THERMOPOROELASTICITY

EXIST. UNIQ.

PART II

J Santos, ~~7/8/19~~

12/8/19

$$+ C\left[\|\dot{u}(0)\|_0^2 + \|u(0)\|_V^2 + \|\gamma^{1/2}\theta(0)\|_1^2 + \|\tau^{1/2}\dot{\theta}(0)\|_0^2 + \tau T_0 \|a^{1/2}\ddot{u}(0)\|_0^2 + \tau T_0 C_2 \|\dot{u}(0)\|_V^2\right] \qquad (25)$$

Next we bound the last ~~q~~ integral terms in the RHS of (25):

$$\left|\int_0^t (\beta\theta, e(\dot{u}^s))(s)\, ds\right|$$

$$\leq C\int_0^t \|\theta(s)\|_0 \|e(\dot{u}^s)\|(s)_0\, ds$$

$$\leq C\int_0^t \|\theta(s)\|_0 \|\dot{u}^s(s)\|_1\, ds$$

$$\leq C\left[\int_0^t \|\theta(s)\|_0^2\, ds + \int_0^t \|\dot{u}(s)\|_V^2\, ds\right]$$

$$\equiv C\; T_1 + T_2$$

Now T_1 dissapears using GRONWALL because of the term $\|\gamma^{1/2}\theta(t)\|_1^2$ in the LHS of (25)

and T_2 also using GRONWALL because of the term $\tau T_0 C_1 \|\mathring{u}(t)\|_V^2$ in the LHS of (25).

Next,

$$\left| \int_0^t (\beta_f \theta, e(\mathring{u}^f))(s)\, ds \right|$$

$$\leq C \int_0^t \|\theta(s)\|_0 \, \|e(\mathring{u}^f)\|_0(s)\, ds$$

$$\leq C \int_0^t \|\theta(s)\|_0 \, \|\mathring{u}(s)\|_V \, ds$$

$$\leq C \left[\int_0^t \|\theta(s)\|_0^2\, ds + \int_0^t \|\mathring{u}(s)\|_V^2\, ds \right]$$

$$\equiv C (T_1 + T_2)$$

as in previous terms, dissapear using GRONWALL'S –

$$\left|\int_0^t (T_0\beta\, e(\mathring{u}^f), \mathring{\theta})(s)\, ds\right|$$

$$\leq C \int_0^t \|e(\mathring{u}^f)_{(s)}\|_0 \|\mathring{\theta}(s)\|_0\, ds$$

$$\leq C \int_0^t \|\mathring{u}(s)\|_V \|\mathring{\theta}(s)\|_0\, ds$$

$$\leq C \left[\int_0^t \|\mathring{u}(s)\|_V^2\, ds + \int_0^t \|\mathring{\theta}(s)\|_0^2\, ds\right]$$

$$\equiv C\,[T_1 + T_3]$$

Similarly $\left|\int_0^t (T_0\beta\, e(\mathring{u}^s), \mathring{\theta}(s))\right| \leq C\,[T_1 + T_3]$

Now T_3 dissapears using GRONWALL'S because of the term $\|\tau^{1/2}\, \mathring{\theta}(t)\|_0^2$ in the LHS of (25)

Next using that $\|v\|_{L^2(\partial\Omega)} \leq \mathring{C}\, \|v\|_{0,\Omega}^{1/2} \|v\|_{1,\Omega}^{1/2}$)

$$\left|\int_0^t \langle g, \mathring{u}^s\rangle_\Gamma(s)\, ds\right| \leq C \int_0^t \|g(s)\|_1 \|\mathring{u}^s(s)\|_1\, ds$$

$$\leq C \left[\int_0^t \|g(s)\|_1^2\, ds + \int_0^t \|\mathring{u}(s)\|_V^2\, ds\right]$$

