THERMO PORO ELASTICITY

EXIST. UNIQ.

PART III

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$$\frac{1}{2}\frac{d}{dt}\Big[(a\ddot{u},\ddot{u}) + B_{\xi_1}(\dot{u},\dot{u}) + (c\ddot{\theta},\ddot{\theta}) + \|\gamma^{1/2}\dot{\theta}\|_1^2$$

$$+ \tau T_0\,(a_\beta \dddot{u}, \dddot{u})$$

$$+ \tau T_0\, B_{\beta,\xi_2}(\ddot{u},\ddot{u})\Big] + (c\ddot{\theta},\ddot{\theta})$$

$$\leq C\big[\|\dot{u}\|_0^2 + \|\ddot{u}\|_0^2 + \|\dddot{u}\|_0^2 + \|\dot{\theta}\|_0^2 + \|\ddot{\theta}\|_0^2\big]$$

$$+ (\beta\dot{\theta}, e(\ddot{u}^s_v)) + (\beta_f\dot{\theta}, e(\ddot{u}^f_v))$$

$$- (T_0\beta\, e(\ddot{u}^s_x), \ddot{\theta}) - (T_0\beta\, e(\ddot{u}^f_x), \ddot{\theta}) \qquad (40)$$

$$+ (\dot{f}, \ddot{u}_x) - (\dot{q}, \ddot{\theta}_x) + \langle \dot{g}, \ddot{u}^s_x\rangle_\Gamma$$

$$+ \langle \dot{\chi}, \ddot{u}^f_x\cdot\nu\rangle_\Gamma + \langle \dot{h}, \ddot{\theta}_x\rangle_\Gamma$$

$$+ T_0\tau\left\langle \frac{\beta}{\beta_f}\ddot{\chi}, \dddot{u}^f_x\cdot\nu\right\rangle_\Gamma + T_0\tau\langle \ddot{g}, \dddot{u}^s\rangle_\Gamma$$

$$+ (\ddot{f}^s_x, \dddot{u}^s)$$

Let us bound the integrals of the last 12 terms in the RHS of (40)

$$\left| \int_0^T \left[(\overset{\circ}{f}, \overset{\circ\circ}{\underset{\sim}{u}})(s) - (\overset{\circ}{q}, \overset{\circ\circ}{\underset{\sim}{\theta}})(s) + (\overset{\circ\circ}{f}{}^s, \overset{\circ\circ\circ}{\underset{\sim}{u}}{}^s)(s) \right] ds \right.$$

$$\leq C \left[\|\overset{\circ}{f}\|^2_{L^2(J,L^2)} + \|\overset{\circ}{q}\|^2_{L^2(J,L^2)} + \|\overset{\circ\circ}{f}\|^2_{L^2(J,L^2)} \right. \quad (41)$$

$$\left. + \int_0^t \left(\|\ddot{u}(s)\|_0^2 + \|\overset{\circ\circ\circ}{u}(s)\|_0^2 + \|\overset{\circ\circ}{\theta}(s)\|_0^2 \right) ds \right]$$

$$\left| \int_0^t \left[(\beta \overset{\circ}{\theta}, e(\ddot{u}^s))(s) + (\beta_f \overset{\circ}{\theta}, e(\ddot{u}^f))(s) \right] ds \right|$$

$$\leq C \left[\int_0^t \|\overset{\circ}{\theta}(s)\|_0^2 \, ds + \int_0^t \left(\|\ddot{u}^s(s)\|_1^2 + \|\ddot{u}^f(s)\|^2_{H(\mathrm{div},\Omega)} \right) ds \right.$$

$$\leq C \left[\int_0^t \|\overset{\circ}{\theta}(s)\|_0^2 \, ds + \int_0^t \|\ddot{u}(s)\|_V^2 \, ds \right] \quad (42)$$

$$= T_{10} + T_{11}$$

T_{10} dissapears using GRONWALL'S with $\|\gamma^{1/2} \overset{\circ}{\theta}\|_1^2$

T_{11} dissapears " " " $\mathcal{B}_{\beta,\xi_2}(\ddot{u}, \ddot{u})$

$$\left| \int_0^t \left[(T_0 \beta e(\ddot{\mathring{u}}^s), \ddot{\mathring{\theta}})(s) + (T_0 \beta e(\ddot{\mathring{u}}^f), \ddot{\mathring{\theta}})(s) \right] ds \right|$$

$$\leq C \left[\int_0^t \| \ddot{\mathring{u}} \|_{V}^2(s)\, ds + \int_0^t \| \ddot{\mathring{\theta}}(s) \|_0^2\, ds \right] \qquad (43)$$

$$= T_{11} + T_{12}$$

Now T_{12} disappear from Gronwall's and after integration —

$$(\tau \ddot{\mathring{\theta}}, \ddot{\mathring{\theta}})(t)$$

$$\left| \int_0^t \langle \dot{g}, \ddot{\mathring{u}}^s \rangle_\Gamma (s)\, ds \right| \leq C \int_0^t \| \dot{g}(s) \|_1 \, \| \ddot{\mathring{u}}^s(s) \|_1 \, ds$$

