^s), \varepsilon_{13}(v)) + (2\mu\, \varepsilon_{31}(u^s), \varepsilon_{31}(v))$$

$$+ (\lambda_u(\varepsilon_{11}(u^s) + \varepsilon_{33}(u^s)) + B\, e(u^f), \varepsilon_{11}(v) + \varepsilon_{33}(v))$$

$$- (\beta\theta, e(v)) - \langle \sigma_{ij} \nu_j, v_i \rangle$$

$$= (2\mu\, \varepsilon_{11}(u^s), \varepsilon_{11}(v)) + (2\mu\, \varepsilon_{33}(u^s), \varepsilon_{33}(v))$$

$$+ 4(\mu\, \varepsilon_{13}(u^s), \varepsilon_{13}(v)) + (\lambda_\mu\, \varepsilon_{11}(u^s), \varepsilon_{11}(v))$$

$$+ (\lambda_\mu\, \varepsilon_{33}(u^s), \varepsilon_{11}(v)) + (\lambda_\mu\, \varepsilon_{11}(u^s), \varepsilon_{33}(v))$$

$$+ (\lambda_\mu\, \varepsilon_{33}(u^s), \varepsilon_{33}(v)) - (\beta\theta, e(v))$$
$$+ (B\, e(u^f), \varepsilon_{11}(v)) + (B\, e(u^f), \varepsilon_{33}(v))$$
$$- \langle \sigma\nu, v \rangle$$

$$= ((\lambda_\mu + 2\mu)\, \varepsilon_{11}(u^s), \varepsilon_{11}(v)) + ((\lambda_\mu + 2\mu)\varepsilon_{33}(u^s), \varepsilon_{33}(v))$$

$$+ (\lambda_\mu\, \varepsilon_{33}(u^s), \varepsilon_{11}(v)) + (\lambda_\mu\, \varepsilon_{11}(u^s), \varepsilon_{33}(v))$$

$$+ 4(\mu\, \varepsilon_{13}(u^s), \varepsilon_{13}(v)) - (\beta\theta, e(v))$$

$$- \langle \sigma\nu, v \rangle + (B\, e(u^f), \varepsilon_{11}(v)) \quad e(v)$$
$$+ (B\, e(u^f), \varepsilon_{33}(v))$$

$$(\nabla p_f, w) \overset{\text{Testing against } w \in H(\mathrm{div}, \Omega) \quad w = v^f,}{=} -(p_f, \nabla\cdot w) + \langle w\cdot\nu, p_f\rangle$$ 5

$$= (\alpha M\, e(u^s) + M\, e(u^f) \overset{-\beta_f\theta}{,} e(w))$$

$$+ \langle w\cdot\nu,\ p_f\rangle$$

$$= (B\, e(u^s), e(w)) + (M\, e(u^f), e(w))$$

$$+ \langle w\cdot\nu, p_f\rangle - (\beta_f\theta, e(w)).$$

Then

$$-(\nabla\cdot\sigma, v) + (\nabla p_f, w)$$

$$= ((\lambda_\mu + 2\mu)\, \varepsilon_{11}(u^s), \varepsilon_{11}(v)) + ((\lambda_\mu+2\mu)\,\varepsilon_{33}(u^s), \varepsilon_{33}(v))$$

$$+ (\lambda_\mu\, \varepsilon_{33}(u^s), \varepsilon_{11}(v)) + (\lambda_\mu\, \varepsilon_{11}(u^s), \varepsilon_{33}(v))$$

$$+ (4\mu\, \varepsilon_{13}(u^s), \varepsilon_{13}(v)) + (B\, e(u^f), e(v))$$

$$+ (B\, e(u^s), e(w)) + (M\, e(u^f), e(w))$$

$$- (\beta\theta, e(v)) - \langle\sigma\nu, v\rangle + \langle p_f, w\cdot\nu\rangle$$