$$\leq C \left[T_1 + \|g\|_{L^2(J, H^1(\Omega))}^2\right]$$

GRONWALL'S HERE

Next recall that

$$|q\cdot\nu|_{-1/2,\Gamma} \le C\|q\|_{H(div,\Omega)}.$$

Then,

$$\left|\int_0^t \langle \chi, \mathring{u}^f\cdot\nu\rangle(s)\,ds\right|$$

$$\le \int_0^t \|\chi(s)\|_{1/2}\, \|(\mathring{u}^f\cdot\nu)(s)\|_{-1/2}\,ds$$

$$\le C\left[\int_0^t \|\chi\|^2_{1/2}(s)\,ds + \int_0^t \|\mathring{u}^f(s)\|^2_{H(div,\Omega)}\,ds\right]$$

$$\le C\left[\|\chi\|^2_{L^2(J,H^{1/2}(\partial\Omega))} + \int_0^t \|\mathring{u}(s)\|^2_V\,ds\right]$$

$$\equiv \left[\|\chi\|^2_{L^2(J,H^{1/2}(\partial\Omega))} + \underbrace{T_1}_{\text{GRONWALL'S}}\right]$$

Next,

$$\left|\int_0^t \langle \underbrace{h}_{u}, \underbrace{\dot\theta\rangle(s)\,ds}_{dv}\right| =$$

$$= \left|\langle h,\theta\rangle(t) - \langle h,\theta\rangle(0) - \int_0^t \langle \mathring{h},\theta\rangle(s)\,ds\right|$$

$$\leq \|h(t)\|_{L^2(\partial\Omega)} \, \|\theta(t)\|_{L^2(\partial\Omega)}$$

$$+ \|h(0)\|_{L^2(\Gamma)} \, \|\theta(0)\|_{L^2(\Gamma)}$$

$$+ \int_0^t |\dot{h}(s)|_{0,\Gamma} \, |\theta(s)|_{L^2(\Gamma)} \, ds$$

$$\leq C\Big[\|h(t)\|_1 \, \|\theta(t)\|_1 + \|h(0)\|_1 \, \|\theta(0)\|_1$$

$$+ \int_0^t \|\dot{h}(s)\|_1^2 \, ds + \int_0^t \|\theta(s)\|_1^2 \, ds \Big]$$

$$\leq \varepsilon \|\theta(t)\|_1^2 + C\Big[\|h\|_{L^\infty(J,\, H^1)}$$

$$+ \|\dot{h}\|_{L^2(J,H^1)} + \int_0^t \|\theta(s)\|_1^2 \, ds \Big]$$

T_3, use GRONWALL with $\|\gamma^{1/2}\theta(t)\|_1^2$ in LHS of (25)

Next,

$$\left| \int_0^t T_0 \tau \, \langle \underbrace{\dot{g}}_{u}, \underbrace{\ddot{u}^s \rangle(s) \, ds}_{dV} \right|$$

$$= \left| \langle \dot{g}, \dot{u}^s \rangle(t) - \langle \dot{g}, \dot{u}^s \rangle(0) - \int_0^t \langle \ddot{g}, \dot{u}^s \rangle(s) \, ds \right| T_0 \tau$$

$$\leq C\Big[\|\dot g(t)\|_1\,\|\mathring{u}^s(t)\|_1 + \|\dot g(0)\|_1\,\|\mathring{u}^s(0)\|_1$$

$$+ \int_0^t \|\ddot g(s)\|_1^2\,ds + \int_0^t \|\mathring{u}^s(s)\|_1^2\,ds\Big]$$

$$\leq C\Big[\|\dot g\|^2_{L^\infty(J,H^1)} + \|\ddot g\|^2_{L^2(J,H^1)}$$

$$+ \underbrace{\int_0^t \|\mathring{u}(s)\|_V^2\,ds}_{\downarrow\, T_2}\Big] + \underbrace{\varepsilon\,\|\mathring{u}(t)\|_V^2}_{\text{Absorbed in LHS of (25) with } \tau T_0 C_1 \|\mathring{u}(t)\|_V^2 .}$$