$$\leq C \left[\int_0^t \| \dot{g}(s) \|_1^2\, ds + \int_0^t \| \ddot{\mathring{u}}(s) \|_V^2\, ds \right]$$

$$\leq C \left[\| \dot{g} \|^2_{L^2(J, H^1)} + \int_0^t \| \ddot{\mathring{u}}(s) \|_V^2\, ds \right] \qquad (44)$$

$$= C \left[\| \dot{g} \|^2_{L(J, H^1)} + T_{11} \right]$$

(40)

$$\left| \int_0^t \langle \mathring{\chi}, \ddot{u}^f \cdot \nu \rangle_\Gamma (s)\, ds \right|$$

$$\leq C \int_0^t \| \mathring{\dot{\chi}}(s) \|_{1/2,\Gamma} \, \| \ddot{u}^f \cdot \nu \|(s)_{-1/2,\Gamma} \, ds$$

$$\leq C \int_0^t \| \dot{\chi}(s) \|_{1/2,\Gamma} \, \| \ddot{u}^f(s) \|_{H(\mathrm{div},\Omega)} \, ds$$

$$\leq C \left[\| \mathring{\chi} \|^2_{L^2(J, H^{1/2}(\Gamma))} + \int_0^t \| \ddot{u}(s) \|^2_V \, ds \right]$$

$$= C \left[\| \mathring{\chi} \|^2_{L^2(J, H^{1/2}(\Gamma))} + T_{11} \right] \quad (45)$$

$$\left| \int_0^t \langle \underset{u}{\mathring{h}}, \underbrace{\ddot{\theta} \rangle_\Gamma (s)}_{dV} \right| = \left| \int_0^t \langle \mathring{h}, \dot{\theta} \rangle_\Gamma (t) - \langle \mathring{h}, \dot{\theta}(0) \rangle_\Gamma - \int_0^t \langle \ddot{h}, \dot{\theta} \rangle_\Gamma (s)\, ds \right|$$

$$\leq C \Big[\| \mathring{h}(t) \|_1 \, \| \dot{\theta}(t) \|_1 + \| \mathring{h}(0) \|_1 \, \| \dot{\theta}(0) \|_1 + \int_0^t \Big(\| \ddot{h}(s) \|_1^2 + \int_0^t \| \dot{\theta}(s) \|_1^2 \Big) ds \Big]$$

$$\leq C\Big[\|\mathring{h}\|^2_{L^\infty(J,H^1)} + \|\overset{\circ\circ}{h}\|^2_{L^2(J,H^1)}$$

$$+ \underbrace{\int_0^t \|\mathring{\theta}(s)\|_1^2\,ds}\Big] + \varepsilon\|\mathring{\theta}(t)\|_1^2 \qquad (46)$$

$$= C\Big[\|\mathring{h}\|^2_{L^\infty(J,H^1)} + \|\overset{\circ\circ}{h}\|^2_{L^2(J,H^1)} + \underbrace{T_{10}}_{\text{GRONWALL'S}}\Big]$$

$$+ \underbrace{\varepsilon\,\|\mathring{\theta}(t)\|_1^2}$$

Absorbed in LHS of 40) in the term $\|\gamma^{1/2}\theta(t)\|_1^2$ after integration in time of (40).

$$\left|\int_0^t T_0\tau\,\Big\langle \underbrace{\frac{\beta}{\beta_f}\overset{\circ\circ}{x}}_{u},\ \underbrace{\overset{\circ\circ\circ}{u}{}^f\cdot\nu\Big\rangle_\Gamma(s)\,ds}_{dv}\right|$$

$$\cancel{\leq C\int_0^t \|\overset{\circ\circ}{x}\|_{1/2}(s)\ldots}$$

$$= \left|\tau T_0\left(\frac{\beta}{\beta_f}\right)\Big[\langle \overset{\circ\circ}{x}, \overset{\circ\circ}{u}{}^f\cdot\nu\rangle(t) - \langle \overset{\circ\circ}{x}, \overset{\circ\circ}{u}{}^f\cdot\nu\rangle(0)\right.$$

$$\left. - \int_0^t \langle \overset{\circ\circ\circ}{x}, \overset{\circ\circ}{u}{}^f\cdot\nu\rangle_\Gamma(s)\,ds\right|$$

$$\leq C \Big[\|\ddot{\chi}(t)\|_{1/2} \, \|\ddot{u}^f(t)\cdot\nu\|_{-1/2} + \|\ddot{\chi}(0)\|_{1/2} \, \|\ddot{u}^f(0)\cdot\nu\|_{-1/2}$$

$$+ \int_0^t \|\dddot{\chi}(s)\|_{1/2} \, \|\ddot{u}^f(s)\cdot\nu\|_{-1/2} \, ds \Big]$$

$$\leq C \Big[\|\ddot{\chi}(t)\|_{1/2} \, \|\ddot{u}^f(t)\|_{H(\mathrm{div},\Omega)}$$