$$- (\beta_f\theta, e(w))$$

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$$= \left(\underbrace{\begin{pmatrix} \lambda_\mu + 2\mu & \lambda_\mu & B & 0 \\ \lambda_\mu & \lambda_\mu + 2\mu & B & 0 \\ B & B & M & 0 \\ 0 & 0 & 0 & 4\mu \end{pmatrix}}_{\mathbb{E}\ (\text{pos-def})} \underbrace{\begin{pmatrix} \varepsilon_{11}(u^s) \\ \varepsilon_{33}(u^s) \\ e(u^f) \\ \varepsilon_{13}(u^s) \end{pmatrix}}_{\tilde{\varepsilon}(u)}, \begin{pmatrix} \varepsilon_{11}(v) \\ \varepsilon_{33}(v) \\ e(w) \\ \varepsilon_{13}(v) \end{pmatrix} \right)$$

$$- (\beta\theta, e(v)) + \langle \sigma\nu, v \rangle$$

$$+ \langle p_f, w \cdot \nu \rangle - (\beta_f\theta, e(w))$$

$$\equiv B((u^s, u^f), (v, w)) - (\beta\theta, e(v)) - (\beta_f\theta, e(w)) \quad (5)$$

$$- \langle \sigma\nu, v \rangle + \langle p_f, w \cdot \nu \rangle$$

Let

$$\cancel{(u^s, u^f) = (-\nabla\cdot\sigma(u,T)}$$

$$u = (u^s, u^f) \qquad V = (v^s, v^f) \quad [v^f = w]$$

$$\mathcal{L}(u, \theta) = (-\nabla\cdot\sigma(u,\theta), \nabla p_f(u,\theta)) \quad (6)$$

Then

$$\left(\alpha(u,\theta), v\right) = B(u,v) - (\beta\theta, e(v^s)) - (\beta_f\theta, e(v^f)) - \langle \sigma(u,\theta)\cdot\nu, v^s\rangle_\Gamma + \langle p_f(u,\theta), v^f\cdot\nu\rangle_\Gamma$$

HEAT EQUATION FOR θ:

$$\tau\ddot{\theta} + c\dot{\theta} - \nabla\cdot(\gamma\nabla\theta) + \beta T_0\, e(\dot{u}^s) + \tau\beta T_0\, \cancel{e(\ddot{u}^s)} + \beta T_0\, e(\dot{u}^f) + \tau\beta T_0\, \cancel{e(\ddot{u}^f)} = -q, \ \Omega \qquad (7)$$

Testing against $w \in H^1(\Omega)$:

$$(\tau\ddot{\theta}, w) + (c\dot{\theta}, w) + (\gamma\nabla\theta, \nabla w) - \langle \gamma\nabla\theta\cdot\nu, w\rangle_\Gamma + (\beta T_0\, e(\dot{u}^s), w) + (\beta T_0\,\tau\, \cancel{e(\ddot{u}^s)}, w) + (\beta T_0\, e(\dot{u}^f), w) + (\tau\beta T_0\, \cancel{e(\ddot{u}^f)}, w) = -(q, w), \qquad (8)$$

$w \in H^1(\Omega)$.

The Initial B.V.P. (8)

$$\rho \ddot{u}^s + \rho_f \ddot{u}^f - \nabla\cdot\sigma(u,\theta) = f^s, \Omega \quad (9)$$

$$\rho_f \ddot{u}^s + g\ddot{u}^f + \frac{\eta}{K}\dot{u}^f + \nabla p_f(u,\theta) = f^f, \Omega, \quad (10)$$

+ HEAT EQ'N (7) for θ

$$\sigma(u,\theta)\cdot\nu = g, \quad \Gamma = \partial\Omega \quad (11)$$

$$-p_f(u,\theta) = x$$

$$\gamma\nabla\theta\cdot\nu = h, \quad \Gamma \quad (12)$$

$$u(0) = u^0, \dot{u}(0) = u^1, \quad (13)$$

$$\theta(0) = \theta^0, \dot{\theta}(0) = \theta^1 \quad (14)$$

Multiply (9) by $v^s \in [H^1(\Omega)]^2$, (10) by $v^f \in H(div,\Omega)$, and (7) by $w \in H^1(\Omega)$, use (see (8)) integration by parts as before and add the resulting equations to get the WEAK FORM:

