THERMOPOROELASTICITY

EXIST. UNIQ. [INITIAL COND. BOUNDS ELLIPTIC REG.]

PART V

J. SANTOS

12/8/19

BOUND FOR $\| \ddot{\theta}_m(0) \|_0$:

Choose $t=0$, $v=0$, $w=\ddot{\theta}_m(0)$ in (55):

$$\left(\tau \ddot{\theta}_m(0), \ddot{\theta}_m(0)\right) + \left(c\, \dot{\theta}_m(0), \ddot{\theta}_m(0)\right)$$
$$- \left(\nabla\cdot(\gamma \nabla \theta_m(0), \ddot{\theta}_m(0)\right) + \left(\beta T_0\, e(\dot{u}^s_m(0)), \ddot{\theta}_m(0)\right)$$
$$+ \left(\beta \tau T_0\, e(\ddot{u}^s_m(0)), \ddot{\theta}_m(0)\right)$$
$$+ \left(\beta T_0\, e(\dot{u}^f_m(0)), \ddot{\theta}(0)\right) + \left(\beta \tau T_0\, e(\ddot{u}^f_m(0)), \ddot{\theta}_m(0)\right)$$
$$= - \left(q(0), \ddot{\theta}(0)\right)$$

Then,

$$\| \ddot{\theta}_m(0) \|_0^2 \leq C \Big[\| \dot{\theta}_m(0) \|_0 \, \| \ddot{\theta}_m(0) \|_0$$
$$+ \| \theta_m(0) \|_2 \, \| \ddot{\theta}_m(0) \|_0 + \| \dot{u}^s_m(0) \|_1 \, \| \ddot{\theta}_m(0) \|_0$$
$$+ \| \ddot{u}^s_m(0) \|_0 \, \| \ddot{\theta}_m(0) \|_0 + \| \dot{u}^f_m(0) \|_{H(div,\Omega)} \, \| \ddot{\theta}_m(0) \|_0$$
$$+ \| \ddot{u}^f_m(0) \|_{H(div,\Omega)} \, \| \ddot{\theta}_m(0) \|_0 + \| q(0) \|_0 \, \| \ddot{\theta}_m(0) \|_0 \Big]$$

Then,

$$
\begin{aligned}
\|\ddot{\theta}_m(0)\|_0 \le C \Big[& \|\dot{\theta}_m(0)\|_0 + \|\theta_m(0)\|_2 \\
& + \|\dot{u}_m(0)\|_V + \|\ddot{u}_m(0)\|_0 + \|u_m(0)\|_V \\
& + \|\ddot{u}_m(0)\|_V + \|\underline{g}(0)\|_0 \Big] \\
\le C \Big[& \|\dot{\theta}_m(0)\|_0 + \|\theta_m(0)\|_2 + \|\dot{u}_m(0)\|_1 \\
& + \|\ddot{u}_m(0)\|_0 + \|\dot{u}_m(0)\|_1 + \|\ddot{u}_m(0)\|_1 \Big]
\end{aligned}
$$

ELLIPTIC REGULARITY BOUND FOR $\|\ddot{\ddot{u}}_m(0)\|_2 + \|\ddot{\ddot{\theta}}_m(0)\|_{H^1(0,1;\Omega)}$

$$\rho \ddot{u}_m - \mathcal{L}(u_m, \theta_m) = f$$

$$\rho \ddot{u}_m(0) - \mathcal{L}(u_m(0), \theta_m(0)) = f(0)$$

$\ddot{u}_m(0) \in L^2$, $\mathcal{L}(u_m(0), \theta_m(0)) \in L^2$, $f(0) \in L^2$

$$\|\mathcal{L}(u_m(0), \theta_m(0))\|_0 = \nabla\cdot\sigma(u_m, \theta_m)_{(0)} - \nabla f(\theta_m, \theta_m)_{(0)}$$

$$\leq C\left[\|u_m(0)\|_2 + \|\theta_m(0)\|_1\right] \leq C$$

$$\rho \ddot{\ddot{u}}_m(0) - \mathcal{L}(\ddot{u}_m(0), \ddot{\theta}_m(0)) = \ddot{f}(0)$$

$\ddot{\ddot{u}}_m(0) \in L^2$, $\mathcal{L}(\ddot{u}_m(0), \ddot{\theta}_m(0)) \in L^2$, $\ddot{f}(0) \in L^2$

~~$\Rightarrow \|\ddot{u}_m(0)\| + \|\ddot{\theta}_m(0)\|_1 \leq C$~~

~~$(\rho \ddot{u}_m(0), \dot{u}_m(0)) - (\mathcal{L}(\ddot{u}_m(0), \ddot{\theta}_m(0)) = \ddot{f}(0))$~~

Elliptic regularity

$$\|\ddot{u}_m(0)\|_2 + \|\ddot{\theta}_m(0)\|_1 \leq C\left(\|\ddot{f}(0)\|_0 + \|\ddot{\ddot{u}}(0)\|_0\right)$$

