Geophysical Prospecting **39**, 219–240, 1991

LAYERED PERMEABLE SYSTEMS[1]

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ABSTRACT

SCHOENBERG, M. 1991. Layered permeable systems. *Geophysical Prospecting* **39**, 219–240.

Permeability is a second rank tensor relating flow rate to pressure gradient in a porous medium. If the permeability is a constant times the identity tensor the permeable medium is isotropic; otherwise it is anisotropic. A formalism is presented for the simple calculation of the permeability tensor of a heterogeneous layered system composed of interleaved thin layers of several permeable constituent porous media in the static limit. Corresponding to any cumulative thickness H of a constituent is an element consisting of scalar H and a matrix which is H times a hybrid matrix function of permeability. The calculation of the properties of a medium equivalent to the combination of permeable constituents may then be accomplished by simple addition of the corresponding scalar/matrix elements. Subtraction of an element removes a permeable constituent, providing the means to decompose a permeable medium into many possible sets of permeable constituents, all of which have the same flow properties. A set of layers of a constituent medium in the heterogeneous layered system with permeability of the order of $1/h$ as $h \to 0$, where h is that constituent's concentration, acts as a set of infinitely thin channels and is a model for a set of parallel cracks or fractures. Conversely, a set of layers of a given constituent with permeability of the order of h as $h \to 0$ acts as a set of parallel flow barriers and models a set of parallel, relatively impermeable, interfaces, such as shale stringers or some faults. Both sets of channels and sets of barriers are defined explicitly by scalar/matrix elements for which the scalar and three of the four submatrices vanish. Further, non-parallel sets of channels *or* barriers can be 'added' and 'subtracted' from a background homogeneous anisotropic medium commutatively and associatively, but not non-parallel sets of channels *and* barriers reflecting the physical reality that fractures that penetrate barriers will give a different flow behaviour from barriers that block channels. This analysis of layered media, and the representations of the phenomena that can occur as the thickness of a constituent is allowed to approach zero, are applicable directly to layered heat conductors, layered electrostatic conductors and layered dielectrics.

INTRODUCTION

Permeability is a second rank tensor relating the fluid flow rate vector in a porous solid to the macroscopic pressure gradient in the medium. It is a fundamental

[1] Based on a paper read at the 51st EAEG meeting, Berlin, May–June 1989; received September 1989, revision accepted August 1990.

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property of a porous medium, indicating how easily fluids move, for example, through rock in a hydrocarbon reservoir. However, permeability in the earth is almost everywhere anisotropic, sometimes by an order of magnitude or more. Figure 1 shows a piece of Navajo sandstone for which the horizontal permeability (parallel to the thin bands) is greater than 250 times the vertical permeability (perpendicular to the bands). The dark bands are layers with much finer grains and narrower pore throats than the lighter layers. The layers exhibit very little textural variation.

Basically, measurements are often made over distances large with respect to the width of individual layers in a finely-layered region, so the permeability observed is an average of the permeabilities of the individual constituent media (hereafter called constituents). Each layer is one of those constituents, and one must envisage perhaps many layers with only several constituents (see Fig. 2). Typically an alternating sequence of layers (not necessarily periodic) consists of many layers of only two constituents. Generally, each constituent may itself be anisotropic.

This situation was considered by Schoenberg and Muir (1989) with reference to elastic stiffness moduli. They constructed a calculus to deal efficiently with the calculation of stiffness moduli and thus plane wave phase velocities of a medium equivalent to the layered medium in the long wavelength limit. In addition, they showed how the calculus could be used to decompose an equivalent medium into possible constituents, and to handle in a coherent manner certain constituents, such as parallel fractures, that were limiting cases of layers of a given constituent.

The basic ideas of calculating the properties of equivalent media in the static limit are applied here to the less complicated situation of analysing the permeability tensor of a stratified medium under constant or slowly varying pressure gradients.

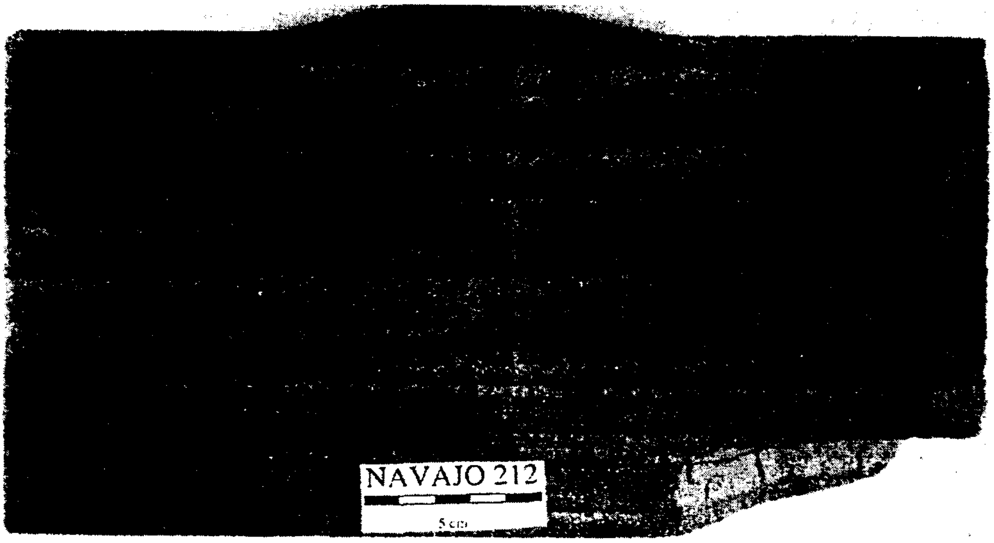

FIG. 1. The Navajo sandstone shown here exhibits highly anisotropic permeability. The permeability anisotropy is thought to be caused by the thin dark bands which consist of much smaller grains. The markings on the specimen denote the location of the cores which were used in the permeability experiments. Photo courtesy of Stefan M. Luthi, Schlumberger-Doll Research.

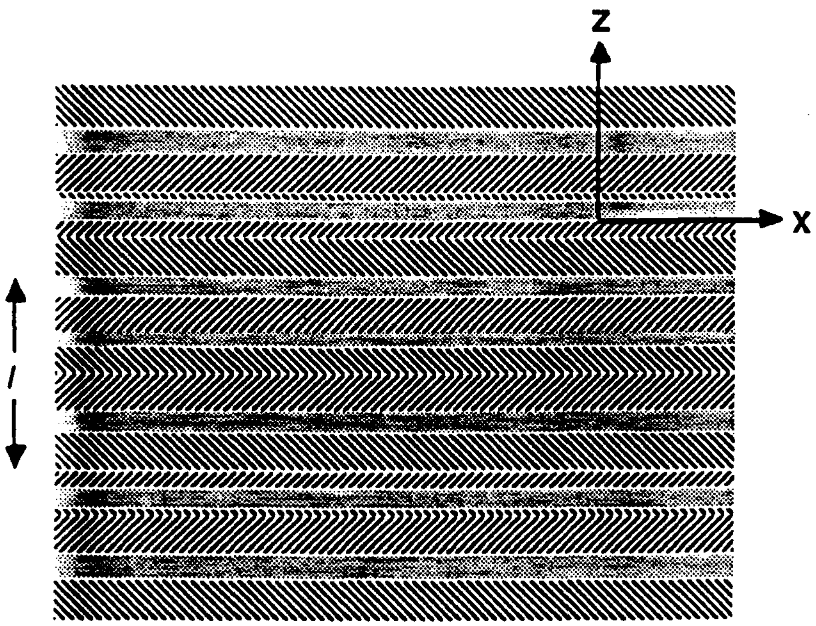

FIG. 2. A stack of permeable layers, in this case consisting of three constituents. Each constituent may be anisotropic. In any interval of thickness ℓ or larger, where ℓ is much smaller than a wavelength, the percentage of each constituent is assumed to be stationary with respect to the vertical coordinate z.