Next,

$$\Big|T_0\tau \int_0^t \langle \underbrace{\ddot x}_{u}, \underbrace{\ddot u^f\cdot\nu}_{dv}\rangle_\Gamma(s)\,ds$$

$$= \Big|T_0\tau\Big(\langle \dot x, \mathring{u}^f\cdot\nu\rangle(t) - \langle \dot x, \mathring{u}^f\cdot\nu\rangle(0) - \int_0^t \langle \ddot x, \mathring{u}^f\cdot\nu\rangle(s)\,ds\Big|$$

$$\leq C\Big[\|\dot x(t)\|_{1/2,\Gamma}\,\|\mathring{u}^f\cdot\nu\|_{-1/2,\Gamma} + \|\dot x(0)\|_{1/2,\Gamma}\,\|\mathring{u}^f\cdot\nu(0)\|_{-1/2,\Gamma}$$

$$+ \int_0^t \|\ddot x(s)\|_{+1/2,\Gamma}\,\|\mathring{u}^f\cdot\nu(s)\|_{-1/2}\,ds\Big]$$

$$\leq \epsilon \, \| \mathring{u}^f \|^2_{H(\mathrm{div},\Omega)} + C \Big[\| \dot{\chi} \|^2_{L^\infty(J, H^{1/2}(\Gamma))}$$

$$+ \| \ddot{\chi} \|^2_{L^2(J, H^{1/2}(\Gamma))} + \int_0^t \| \mathring{u}^f(s) \|^2_{H(\mathrm{div},\Omega)} \Big]$$

$$\leq \underbrace{\epsilon \, \| \mathring{u} \|^2_V}_{\text{Absorbed in LHS of (25)}} + C \Big[\| \dot{\chi} \|^2_{L^\infty(J, H^{1/2})}$$

$$+ \| \ddot{\chi} \|^2_{L^2(J, H^{1/2})} + \int_0^t \| \mathring{u}(s) \|^2_V \, ds \Big]$$

$\downarrow T_2$, GRONWALL.

Collecting all bounds, from (25) we get

$$\|a^{1/2}\dot{u}(t)\|_0^2 + C_1\|u(t)\|_V^2 + \|\gamma^{1/2}\theta(t)\|_1^2$$

$$+ \|\tau^{1/2}\dot{\theta}(t)\|_0^2 + \tau T_0\|a^{1/2}\ddot{u}(t)\|_0^2$$

$$+ \tau T_0 C_1 \|\dot{u}(t)\|_V^2 + \int_0^t (c\dot{\theta},\dot{\theta})(s)\,ds$$

$$\le \varepsilon\left(\|\theta(t)\|_1^2 + \|\dot{u}(t)\|_V^2\right)$$

contained here

$$+ C\Big[\int_0^t \|\theta(s)\|_0^2\,ds + \int_0^t \|\dot{u}(s)\|_V^2\,ds + \int_0^t \|\theta(s)\|_1^2\,ds$$

contained here

$$+ \int_0^t \Big(\|u(s)\|_0^2 + \|\dot{u}(s)\|_0^2 + \|\ddot{u}(s)\|_0^2 + \|\dot{\theta}(s)\|_0^2\Big)\,ds \qquad (26)$$

$$\boxed{\begin{aligned}&+ \|f\|^2_{L^{\cancel{\infty}}(J,L^2)} + \|\dot{f}\|^2_{L^{\cancel{\infty}}(J,L^2)} + \|q\|^2_{L^{\cancel{\infty}}(J,L^2)}\\ &+ \|g\|^2_{L^{\cancel{\infty}}(J,H^1(\Omega))} + \|\chi\|^2_{L^{\cancel{\infty}}(J,H^{1/2}(\Gamma))} \equiv M(f,q,\chi,h,g)\\ &+ \|\dot{h}\|^2_{L^\infty(J,H^1(\Omega))} + \|\dot{g}\|^2_{L^\infty(J,H^1(\Omega))} + \cancel{\|\ddot{g}\|^2_{L^2(J,(H^1(\Omega))')}}\\ &+ \|\dot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))} + \cancel{\|\ddot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))}}\end{aligned}}$$

$$+ \|\mathring{u}(0)\|_V^2 + \|u(0)\|_V^2 + \|\gamma^{1/2}\theta(0)\|_1^2$$ (26)

$$+ \|\tau^{1/2}\mathring{\theta}(0)\|_0^2 + \tau T_0 \|a^{1/2}\ddot{u}(0)\|_0^2$$