Find $(u^s, u^f) \in [H^1(\Omega)]^2 \times H(div,\Omega)$ such that

$$(\rho \ddot{u}^s, v^s) + (\rho_f \ddot{u}^f, v^s) + \left(\frac{\eta}{\kappa}\dot{u}^f, v^f\right)$$
$$+ (\rho_f \ddot{u}^s, v^f) + (g \ddot{u}^f, v^f) + B(u, v)$$
$$- (\beta\theta, e(v^s)) - (\beta_f \theta, e(v^f))$$
$$+ (\tau \ddot{\theta}, w) + (c\dot{\theta}, w) + (\gamma \nabla\theta, \nabla w) \qquad (15)$$
$$+ (T_0 \beta e(\dot{u}^s), w) + (T_0 \beta \tau e(\ddot{u}^s), w)$$
$$+ (T_0 \beta e(\dot{u}^f), w) + (T_0 \beta \tau e(\ddot{u}^f), w)$$
$$- \langle g, v^s \rangle_\Gamma - \langle \chi, v^f \cdot \nu \rangle_\Gamma - \langle h, w \rangle_\Gamma$$
$$= (f^s, v^s) + (f^f, v^f) - (q, w),$$
$$\forall \ (v^s, v^f, w) \in [H^1(\Omega)]^3$$

CHOOSE $v^s = \dot{u}^s$, $v^f = \dot{u}^f$, $w = \dot{\theta}$

in (15).

Set
$$\mathcal{U} = \begin{pmatrix} \rho I & \rho_f I \\ \rho_f I & g I \end{pmatrix}, \quad I = \text{identity in } R^{2\times 2}.$$

Then for the mass terms:
$$\mathcal{D} = \begin{pmatrix} 0 I & 0 I \\ 0 I & \frac{\eta}{\kappa} I \end{pmatrix}$$

$$(\varrho \ddot{u}^s, \dot{u}^s) + (\varrho_f \ddot{u}^f, \dot{u}^s)$$

$$+ (\varrho_f \ddot{u}^s, \dot{u}^f) + (g \ddot{u}^f, \dot{u}^f)$$

$$= \frac{1}{2} \frac{d}{dt} (\varrho \dot{u}^s, \dot{u}^s) + \frac{d}{dt} \frac{1}{2} (\varrho_f \dot{u}^f, \dot{u}^s)$$

$$+ \frac{1}{2} \frac{d}{dt} (\varrho_f \dot{u}^s, \dot{u}^f) + \frac{1}{2} \frac{d}{dt} (g \dot{u}^f, \dot{u}^f)$$

$$= \frac{1}{2} \frac{d}{dt} \left(a \begin{pmatrix} \dot{u}^s \\ \dot{u}^f \end{pmatrix}, \begin{pmatrix} \dot{u}^s \\ \dot{u}^f \end{pmatrix} \right) = \frac{1}{2} \frac{d}{dt} (a \dot{u}, \dot{u})$$

$$= \frac{1}{2} \frac{d}{dt} \Big((\varrho \dot{u}^s, \dot{u}^s) + (\varrho_f \dot{u}^f, \dot{u}^s)$$

$$+ (\varrho_f \dot{u}^f, \dot{u}^s) + (g \dot{u}^f, \dot{u}^f) \Big)$$

Also

$$B(u, \dot{u}) = \frac{1}{2} \frac{d}{dt} B(u, u), \quad (\tau \ddot{\theta}, \dot{\theta}) = \frac{1}{2} \frac{d}{dt} (\tau \dot{\theta}, \dot{\theta}),$$

$$(\gamma \nabla \theta, \nabla \dot{\theta}) = \frac{1}{2} \frac{d}{dt} (\gamma \nabla \theta, \nabla \theta)$$