The purpose is to show how the permeability can be analysed in a layered porous reservoir and to expose the relevant parameters needed to specify flow channels and flow barriers. A secondary purpose is to show that this approach to layered media is useful in considering a broad class of linear constitutive relations, and the Appendix contains results for linear relations of arbitrary dimension. The particular example of the constitutive relation of a permeable solid, where a 3D vector field is linearly related to another vector field by a real symmetric second rank tensor is only one example of a class of problems including those of: (1) heat conducting solids where the heat flux vector is related to the temperature gradient by the heat conductivity tensor; (2) electrical conductors in the static limit where the conduction current density is related to the gradient of the potential (which is the electric field) by a real conductivity tensor; and (3) dielectrics in the static limit where the charge displacement vector is related to the electric field vector by the permittivity matrix. In general, heat conductivity, electrical conductivity and electrical permittivity tensors are anisotropic. All the ideas developed with the use of the calculus for permeable layered media have their exact analogue in the areas of heat conductivity and static electrical properties of layered media. In addition, a set of parallel flow channels in a rock mass, which may be modelled as a set of very thin layers of high permeability can perhaps be identified with very thin layers of high conductivity if the fluid flowing in the channels is an electrolyte, and with long parallel fractures or microcracks if they are open enough to change substantially the overall elastic compliance of the medium (Crampin 1984).

A porous medium is obviously extremely inhomogeneous at a level of the grain and pore size. However, for a porous medium homogeneous down to a scale covering many grains, the generalized Darcy's law states that the macroscopic pressure gradient ∇p and $\nu\rho\mathbf{q}$ are linearly related by a second rank permeability tensor $\mathbf{K}$ of dimension *length*^{2}. ν is the kinematic viscosity; ρ is fluid density; $\mathbf{q}$ is the flow rate of dimension *velocity* defined so that $\rho\mathbf{q}$ is the volume integral of the point-wise momentum of the fluid over the pore space in a volume divided by the volume. Thus $\rho\mathbf{q}$ is porosity times the volume average, in the Biot sense, of $\rho\mathbf{v}$ over the pore space, where $\mathbf{v}$ is the point-wise fluid velocity. Therefore the generalized Darcy's law may be written as

$$\mathbf{q} = -\frac{1}{\nu\rho}\mathbf{K}\cdot\nabla p, \tag{1}$$

or in matrix notation,

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = -\frac{1}{\nu\rho}\begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}\begin{bmatrix} p_{,x} \\ p_{,y} \\ p_{,z} \end{bmatrix},$$

with the comma [,] denoting partial differentiation.

In the very long wavelength, low-frequency range (quasi-steady state), the assumptions that the permeability matrix $\mathbf{K}$ be real and that the fluid be incompressible, i.e. $\nabla\cdot\mathbf{q} = 0$, are very good approximations even for gas-saturated media (Biot 1956; Schoenberg and Sen 1987). The condition that $-\nabla p\cdot\mathbf{q}$ be positive (thus assuming there is always some flow given sufficient pressure) implies $\mathbf{K}$ is positive definite. I further assume that reciprocity holds, which is equivalent to $\mathbf{K}$ being symmetric. Under these conditions, there is always a rectangular coordinate system in which $\mathbf{K}$ is diagonal, the diagonal elements being the eigenvalues which are real and positive. In general all three eigenvalues are different. The two more restrictive cases are when two eigenvalues are the same and when all three are the same, the isotropic case.

Note that the pressure gradient ∇p can be expressed in terms of $\mathbf{q}$ using the inverse of the permeability matrix, $\mathbf{L} = \mathbf{K}^{-1}$. $\mathbf{L}$ is the flow resistivity matrix, or the impermeability matrix, or simply the impermeability. As $\mathbf{K}$ has dimension *length*^{2}, $\mathbf{L}$ has dimension *length*$^{-2}$, and it too is symmetric, positive definite with eigenvalues equal to the inverses of those of $\mathbf{K}$. Equation (1) inverted is

$$\nabla p = -\nu\rho\mathbf{L}\cdot\mathbf{q}. \tag{2}$$

Its use greatly simplifies the insertion and removal of flow barriers, while fractures are easier to handle permeability. This is analogous to the fact that the elastic effects of fractures are much easier to analyse using elastic compliance instead of elastic stiffness moduli, a fact that was not appreciated in the original Schoenberg-Muir paper, but that has been used subsequently by Nichols, Muir and Schoenberg (1989) for elastic layers.

In addition to a constitutive relation, interface conditions on the field variables between homogeneous regions must be posited. Perfect contact at an interface $z = 0$

is defined as: (a) pressure p is continuous across $z = 0$ implying that any tangential derivative of p is continuous, or in vector form $\nabla_T p(0^-) = \nabla_T p(0^+)$; and (b) there are no sources or sinks for fluid in the interface between the different media so that the normal component of $\mathbf{q}$ is continuous across the interface, i.e. $q_z(0^-) = q_z(0^+)$. Interface condition (a) holds for temperature in the heat conduction problem and for the electric potential in the electrical conductivity problem, while (b) holds for the heat flux vector and the conduction current density vector. In all cases with constitutive relations and interface conditions of this same form, the analysis below applies.

In Section 1, the properties of a homogeneous medium equivalent to a layered permeable medium are formulated using submatrices of the permeability and impermeability matrices following the approach used in the Appendix of Helbig and Schoenberg (1987) which was for elastic equivalent medium properties. For any set of n constituent media, there exists a homogeneous anisotropic medium that behaves, in the quasi-static limit, exactly as does the finely-layered medium consisting of many layers, each layer being one of the n constituents. This means, in this case, that on a scale much larger than the scale of the layering, the equivalent medium flows exactly as does the layered medium under the same applied pressure gradients. The derivation and the appearance of the formulae for the equivalent medium properties are not dependent on the number of variables in the constitutive relation or the sizes of the submatrices. The approach, applicable to a broad range of problems of arbitrary dimension, is presented in the Appendix, which also includes a discussion of matrix inversion using submatrices and general equivalent media formulae.