$$+ \tau T_0 C_1 \|\dot{u}(0)\|_V^2 \Big]$$

Using GRONWALL'S in (26) and absorbing the ε - terms in the LHS of (26) we get (here I use u_m, θ_m instead of u, θ)

$$
\begin{aligned}
&\|\mathring{u}_m\|_{L^\infty(J,\,L^2(\Omega))} + \|u_m\|_{L^\infty(J,V)} \\
&+ \|\theta_m\|_{L^\infty(J,H^1(\Omega))} + \|\mathring{\theta}_m\|_{L^\infty(J,(L^2(\Omega)))} \qquad (27) \\
&+ \|\mathring{\mathring{u}}_m\|_{L^\infty(J,(L^2(\Omega))^4)} + \|\mathring{u}_m\|_{L^\infty(J,V)} \\
&\le C\Big[M(f,q,x,h,g) + \|\mathring{u}_m(0)\|_0 + \|u_m(0)\|_V \\
&+ \|\theta_m(0)\|_1 + \|\mathring{\theta}_m(0)\|_0 + \|\mathring{\mathring{u}}_m(0)\|_0 \\
&+ \|\mathring{u}_m(0)\|_V \Big]
\end{aligned}
$$

combined here

Next we take time derivative in (7)

(17):

$$\mathcal{U}\,\dddot{u} - \mathcal{L}(\dot{u}, \dot{\theta}) = \dot{f} \qquad (28)$$

$$\tau\,\dddot{\theta} + c\,\ddot{\theta} - \nabla\cdot(\gamma\,\nabla\dot{\theta}) + \beta T_0\, e(\ddot{u}^s)$$
$$+ \tau\beta T_0\, e(\dddot{u}^s) + \beta T_0\, e(\ddot{u}^f) \qquad (29)$$
$$+ \tau\beta T_0\, e(\dddot{u}^f) = -\dot{q}$$

Testing (28) and (29) ~~as in (8)~~ ~~we get~~
as in (8) and (15) we get:

$$(\alpha \dddot{u}, v) + B(\dot{u}, v)$$

$$- (\beta \dot{\theta}, e(v^s)) - (\beta_f \dot{\theta}, e(v^f))$$

$$+ (\tau \dddot{\theta}, w) + (c \ddot{\theta}, w) + (\gamma \nabla \dot{\theta}, \nabla w)$$

$$+ (T_0 \beta\, e(\ddot{u}^s), w) + (\tau T_0 \beta\, \cancel{e(\dddot{u}^s)}, w)$$

$$+ (T_0 \beta\, e(\ddot{u}^f), w) + (\tau T_0 \beta\, \cancel{e(\dddot{u}^f)}, w) \quad (30)$$

$$- \langle \dot{g}, v^s \rangle_\Gamma - \langle \dot{\chi}, v^f \cdot \nu \rangle_\Gamma - \langle \dot{h}, w \rangle_\Gamma$$

$$= (\dot{f}^s, v^s) + (\dot{f}^f, v^f) - (\dot{q}, w)$$

CHOOSE $v = \ddot{u}$, $w = \ddot{\theta}$ in (30):

$$(\alpha \dddot{u}, \ddot{u}) + B(\dot{u}, \ddot{u}) - (\beta \ddot{\theta}, e(\ddot{u}^s)) \quad (30)$$

$$- (\beta_f \ddot{\theta}, e(\ddot{u}^f)) + (\tau \dddot{\theta}, \ddot{\theta})$$

$$+ (c\ddot{\theta}, \ddot{\theta}) + (\gamma \nabla \dot{\theta}, \nabla \ddot{\theta})$$

$$+ (T_0 \beta\, e(\ddot{u}^s_{xx}), \ddot{\theta}) + (\tau T_0 \beta\, e_x(\dddot{u}^s), \ddot{\theta})$$

$$+ (T_0 \beta\, e(\ddot{u}^f_{xx}), \ddot{\theta}) + (\tau T_0 \beta\, e_x(\dddot{u}^f), \ddot{\theta}) \quad (31)$$

$$- \langle \dot{g}, \dddot{u}^s \rangle_\Gamma - \langle \dot{\chi}, \ddot{u}^f \cdot \nu \rangle_\Gamma - \langle \dot{h}, \ddot{\theta} \rangle_\Gamma$$

$$= (\dot{f}^s, \ddot{u}^s) + (f^f, \ddot{u}^f) - (\dot{q}, \ddot{\theta})$$

To handle the terms with $e(\dddot{u}^s)$, $e(\dddot{u}^f)$ in (31) we take time derivative in (28)

$$\alpha \ddddot{u} - \mathcal{L}(\ddot{u}, \ddot{\theta}) = \ddot{f} \quad (32)$$