$$+ \|\ddot{\chi}(0)\|_{1/2} \, \|\ddot{u}(0)\|_{H(\mathrm{div},\Omega)}$$

$$+ \int_0^t \|\dddot{\chi}(s)\|^2_{1/2,\Gamma} \, ds + \int_0^t \|\ddot{u}^f(s)\|^2_{H(\mathrm{div},\Omega)} \, ds \Big]$$

$$\leq C \Big[\|\ddot{\chi}(t)\|_{1/2} \, \|\ddot{u}(t)\|_V + \|\ddot{\chi}(0)\|_{1/2} \, \|\ddot{u}(0)\|_V$$

$$+ \int_0^t \|\dddot{\chi}(s)\|^2_{1/2,\Gamma} \, ds + \int_0^t \|\ddot{u}(s)\|^2_V \, ds \Big]$$

$$\leq C \Big[\|\ddot{\chi}\|^2_{L^\infty(J, H^{1/2}(\Gamma))} + \|\dddot{\chi}\|^2_{L^2(J, H^{1/2}(\Gamma))}$$

$$+ \|\ddot{u}(0)\|^2_V + \int_0^t \|\ddot{u}(s)\|^2_V \, ds \Big] + \varepsilon \|\ddot{u}(t)\|^2_V \qquad (47)$$

Absorbed in LHS of (40) in the term $\tau T_0 B_{\beta, s_2}(\ddot{u}, \ddot{u})$

$$\left| \int_0^t T_0\tau \,\langle \underbrace{\overset{\circ\circ}{\ddot g}}_{u}, \underbrace{\overset{\circ\circ\circ}{\ddot u}{}^s \rangle(s)\, ds}_{dv} \right|$$

$$= \left| T_0\tau \left(\langle \overset{\circ\circ}{\ddot g}, \overset{\circ\circ}{\ddot u}{}^s \rangle_\Gamma(t) - \langle \overset{\circ\circ}{\ddot g}, \overset{\circ\circ}{\ddot u}{}^s \rangle_\Gamma(0) - \int_0^t \langle \overset{\circ\circ\circ}{\ddot g}, \overset{\circ\circ}{\ddot u}{}^s \rangle_\Gamma(s)\, ds \right| \right.$$

$$\leq C \Big[\|\overset{\circ\circ}{\ddot g}\|_1(t)\, \|\overset{\circ\circ}{\ddot u}{}^s(t)\|_1 + \|\overset{\circ\circ}{\ddot g}(0)\|_1 \|\overset{\circ\circ}{\ddot u}{}^s(0)\|_1$$

$$+ \int_0^t \|\overset{\circ\circ\circ}{\ddot g}(s)\|_1 \, \|\overset{\circ\circ}{\ddot u}{}^s(s)\|_1 \, ds \Big] \qquad (48)$$

$$\leq C \Big[\|\overset{\circ\circ}{\ddot g}\|^2_{L^\infty(J,H^1)} + \|\overset{\circ\circ\circ}{\ddot g}\|^2_{L^2(J,H^1)} + \int_0^t \|\overset{\circ\circ}{\ddot u}(s)\|_V^2 \, ds$$

$$+ \|\ddot u(0)\|_V^2 \Big] + \underbrace{\varepsilon \|\ddot u(t)\|_V^2}_{\text{Absorbed in the LHS of (40)}} + \cancel{\int_0^t \|\overset{\circ}{\ddot u}(s)\|_1^2} \, ds$$

in the term $\tau T_0 B_{\beta,\xi_2}(\overset{\circ\circ}{\ddot u}, \overset{\circ\circ}{\ddot u})$.

Integrate (40) from 0 to t and Using the bounds (41) – (48) in (40) to get:

$$
\begin{aligned}
&(a\ddot{u},\ddot{u})(t) + C_1\,\|\dot{u}(t)\|_V^2 + \|\tau^{1/2}\ddot{\theta}(t)\|_0^2 \\
&\quad + \|\gamma^{1/2}\dot{\theta}(t)\|_1^2 \\
&\quad + \tau T_0\,(a_\beta\dddot{u},\dddot{u})(t) + \tau T_0 C_2\,\|\ddot{u}(t)\|_V^2 \\
&\quad + \int_0^t \|C^{1/2}\dot{\theta}(s)\|_0^2\,ds \\
&\le C\Big[\int_0^t\Big(\|\dot{u}(s)\|_0^2 + \|\ddot{u}(s)\|_0^2 + \|\dddot{u}(s)\|_0^2 \\
&\quad + \|\ddot{\theta}(s)\|_0^2 + \|\dot{\theta}(s)\|_0^2 + \|\ddot{u}(s)\|_V^2\Big)\,ds \\
&\quad \boxed{\begin{aligned}
&+ \|\dot{f}\|^2_{L^2(J,L^2)} + \|\dot{q}\|^2_{L^2(J,L^2)} + \|\ddot{f}\|^2_{L^2(J,L^2)} \\
&+ \|\dot{g}\|^2_{L^2(J,H^1)} + \|\dot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))} \\
&+ \|\dot{h}\|^2_{L^\infty(J,H^1)} + \|\ddot{h}\|^2_{L^2(J,H^1)} \\
&+ \|\ddot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))} + \|\dddot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))} \\
&+ \|\ddot{g}\|^2_{L^\infty(J,H^1)} + \|\dddot{g}\|^2_{L^2(J,H^1)}\Big]
\end{aligned}} \quad M_1(f,g,q,h,\chi) \\
&\quad + \varepsilon\left(\|\ddot{u}(t)\|_V^2 + \|\dot{\theta}(t)\|_1^2\right)
\end{aligned}
\tag{49}
$$