In Section 2 the ideas of the Schoenberg–Muir calculus (1989) are applied, developed for elastic layers, to the problem of permeable layers under consideration here. Essentially, one mirrors the physical construction of a section of a given thickness of a layered medium composed of several constituents by associating with each constituent an element consisting of the cumulative thickness of the constituent and that thickness times the hybrid matrix function, its permeability. Then as one constructs the physical model by interleaving thin layers of each of the constituents, mathematically all one does is simply add these elements, giving a new element corresponding to the total thickness of the homogeneous medium equivalent, in the static limit, to the section of layered media just constructed. The order or way in which the constituents are inserted does not affect the result. The advantage of this approach is that removal of an amount of a given constituent is mathematically equivalent to subtraction of the element corresponding to that amount of the constituent, thereby providing the means to decompose a section of a permeable medium into a set of permeable constituents and their thicknesses. As each of the elements is merely a scalar and a matrix with certain specifiable properties, the set of all such elements is a commutative group under addition, called G, formalizing the operations that are allowed, both mathematically and physically.

The constituent properties which always carry over to the equivalent medium properties, i.e. for which properties is the set of all elements corresponding to layers with those properties a subgroup of G, is discussed in Section 3. Special attention is devoted to symmetry properties of the permeability tensor.

In Section 4, parallel cracks or fractures are characterized as infinitesimally thin, but free flowing, channels, while conversely, thin but highly impermeable layers, such as shale stringers or faults at which the pores are misaligned and clogged, are characterized as planar barriers to flow. Both these phenomena have simple explicit representations in the group domain. The insertion or removal of channels or barriers at any orientation becomes a simple arithmetic order-independent operation. Only when there are intersecting sets of channels and barriers, does the order in which they were introduced influence the properties of the equivalent medium. In addition, an arbitrary anisotropic permeable medium is shown to be equivalent to an isotropic background with a single set of flow channels, or, to an isotropic medium with two sets of flow barriers intersecting at right angles. In some sense these are minimal representations for an arbitrary medium.

1. Systems of Anisotropic Permeable Layers

Consider a region of porous, homogeneous (over a scale much larger than pore or grain size), but in general anisotropic layers, composed of n constituents, each with concentration h_i, so that $\sum_{i=1}^{n} h_i = 1$, and permeability $\mathbf{K}_i$, saturated with the same single fluid in all layers. Set Cartesian coordinates so that the z-axis is perpendicular to the layering, and the x- and y-axes lie in the plane of the layering (Fig. 2).

Assumption (1) is that the concentrations h_i of the finely-layered constituents are approximately the same in any interval in z of width ℓ or larger, i.e. the layered medium is stationary down to length scale ℓ, the stationarity length.
Assumption (2), that of slow variation over a length scale $L \gg \ell$, is that all layers of the same constituent encounter the same environment and thus have the same values of the field variables, ∇p and $\mathbf{q}$.

At any boundary between layers, which must be a plane of constant z, q_z (expressing the flow per unit area across the boundary) must be continuous, and the pressure p, and hence the derivatives of p parallel to the layering, must be continuous. Thus q_z, $p_{,x}$ and $p_{,y}$ are constant throughout the region while the components of $\mathbf{q}$ parallel to the layering and the derivative of p perpendicular to the layering depend on the local properties of the layered permeable system. Equation (1) in the ith constituent can be rewritten to separate field variables that are constant over long distances from those that vary with i as

$$\mathbf{q}_{T_i} = -\frac{1}{v\rho}\,[\mathbf{K}_{TT_i}\nabla_T p + \mathbf{k}_{TN_i} p_{,z_i}],$$
$$q_z = -\frac{1}{v\rho}\,[\mathbf{k}_{NT_i}\nabla_T p + K_{NN_i} p_{,z_i}], \tag{3}$$

where

$$\mathbf{q}_{T_i} = \begin{bmatrix} q_{x_i} \\ q_{y_i} \end{bmatrix}, \qquad \nabla_T p = \begin{bmatrix} p_{,x} \\ p_{,y} \end{bmatrix},$$

and

$$K_{\mathrm{NN}_i} = K_{zz_i}, \qquad \mathbf{k}_{\mathrm{TN}_i} = \begin{bmatrix} K_{xz_i} \\ K_{yz_i} \end{bmatrix}, \qquad \mathbf{k}_{\mathrm{NT}_i} = \mathbf{k}^{\mathrm{t}}_{\mathrm{TN}_i}, \qquad \mathbf{K}_{\mathrm{TT}_i} = \begin{bmatrix} K_{xx_i} & K_{xy_i} \\ K_{xy_i} & K_{yy_i} \end{bmatrix}.$$

$\mathbf{q}_{\mathrm{T}_i}$ is the velocity tangent to the layering and $\nabla_{\mathrm{T}} p$ is the tangential gradient of the pressure. Superscript t denotes the transpose. The convention used is that simple italics denote a scalar or a 1×1 submatrix, bold face, lower case denotes a vector or a 1×2 or 2×1 submatrix, and boldface capital denotes a matrix or a 2×2 submatrix.

Before these equations can be averaged to find an equivalent permeable medium, one has to solve for the variables that vary from layer to layer. Solving the second of (3) for $p_{,z_i}$ and substituting into the first of (3) gives

$$\begin{aligned} -\nu\rho\mathbf{q}_{\mathrm{T}_i} &= (\mathbf{K}_{\mathrm{TT}_i} - \mathbf{k}_{\mathrm{TN}_i} K^{-1}_{\mathrm{NN}_i} \mathbf{k}_{\mathrm{NT}_i})\nabla_{\mathrm{T}} p + \mathbf{k}_{\mathrm{TN}_i} K^{-1}_{\mathrm{NN}_i}(-\nu\rho q_z), \\ p_{,z_i} &= -K^{-1}_{\mathrm{NN}_i}\mathbf{k}_{\mathrm{NT}_i}\nabla_{\mathrm{T}} p + K^{-1}_{\mathrm{NN}_i}(-\nu\rho q_z). \end{aligned} \tag{4}$$

This is the hybrid form of the flow–pressure gradient relation, and the coefficients of $\nabla_{\mathrm{T}} p$ and $(-\nu\rho q_z)$ on the right-hand side of (4) are the submatrices of the hybrid modulus matrix (for short, the hybrid submatrices) of the ith permeable medium in terms of the submatrices of the permeability matrix.