Testing (32)

$$(\alpha \ddddot{u}, v) + B((\ddot{u}, \ddot{\theta}), v) - (\beta \ddot{\theta}, e(v^s)) \quad (33)$$

$$- (\beta_f \ddot{\theta}, e(v^f)) = (\ddot{f}, v) + \langle \ddot{g}, v^s \rangle_\Gamma$$

$$+ \langle \ddot{\chi}, v^f \cdot \nu \rangle_\Gamma$$

Choose $v^s = \dddot{u}^s$, $v^f = 0$ in (33)

$$(\rho \ddddot{u}^s, \dddot{u}^s) + (\rho_f \ddddot{u}^f, \dddot{u}^s)$$

$$+ \left((\lambda_\mu + 2\mu)\, \varepsilon_{11}(\ddot{u}^s), \varepsilon_{11}(\dddot{u}^s)\right) +$$

$$\left((\lambda_\mu + 2\mu)\, \varepsilon_{33}(\ddot{u}^s), \varepsilon_{33}(\dddot{u}^s)\right)$$

$$+ \left(\lambda_\mu\, \varepsilon_{33}(\ddot{u}^s), \varepsilon_{11}(\dddot{u}^s)\right) + \left(\lambda_\mu\, \varepsilon_{11}(\ddot{u}^s), \varepsilon_{33}(\dddot{u}^s)\right)$$

$$+ \left(4\mu\, \varepsilon_{13}(\ddot{u}^s), \varepsilon_{13}(\dddot{u}^s)\right) + \cancel{\beta e(\ddot{u}^f)}$$

$$+ \left(\beta e(\ddot{u}^f), e(\dddot{u}^s)\right) - \left(\beta \ddot{\theta}, e(\dddot{u}^s)\right)$$

$$= (\ddot{f}^s, \dddot{u}^s) + \langle \ddot{g}, \dddot{u}^s \rangle_\Gamma \qquad (34)$$

Also, choose $v^s = 0$, $v^f = \dddot{u}^f$ in (33)

$$(\rho_f \ddddot{u}^s, \dddot{u}^f) + (g \ddddot{u}^f, \dddot{u}^f) + \left(\beta e(\ddot{u}^s), e(\dddot{u}^f)\right)$$

$$+ \left(M e(\ddot{u}^f), e(\dddot{u}^f)\right) - \left(\beta_f \ddot{\theta}, e(\dddot{u}^f)\right)$$

$$= \langle \ddot{\chi}, \dddot{u}^f \cdot \nu \rangle_\Gamma + (\ddot{f}^f, \dddot{u}^f) \qquad (35)$$

$$(\beta_f \ddot{\theta}, e(\dddot{u}^f)) = (\rho_f \ddddot{u}^s, \dddot{u}^f)$$

$$+(g \ddddot{u}^f, \dddot{u}^f) + (B e(\ddot{u}^s), e(\dddot{u}^f))$$

$$+(M e(\ddot{u}^f), e(\dddot{u}^f))$$

$$- \langle \ddot{\lambda}, \dddot{u}^f \cdot \nu \rangle_{\Gamma}$$

Then

$$(\beta \ddot{\theta}, e(\dddot{u}^f)) = \frac{\beta}{\beta_f}(\beta_f \ddot{\theta}, e(\dddot{u}^f))$$

$$= \left(\frac{\beta}{\beta_f} \rho_f \ddddot{u}^s, \dddot{u}^f\right) + \left(g \frac{\beta}{\beta_f} \ddddot{u}^f, \dddot{u}^f\right)$$

$$+ \left(\frac{\beta}{\beta_f} B e(\ddot{u}^s), e(\dddot{u}^f)\right)$$

$$+ \left(\frac{\beta}{\beta_f} M e(\ddot{u}^f), e(\ddot{u}^f)\right) \qquad (36)$$

$$- \left\langle \frac{\beta}{\beta_f} \ddot{\lambda}, \dddot{u}^f \cdot \nu \right\rangle_{\Gamma} - (f, \dddot{u}^f)$$