$$+ C\Big[\|\ddot{u}(0)\|_0^2 + \|\dot{u}(0)\|_V^2 + \|\ddot{\theta}(0)\|_0^2 + \|\dot{\theta}(0)\|_1^2 + \|\dddot{u}(0)\|_0^2 + \|\ddot{u}(0)\|_V^2 \Big]$$

Now using GRONWALL'S, absorbing the ε-term in the LHS of (49) and changing u, θ by u_m, θ_m we get

$$\|\ddot{u}_m(t)\|_0^2 + \|\dot{u}_m(t)\|_V^2 + \|\dot{\theta}_m(t)\|_0^2 + \|\theta_m(t)\|_1^2 + \|\ddot{u}_m(t)\|_V^2 + \|\dddot{u}_m(t)\|_0^2$$

$$\leq C\Big[M_1(f, q, g, h, \chi) + \|\ddot{u}_m(0)\|_0^2 + \|\dot{u}_m(0)\|_V^2 + \|\ddot{\theta}_m(0)\|_0^2 + \|\dot{\theta}_m(0)\|_1^2 + \|\dddot{u}_m(0)\|_0^2 + \|\ddot{u}_m(0)\|_V^2 \Big] \quad (50)$$

Let us bound the initial terms in the RHS of (27) and (50).

Recall we assume that

$$u_m(0) \longrightarrow u^0 \text{ in } H^2(\Omega) \quad (51) \quad u_m \in S_m^u$$

$$\dot{u}_m(0) \longrightarrow u^1 \text{ in } H^2(\Omega) \quad (52)$$

$$\theta_m(0) \longrightarrow \theta^0 \text{ in } H^2(\Omega) \quad (53) \quad \theta_m \in S_m^\theta$$

$$\dot{\theta}_m(0) \longrightarrow \theta^1 \text{ in } H^2(\Omega) \quad (54)$$

To bound $\|\ddot{u}_m(0)\|_0$, use (15) for $u = u_m \in S_m^u, v \in S_m^v$, $w \in S_m^\theta$, $v = (v^s, v^f)$, $\theta_m \in S_m^\theta$

Using integration by parts in the $B(u_m, v)$ and $(\gamma \nabla \theta_m, \nabla w)$ we get

$$(\mathcal{A}\,\ddot{u}_m, v) - (\mathcal{L}(u_m, \theta_m), v)$$

$$+ (\tau\,\ddot{\theta}_m, w) + (c\,\dot{\theta}, w) - (\nabla\cdot(\gamma\nabla\theta_m), w)$$

$$+(\beta T_0\, e(\dot{u}^s_m), w) + (\beta\tau T_0\, e(\ddot{u}^s), w) \tag{55}$$

$$+ (\beta T_0\, e(\dot{u}^f), w) + (\beta\tau T_0\, e(\ddot{u}^f), w)$$

$$= (-q, w) + (f, v), \quad t \in J.$$

Choose $t = 0$, $v = \ddot{u}_m(0)$, $w = 0$ in (55):

$$(\mathcal{A}\,\ddot{u}_m(0), \ddot{u}_m(0)) - (\mathcal{L}(u_m(0), \theta_m(0)), \ddot{u}_m(0)) \tag{56}$$

$$= (f(0), \ddot{u}_m(0))$$

Since

$$\mathcal{L}((u_m(0), \theta_m(0)) = (-\nabla\cdot\sigma(u_m(0), \theta_m(0)), \nabla p_f(u_m(0), \theta_m(0)))$$

and

$$\nabla p_f = \nabla(-\alpha M\, e(u^s) - M\, e(u^f) + \beta_f \theta)$$

$$\nabla\cdot\sigma = \nabla\cdot\big(2\mu\,\varepsilon_{ij}(u) + \delta_{ij}\,[\lambda_u\, e(u^s) + \beta\, e(u^f) - \beta\theta]\big),$$

$$\left(\mathcal{L}(u_m(0), \theta_m(0)), \ddot{u}_m(0)\right) \leq$$

$$C\left[\left(\|u_m(0)\|_2 + \|\theta_m(0)\|_1\right)\|\ddot{u}_m(0)\|_0\right]$$