Due to assumptions (a) and (b), the pressure drop for the equivalent homogeneous medium in the z-direction over any width $H \geq \ell$ but smaller than L must equal the sum over all the constituents of their z derivatives of pressure times their respective cumulative thickness in the width H. Dividing such a sum by H implies that $p_{,z}$ is given by the thickness-weighted average of the $p_{,z_i}$, i.e. $\sum_{i=1}^{n} h_i p_{,z_i} \equiv \langle p_{,z}\rangle$. A similar argument about the horizontal flow through a vertical section of width H implies that $\mathbf{q}_{\mathrm{T}}$ is given by the thickness-weighted average of the $\mathbf{q}_{\mathrm{T}_i}$, i.e. $\sum_{i=1}^{n} h_i \mathbf{q}_{\mathrm{T}_i} \equiv \langle \mathbf{q}_{\mathrm{T}}\rangle$. Thus thickness-weighted averaging of (4) gives, at length scales at least of the order of ℓ,

$$\begin{aligned} -\nu\rho\langle\mathbf{q}_{\mathrm{T}}\rangle &= (\langle\mathbf{K}_{\mathrm{TT}}\rangle - \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle)\nabla_{\mathrm{T}} p + \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle(-\nu\rho q_z), \\ \langle p_{,z}\rangle &= -\langle K^{-1}_{\mathrm{NN}}\mathbf{k}_{\mathrm{NT}}\rangle\nabla_{\mathrm{T}} p + \langle K^{-1}_{\mathrm{NN}}\rangle(-\nu\rho q_z). \end{aligned} \tag{5}$$

This is the hybrid form of the averaged anisotropic flow–pressure gradient relation in the layered medium, which is precisely the flow–pressure gradient relation of the homogeneous medium that is equivalent to the heterogeneous layered medium at length scales of the order of ℓ or larger. The coefficients on the right-hand side of (5) are the hybrid submatrices of the equivalent medium, which are only the thickness-weighted averages of the hybrid submatrices of the constituents. From the hybrid submatrices, the permeability matrix of the equivalent medium is returned by solving (5) for $\langle\mathbf{q}_{\mathrm{T}}\rangle$ and q_z and identifying the coefficients of $\nabla_{\mathrm{T}} p$ and $\langle p_{,z}\rangle$ with the appropriate permeability submatrices, yielding

$$\begin{bmatrix} \mathbf{K}_{\mathrm{TT}} & \mathbf{k}_{\mathrm{TN}} \\ \mathbf{k}_{\mathrm{NT}} & K_{\mathrm{NN}} \end{bmatrix} = \begin{bmatrix} \begin{matrix} \langle\mathbf{K}_{\mathrm{TT}}\rangle - \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\mathbf{k}_{\mathrm{NT}}\rangle \\ + \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle\langle K^{-1}_{\mathrm{NN}}\rangle^{-1}\langle K^{-1}_{\mathrm{NN}}\mathbf{k}_{\mathrm{NT}}\rangle \end{matrix} & \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle\langle K^{-1}_{\mathrm{NN}}\rangle^{-1} \\ \langle K^{-1}_{\mathrm{NN}}\rangle^{-1}\langle K^{-1}_{\mathrm{NN}}\mathbf{k}_{\mathrm{NT}}\rangle & \langle K^{-1}_{\mathrm{NN}}\rangle^{-1} \end{bmatrix}. \tag{6}$$

Equation (6) is the full anisotropic equivalent of the fact that, when layers are stacked together, permeability normal to the layering is the harmonic average of the constituent permeabilities (connection of conductors in series) while permeability tangential to the layering is the arithmetic average (connection of conductors in parallel).

The above analysis using (3) to (6) can be carried out in the same way using the impermeability rather than the permeability. Analogous to (3), **L** can be broken into submatrices for the purpose of separating variables which change from layer to layer from those that are constant over may layers, yielding for the ith constituent,

$$\begin{aligned} \nabla_T p &= -\nu\rho[\mathbf{L}_{TT_i} q_{T_i} + \mathbf{l}_{TN_i} q_z], \\ p_{,z_i} &= -\nu\rho[\mathbf{l}_{NT_i} \mathbf{q}_{T_i} + L_{NN_i} q_z]. \end{aligned} \tag{7}$$

The definitions of the submatrices of **L** are analogous to those of **K** of (3). Now solving (7) for the quantities which vary with i yields

$$\begin{aligned} -\nu\rho\mathbf{q}_{T_i} &= \mathbf{L}_{TT_i}^{-1}\nabla_T p - \mathbf{L}_{TT_i}^{-1}\mathbf{l}_{TN_i}(-\nu\rho q_z), \\ p_{,z_i} &= \mathbf{l}_{NT_i}\mathbf{L}_{TT_i}^{-1}\nabla_T p + (L_{NN_i} - \mathbf{l}_{NT_i}\mathbf{L}_{TT_i}^{-1}\mathbf{l}_{TN_i})(-\nu\rho q_z). \end{aligned} \tag{8}$$

This is the hybrid form of the flow–pressure gradient relation as is (4), but here the hybrid submatrices of the ith permeable medium are expressed in terms of the submatrices of the impermeability matrix. As above, the hybrid submatrices of the equivalent medium are the average of the hybrid submatrices of the constituents.

Then solving for $\nabla_T p$ and $\langle p_{,z}\rangle$ and identifying the coefficients of $\langle \mathbf{q}_T\rangle$ and q_z with the appropriate impermeability submatrices, yields for the impermeability matrix of the equivalent homogeneous medium, in submatrix form,

$$\begin{bmatrix} \mathbf{L}_{TT} & \mathbf{l}_{TN} \\ \mathbf{l}_{NT} & L_{NN} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{L}_{TT}^{-1}\rangle^{-1} & \langle \mathbf{L}_{TT}^{-1}\rangle^{-1}\langle \mathbf{L}_{TT}^{-1}\mathbf{l}_{TN}\rangle \\ \langle \mathbf{l}_{NT}\mathbf{L}_{TT}^{-1}\rangle\langle \mathbf{L}_{TT}^{-1}\rangle^{-1} & \begin{matrix} \langle L_{NN}\rangle - \langle \mathbf{l}_{NT}\mathbf{L}_{TT}^{-1}\mathbf{l}_{TN}\rangle \\ + \langle \mathbf{l}_{NT}\mathbf{L}_{TT}^{-1}\rangle\langle \mathbf{L}_{TT}\rangle^{-1}\langle \mathbf{L}_{TT}^{-1}\mathbf{l}_{TN}\rangle \end{matrix} \end{bmatrix}, \tag{9}$$

the impermeability of the equivalent medium in terms of the impermeabilities of the constituent media. It is just as easy to find the permeability of the equivalent medium in terms of the impermeabilities of the constituent media, and the impermeability of the equivalent medium in terms of the permeabilities of the constituent media.

Since the hybrid submatrices are expressed both in terms of permeability and impermeability, they are a convenient point to derive a matrix inversion in terms of submatrices, which is shown in the Appendix for an m-dimensional linear constitutive relation in terms of p- and q-dimensional submatrices, where $p + q = m$.