(1)

$$\Lambda_1(u,v) = \left(\sigma_{ij}(u), \frac{\partial v_i}{\partial x_j}\right)$$

$$= \left(2\mu\, \varepsilon_{ij}(u^s) + \delta_{ij}\, \lambda \frac{\partial u_i^s}{\partial x_j} + \delta_{ij}\, B \frac{\partial u_i^t}{\partial x_j} - \beta\theta\delta_{ij}\, \frac{\partial v_i}{\partial x_j}\right)$$

$$= \left(2\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \frac{\partial v_i}{\partial x_j}\right)$$

$$+ \left(\lambda\, \delta_{ij} \frac{\partial u_i^s}{\partial x_j}, \frac{\partial u_i^t}{\partial x_j}\right) - \left(\beta\theta\,\delta_{ij}, \frac{\partial v_i}{\partial x_j}\right)$$

$$+ \left(B\, \delta_{ij} \frac{\partial u_i^t}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)$$

$$\cancel{+\left(\frac{\partial}{\partial x_j}\left(2\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)\right)}$$

$$- \left(\frac{\partial}{\partial x_j}\left(2\mu \frac{\partial u_i}{\partial x_j}\right), v_i\right) - \left(\frac{\partial}{\partial x_j}\left(2\mu \frac{\partial u_j}{\partial x_i}\right), v_i\right)$$

$i \to j$

$$- \left(\frac{\partial}{\partial x_i}\left(\delta_{ij}\left(\lambda \frac{\partial u_i^s}{\partial x_j} + B \frac{\partial u_i^t}{\partial x_j} - \beta\theta\right), v_i\right)\right)$$

$$\cancel{-\left(\frac{\partial}{\partial x_j}\left(2\mu \frac{\partial u_i}{\partial x_j}\right), v_i\right) - \left(\frac{\partial}{\partial x_i}\left(2\mu \frac{\partial u_i}{\partial}\right.\right.}$$

$$= -\left(\frac{\partial}{\partial x_j}\left(2\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), v_i\right)\right) \quad (2)$$

$$- \left(\frac{\partial}{\partial x_j}\delta_{ij}\left(\lambda\frac{\partial u_i^s}{\partial x_j} + \beta\frac{\partial u_i^f}{\partial x_j} - \beta\theta\right), v_i\right)$$

$$= -\left(\frac{\partial}{\partial x_j}\left[2\mu\,\varepsilon_{ij}(u) + \delta_{ij}\left(\lambda\frac{\partial u_i^s}{\partial x_j} + \beta\frac{\partial u_i^f}{\partial x_j}\right]\right.\right.$$

$$\left. - \beta\theta\right], v_i\Big)$$

$$= -\left[\left(\frac{\partial}{\partial x_j}\left[2\mu\,\varepsilon_{ij}(u) + \delta_{ij}\left(\lambda\nabla\cdot u^s + \beta\nabla\cdot u^f - \beta\theta\right), v_i\right)\right.\right.$$

$$= -(\nabla\cdot\sigma(u,\theta), v) + \langle\sigma(u,\theta)\nu, v\rangle$$

$$A_2(u, w) = (p_f(u,\theta), \nabla\cdot w)$$

$$= \left(B\frac{\partial u_i^s}{\partial x_j}\delta_{ij} + M\frac{\partial u_i^f}{\partial x_j}\delta_{ij} - \beta_f\theta, \frac{\partial w_k}{\partial x_k}\right)$$

$$= -\left(\frac{\partial}{\partial x_k}\left(B\frac{\partial u_i^s}{\partial x_j}\delta_{ij} + M\frac{\partial u_i^f}{\partial x_j}\delta_{ij} - \beta_f\theta\right), w_k\right)$$

$$= -(\nabla p_f(u,\theta), w) + \langle p_f, w\cdot\nu\rangle$$

③

$$\Lambda(u_m, v) - (\beta\theta_m, \nabla\cdot v^s) - (\beta_f\theta_m, \nabla\cdot v^f)$$

$$= \left(\sigma_{ij}(u_m, \theta), \varepsilon_{ij}(v^s)\right) + \left(\beta\nabla\cdot u^s, \nabla\cdot v^f\right)$$

$$+ (M\nabla\cdot u^f, \nabla\cdot v^f)$$

$u_m \to u$

$$= \left(2\mu\frac{1}{2}\left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i}\right), \frac{\partial v_i}{\partial x_j}\right)$$

$$+ \left[\left(\beta\,\delta_{ij}\frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_i^f}{\partial x_j}\right) + \left(M\delta_{ij}\frac{\partial u_i^f}{\partial x_j}, \frac{\partial v_i^f}{\partial x_j}\right)\right]$$

$$= \left(2\mu\frac{1}{2}\frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right) + \left(2\mu\frac{1}{2}\frac{\partial u_j}{\partial x_i}, \frac{\partial v_i}{\partial x_j}\right)$$