Then from (56)

$$\|a^{1/2}\ddot{u}_m(0)\|_0^2 \leq C\left[\left(\|u_m(0)\|_2 + \|\theta_m(0)\|_1\right)\|\ddot{u}_m(0)\|_0 + \|f(0)\|_0\|\ddot{u}_m(0)\|_0\right]$$

and

$$\|\ddot{u}_m(0)\|_0 \leq C\left[\|u_m(0)\|_2 + \|\theta_m(0)\|_1 + \|f(0)\|_0\right] \quad (57)$$

But

$$\|u_m(0)\|_2 \leq \|u_m(0) - u^0\|_2 + \|u^0\|_2$$
$$\leq \varepsilon + \|u^0\|_2, \quad m \geq m_0(\varepsilon)$$

$$\|\theta_m(0)\|_1 \leq \|\theta_m(0)\|_2 \leq \|\theta_m(0) - \theta^0\|_2 + \|\theta^0\|_2$$
$$\leq \varepsilon + \|\theta^0\|_2, \quad m \geq m_0(\varepsilon). \rightarrow (57\text{-}1)$$

Then,

$$\|\ddot{u}_m(0)\|_0 \leq C\left[\|u^0\|_2 + \|\theta^0\|_2 + 1 + \|f(0)\|_0\right], \quad m \geq m_0(\varepsilon) \quad (58)$$

Also,

$$\|\mathring{u}_m(0)\|_0 \le \|\mathring{u}_m(0) - u^1\|_2 + \|u^1\|_2$$
$$\le \varepsilon + \|u^1\|_2, \quad m \ge m_0(\varepsilon) \qquad (59)$$

$$\|u_m(0)\|_V \le \|u_m(0)\|_2 \le \varepsilon + \|u^0\|_2, \quad m \ge m_0(\varepsilon) \qquad (60)$$

$$\|\theta_m(0)\|_1 \le \|\theta_m(0) - \theta^1\|_2 + \|\theta^1\|_2$$
$$\le \varepsilon + \|\theta^1\|_2, \quad m \ge m_0(\varepsilon) \qquad (61)$$

$$\|\mathring{u}_m(0)\|_V \le \|\mathring{u}_m(0)\|_2 \le \|\mathring{u}_m(0) - u^1\|_2 + \|u^1\|_2$$
$$\le \varepsilon + \|u^1\|_2, \quad m \ge m_0(\varepsilon) \qquad (62)$$

~~Then, from (27) and (51)–(61)~~

$$\|\mathring{\theta}_m(0)\|_0 \le \|\mathring{\theta}_m(0) - \theta^1\|_0 + \|\theta^1\|_0 \qquad (62\text{-}1)$$
$$\le \varepsilon + \|\theta^1\|_2$$

For $\ddot{\theta}_m(0)$: Testing (7) against $\ddot{\theta}_m(0)$:

(49-1)

$$(\tau \ddot{\theta}_m(0), \ddot{\theta}_m(0)) + (C \dot{\theta}_m(0), \ddot{\theta}_m(0))$$

$$- (\nabla\cdot(\gamma \nabla\theta_m(0)), \ddot{\theta}_m(0))$$

$$+ (\beta T_0 [\nabla\cdot \dot{u}^s_m(0) + \nabla\cdot \dot{u}^f_m(0)], \ddot{\theta}_m(0)) = -(q(0), \ddot{\theta}(0))$$

Then

$$\|\tau^{1/2} \ddot{\theta}_m(0)\|_0^2 \le C \Big[\|\dot{\theta}_m(0)\|_0 + \|\theta_m(0)\|_2$$

$$+ \{ \|\nabla\cdot \dot{u}^s_m(0)\|_0 + \|\nabla\cdot \dot{u}^f_m(0)\|_0 \}$$

$$+ \|q(0)\|_0 \Big] \|\ddot{\theta}_m(0)\|_0$$

$$\le C \Big[\underset{(62\text{-}1)}{\|\dot{\theta}_m(0)\|_0} + \underset{\to(57\text{-}1)}{\|\theta_m(0)\|_2} + \underset{\to(62)}{\|\dot{u}_m(0)\|_1}$$

$$+ \|q(0)\|_0 \Big] \|\ddot{\theta}_m(0)$$

Then

$$\|\ddot{\theta}_m(0)\|_0 \le C \Big[\varepsilon_1 + \|\theta^1\|_2 + \|\theta^0\|_2 + \|u^1\|_2 \Big]$$

For $\|\ddot{u}_m(0)\|_V$ (49-2)

Taking ~~the~~ time derivative in (28)

$$\mathcal{O}\ \ddot{\ddot{u}}(0) - \mathcal{L}(\ddot{u}, \ddot{\theta})_{(0)} = \ddot{f}(0)$$

$$\ddot{u}_{(0)} = (\ddot{u}^s, \ddot{u}^f)(0)$$

$\ddot{u}^s$ has 2 derivatives in space (H^2)

$\ddot{u}^f$ has 1 space derivative in $H(\mathrm{div})$

θ has 1 space derivative in L^2

Then

$$\|\ddot{u}^s(0)\|_2 + \|\ddot{u}^f(0)\|_{H^1(\mathrm{div},\Omega)} + \|\ddot{\theta}(0)\|_1 \le C\|\ddot{f}(0)\|_0$$