2. Model Building by Addition and Subtraction

The Schoenberg–Muir calculus is a formal way of examining the elastic quantities that are unchanged in the replacement of a section of thickness H of one stratified medium by a section of another stratified medium of the same thickness when both stratified media are equivalent in the long wavelength limit. Applied to permeability, two stratified media equivalent means that in the static limit, they both have the same tangential flow, $\sum \mathbf{q}_{\mathrm{T}_i}$, and the same normal pressure drop, $\sum p_{,z_i}$, when q_z and $\nabla_{\mathrm{T}} p$ are the same across both sections. From (4) or (8), it is clear that, in addition to thickness being the same to preserve the geometry, the sums of thickness times hybrid submatrices must be the same. Thus associated with a homogeneous section of thickness H and permeability $\mathbf{K}$ will be an element consisting of a scalar and a matrix

$$\mathbf{G} = \left\{H, H\begin{bmatrix} \mathbf{K}_{\mathrm{TT}} - \mathbf{k}_{\mathrm{TN}} K_{\mathrm{NN}}^{-1} \mathbf{k}_{\mathrm{NT}} & \mathbf{k}_{\mathrm{TN}} K_{\mathrm{NN}}^{-1} \\ -K_{\mathrm{NN}}^{-1} \mathbf{k}_{\mathrm{NT}} & K_{\mathrm{NN}}^{-1} \end{bmatrix}\right\}$$

$$= \left\{H, H\begin{bmatrix} \mathbf{L}_{\mathrm{TT}}^{-1} & -\mathbf{L}_{\mathrm{TT}}^{-1} \mathbf{l}_{\mathrm{TN}} \\ \mathbf{l}_{\mathrm{NT}} \mathbf{L}_{\mathrm{TT}}^{-1} & L_{\mathrm{NN}} - \mathbf{l}_{\mathrm{NT}} \mathbf{L}_{\mathrm{TT}}^{-1} \mathbf{l}_{\mathrm{TN}} \end{bmatrix}\right\}. \tag{10}$$

If that homogeneous section is to be equivalent to a stratified section composed of n constituents, each of cumulative thickness H_i,

$$\mathbf{G} = \sum_{i=1}^{n} \mathbf{G}_i, \tag{11}$$

where $\mathbf{G}_i$ is the element corresponding to the ith constituent. If an element $\mathbf{G}_a$ is subtracted from $\mathbf{G}$, a new element is generated

$$\mathbf{G}_b = \mathbf{G} - \mathbf{G}_a, \tag{12}$$

corresponding to the section $\mathbf{G}$ with the thickness H_a of constituent a removed. The set of all such elements, i.e. the set of all elements consisting of a scalar of dimension *length*, a 3×3 matrix with its upper left (UL) 2×2 submatrix symmetric and of dimension *length*^{3}, its lower right (LR) 1×1 submatrix of dimension *length*$^{-1}$, its upper right (UR) 1×2 submatrix of dimension *length*, and its lower left (LL) 2×1 submatrix equal to the negative transpose of the UR submatrix, is closed under addition, and is a commutative group, called G; and hence, the elements are called group elements. This means nothing more than that the addition is associative and commutative; there exists a zero element $\{0, \mathbf{0}\}$; and every element $\mathbf{G}$ has its inverse $-\mathbf{G}$.

Clearly, dividing the 3×3 matrix by the constant gives the hybrid submatrices from which either the permeability, following the general derivation (A5), or the impermeability, following (A6), is easily found. The components $g(1)$ to $g(5)$ used by Schoenberg and Muir (1989) were the thickness, the mass (thickness times density), and the thickness times the 3×3 LR, 3×3 UR and 3×3 UL hybrid submatrices, respectively, the hybrid submatrices having been derived from the 6×6 elastic modulus matrix.

Summarizing: to combine sections with one another, add their group elements; to remove a given thickness of a constituent from a section, subtract its group element from that of the section. It is proved (see (A4)) that positive definite $\mathbf{K}$ is equivalent to positive definite UL and LR hybrid submatrices so, for a group element to correspond to a positive thickness of a realizable permeable medium, the scalar and the 1×1 LR submatrix must be positive, and the UL submatrix must be positive definite. Thus, combining constituents always yields a realizable medium, as can be seen both from physical grounds and mathematically, as the sum of positive numbers is positive and the sum of positive definite matrices is positive definite. If after removal (subtraction) of a constituent medium, one or more of these conditions is violated, the remaining group element does not correspond to a positive thickness of a realizable medium.

3. Subgroups of G

A subset of G may be defined by a set of distinguishing properties. When such a subset is closed, i.e. a subgroup, the sum of any elements in the subgroup retains the distinguishing properties of the subgroup. These distinguishing properties are reducible to the vanishing of a set of linear combinations of quantities in the group element, i.e. the leading scalar and the matrix components. But what quantities exactly? From dimensional considerations, any linear constraint in the specification of a subgroup must be either:

(a) a linear combination of H and the elements of the UR submatrix vanishing, i.e.

$$c_1 H K_{\mathrm{TN}_1} K_{\mathrm{NN}}^{-1} + c_2 H K_{\mathrm{TN}_2} K_{\mathrm{NN}}^{-1} + c_3 H$$
$$= c_1 H K_{xz}/K_{zz} + c_2 H K_{yz}/K_{zz} + c_3 H = 0;$$

(b) a linear combination of the elements of the UL submatrix vanishing, i.e.

$$d_1 H(K_{\mathrm{TT}_{11}} - K_{\mathrm{TN}_1}^2 K_{\mathrm{NN}}^{-1}) + d_2 H(K_{\mathrm{TT}_{22}} - K_{\mathrm{TN}_2}^2 K_{\mathrm{NN}}^{-1})$$
$$+ d_3 H(K_{\mathrm{TT}_{12}} - K_{\mathrm{TN}_1} K_{\mathrm{TN}_2} K_{\mathrm{NN}}^{-1})$$
$$= d_1 H(K_{xx} - K_{xz}^2/K_{zz}) + d_2 H(K_{yy} - K_{yz}^2/K_{zz}) + d_3 H(K_{xy} - K_{xz} K_{yz}/K_{zz}) = 0;$$

or (c) the 1×1 LR submatrix vanishing, i.e. $HK_{\mathrm{NN}}^{-1} = 0$.

All such constraints can equally well be written in terms of components of $\mathbf{L}$. Four subgroups corresponding to permeable layers with certain symmetry properties are:

(i) The group elements corresponding to constituents that have one of their eigenvectors of the permeability tensor in the z-direction normal to the layering. The permeability of these constituents is symmetric with respect to the plane of the layering. For this to occur, $\mathbf{k}_{\mathrm{TN}}$ must vanish leaving four independent permeability components instead of six for the general permeable medium. The vanishing of $\mathbf{k}_{\mathrm{TN}}$ can be written as two linear homogeneous linear constraints of type (a), $HK_{xz}/K_{zz} = 0$ and $HK_{yz}/K_{zz} = 0$, proving that such constituents form a subgroup, call it $EZ \subset G$ (EZ for eigenvector in z-direction) and if all constituents in a section have one of their eigenvectors normal to the layering, the equivalent medium does too. This is easily seen also from (6), which reduces, for constituents in subgroup EZ, to

$$\begin{bmatrix} \mathbf{K}_{TT} & \mathbf{k}_{TN} \\ \mathbf{k}_{NT} & K_{NN} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{K}_{TT} \rangle & \mathbf{0} \\ \mathbf{0} & \langle K_{NN}^{-1} \rangle^{-1} \end{bmatrix}. \tag{13}$$

Since $\mathbf{k}_{TN} = \mathbf{0}$, there is no coupling between the normal and tangential permeability, and the normal permeability is given, as always, by the harmonic average of the normal permeabilities whereas the tangential permeability components are given by the arithmetic averages of their corresponding components.