Then from (15) for $v^s = \dot{u}^s$, $v^f = \dot{u}^f$

$w = \dot{\theta}$ we get

$$\frac{1}{2}\frac{d}{dt}\Big[(a\dot{u},\dot{u}) + B(u,u) + (\gamma\nabla\theta,\nabla\theta) + (\gamma\dot{\theta},\dot{\theta})\Big] + \left(\frac{\eta}{\kappa}\dot{u}^f,\dot{u}^f\right)$$

$$+ (c\dot{\theta},\dot{\theta}) - (\beta\theta, e(\dot{u}^s))$$

$$- (\beta_f\theta, e(\dot{u}^f)) \qquad (16)$$

$$+ (T_0\beta\, e(\dot{u}^s),\dot{\theta}) + (\tau T_0\beta\, e(\ddot{u}^s),\dot{\theta})$$

$$+ (T_0\beta\, e(\dot{u}^f),\dot{\theta}) + (\tau T_0\beta\, e(\ddot{u}^f),\dot{\theta})$$

$$= \underbrace{(f^s,\dot{u}^s) + (f^f,\dot{u}^f)}_{(f,\dot{u})} - (q,\dot{\theta})$$

$$+ \langle g,\dot{u}^s\rangle_\Gamma + \langle x,\dot{u}^f\cdot\nu\rangle_\Gamma$$

$$+ \langle h,\dot{\theta}\rangle_\Gamma$$

Next we need to control the terms

$e(\ddot{u}^s)$, $e(\ddot{u}^f)$ in the RHS of (16)

FOR THAT PURPOSE WE TAKE TIME DERIVATIVE in (9)-(10)

First we write (9)–(10) as

$$\mathcal{A}\ddot{u} - \mathcal{L}(u,\theta) = f \qquad (17)$$

Then,

$$\mathcal{A}\dddot{u} + \frac{\eta}{\kappa}\ddot{u}^f - \mathcal{L}(\dot{u},\dot{\theta}) = \dot{f} \qquad (18)$$

TESTING (18) with $v = (v^s, v^f) \in (H^1(\Omega))^2 \times H(\mathrm{div};\Omega)$,

$$(\mathcal{A}\dddot{u}, v) + B(\dot{u}, v) - (\beta\dot{\theta}, e(v^s)) + \left(\frac{\eta}{\kappa}\ddot{u}^f, v\right) - (\beta_f\dot{\theta}, e(v^f)) - \langle \sigma(\dot{u},\dot{\theta})\cdot\nu, v^s\rangle_\Gamma + \langle p_f(\dot{u},\dot{\theta}), v^f\cdot\nu\rangle_\Gamma = (\dot{f}, v) \qquad (19)$$

CHOOSE $v^f = 0, v^s = \ddot{u}^s$ in (19) to get

$$(\mathcal{A}\dddot{u}, (\ddot{u}^s, 0)) + B(\dot{u}, (\ddot{u}^s, 0)) - (\beta\dot{\theta}, e(\ddot{u}^s)) = (\dot{f}^s, \ddot{u}^s) + \langle \dot{g}, \ddot{u}^s\rangle_\Gamma \qquad (19\text{-}1)$$

Expanding (19-1):

(12-1)

$$(\rho \overset{\circ\circ\circ}{u}^s, \overset{\circ\circ}{u}^s) + (\rho_f \overset{\circ\circ\circ}{u}^f, \overset{\circ\circ}{u}^s)$$

$$+ ((\lambda_\mu + 2\mu)\varepsilon_{11}(\overset{\circ}{u}^s), \varepsilon_{11}(\overset{\circ\circ}{u}^s))$$

$$+ ((\lambda_\mu + 2\mu)\varepsilon_{33}(\overset{\circ}{u}^s), \varepsilon_{33}(\overset{\circ\circ}{u}^s))$$

$$+ (\lambda_\mu \varepsilon_{33}(\overset{\circ}{u}^s), \varepsilon_{11}(\overset{\circ\circ}{u}^s)) + (\lambda_\mu \varepsilon_{11}(\overset{\circ}{u}^s), \varepsilon_{33}(\overset{\circ\circ}{u}^s))$$

$$+ (4\mu\, \varepsilon_{13}(\overset{\circ}{u}^s), \varepsilon_{13}(\overset{\circ\circ}{u}^s)) \qquad (19\text{-}2)$$

$$+ (B e(\overset{\circ}{u}^f), e(\overset{\circ\circ}{u}^s))$$

$$- (\beta \overset{\circ}{\theta}, e(\overset{\circ\circ}{u}^s)) = (\overset{\circ}{f}^s, \overset{\circ\circ}{u}^s) + \langle \overset{\circ}{g}, \overset{\circ\circ}{u}^s \rangle_\Gamma$$