$i \to j$

$$+ [\qquad]$$

$$= \left(2\mu\frac{1}{2}\frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right) + \left(2\mu\frac{1}{2}\frac{\partial u_i}{\partial x_j}, \frac{\partial v_j}{\partial x_i}\right)$$

$$+ [\qquad]$$

$$\cancel{= \left(2\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \frac{\partial v_i}{}\right)}$$

$$= \left(2\mu\frac{1}{2}\frac{\partial u_i^s}{\partial x_j}, \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$$

$$+ [\qquad]$$

$$= \left(\frac{\partial u_i^s}{\partial x_j}, 2\mu \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right)$$

$$]$$

$$+ \quad [$$

$$= -\left(u_i^s, \frac{\partial}{\partial x_j}\left(2\mu \, \varepsilon_{ij}(v) \right) \right)$$

$$+ \left(\frac{\partial u_i^s}{\partial x_j}, B \frac{\partial v_i^f}{\partial x_j} \delta_{ij} \right) + \left(\frac{\partial u_i^f}{\partial x_j}, M \frac{\partial v_i^f}{\partial x_j} \right)$$

$$= -\left(u_i^s, \frac{\partial}{\partial x_j}\left(2\mu \, \varepsilon_{ij}(v) \right) \right)$$

$$- \left(u_i^s, \frac{\partial}{\partial x_j}\left(B \frac{\partial v_i^f}{\partial x_j} \delta_{ij} \right) \right)$$

$$- \left(u_i^f, \frac{\partial}{\partial x_j}\left(M \left(\frac{\partial v_i^f}{\partial x_j} \delta_{ij} \right) \right) \right)$$

$$= -\left(u_i^s, \frac{\partial}{\partial x_j}\left(2\mu \, \varepsilon_{ij}(v^s) \right) + \frac{\partial}{\partial x_j}\left(B \nabla \cdot v^f \right) \right)$$

$$- \left(u_i^f, \frac{\partial}{\partial x_j}\left(M \nabla \cdot v^f \right) \right)$$

(6)

$$-\left(\mathcal{L}(v,w), (u_m^s\ u_m^f) \right) \quad v=(v^s, v^f)$$

$$= -\left(\nabla\cdot\sigma(v,w,\theta), -\nabla p_f(v,w,\theta) \right), (u_m^s, u_m^f))$$

$$= -\left(\frac{\partial}{\partial x_j}\sigma(v^s, v^f,\theta), u_m^s \right) + \left(\frac{\partial}{\partial x_j} p_f(v^s v^f, \theta), u_m^f \right)$$

$$u_m \equiv u.$$

$$= \left(\sigma_{ij}(v^s, v^f, \theta), \frac{\partial u_i^s}{\partial x_j} \right) - \left(p_f(v^s, v^f, \theta), \frac{\partial}{\partial x_j} u_m^f \right)$$

$$+ \underbrace{\langle \sigma(v,w)\cdot\nu, u_m^s \rangle + \langle p_f(v^s, v^f, \theta), u_m^f\cdot\nu \rangle}_{\text{bdry.}}$$

$$= \left(2\mu \frac{1}{2}\left(\frac{\partial v_i^s}{\partial x_j} + \frac{\partial v_j^s}{\partial x_i} \right), \frac{\partial u_i^s}{\partial x_j} \right)$$

$$+ \delta_{ij}\left(\lambda \frac{\partial v_i^s}{\partial x_j} + B\frac{\partial v_i^f}{\partial x_j} - \beta\theta \right), \frac{\partial u_i^s}{\partial x_j} \Big)$$

$$+ \left(B\frac{\partial v_i^s}{\partial x_j}\delta_{ij} + M\frac{\partial v_i^f}{\partial x_j}\delta_{ij} + \beta_f\theta, \frac{\partial u_m^f}{\partial x_j} \right)$$

$$= \left(2\mu\ \varepsilon_{ij}(v^s), \varepsilon_{ij}(u^s) \right)$$

$$+ \left(\lambda\nabla\cdot v^s + B\nabla\cdot v^f - \beta\theta, \nabla\cdot u^s \right)$$

$$+ \left(B\nabla\cdot v^s + M\nabla\cdot v^f + \beta_f\theta, \nabla\cdot u^f \right)$$

$$-(\nabla\cdot\sigma, v) = -\left(\frac{\partial}{\partial x_j}\sigma_{ij}, v_i\right)$$

$$= \left(\sigma_{ij}, \frac{\partial v_i}{\partial x_j}\right) - \cancel{\langle \sigma_{ij}(u)\nu_j, v_i\rangle}$$

$$= \left(2\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \frac{\partial v_i}{\partial x_j}\right)$$

$$+\left(\lambda\frac{\partial u_i}{\partial x_j}\delta_{ij}, \frac{\partial v_i}{\partial x_j}\right)$$

$$\cancel{=(\sigma_{ij}, \nabla v) = (}$$

$$\cancel{\frac{\partial u_i}{\partial x_j} = \nabla u \qquad \frac{\partial u_j}{\partial x_i} = \nabla u^T = \nabla\begin{pmatrix}u_2\\u_1\end{pmatrix}}$$

$$\cancel{\frac{\partial u_i}{\partial x_j} = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}\right) \qquad \frac{\partial u_j}{\partial x_i} = \left(\frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}\right)}$$

$$\cancel{\nabla u + \nabla u^T = \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}+\frac{\partial u_1}{\partial x_2}\right)}$$

$$\cancel{\nabla u + \nabla u^T = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}\right)}$$