(ii) The group elements corresponding to constituents that not only have one of their eigenvectors of the permeability tensor in the z-direction, and thus are already in EZ, but have their xy-plane eigenvectors in fixed directions, with no loss of generality, say the x- and y-directions. Then not only must $\mathbf{k}_{TN}$ vanish but $\mathbf{K}_{TT}$ must be diagonal leaving only three independent permeability components. Constraints of type (b) for elements in EZ reduce to $d_1 HK_{xx} + d_2 HK_{yy} + d_3 HK_{xy} = 0$ and $\mathbf{K}_{TT}$ diagonal can be written as a constraint of type (b), $HK_{xy} = 0$, proving such elements form a subgroup, call it $OR \subset EZ$ (OR for 'orthorhombic'). If each constituent of a section has one of its eigenvectors perpendicular to the layering and the other two parallel to some fixed in-plane directions, then so does the equivalent medium.

(iii) The group elements corresponding to constituents that not only have one of their eigenvectors of the permeability tensor in the z-direction but have equal in-plane (xy-plane) eigenvectors. For this to occur, not only must $\mathbf{k}_{NT}$ vanish and $\mathbf{K}_{TT}$ be diagonal but $\mathbf{K}_{TT}$ must be proportional to $\mathbf{I}_2$, the 2×2 identity matrix, leaving two independent permeability components. This additional condition can be written as a constraint of type (b), $HK_{xx} - HK_{yy} = 0$, proving that such elements form a subgroup, denoted by $TI \subset OR$ (TI for 'transversely isotropic'). The permeability is rotationally symmetric about the z-axis.

(iv) The group elements corresponding to constituents that have an eigenvector in a fixed direction in the plane of the layering. Without loss of generality, let that direction be the y-axis. This requires that $K_{TN_2} = 0$ and $\mathbf{K}_{TT}$ be diagonal leaving four independent permeability components. These can be written as two linear homogeneous linear constraints, one of type (a), $HK_{yz}/K_{zz} = 0$, and one of type (b), $H(K_{xy} - K_{xz}K_{yz}/K_{zz}) = 0$, which, due to the first constraint, becomes $HK_{xy} = 0$, again proving that such constituents form a subgroup, call it $EY \subset G$ (for eigenvector in y-direction).

These relationships between these subgroups connected to the symmetry of the permeability matrix can be written as

$$TI \subset OR \subset \begin{Bmatrix} EZ \\ EY \end{Bmatrix} \subset G, \qquad OR \equiv EY \cap EZ. \tag{14}$$

If a constituent is isotropic, $\mathbf{K}$ is proportional to $\mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 identity matrix. Thus any isotropic constituent belongs to subgroup TI. From (13), noting the difference between arithmetic and harmonic means, it is clearly seen that even if *all* constituents in a section are isotropic, the equivalent medium is *not* isotropic but transversely isotropic. This is of course reinforced by the fact that the isotropy constraint (beyond those required for a constituent to be a member of TI) is that $K_{xx} - K_{zz} = 0$ which is not a constraint of either type (a), (b) or (c), and thus the set of elements corresponding to isotropic constituents does not form a subgroup of TI. It can also be shown that elements corresponding to constituents with a principal direction parallel to the layering but arbitrarily aligned in that plane also do not form a subgroup. The coefficients in the linear constraints specifying a given direction depend on the direction so different layers would satisfy different constraints.

4. Flow Channels and Flow Barriers

Parallel fractures as sets of flow channels

Consider a particular constituent denoted by subscript c, of a layered region of total thickness H, that has cumulative thickness H_c and hence relative thickness $h_c \equiv H_c/H$ and let the permeability become large while the thickness of this constituent, H_c, becomes small, allowing an infinitesimally small thickness of this constituent to flow a finite amount of fluid. The layers of this constituent can be thought of as a set of parallel free-flowing interfaces or open long planar cracks, called here 'channels'. Across each layer, q_z and $\nabla_T p$ are continuous, as across any single layer in the long wavelength limit. It is clear from the first of (4) that at least some components of $\mathbf{q}_{T_c}$ will become infinite but the total flow along the channels, $h_c H\mathbf{q}_{T_c}$, will remain finite. Parallel fractures in a rock mass, a set of open parallel faults, a set of aligned microcracks, or the set of fractures generated by a 'hydrofrac' could all be modelled by a set of such channels.

Instead of letting the permeability of the c medium become large, it is more convenient to let the impermeability, or in particular the TT submatrix of the impermeability, tend to zero as $h_c \to 0$, i.e. $\mathbf{L}_{TT_c}$ must be order h_c as $h_c \to 0$ and the positive definiteness of $\mathbf{L}$ requires that $\mathbf{l}_{TN_c}$ be of order h_c also. Define $\mathbf{L}_{TT_c} \equiv h_c \tilde{\mathbf{L}}_{TT}$ and $\mathbf{l}_{TN_c} \equiv h_c \tilde{\mathbf{l}}_{TN}$. Clearly, $\mathbf{L}_{TT_c}^{-1} = (1/h_c)\tilde{\mathbf{L}}_{TT}^{-1}$. The group element of such channels, i.e. the thickness and the thickness times the matrix of the hybrid submatrices, is, from (10),

$$\mathbf{G}_c = \lim_{h_c \to 0} \left\{ h_c H, \begin{bmatrix} H\tilde{\mathbf{L}}_{TT}^{-1} & -h_c H\tilde{\mathbf{L}}_{TT}^{-1}\tilde{\mathbf{l}}_{TN} \\ h_c H\tilde{\mathbf{l}}_{NT}\tilde{\mathbf{L}}_{TT}^{-1} & h_c H[L_{NN_c} - h_c\tilde{\mathbf{l}}_{NT}\tilde{\mathbf{L}}_{TT}^{-1}\tilde{\mathbf{l}}_{TN}] \end{bmatrix} \right\}$$

$$\equiv \left\{ 0, \begin{bmatrix} H\mathbf{Y} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \right\}, \tag{15}$$

where $\mathbf{Y} \equiv \tilde{\mathbf{L}}_{TT}^{-1}$, a 2×2 symmetric submatrix characterizing the set of channels and which I call the 'excess permeability matrix'. All group elements of the form (15) are a subgroup of G, implying that combining parallel sets of channels yields a set of channels.