Also, choose $v^s = 0$, $v^f = \overset{\circ\circ}{u}^f$ in (19):

$$(\rho_f \overset{\circ\circ\circ}{u}^s, \overset{\circ\circ}{u}^f) + (g \overset{\circ\circ\circ}{u}^f, \overset{\circ\circ}{u}^f) + (\tfrac{\eta}{k}\overset{\circ\circ}{u}^f, \overset{\circ\circ}{u}^f) + (B e(\overset{\circ}{u}^s), e(\overset{\circ\circ}{u}^f))$$

$$+ (M e(\overset{\circ}{u}^f), e(\overset{\circ\circ}{u}^f)) - (\beta_f \overset{\circ}{\theta}, e(\overset{\circ\circ}{u}^f))$$

$$= \langle \overset{\circ}{\chi}, \overset{\circ\circ}{u}^f \cdot \nu \rangle_\Gamma + (\overset{\circ}{f}^f, \overset{\circ\circ}{u}^f) \qquad (19\text{-}3)$$

From ~~[illegible]~~ (19-3):

$$\left(\beta \mathring{\theta}, e(\ddot{u}^f)\right) = \frac{\beta}{\beta_f}\left(\beta_f \mathring{\theta}, e(\ddot{u}^f)\right) \quad (12\text{-}2)$$

$$= \left(\frac{\beta}{\beta_f}\rho_f \dddot{u}^s, \ddot{u}^f\right) + \left(\frac{\beta}{\beta_f} g\, \dddot{u}^f, \ddot{u}^f\right) + \frac{\beta}{\beta_f}\left(\frac{\eta}{K}\ddot{u}^f, \ddot{u}^f\right)$$

(19-4)

$$+ \left(\frac{\beta}{\beta_f} B\, e(\mathring{u}^s), e(\ddot{u}^f)\right)$$

$$+ \left(M \frac{\beta}{\beta_f} e(\mathring{u}^f), e(\ddot{u}^f)\right) - \frac{\beta}{\beta_f}\langle \mathring{\chi}, \ddot{u}^f\cdot\nu\rangle_{\Gamma}$$

(19-4)

$$- \frac{\beta}{\beta_f}\left(\mathring{f}^f, \ddot{u}^f\right)$$

Using (19-2) and (19-4)

$$\left(\beta\mathring{\theta}, e(\ddot{u}^s)\right) + \left(\beta\mathring{\theta}, e(\ddot{u}^f)\right)$$

$$= \left((\lambda_\mu+2\mu)\,\varepsilon_{11}(\mathring{u}^s), \varepsilon_{11}(\ddot{u}^s)\right)$$

$$+ \left((\lambda_\mu+2\mu)\,\varepsilon_{33}(\mathring{u}^s), \varepsilon_{33}(\ddot{u}^s)\right)$$

$$+ \left(\lambda_\mu\,\varepsilon_{33}(\mathring{u}^s), \varepsilon_{11}(\ddot{u}^s)\right) + \left(\lambda_\mu\,\varepsilon_{11}(\mathring{u}^s), \varepsilon_{33}(\ddot{u}^s)\right)$$

$$+ \left(4\mu\,\varepsilon_{13}(\mathring{u}^s), \varepsilon_{13}(\ddot{u}^s)\right) + \left(B\, e(\mathring{u}^f), e(\ddot{u}^s)\right)$$

$$+ \left(\frac{\beta}{\beta_f} B\, e(\mathring{u}^s), e(\ddot{u}^f)\right) + \left(\frac{\beta}{\beta_f} M\, e(\mathring{u}^f), e(\ddot{u}^f)\right)$$

$$+\left(\rho\,\dddot{u}^{s},\ddot{u}^{s}\right)+\left(\rho_f\,\dddot{u}^{f},\ddot{u}^{s}\right) \qquad \text{(12-3)}$$