By elliptic regularity

Then $\underline{\|\ddot{u}_m(0)\|_V} \le C\left[\|\ddot{u}^s_m(0)\|_2 + \|\ddot{u}^f_m(0)\|_{H^1(\mathrm{div})}\right. \le \|\ddot{f}(0)\|_0$

~~and is bounded~~

and $\|\ddot{u}_m(0)\|_V$ is bounded ✓

~~$\|\ddot{u}_m(0)\|_2 \le \|\ddot{u}_m(0)\|_2 - \ddot{u}_m(0) - \ddot{u}(0)$~~

$$\|\dot{u}_m\|_{L^\infty(J,L^2(\Omega))} + \|u_m\|_{L^\infty(J,V)}$$

$$+ \|\theta_m\|_{L^\infty(J,H^1(\Omega))} + \|\dot{\theta}_m\|_{L^\infty(J,L^2(\Omega))}$$

$$+ \|\ddot{u}_m\|_{L^\infty(J,(L^2(\Omega))^4)} + \|\dot{u}_m\|_{L^\infty(J,V)} \qquad (63)$$

$$\le C\left[M(f,q,\chi,h,g) + \|u^0\|_2 + \|\theta^0\|_2 + \|u^1\|_2 + \|\theta^1\|_2 + \|f(0)\|_0\right]$$

OK up to here

Next, we bound the initial terms in (50)

$\|\ddot{u}_m(0)\|_0$ is bounded in (58)

$\|\dot{u}_m(0)\|_V$ is bounded in (60)

$$\|\dot{\theta}_m(0)\|_1 \le \|\dot{\theta}_m(0) - \theta^1\|_2 + \|\theta^1\|_2$$

$$\le \varepsilon + \|\theta^1\|_2, \quad m \ge m_0(\varepsilon)$$

We need to bound $\|\ddot{\theta}_m(0)\|_0$, $\|\dddot{u}_m(0)\|_V$ ✓,

$\|\ddot{u}_m(0)\|_V$

BOUND FOR $\|\dddot{u}_m(0)\|_0$.

Take time derivative in (55) for $w=0$

$$(a\,\dddot{u}_m, v) - (\mathcal{L}(\dot{u}_m, \dot{\theta}_m), v) = (\dot{f}, v) \qquad (64)$$

Take $v = \dddot{u}_m(0)$, $t=0$

$$(a\,\dddot{u}_m(0), \dddot{u}_m(0)) - \mathcal{L}(\dot{u}_m(0)\,\dot{\theta}_m(0), \dddot{u}_m(0)) = (\dot{f}(0), \dddot{u}_m(0)) \qquad (65)$$

Then,

$$\|a^{1/2}\,\dddot{u}_m(0)\|_0^2 \le C\Big[\big(\|\dot{u}_m(0)\|_2 + \|\dot{\theta}_m(0)\|_1\big)\cdot\|\dddot{u}_m(0)\|_0 + \|\dot{f}(0)\|_0\,\|\dddot{u}_m(0)\|_0\Big]$$

Then,

$$\|\dddot{u}_m(0)\|_0 \le C\big[\|\dot{u}_m(0)\|_2 + \|\dot{\theta}_m(0)\|_1 + \|\dot{f}(0)\|_0\big] \qquad (66)$$

$$\le C\big[\|\dot{u}_m(0) - u'\|_2 + \|u'\|_2 + \|\dot{\theta}_m(0) - \theta'\|_1 + \|\theta'\|_1 + \|\dot{f}(0)\big] \le C\big[\|u'\|_2 + \|\theta'\|_1 + \|\dot{f}(0)\|_0 + 1\big]$$

Let us rewrite (15) : (51-1)

$$
\begin{aligned}
&(\mathcal{O}\ddot{u}_m, v) + B(u_m, v) - (\beta\theta_m, e(v^s)) \\
&- (\beta_f \theta_m, e(v^f)) + (\tau\ddot{\theta}_m, w) \\
&+ (C\dot{\theta}_m, w) + (\gamma\nabla\theta_m, \nabla w) \qquad (67) \\
&+ (T_0\beta\, e(\dot{u}^s_m), w) + (T_0\beta\tau\, e(\ddot{u}^s_m), w) \\
&+ (T_0\beta\, e(\dot{u}^f_m), w) + (T_0\beta\tau\, e(\ddot{u}^f_m), w) \\
&= (f, v) - (q, w) + \langle g, v^s\rangle_\Gamma \\
&\quad + \langle \chi, v^f\cdot\nu\rangle_\Gamma + \langle h, w\rangle_\Gamma, \\
&\quad v_m = (v^s, v^f) \in S^u_m, \quad w \in S^\theta_m
\end{aligned}
$$