To introduce a set of channels with flow behaviour specified by excess permeability matrix $\mathbf{Y}$, into a background rock, with group element $\mathbf{G}_b$, of thickness H and permeability $\mathbf{K}$, add group elements $\mathbf{G}_b$ and $\mathbf{G}_c$, yielding the group element of the model of the cracked rock,

$$\mathbf{G}_b + \mathbf{G}_c = \left\{H, \begin{bmatrix} H(\mathbf{K}_{TT} - \mathbf{k}_{TN} K_{NN}^{-1} \mathbf{k}_{NT} + \mathbf{Y}) & H\mathbf{k}_{TN} K_{NN}^{-1} \\ -HK_{NN}^{-1} \mathbf{k}_{NT} & HK_{NN}^{-1} \end{bmatrix}\right\}. \tag{16}$$

Then, from (A5), given the hybrid matrix, the permeability of the cracked rock is given by

$$\begin{bmatrix} \mathbf{K}_{TT} + \mathbf{Y} & \mathbf{k}_{TN} \\ \mathbf{k}_{NT} & K_{NN} \end{bmatrix}. \tag{17}$$

Only the TT submatrix of the permeability is changed due to the addition of the channels, and the change is independent of the background. The change of the full permeability matrix due to the channels may be written

$$\Delta\mathbf{K} = \mathbf{E}_T^t \mathbf{Y} \mathbf{E}_T, \quad \text{where} \quad \mathbf{E}_T \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{18}$$

Note that the change in impermeability due to the channels is $\Delta\mathbf{L} = (\mathbf{K} + \Delta\mathbf{K})^{-1} - \mathbf{K}^{-1}$ which is strongly dependent on the background permeability and in general has no vanishing submatrices, which is why it is preferable to work with permeability when adding or subtracting channels.

Flow channels in other directions can easily be added by rotating the coordinate system to a primed coordinate system, with the z'-axis normal to the channels. In this coordinate system, the background permeability tensor is given by $\mathbf{AKA}^t$, where $\mathbf{A}$ is the direction cosine matrix of the primed coordinates relative to the original unprimed coordinates. Thus, assuming the channels have excess permeability matrix $\mathbf{Y}'$, the permeability of the rock with channels in the primed coordinates is, according to (18),

$$\mathbf{AKA}^t + \mathbf{E}_T^t \mathbf{Y}' \mathbf{E}_T, \tag{19}$$

and rotating back to the original coordinates gives the permeability as

$$\mathbf{A}^t(\mathbf{AKA}^t + \mathbf{E}_T^t \mathbf{Y}' \mathbf{E}_T)\mathbf{A} = \mathbf{K} + [\mathbf{E}_T \mathbf{A}]^t \mathbf{Y}' [\mathbf{E}_T \mathbf{A}] = \mathbf{K} + \mathbf{A}^t \begin{bmatrix} \mathbf{Y}' & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \mathbf{A}. \tag{20}$$

From (20), it is clear that the change in permeability due to channels at any orientation is independent of the properties of the background. Thus any number of sets of channels with arbitrary non-parallel orientations can be inserted into (by addition) or removed from (by subtraction) any background and in any order without ambiguity. The permeability of a medium with n sets of intersecting channels is given by

$$\mathbf{K} + \sum_{j=1}^{n} [\mathbf{E}_T \mathbf{A}_j]^t \mathbf{Y}_j [\mathbf{E}_T \mathbf{A}_j], \tag{21}$$

where $\mathbf{A}_j$ is the direction cosine matrix of the coordinate system associated with the jth set of channels.

Figure 3 is a representation of two intersecting sets of channels in an otherwise isotropic medium of permeability $K\mathbf{I}_3$. The thick channels are parallel to the

xy-plane and are assumed to have excess permeability matrix $\tilde{\mathbf{L}}_{TT}^{-1} = Y_1 \mathbf{I}_2$. The thin channels are at angle θ to the thick channels and are assumed to have excess permeability matrix $\tilde{\mathbf{L}}_{TT}'^{-1} = Y_2 \mathbf{I}_2$. The permeability of this system, provided flow disruption at the intersections can be neglected, is

$$K\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + Y_1\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

$$+ Y_2\begin{bmatrix}\cos\theta&0\\0&1\\\sin\theta&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\cos\theta&0&\sin\theta\\0&0&0\end{bmatrix}$$

$$= K\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + Y_1\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} + Y_2\begin{bmatrix}\cos^2\theta&0&\cos\theta\sin\theta\\0&1&0\\\cos\theta\sin\theta&0&\sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}K+Y_1+Y_2\cos^2\theta&0&Y_2\cos\theta\sin\theta\\0&K+Y_1+Y_2&0\\Y_2\cos\theta\sin\theta&0&K+Y_2\sin^2\theta\end{bmatrix}. \tag{22}$$

Because of the thin channels, $\mathbf{k}_{TN} \neq 0$ unless $\theta = 0, \pi/2$. The group element of this medium is in subgroup EY. To provide numbers for this example, let the background permeability $K = 0.1$, the thick channel $Y_1 = 0.6$ and the thin channel $Y_2 = 0.3$. Then, referring to (22), the permeability has an eigenvalue equal to 1.0 with a corresponding eigenvector in the y-direction for all θ. The larger eigenvalue in the xz-plane, the ratio of the larger to smaller of the two xz-plane eigenvalues, and the angle between the principal direction corresponding to the larger eigenvalue and the x-axis are shown as a function of θ in Table 1 for this example. The maximum value of the angle for this example is 15° when $\theta = 60°$.

TABLE 1. Flow in xz-plane as a function of θ, corresponding to Fig. 3.

$\theta°$	0	10	20	30	40	50	60	70	80	90
Larger eigenvalue	1.00	0.99	0.98	0.95	0.91	0.86	0.81	0.76	0.72	0.70
Ratio larger/smaller	10.00	9.37	7.87	6.18	4.73	3.61	2.79	2.22	1.87	1.75
Angle ° from x-axis	0.0	3.3	6.5	9.6	12.3	14.2	15.0	13.8	8.9	0.0

Note that the vanishing of the determinant of the excess permeability matrix $\mathbf{Y}$ implies the channels can flow in one direction only, thus simulating the flow behaviour of a set of parallel needle-like cracks, or flow tubes.