From (35) and (36):

$$(\beta\ddot{\theta}, e(\dddot{u}^s)) + (\beta\cancel{f}\ddot{\theta}, e(\dddot{u}^f))$$

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$$= (\rho \ddddot{u}^s, \dddot{u}^s) + (\rho_f \ddddot{u}^f, \dddot{u}^s)$$

$$+ \left(\frac{\beta}{\beta_f}\rho_f \ddddot{u}^s, \dddot{u}^f\right) + \left(g\frac{\beta}{\beta_f}\ddddot{u}^f, \dddot{u}^f\right)$$

$$+ \left((\lambda_\mu + 2\mu)\, \varepsilon_{11}(\ddot{u}^s), \varepsilon_{11}(\dddot{u}^s)\right)$$

(37)

$$+ \left(\lambda_\mu + 2\mu\; \varepsilon_{33}(\ddot{u}^s), \varepsilon_{33}(\dddot{u}^s)\right)$$

$$+ \left(\lambda_\mu\, \varepsilon_{33}(\ddot{u}^s), \varepsilon_{11}(\dddot{u}^s)\right) + \left(\lambda_\mu\, \varepsilon_{11}(\ddot{u}^s), \varepsilon_{33}(\dddot{u}^s)\right)$$

$$+ \left(4\mu\, \varepsilon_{13}(\ddot{u}^s), \varepsilon_{13}(\dddot{u}^s)\right) + \left(B\, e(\ddot{u}^f), e(\dddot{u}^s)\right)$$

$$+ \left(\frac{\beta}{\beta_f} B\, e(\ddot{u}^s), e(\dddot{u}^f)\right)$$

$$+ \left(\frac{\beta}{\beta_f} M\, e(\ddot{u}^f), e(\ddot{u}^f)\right)$$

$$- \left\langle \frac{\beta}{\beta_f}\ddot{\chi}, \dddot{u}^f \cdot \nu \right\rangle_\Gamma - (\ddot{f}^s, \dddot{u}^s) - \langle \ddot{g}, \dddot{u}^s\rangle_\Gamma$$

$$- (\ddot{f}^f, \dddot{u}^f)$$

$$a_\beta = \begin{pmatrix} e I & e_f I \\ \frac{\beta}{\beta_f} e_f I & \frac{\beta}{\beta_f} g I \end{pmatrix}$$

$$E_\beta = \begin{bmatrix} \lambda_\mu + 2\mu & \lambda_\mu & B & 0 \\ \lambda_\mu & \lambda_\mu + 2\mu & B & 0 \\ B\frac{\beta}{\beta_f} & B\frac{\beta}{\beta_f} & \frac{\beta}{\beta_f} M & 0 \\ 0 & 0 & 0 & 4\mu \end{bmatrix}$$

Then

$$\left(E_\beta \begin{pmatrix} \varepsilon_{11}(u^s) \\ \varepsilon_{33}(u^s) \\ e(u^f) \\ \varepsilon_{13}(u^s) \end{pmatrix}, \begin{pmatrix} \varepsilon_{11}(v) \\ \varepsilon_{33}(v) \\ e(w) \\ \varepsilon_{13}(v) \end{pmatrix} \right) = B_\beta(u, v)$$

$$= \left((\lambda_\mu + 2\mu)\varepsilon_{11}(u^s) + \lambda_\mu \varepsilon_{33}(u^s) + B e(u^f), \varepsilon_{11}(v) \right)$$

$$+ \left(\lambda_\mu \varepsilon_{11}(u^s) + (\lambda_\mu + 2\mu)\varepsilon_{33}(u^s) + B e(u^f), \varepsilon_{33}(v) \right)$$