$$+\left(\rho_f\frac{\beta}{\beta_f}\dddot{u}^{s},\ddot{u}^{f}\right)+\left(g\frac{\beta}{\beta_f}\dddot{u}^{f},\ddot{u}^{f}\right)+\left(\frac{\beta}{\beta_f}\frac{\chi}{k}\dddot{u}^{f},\ddot{u}^{f}\right)$$

$$-\left(\dot{f}^{s},\ddot{u}^{s}\right)-\left\langle\dot{g},\ddot{u}^{s}\right\rangle_{\Gamma}-\left(\dot{f}^{f},\ddot{u}^{f}\right)\frac{\beta}{\beta_f}$$

$$-\left\langle\dot{\chi},\ddot{u}^{f}\cdot\nu\right\rangle_{\Gamma}\frac{\beta}{\beta_f}$$

Set

$$\mathcal{A}_\beta=\begin{pmatrix}\rho I & \rho_f I\\ \frac{\beta}{\beta_f}\rho_f I & \frac{\beta}{\beta_f}g I\end{pmatrix}$$

$$E_\beta=\begin{bmatrix}\lambda_\mu+2\mu & \lambda_\mu & B & 0\\ \lambda_\mu & \lambda_\mu+2\mu & B & 0\\ B\frac{\beta}{\beta_f} & B\frac{\beta}{\beta_f} & M\frac{\beta}{\beta_f} & 0\\ 0 & 0 & 0 & 4\mu\end{bmatrix}$$

~~Therefore~~ NOTE THAT $\mathcal{A}_\beta$, E_β are positive definite since $\mathcal{A}$, E are pos. def. and 1 row of $\mathcal{A}$, E are multiplied by $\frac{\beta}{\beta_f}$.

$$B_\beta(u,v)\equiv\left(E_\beta\tilde{\mathcal{E}}(u),\tilde{\mathcal{E}}(v)\right).$$

Then, Computing minors of $\mathcal{A}_\beta$, E_β we get the result

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$$(\beta \dot\theta, e(\ddot u^s)) + (\beta\dot\theta, e(\ddot u^f))$$

$$= (a_\beta \dddot u, \ddot u) + B_\beta(\dot u, \ddot u) \qquad (20)$$

$$- (\dot f^s, \ddot u^s) - \langle \dot g, \ddot u^s\rangle_\Gamma - \langle \dot\lambda, \ddot u^f\cdot\nu\rangle_\Gamma^{\frac{\beta}{\beta_f}}$$

$$-\frac{\beta}{\beta_f}(\dot f^f, \ddot u^f) \qquad +\left(\frac{\gamma}{k}\frac{\beta}{\beta_f}\ddot u^f, \dot u^f\right)$$

$$= \frac{1}{2}\frac{d}{dt}\left[(a_\beta \ddot u, \ddot u) + B_\beta(\dot u, \dot u)\right]$$

$$- (\dot f^s, \ddot u^s) - \langle \dot g, \ddot u^s\rangle_\Gamma - \langle \dot\lambda, \ddot u^f\cdot\nu\rangle_\Gamma^{\frac{\beta}{\beta_f}}$$

$$+\left(\frac{\beta}{\beta_f}\frac{\gamma}{k}\ddot u^f, \ddot u^f\right) - \frac{\beta}{\beta_f}(\dot f^f, \ddot u^f)$$

Using (20) in the LHS of (16):

$$\frac{1}{2}\frac{d}{dt}\Big[(a\dot u, \dot u) + B(u,u) + (\gamma\nabla\theta, \nabla\theta)$$

$$+ (\tau\dot\theta, \dot\theta) + \tau T_0(a_\beta \ddot u, \ddot u) + \tau T_0 B_\beta(\dot u, \dot u)\Big]$$

$$+ (c\dot\theta, \dot\theta) - (\beta\theta, e(\dot u^s)) - (\beta_f\theta, e(\dot u^f))$$

$$+ (T_0\beta\, e(\dot u^s), \dot\theta) + (T_0\beta\, e(\dot u^f), \dot\theta)$$