$$\cancel{= \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}+\frac{\partial u_1}{\partial x_2}\right)}$$

$$= -\left[\left(2\mu u_i , \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right) + \left(u_j , \frac{\partial}{\partial x_i} \frac{\partial v_i}{\partial x_j} \right) \right.$$

$$- \left(\lambda \, u_i , \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} \delta_{ij} \right)$$

$$= - \left(-2\mu u_i , \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} + \left(\overset{-2\mu}{u_i} , \frac{\partial}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right) \right.$$

$$- \left(\lambda \; u_i , \frac{\partial v_i}{\partial x_i} \right)$$

$$= - \left(-2\mu u_i , \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right)$$

$$- \left(\lambda \, u_i , \frac{\partial v_i}{\partial x_i} \right)$$

$$= - \left(2\mu \, u_i , \frac{\partial}{\partial x_j} \varepsilon_{ij}(v) \right)$$

$$- \left(\lambda \, u_i , \frac{\partial}{\partial x_i} \frac{\partial v_i}{\partial x_j} \delta_{ij} \right)$$

$$= - \cancel{(2\mu , u_i}$$

$$- \left(u_i , \frac{\partial}{\partial x_j} 2\mu \, \varepsilon_{ij}(v) \right)$$

$$- \left(u_i , \frac{\partial}{\partial x_j} \lambda \, \delta_{ij} \frac{\partial v_i}{\partial x_i} \right)$$

$$= - \left(u_i , \frac{\partial}{\partial x_j} \left(2\mu \, \varepsilon_{ij}(v) + \lambda \delta_{ij} \frac{\partial v_i}{\partial x_i} \right) \right)$$

$$= - \left(u_i , \frac{\partial}{\partial x_j} \sigma_{ij}(v) \right)$$

9

THERMO POROELASTICITY

EXIST. UNIQ.
PART VI

J. SANTOS

12/8/19

INT. BY PARTS,

SATISFY INITIAL COND, BRY COND.

①

$$-(\nabla\cdot\sigma(u,\theta), -\nabla p_f(u,\theta))$$

$$= -\mathcal{L}(u,\Theta)$$

$$= -\Big(\frac{\partial}{\partial x_j}\big[2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}[\lambda\nabla\cdot u^s + B\nabla\cdot u^f - \beta\Theta]\big],$$

$$-\frac{\partial}{\partial x_j}\big(B\nabla\cdot u^s + M\nabla\cdot u^f - \beta_f\Theta\big)\Big)$$

$$-\big(\mathcal{L}(u,\Theta), (v^s, v^f)\big)$$

$$= -\Big(\frac{\partial}{\partial x_j}\big[2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}[\lambda\nabla\cdot u^s + B\nabla\cdot u^f - \beta\Theta]\big], v_i^s\Big) +$$

$$-\Big(\frac{\partial}{\partial x_j}\overbrace{\big(B\nabla\cdot u^s + M\nabla\cdot u^f - \beta_f\Theta\big)}^{p_f}, v^f\Big)$$

$$= \Big(2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}[\lambda\nabla\cdot u^s + B\nabla\cdot u^f - \beta\Theta], \frac{\partial v_i^s}{\partial x_j}\Big)$$

$$+\Big(B\nabla\cdot u^s + M\nabla\cdot u^f - \beta_f\Theta, \frac{\partial v_j^f}{\partial x_j}\Big)$$

$$-\langle\sigma\cdot\nu, v^s\rangle + \langle p_f, v^f\cdot\nu\rangle$$

$$\left(2\mu \frac{1}{2}\left(\frac{\partial u_i^s}{\partial x_j}+\frac{\partial u_j^s}{\partial x_i}\right), \frac{\partial v_i^s}{\partial x_j}\right)$$

$$= \left(\mu \frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_i^s}{\partial x_j}\right) + \left(\mu \frac{\partial u_j^s}{\partial x_i}, \frac{\partial v_i^s}{\partial x_j}\right)$$

$$i \to j$$

$$\left(\mu \frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_i^s}{\partial x_j}\right) + \left(\mu \frac{\partial u_i^s}{\partial x_j}, \frac{\partial v_j^s}{\partial x_i}\right)$$

$$= \left(\mu \frac{\partial u_i^s}{\partial x_j}, \left(\frac{\partial v_i^s}{\partial x_j}+\frac{\partial v_j^s}{\partial x_i}\right)\right)$$

$$= \left(\mu \frac{\partial u_i^s}{\partial x_j}, 2\, \varepsilon_{ij}(v^s)\right)$$

$$= \left(\mu \frac{\partial u_i^s}{\partial x_j}, \varepsilon_{ij}(v^s)\right) + \left(\mu \frac{\partial u_i^s}{\partial x_j}, \varepsilon_{ij}(v^s)\right)$$

$$i \to j$$

$$= \left(\mu \frac{\partial u_i^s}{\partial x_j}, \varepsilon_{ij}(v^s)\right) + \left(\mu \frac{\partial u_j^s}{\partial x_i}, \underbrace{\varepsilon_{ji}(v^s)}_{= \varepsilon_{ij}}\right)$$