$[H(\mathrm{div},\Omega)]'$ can be identified with a closed subspace of $[L^2(\Omega)]^3$, then elements in $V' = H^{-1}(\Omega) \times (H(\mathrm{div},\Omega))'$ can be identified with a quintuple $(z_1 z_2 z_3 z_4 z_5) \in V'$. If $[\,,\,]$ is the duality between V' and V, ~~$[z, w]$~~ and $w = (w_1, w_2) = ((w_{11}, w_{12}), w_{21}, w_{22})) \in V$ then

$$[z, w] = ((z_1, z_2), w_1) + \int_\Omega ((z_3, z_4) \cdot w_2 + z_5 \nabla \cdot w_2 \, dx$$

$$|[z, w]| \leq \|(z_1, z_2)\|_{-1} \|w_1\|_1 + \|(z_3\ z_4\ z_5)\|_0 \|w_2\|_{H(\mathrm{div},\Omega)}$$

$$\leq \|z\|_{V'} \|w\|_V$$

Thanks to (63) there exists sequences $u_{m,q}$, $\theta_{m,q}$, renamed u_m, θ_m such that

$$u_m \longrightarrow u \quad \text{in } L^\infty(J, V) \text{ weak-}* \quad [L^\infty(J,V) \text{ weak-}*] \quad (67)$$

i.e.

$$\int_0^T [v, u_m](t)\,dt \xrightarrow[m\to\infty]{} \int_0^T [v, u](t)\,dt \quad (68)$$

$$\forall\, v \in L^1(J, V')$$

$$V' = [H^{-1}(\Omega)]^2 \times [H(\mathrm{div},\Omega)]', \quad [L^\infty(J,V)]' = L^1(J, V')$$

~~Then~~

$$\mathring{u}_m \longrightarrow \mathring{u} \quad \text{in } L^\infty(J, V) \text{ weak-}* \quad (69)$$

$$\mathring{\mathring{u}}_m \longrightarrow \mathring{\mathring{u}} \quad \text{in } L^\infty(J, (L^2(\Omega))^4) \text{ weak-}* \quad (70)$$

$$\theta_m \longrightarrow \theta \quad \text{in } L^\infty(J, H^{+1}) \text{ weak-}* \quad (71)$$

$$\mathring{\theta}_m \longrightarrow \mathring{\theta} \quad \text{in } L^\infty(J, L^3) \text{ weak-}* \quad (72)$$

Claim;

$$(a\,\mathring{\mathring{u}}_m, v) \xrightarrow[m\to\infty]{} (a\,\mathring{\mathring{u}}, v) \quad \text{in } \mathcal{D}'(0,T) \quad (73)$$

i.e.

$$(a\,\mathring{\mathring{u}}_m, v)(\varphi) \longrightarrow (a\,\mathring{\mathring{u}}, v)(\varphi) \quad \forall \varphi \in C_0^\infty(0,T)$$

~~Proof~~:

First we show that

$$(a\,\mathring{u}_m, v) \longrightarrow (a\,\mathring{u}, v) \quad \text{in } L^\infty(J) \text{ weak-}* \quad (74)$$

so that

$$\int_0^T (a\,\mathring{u}_m, v)(t)\, g(t)\, dt \longrightarrow \int_0^T (a\,\mathring{u}, v)(t)\, g(t)\, dt$$

$$\forall g \in L^1(0,T)$$

Since $g(t) \in L^1(0,T)$, $v(x) \in (H^1(\Omega))^2$ (54)

$\to g(t)v(x) \in L^1(J, H^1) \subset L^1(J, L^2)$

Then, thanks to (69)

$$\int_0^T [g(t)v(x), a\,\mathring{u}_m](t)\,dt$$

$$= \int_0^T (a\,\mathring{u}_m, v)\, g(t)\,dt$$

$$\to \int_0^T [g(t)v(x), a\,u](t)\,dt$$

$$= \int_0^T (a\,\mathring{u}, v)\, g(t)\,dt.$$

Now $(a\,\mathring{u}_m, v)$, $(a\,\mathring{u}, v)$ define distribution in $\mathcal{D}'(0,T)$:

$$(a\,\mathring{u}_m, v)(\varphi) = \int_0^T (a\,\mathring{u}_m, v)(t)\,\varphi(t)\,dt$$

$$(a\,\mathring{u}, v)(\varphi) = \int_0^T (a\,\mathring{u}, v)_{(t)}\,\varphi(t)\,dt$$

Then,

$$\frac{d}{dt}\left(a\,\mathring{u}_m, v\right)(\varphi) = \Big(a\,\overbrace{\frac{d}{dt}\mathring{u}_m}^{=\mathring{\ddot{u}}_m}, v\Big)(\varphi)$$

$$= -\left(a\,\mathring{u}_m, v\right)\left(\frac{\partial\varphi}{\partial t}\right) = -\int_0^T (a\,\mathring{u}_m, v)\frac{\partial\varphi}{\partial t}\,dt$$