$$-\tau T_0\frac{\beta}{\beta_f}(\dot f^f, \ddot u^f)$$

$$- \tau T_0(\dot f^s, \ddot u^s) - \tau T_0\langle \dot g, \ddot u^s\rangle_\Gamma - \tau T_0\langle\dot\lambda, \ddot u^f\cdot\nu\rangle_\Gamma^{\frac{\beta}{\beta_f}} + \left(\frac{\beta}{\beta_f}\frac{\gamma}{k}\ddot u^f, \ddot u^f\right)$$

$$= (f, \dot u) - (q, \dot\theta) + \langle g, \dot u^s\rangle_\Gamma + \langle\lambda, \dot u^f\cdot\nu\rangle_\Gamma$$

$$+ \langle h, \dot\theta\rangle_\Gamma \,.$$

Set $V = [H^1(\Omega)]^2 \times H(\mathrm{div}, \Omega)$,

$$\|(v^s, v^f)\|_V = \|v\|_V = \left[\|v^s\|_1^2 + \|v^f\|_{H(\mathrm{div},\Omega)}^2\right]^{1/2}$$

Then, choose ξ_1, ξ_2 such that $B_{\xi_1}(u,u) = B(u,u) + \xi_1(u,u)$ ~~$B_{\xi_2}(u,u) = B_\beta(u,u)$~~

$$B_{\xi_1}(u,u) \geq C_1\left(\|u^s\|_1^2 + \|u^f\|_{H(\mathrm{div},\Omega)}\right)$$

~~$B_{\xi_2}(u,u) \geq \xi_2 \|u\|_{\dots}^2$~~ . (Korn's 2nd ineq.)

Next, add to (21) the inequalities

$$\xi_1 \frac{d}{dt}(u,u) \leq \xi_1 \left(\|\dot{u}\|_0^2 + \|u\|_0^2\right) \quad (22)$$

$$\frac{d}{dt}(\gamma\theta,\theta) \leq \|\gamma^{1/2}\dot{\theta}\|_0^2 + \|\gamma^{1/2}\theta\|_0^2 \quad (23)$$

Since

$$\|\gamma^{1/2}\nabla\theta\|_0^2 + \|\gamma^{1/2}\theta\|_0^2 = \|\gamma^{1/2}\theta\|_1^2,$$

from (21) - (23) we get

$$B_{\beta,\xi_2}(u,u) = B_\beta(u,u) + \xi_2(u,u)$$

$$B_{\beta,\xi_2}(u,u) \geq C_2\|u\|_V^2$$

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$$\frac{1}{2}\frac{d}{dt}\Big[\|a^{1/2}\dot{u}\|_0^2 + B_{\xi}(u,u) + \|\gamma^{1/2}\theta\|_1^2$$

$$+ \|\tau^{1/2}\dot{\theta}\|_0^2 + \tau T_0\|a_\beta^{1/2}\ddot{u}\|_0^2 + \tau T_0 B_{\beta,\xi_2}(\dot{u},\dot{u})\Big]$$

$$+ (c\dot{\theta},\dot{\theta})$$

$$\leq c\big(\|u\|_0^2 + \|\dot{u}\|_0^2$$

$$+ \|\theta\|_0^2 + \|\dot{\theta}\|_0^2\big) \qquad (24)$$

$$+ (\beta\theta, e(\dot{u}^s)) + (\beta_f\theta, e(\dot{u}^f))$$

$$- (T_0\beta e(\dot{u}^s),\dot{\theta})$$

$$-(T_0\beta e(\dot{u}^f),\dot{\theta}) + \overset{(f,\dot{u})}{\cancel{(f^s,\dot{u}^s)+(f^f,\dot{u}^f)}}$$

$$-(q,\dot{\theta}) + \langle g,\dot{u}^s\rangle_\Gamma + \langle \chi,\dot{u}^f\cdot\nu\rangle_\Gamma$$

$$+ \langle h,\dot{\theta}\rangle_\Gamma + T_0\tau\langle \dot{g},\ddot{u}^s\rangle_\Gamma + T_0\tau\left(\dot{f}^f,\ddot{u}\right)\frac{\beta}{\beta_f}$$