$$= \left(\mu \left(\frac{\partial u_i^s}{\partial x_j}+\frac{\partial u_j^s}{\partial x_i}\right), \varepsilon_{ij}(v^s)\right)$$

$$= \left(2\mu\, \varepsilon_{ij}(u^s), \varepsilon_{ij}(v^s)\right)$$

$$(\cancel{\delta_{ij}}\,\lambda \nabla\cdot u^s,\ \delta_{ij}\frac{\partial v_i^s}{\partial x_j})$$

$$= (\lambda \nabla\cdot u^s,\ \nabla\cdot v^s)$$

$$(\beta \nabla\cdot u^f,\ \delta_{ij}\frac{\partial v_i^s}{\partial x_j}) = (\beta \nabla\cdot u^f,\ \nabla\cdot v^s)$$

$$-(\beta\theta,\ \delta_{ij}\frac{\partial v_i^s}{\partial x_j}) = -(\beta\theta,\ \nabla\cdot v^s)$$

Then,

$$-\left(\mathcal{L}(u,\theta),\ (v^s, v^f)\right)$$

$$= (2\mu\,\varepsilon_{ij}(u^s),\ \varepsilon_{ij}(v^s))$$
$$+ (\lambda \nabla\cdot u^s + \beta\nabla\cdot u^f - \beta\theta,\ \nabla\cdot u^s)$$
$$+ (\beta\nabla\cdot u^s + M\nabla\cdot u^f - \beta_f\theta,\ \nabla\cdot v^f)$$
$$\cancel{-(\beta\theta,\ \nabla\cdot v^s) - (\beta_f\theta,\ \nabla\cdot v^f)}$$

$$= B(u,v) - (\beta\theta, \nabla\cdot u^s) - (\beta_f\theta, \nabla\cdot u^f)$$
$$- \langle \sigma\nu,\ v^s\rangle + \langle p_f,\ v^f\cdot\nu\rangle$$

From (79), for $v^s, v^f \in C_0^\infty(\Omega)$ (4)

$$B(u,v) - (\beta\theta, \nabla\cdot u^s) - (\beta_f\theta, \nabla\cdot u^f)$$

$$= -\left(\mathcal{L}(u,\theta), (v^s, v^f)\right)$$

Also, for $w \in C_0^\infty(\Omega)$

$$(\gamma\nabla\theta, \nabla w) = -\nabla\cdot(\gamma\nabla\theta, w)$$

Then,

$$(\rho\ddot{u}, v) + \left(\frac{\eta}{K}\dot{u}^f, v^f\right) - (\mathcal{L}(u,\theta), v)$$

$$+ (\tau\ddot{\theta}, w) + (c\dot{\theta}, w) - \nabla\cdot(\gamma\nabla\theta, w)$$

$$+ \left(\beta T_0 (\nabla\cdot\dot{u}^s + \nabla\cdot\dot{u}^f), w\right)$$

$$+ \left(\tau\beta T_0 (\nabla\cdot\ddot{u}^s + \nabla\cdot\ddot{u}^f), w\right) =$$

$$(f, v) - (q, w), \quad (v^s, v^f, w) \in [C_0^\infty(\Omega)]^3$$

$\Longrightarrow$

$$\rho \ddot{u} + \frac{\eta}{k} \dot{u}^f - \mathcal{L}(u,\theta) = f \quad , (A1) \quad (5)$$
$$[\mathcal{D}'(\Omega)]^4$$

$$\tau \ddot{\theta} + c \dot{\theta} - \nabla\cdot(\gamma \nabla\theta) + \beta T_0 (\nabla\cdot\dot{u}^s + \nabla\cdot\dot{u}^f)$$
$$+ \tau\beta T_0 (\nabla\cdot\ddot{u}^s + \nabla\cdot\ddot{u}^f) = -q \quad , \underset{\mathcal{D}'(\Omega)}{(A-2)}$$

Since $\ddot{u} \in L^\infty(J, [L^2(\Omega)]^4)$ (see (7-2))

$\dot{u}^f \in L^\infty(J, H(\mathrm{div}, \Omega))$, $f \in L^2(J, (L^2(\Omega))^4)$

$\rightarrow \mathcal{L}(u,\theta) \in L^\infty(J, [L^2(\Omega)]^4)$.

so that (A1) holds as function

in $L^\infty(J, (L^2(\Omega))^4)$ ($x \in \Omega$, $t \in J$ a.e)

Similarly, $\ddot{\theta} \in L^\infty(J, L^2(\Omega))$

$\dot{\theta} \in L^\infty(J, L^2(\Omega))$

$\nabla\cdot\dot{u}^s, \nabla\cdot\dot{u}^f \in L^\infty(J, L^2(\Omega))$ because

$\dot{u}_m \rightarrow \dot{u}$ in $L^\infty(J, V)$

Also, thanks to (61)

$\ddot{u}_m \to \ddot{u}$ in $L^\infty(J, V)$

also $\nabla\cdot\ddot{u}^s$, $\nabla\cdot\ddot{u}^f \in L^\infty(J, L^2(\Omega))$

Then also (since $q \in L^2(J, L^2(\Omega))$)