Since $\frac{\partial\varphi}{\partial t} \in L^1(\omega)$

$$\lim_m \int_0^T (a\,\mathring{u}_m, v)\,\frac{\partial\varphi}{\partial t} = \int_0^T (a\,\mathring{u}, v)\frac{\partial\varphi}{\partial t}\,dt$$

$$= -\frac{d}{dt}\left(a\,\mathring{u}, v\right)(\varphi)$$

Then

$$\lim_m \left(a\,\mathring{\ddot{u}}_m, v\right)(\varphi) = -\lim_m \int_0^T (a\,\mathring{u}_m, v)\frac{\partial\varphi}{\partial t}\,dt$$

$$= -\frac{d}{dt}\left(a\,\mathring{u}, v\right)(\varphi) = \left(a\,\mathring{\ddot{u}}, v\right)(\varphi) \quad \forall\varphi$$

so that (73) holds —

Next, for $v \in S_m^u$

$\cancel{B(u_m, v) = (E\,\tilde{\varepsilon}(u_m), \tilde{\varepsilon}(v))}$, Since $B(u_m, v) = \left(E\,\tilde{\varepsilon}(u_m),\ \tilde{\varepsilon}(v)\right)$

and $u_m \to u$ in $L^\infty(J, H^1)$ weak-*

$\tilde{\varepsilon}(u_m) \in L^2,\ \tilde{\varepsilon}(v) \in L^2 \to$

$g(t)\,\tilde{\varepsilon}(v) \in L^1(J, L^2(\Omega)) \quad \forall g \in L^1(J)$

$$\to \int_0^T \left(E\,\tilde{\varepsilon}(u_m), \tilde{\varepsilon}(v)\right) g(t)\,dt \xrightarrow[m\to\infty]{} \int_0^T \left(E\,\tilde{\varepsilon}(u), \tilde{\varepsilon}(v)\right) g(t)\,dt \qquad (74)$$

Similarly,

$$\theta_m \to \theta \text{ in } L^\infty(J, H^1) \text{ weak *},$$
$$\nabla\theta_m \to \nabla\theta \text{ in } L^\infty(J, L^2) \text{ weak *} \qquad (75)$$

$\nabla\theta_m \in L^2,\ \nabla w \in L^2 \to$

$\nabla\theta_m, \nabla w \in L^1(\Omega) \to$

$$\int_0^T (\nabla\theta_m, \nabla w)\, g(t)\,dt \to \int_0^T (\nabla\theta_m, \nabla w)\, g(t)\,dt$$

$\forall g \in L^1(J)$,

i.e.

$$(\gamma\nabla\theta_m, \nabla w) \longrightarrow (\gamma\theta, \nabla w) \text{ in } L^\infty(J) \text{ weak *} \qquad (76)$$

Taking limit in m in (67)

$$(\mathcal{O}l\, \ddot{u}, v) + B(u,v) - (\beta\theta, e(v^s))$$
$$- (\beta_f \theta, e(v^f)) + (\tau \ddot{\theta}, w)$$
$$+ (C\dot{\theta}, w) + (\gamma \nabla\theta, \nabla w)$$
$$+ T_0\beta\left[(e\dot{u}^s), w) + (e(\dot{u}^f), w) \right]$$
$$+ T_0\beta\tau\left[(e\ddot{u}^s, w) + (e\ddot{u}^f, w) \right] \quad (77)$$
$$= (f, v) - (q, w) + \langle g, v^s \rangle_\Gamma$$
$$+ \langle \chi, v^f \cdot \nu \rangle + \langle h, w \rangle$$

$v \in S_m^u$, $w \in S_m^\theta$, in $L^\infty(J)$ (a.e. in J)

Now the density of S_m^u, S_m^θ in $(H^2(\Omega))^3$ implies that (77) holds for $(u,\theta) \in (H^2(\Omega))^3$ and the density of $[C^\infty(\Omega)]^3$ in $(H^2(\Omega))^3$, $(H^1(\Omega))^2 \times H(\mathrm{div}, \Omega)$ implies that (77) holds for $(v,w) \in V$

Claims: for $v \in S_m^u$,

$$(P\ddot{u}_m, v) \longrightarrow (P\ddot{u}, v) \text{ in } L^\infty(J)$$

weak-*

ie

$$\int_0^T (P\ddot{u}_m, V)\, g(t)\,dt \to \int_0^T (P\ddot{u}, V)\, g(t)\,dt \quad \forall g \in L^1(J)$$

$$v \in [H^2(\Omega)]^4 \to g(t)\,v \in L^1(J, (H^2(\Omega))^4)$$

We know that $\ddot{u}_m = (\ddot{u}_m^s, \ddot{u}_m^f)$ satisfies

$$\int_0^T [v, \ddot{u}_m](t)\,dt \to \int_0^T [v, u](t)\,dt$$

$\forall\, v \in L^1(J, [L^2(\Omega)]^4)$

$$P v\, g(t) \in L^1(J, [L^2(\Omega)]^4).$$

$$\to \int_0^T [Pvg(t), \ddot{u}_m](t)\,dt \to \int_0^T [Pvg(t), \ddot{u}](t)\,dt$$

$$\int_0^T (P\ddot{u}_m, v)\, g(t)\,dt \longrightarrow \int_0^T (P\ddot{u}, v)\, g(t)\,dt \;\checkmark$$