$$+ T_0\tau(\dot{f}^s,\ddot{u}^s)$$

$$+ T_0\tau\langle \tfrac{\beta}{\beta_f}\dot{\chi},\ddot{u}^f\cdot\nu\rangle_\Gamma$$

$$- \left(\frac{\beta}{\beta_f}\frac{\eta}{K}\dot{u}^f,\ddot{u}^f\right)$$

We integrate (24) from 0 to T, $J=(0,T)$

Use that

$$\int_0^T (f^{\#}, \dot{u}^{\#})(s)\,ds \leq C\int_0^T \|f^{\#}\|_0 \|\dot{u}^{\#}\|_0\,ds$$

$$\leq C\left[\|f\|^2_{L^2(J,L^2(\Omega))} + \int_0^T \|\dot{u}^{\#}(s)\|_0^2\,ds\right]$$

$$\int_0^T (\dot{f}^{\#}, \ddot{u}^{\#})\,ds \leq C\left[\|\dot{f}\|^2_{L^2(J,L^2)} + \int_0^T \|\ddot{u}^{\#}(s)\|_0^2\,ds\right]$$

$$\int_0^T (q, \dot{\theta})(s)\,ds \leq C\left[\|q\|^2_{L^2(J,L^2)} + \int_0^T \|\dot{\theta}(s)\|_0^2\,ds\right]$$

$$\cancel{\int_0^T (f^f, \dot{u}^f)(s)\,ds \leq C\left[\|f^f\|^2_{L^\infty(J,L^2)} + \int_0^T \|\dot{u}^f(s)\|_0^2\,ds\right]}$$

Then (24) ~~becomes, and we have~~

$$\cancel{u(0) = \dot{u}(0) = 0,\ \theta(0) = \dot{\theta}(0) = 0,}$$

$$\int_0^T \left(\frac{\beta}{\beta_f}\ddot{u}^f, \ddot{u}^f\right)(s)\,ds \leq C\int_0^T \|\dot{u}(s)\|_0^2\,ds$$

$$\|a^{1/2}\dot u(t)\|_0^2 + c_1\|u(t)\|_V^2 + \|\gamma^{1/2}\theta(t)\|_1^2$$

$$+\|\tau^{1/2}\dot\theta(t)\|_0^2 + \tau T_0\|a_\beta^{1/2}\ddot u(t)\|_0^2$$

$$+\tau T_0\, c_2\|\dot u(t)\|_V^2 + \int_0^t (c\dot\theta,\dot\theta)(s)\,ds \qquad (25)$$

$$\le C\Big[\|f\|^2_{L^2(J,L^2)} + \|\dot f\|^2_{L^2(J,L^2)} + \|q\|^2_{L^2(J,L^2)}$$

$$+\int_0^t\Big(\|u(s)\|_0^2 + \|\dot u(s)\|_0^2 + \|\ddot u(s)\|_0^2 + \cancel{\|\dot{\ddot u}(s)\|_0^2}$$

$$+\|\theta(s)\|_0^2 \qquad +\|\dot\theta(s)\|_0^2\Big)\,ds$$

$$+\int_0^t(\beta\theta, e(\dot u^s))\,ds + \int_0^t(\beta_f\theta, e(\dot u^f))\,ds - \int_0^t(T_0\beta\, e(\dot u^s),\dot\theta)$$

$$-\int_0^t(T_0\beta\, e(\dot u^f),\dot\theta)\,ds + \int_0^t\langle g,\dot u^s\rangle_\Gamma(s)\,ds$$

$$+\int_0^t\langle \chi, \dot u^f\cdot\nu\rangle_\Gamma(s)\,ds + \int_0^t\langle h,\dot\theta\rangle(s)\,ds$$

$$+\int_0^t T_0\tau\langle\dot g,\ddot u^s\rangle(s)\,ds + T_0\tau\int_0^T\langle\dot\chi,\ddot u^f\cdot\nu\rangle_\Gamma(s)\,ds$$