$\nabla\cdot(\gamma\nabla\theta) \in L^2(J, L^2(\Omega))$ and

holds as function in $L^2(J, L^2(\Omega))$

Boundary conditions: we know that

(79) holds — $v \in V \times H^1(\Omega)$ —

Since $\mathcal{L}(u,\theta) \in L^\infty(J, (L^2(\Omega))^4)$

we can integrate by parts

$$\mathcal{B}(u,v) - (\beta\theta, \nabla\cdot v^s) - (\beta_f\theta, \nabla\cdot v^f)$$

$$= -(\mathcal{L}(u,\theta), (v^s, v^f)) + \langle\sigma\nu, v^s\rangle$$

$$- \langle p_f, v^f\cdot\nu\rangle$$

Now for (79) for $\omega = 0$ (7)

$$(P\ddot{u}, v) + \left(\frac{\eta}{k}\dot{u}^f, v^f\right) - \left(\mathcal{L}(u, \theta), (v^s, v^f)\right)$$

$$+ \langle \sigma\nu, v^s\rangle - \langle p_f, v^f \cdot \nu\rangle$$

$$= (A-1) \quad \cancel{(f, v)} + \langle \sigma\nu, v^s\rangle$$

$$- \langle p_f, v^f \cdot \nu\rangle$$

$$= \cancel{(f\, v)} + \langle g, v^s\rangle + \langle \chi, v^f \cdot \nu\rangle \rightarrow$$

$\rightarrow \langle \sigma\nu, v^s\rangle = \langle g, v^s\rangle$, $v^s \in (H^1(\Omega))^2$

$\sigma\nu = g$ in $H^{1/2}(\Gamma)$

$- \langle p_f, v^f \cdot \nu\rangle = \langle \chi, v^f \cdot \nu\rangle$, $v^f \in H(\mathrm{div}, \Omega)$

$\rightarrow \sigma\nu = g$, in

for any $\cancel{v^f} \in \cancel{H(\mathrm{div}, \Omega)}$ $H^{-1/2}(\Gamma)$ $\exists$

$q \in H(\mathrm{div}$ such that $q \cdot \nu = z$

$\rightarrow p_f = -\chi$ in $H^{1/2}(\Gamma)$

Initial condition

let $\varphi \in C^\infty[0,T]$, $\varphi(0)=1$, $\varphi(T)=0$

We know that

$$\lim_{m\to\infty} \int_0^T (P\mathring{u}_m, v)\,\varphi(t)\,dt$$

$$= \int_0^T (P\mathring{u}, v)\,\varphi(t)\,dt \qquad (B1)$$

We know that $(u,v)\,(u_m, v)$ is continuous

for $[0\,T]$ and we can develop by part

$$\int_0^T (P\mathring{u}_m, v)\,\varphi(t) = (u_m, v)\,\varphi\Big|_0^T$$

$$- \int_0^T (u_m, v)\,\varphi'(t)\,dt \qquad (B2)$$

$$= -(u_m(0), v) - \int_0^T (u_m, v)\,\varphi'(t)\,dt$$

$$\int_0^T (P\mathring{u}, v)\,\varphi(t) = -(u(0), v) - \int_0^T (u, v)\,\varphi'\,dt \qquad (B3)$$

Take limit in m in (B_2)

$$\lim_{m\to\infty} \int_0^T (\rho \ddot{u}_m, v)\varphi(t) = \int_0^T (\rho \ddot{u}, v)\varphi(t)\,dt$$

$$= -\lim_{m} (u_m(0), v) - \int_0^T (u, v)\, \varphi'(t)\,dt \qquad (B4)$$

From (B3) (B4)

$$\lim_{m} (u_m(0), v) = (u(0), v) \quad \forall\, v \in [L^2(\Omega)]^4$$

(weakly in L^2

But also

$$u_m(0) \longrightarrow u^0 \text{ in } [L^2(\Omega)]^4 \text{ (strongly)}$$

$\rightarrow u(0) = u^0$ by uniqueness of weak limit.

Similarly for $u'(0) = u^1$,

$\theta(0) = \theta^0$

$\theta'(0) = \theta^1$

THERMOPOROELASTICITY

VIRTUAL WORK PRNCIPLE

PART VII

J. SANTOS

7/8/19

$$\sigma_{11} = 2\mu\,\varepsilon_{11} + \lambda(\varepsilon_{11}+\varepsilon_{33})$$ $u^f - \phi(\tilde{u}^f - u^s)$ 1

$$+\alpha M\left(\alpha(\varepsilon_{11}+\varepsilon_{33}) + \nabla\cdot u^f\right) - \beta T$$

$T \Leftrightarrow \Theta$ en BIOT

$$\sigma_{33} = 2\mu\,\varepsilon_{33} + \lambda(\varepsilon_{11}+\varepsilon_{33})$$

$$+\alpha M\left(\alpha(\varepsilon_{11}+\varepsilon_{33}) + \nabla\cdot u^f\right) - \beta T$$

$$-\phi p_f = \alpha M\phi\,\varepsilon_{11} + \alpha M\phi\,\varepsilon_{33} + \phi^2 M\,\nabla\cdot u^f$$

$$-\beta_f T$$

$$\frac{\nabla\cdot(\gamma T)}{T_0} - \beta T_0\,\varepsilon_{11} - \beta T_0\,\varepsilon_{33} - \beta T_0\,\nabla\cdot u^f = 0$$

$$\sigma_{13} = 2\mu\,\varepsilon_{13}$$

$\cancel{\nabla\cdot(\gamma T) - T\beta(\varepsilon_{11}+\varepsilon_{33}+\nabla\cdot u^f)}$