$$+ \left(\frac{\beta}{\beta_f} B \big(\underbrace{\varepsilon_{11}(u^s) + \varepsilon_{33}(u^s)}_{e(u^s)} \big), e(w) \right)$$

$$+ \left(\frac{\beta}{\beta_f} M e(u^f), e(w) \right) + \left(4\mu \varepsilon_{13}(u^s), \varepsilon_{13}(v) \right)$$

and (37) becomes

$$\left(\beta\,\ddot{\theta}, e(\ddddot{u}^{s})\right) + \left(\beta\cancel{f}\,\ddot{\theta}, e(\ddddot{u}^{f})\right) \quad (35)$$

$$= \left(a_\beta \ddddot{u}, \dddot{u}\right) + B^{\beta}(\ddot{u}, \dddot{u})$$

$$- \left\langle \frac{\beta}{\beta_f}\ddot{\lambda}, \dddot{u}^{f}\cdot\nu \right\rangle_\Gamma - \left\langle \ddot{g}, \dddot{u}^{s} \right\rangle_\Gamma$$

$$- \left(\ddot{f}^{s}, \dddot{u}^{s}\right) \qquad (38)$$

$$= \frac{1}{2}\frac{d}{dt}\Big[\left(a_\beta \dddot{u}, \dddot{u}\right) + B^{\#}_{\beta}(\ddot{u}, \ddot{u})\Big)$$

$$- \left\langle \frac{\beta}{\beta_f}\ddot{\lambda}, \dddot{u}^{f}\cdot\nu \right\rangle_\Gamma - \left\langle \ddot{g}, \dddot{u}^{s} \right\rangle_\Gamma$$

$$- \left(\ddot{f}^{s}, \dddot{u}^{s}\right)$$

Now we NOTE THAT ~~NEED TO ASSUME~~ THAT

E_β is positive definite $\Rightarrow$

a_β is positive definite.

Using (38) in (31):

$$\frac{1}{2}\frac{d}{dt}\Big[(a\ddot{u},\ddot{u}) + B(\dot{u},\dot{u}) + (\tau\ddot{\theta},\ddot{\theta}) \quad (36)$$

$$+ (\gamma\nabla\dot{\theta},\nabla\dot{\theta}) + \tau T_0\cancel{(a_\beta\dddot{u},\dddot{u})}$$

$$+\tau T_0 B_\beta(\ddot{u},\ddot{u})\Big] + (c\ddot{\theta},\ddot{\theta})$$

$$-(\beta\dot{\theta},e(\ddot{u}^s)) - (\beta_f\dot{\theta},e(\ddot{u}^f)) \quad (39)$$

$$+(T_0\beta e(\ddot{u}^s),\ddot{\theta}) + (T_0\beta e(\ddot{u}^f),\ddot{\theta})$$

$$= (\dot{f},\ddot{u}) - (\dot{q},\ddot{\theta}) + \langle\dot{g},\ddot{u}^s\rangle_\Gamma$$

$$+\langle\dot{\chi},\ddot{u}^f\cdot\nu\rangle_\Gamma + \langle\dot{h},\ddot{\theta}\rangle_\Gamma$$

$$+T_0\tau\cancel{\langle\tfrac{\beta}{\beta_f}\ddot{\chi},\dddot{u}^f\cdot\nu\rangle_\Gamma} + T_0\tau\cancel{\langle\ddot{g},\dddot{u}^s\rangle_\Gamma}$$

$$+(\ddot{f}^s,\dddot{u}^s)$$

Next add to (39)

$$\xi_1\frac{d}{dt}(\dot{u},\dot{u}) \le \xi_1(\|\dot{u}\|_0^2 + \|\ddot{u}\|_0^2),$$

$$T_0\tau\frac{d}{dt}(\ddot{u},\ddot{u}) \le \xi_1 T_0\tau(\|\ddot{u}\|_0^2 + \|\dddot{u}\|_0^2)$$

$$\frac{d}{dt}(\gamma\dot{\theta},\dot{\theta}) \le \|\gamma\dot{\theta}\|_0^2 + \|\gamma\ddot{\theta}\|_0^2.$$