$\cancel{\nabla\cdot u}$

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ -\phi\beta_f \\ \dfrac{-\nabla\cdot(\gamma T)}{T_0} \\ \sigma_{13} \end{pmatrix}
=
\begin{pmatrix}
(\lambda+2\mu+\alpha^2 M) & (\lambda+\alpha M) & \alpha M\phi & -\beta & 0 \\
(\lambda+\alpha M) & (\lambda+2\mu+\alpha^2 M) & \alpha M\phi & -\beta & 0 \\
\alpha M\phi & \alpha M\phi & \cancel{\alpha} M\phi^2 & -\beta_f & 0 \\
-\beta & -\beta & \cancel{\phi}\,\beta_f & 0 & 0 \\
0 & 0 & 0 & 0 & 2\mu
\end{pmatrix}
\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{33} \\ \nabla\cdot u^f \\ T \\ \varepsilon_{13} \end{pmatrix}
$$

$$\int_\Omega \delta W \, \cancel{d\Omega} = \int_{\partial\Omega} \Big[(\tau_{ij} + \delta_{ij} \phi p_f) \, \delta u_i^s \, \nu_j \quad (1)$$

$$- \phi \, p_f \, \delta_{ij} \, \delta \tilde{u}_i^f \, \nu_j + \theta \, \delta_{ij} \, \nu_j \, \delta S_i \Big] d\Omega$$

$$u_f = \phi \, (\tilde{u}_i^f - u^s)$$

$$\int_\Omega \delta W \, d\Omega = \int_{\partial\Omega} \Big(\tau_{ij} \, \delta u_i^s \, \nu_j - p_f \delta_{ij} \, \delta u_i^f \, \nu_j$$

$$+ \theta \, \delta S_i \, \delta_{ij} \, \nu_j \Big) \, d\Omega$$

$$= \int_\Omega \frac{\partial}{\partial x_j} \left(\tau_{ij} \, \delta u_i^s \right) d\Omega$$

$$- \int_\Omega \frac{\partial}{\partial x_j} \left(p_f \, \delta_{ij} \, \delta u_i^f \right) d\Omega$$

$$+ \int_\Omega \frac{\partial}{\partial x_j} \left(\theta \, \delta S_i \, \delta_{ij} \right) d\Omega$$

$$\frac{\partial}{\partial x_j}\left(\theta\,\delta_{ij}\,\delta S_i\right)$$

$$= \frac{\partial}{\partial x_j}\left(\theta\,\delta_{ij}\right)\cdot\delta S + \theta\frac{\partial\delta S_i}{\partial x_j}\delta_{ij}$$

$$= \underset{\substack{\shortparallel\\ 0}}{\nabla\theta}\cdot\delta S + \theta\,\nabla\cdot\delta S_i$$

$$\frac{\partial}{\partial x_j}\left(p_f\,\delta_{ij}\,\delta u_i^f\right) = \frac{\partial}{\partial x_j}\left(p_f\delta_{ij}\right)\delta u_i^f$$

$$+\; p_f\frac{\partial\,\delta u_i^f}{\partial x_j}\delta_{ij}$$

$$= \left(\nabla p_f\right)_i\delta u_i + p_f\,\nabla\cdot\delta u_f$$

③

$$\int_\Omega \delta W \, d\Omega = \int_\Omega \Big(\tau_{ij} \, \delta \varepsilon_{ij}(u^s)$$

$$- \rho_f \nabla \cdot \delta u^f + \theta \, \nabla \cdot \delta S_i \Big) \, d\Omega$$

$$\delta W = \tau_{ij} \, \delta \varepsilon_{ij}(u_s) + \rho_f \, \delta \xi$$

$$+ \theta \, \delta \underbrace{\nabla \cdot S}$$

$$\nabla \cdot S = \chi$$

(2.14)

$$s = \frac{c\theta}{T} + \beta e$$

$$-\nabla \cdot S = s = -\frac{c\theta}{T} - \beta e$$

$$\theta = \frac{T}{c}(s - \beta e) = -\frac{T}{c}\left(\nabla \cdot S + \beta \nabla \cdot u^s\right)$$

$$= \frac{T}{c}\left(-\nabla \cdot S - \beta e\right)$$

$$W = \frac{1}{2}\Big(E_\mu (e^s)^2 + \mu I_2' - 2Be\xi$$

$$+ M\xi^2 + 2Ae\chi + 2D\xi\chi$$

$$+ F(\chi)^2 \Big)$$

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ij} = 2\mu\, \varepsilon_{ij}\,, \quad i \neq j$$

$$\frac{\partial W}{\partial \varepsilon_{11}} = \tau_{11} = E_\mu e^s + \mu(-2\varepsilon_{33} - 2\varepsilon_{22})$$

$$- B\xi + A\chi$$

$$\frac{\partial W}{\partial \varepsilon_{22}} = \tau_{22} = E_\mu e^s + \mu(-2\varepsilon_{11} - 2\varepsilon_{33})$$

$$- B\xi + A\chi$$

$$\frac{\partial W}{\partial \varepsilon_{33}} = \tau_{33} = E_\mu e^s + \mu(-2\varepsilon_{22} - 2\varepsilon_{11})$$

$$- B\xi + A\chi$$

$$E_\mu = \lambda_\mu + 2\mu \qquad \lambda_\mu = \lambda + \alpha^2 M$$

$$\tau_{11} = \lambda_\mu e^s + 2\mu(\varepsilon_{11} + \cancel{\varepsilon_{22}} + \cancel{\varepsilon_{33}})$$

$$- \mu(2\cancel{\varepsilon_{33}} + \cancel{\varepsilon_{22}}) - B\xi + A\chi$$

$$= \lambda_\mu e^s + 2\mu\,\varepsilon_{11} - \beta\xi + A\chi$$

$$\frac{\partial W}{\partial \xi} = p_f = -Be + D\chi + M\xi$$

$$\frac{\partial W}{\partial \chi} = \theta = Ae + D\xi + F\chi$$

~~From (6.6) BIOT~~

$$\cancel{\theta = -\frac{1}{c}\nabla(S + \beta u)}$$

$$\cancel{= -\frac{1}{c}(\chi + \beta)}$$

From (2.13)

$$S = c \log \left(1 + \frac{\theta}{T}\right) + \beta e$$

(2.14)

$$S = \frac{c\theta}{T} + \beta e = -D_\phi \$ = -\chi$$

$$\chi = -\frac{c\theta}{T} - \beta e$$

En nuestra notación

$$\theta = T \qquad T = T_0$$

$$\chi = -\frac{cT}{T_0} - \beta e$$

$$\rho f = -Be + D\left(-\frac{c\theta}{T_0} - \beta e\right) + M\xi$$

$$= -(B+\beta)e - \frac{Dc\theta}{T_0} + \underline{M}\xi$$

Hay que determinar A D y F usando Gedanken

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$$\tau_{11} = \lambda_\mu e^s + 2\mu\,\varepsilon_{11} - \beta\xi + A\chi$$

$$= \lambda_\mu e^s + 2\mu\,\varepsilon_{11} - B\xi + A\left(-\frac{CT}{T_0} - \beta e^s\right)$$

$$= (\lambda_\mu - \beta)\, e^s + 2\mu\,\varepsilon_{11} - B\xi - \underbrace{A\frac{CT}{T_0}}_{\beta^2}$$

$$p_f = -Be^s + M\xi + \underbrace{D\left(-\frac{CT}{T_0} - \beta e^s\right)}_{\chi}$$

$$= -(B+\beta)\, e^s + M\xi - \underbrace{D\frac{CT}{T_0}}_{\beta}$$

$$\frac{\partial W}{\partial \chi} = A e^s + D\xi + F\chi$$

$$= T$$ ← Constitutive para $\theta = T$

$$\tau_{11} = (\lambda+2\mu)\,\varepsilon_{11}^{s} + \underbrace{\alpha^2 M}_{\tilde{M}}(\varepsilon_{11}+\varepsilon_{33}) + \lambda\varepsilon_{33}^{s} + \underbrace{\alpha M}_{B}(W_{11}+W_{33}) - \beta T + f_{1}$$

$$\tau_{33} = (\lambda+2\mu)\,\varepsilon_{33}^{s} + \alpha^2 M(\varepsilon_{11}+\varepsilon_{33}) + \alpha M(W_{11}+W_{33}) + \lambda\varepsilon_{11}^{s} - \beta T + f_3$$

$$\tau_{13} = 2\mu\,\varepsilon_{13}^{s}$$

$$P_f = -\underbrace{\alpha M}_{B}(\varepsilon_{11}+\varepsilon_{33}) - M(W_{11}+W_{33}) - \beta T + f_f$$

$$\tau_{11} = (\lambda+2\mu)\,\varepsilon_{11} + \lambda\varepsilon_{33} + \alpha^2 M\,\nabla\cdot u^{s} + \underbrace{\alpha M}_{B}\,\nabla\cdot u^{f}$$

$$\begin{pmatrix} \lambda+2\mu & \lambda & B & 0 \\ \lambda & \lambda+2\mu & B & 0 \\ B & B & M & 0 \\ 0 & 0 & 0 & 4\mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{33} \\ \nabla\cdot u^{f} \\ \varepsilon_{13} \end{pmatrix}$$

$$= \big((\lambda+2\mu)\,\varepsilon_{11} + \lambda\varepsilon_{33} + B\,\nabla\cdot u^{f},\ \lambda\varepsilon_{11} + (\lambda+2\mu)\,\varepsilon_{33} + B\,\nabla\cdot u^{f},\ B(\varepsilon_{11}+\varepsilon_{33}) + M\,\nabla\cdot u^{f},\ 4\mu\,\varepsilon_{13}\big)$$