Planar flow-impeding interfaces as sets of flow barriers

Consider a particular permeable constituent, denoted by subscript r (for resistance to flow), of a layered region of total thickness H having a cumulative thickness H_r and hence relative thickness $h_r \equiv H_r/H$. Let the permeability become small as

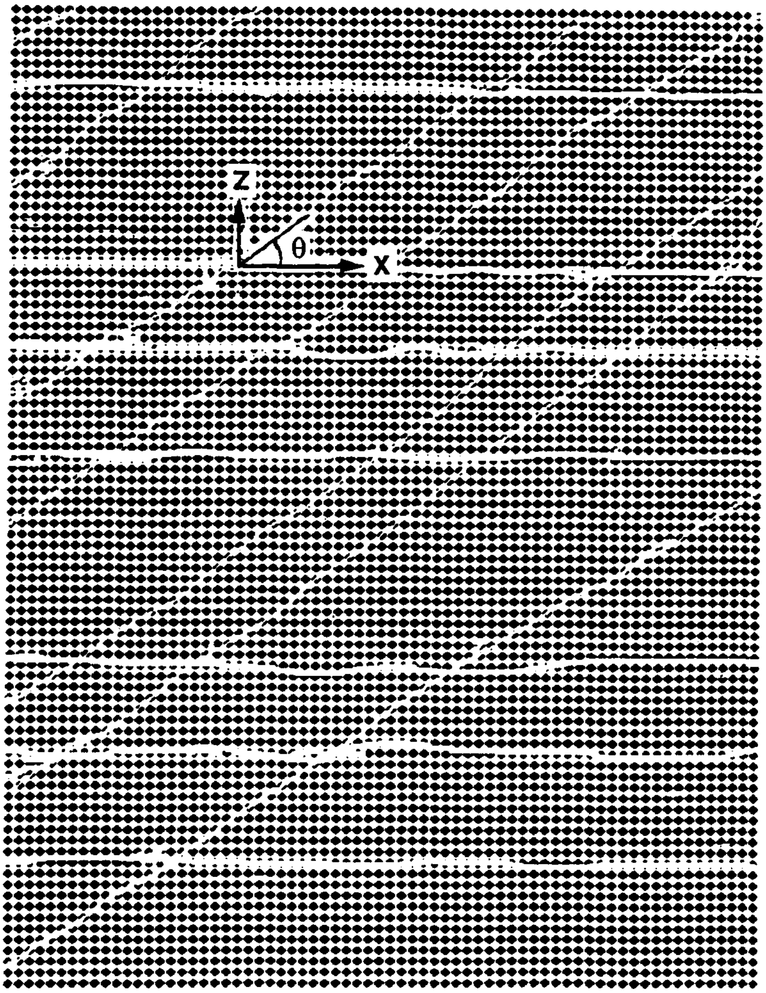

FIG. 3. A representation of a set of thick channels roughly parallel to the xy-plane (y-axis out of the paper), intersected by a thinner set of channels at an angle θ.

the total thickness of this constituent becomes small, so that a smaller and smaller thickness can still impede flow normal to the layering. Such impermeable planar interfaces can be thought of as relatively impermeable membranes or barriers to flow, called here simply 'barriers'. From the second of (4), any flow across a barrier must be accompanied by a pressure jump, i.e. $p_{,z_r}$ approaches infinity, but $h_r H p_{,z_r}$ will remain finite. A set of closed parallel faults across which tangential slip has taken place in an otherwise homogeneous porous medium may act as flow barriers due to pore misalignment or pore clogging that occurred at the time of slip. Thin parallel shale stringers can also be modelled as a set of flow barriers. Barriers occur commonly in tidal flats. Due to tidal deposition, thin sheets of fine clay are embedded in the sand often separated by as little as several centimetres. Such structures can continue to depths of tens of metres.

To model a set of barriers, let the 1×1 submatrix K_{nn} of the permeability tensor of this constituent be of order h_r as $h_r \to 0$. Positive definiteness requires that $\mathbf{k}_{TN_r}$ be also of order h_r. Define $K_{NN_r} \equiv h_r \tilde{K}_{NN}$ and $\mathbf{k}_{TN_r} \equiv h_r \tilde{\mathbf{k}}_{TN}$. The group element of the barriers are, from (10),

$$\mathbf{G}_r = \lim_{h_r \to 0} \left\{ h_r H, \begin{bmatrix} h_r H[\mathbf{K}_{TT_r} - h_r \tilde{\mathbf{k}}_{TN} \tilde{K}_{NN}^{-1} \tilde{\mathbf{k}}_{NT}] & h_r H \tilde{\mathbf{k}}_{TN} \tilde{K}_{NN}^{-1} \\ h_r H \tilde{K}_{NN}^{-1} \tilde{\mathbf{k}}_{NT} & H \tilde{K}_{NN}^{-1} \end{bmatrix} \right\}$$

$$\equiv \left\{ 0, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & HZ \end{bmatrix} \right\}, \quad (23)$$

where $Z \equiv \tilde{K}_{NN}^{-1}$ is a scalar characterizing the set of barriers, and which I call the 'flow impedance'. All group elements of this form are a subgroup of G.

To introduce barriers specified by flow impedance Z into a background rock, denoted by subscript b, of thickness H and impermeability matrix $\mathbf{L}$, add the group elements of the background and of the barriers, yielding the group element of the rock with barriers

$$\mathbf{G}_b + \mathbf{G}_r = \left\{ H, \begin{bmatrix} H\mathbf{L}_{TT}^{-1} & -H\mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} \\ H\mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} & H[L_{NN} - \mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} + Z] \end{bmatrix} \right\}, \quad (24)$$

and using (A6), from the hybrid matrix, $\mathbf{L}$ of the rock with barriers is

$$\begin{bmatrix} \mathbf{L}_{TT} & \mathbf{l}_{TN} \\ \mathbf{l}_{NT} & L_{NN} + Z \end{bmatrix}. \quad (25)$$

Only the NN submatrix of $\mathbf{L}$, a scalar, is changed by the addition of barriers, and the change is independent of the background. The total change in the impermeability due to the barriers may be written in matrix form,

$$\Delta \mathbf{L} = \mathbf{E}_N^t Z \mathbf{E}_N, \quad \text{where} \quad \mathbf{E}_N \equiv [0 \quad 0 \quad 1], \quad (26)$$

analogous to the changes of the permeability tensor due to the channels given in (18). I have used the impermeability here because the permeability change due to the barriers, $\Delta \mathbf{K} = (\mathbf{L} + \Delta \mathbf{L})^{-1} - \mathbf{L}^{-1}$, is again dependent on the properties of the background and in general has no vanishing submatrices.

As above with flow channels, a set of barriers with arbitrary orientation can be added to any background by rotating to a primed coordinate system with its z'-axis normal to the barriers to be added. Let the flow impedance of the barriers be Z'. Then the total flow impedance in the primed system is

$$\mathbf{A}\mathbf{L}\mathbf{A}^t + \mathbf{E}_N^t Z' \mathbf{E}_N \quad (27)$$

and rotating back to the original coordinates gives the flow impedance as

$$\mathbf{A}^t(\mathbf{A}\mathbf{L}\mathbf{A}^t + \mathbf{E}_N^t Z' \mathbf{E}_N)\mathbf{A} = \mathbf{L} + [\mathbf{E}_N \mathbf{A}]^t Z' [\mathbf{E}_N \mathbf{A}] = \mathbf{L} + \mathbf{A}^t \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z' \end{bmatrix} \mathbf{A}. \quad (28)$$

Note that $[\mathbf{E}_N \mathbf{A}]$ depends only on the direction cosines of the z'-axis. As with channels, the change in impermeability due to the barriers is independent of the properties of the background. Thus sets of barriers with arbitrary orientation can also be introduced to any background in any order. The impermeability of a medium with n sets of intersecting barriers is given by

$$\mathbf{L} + \sum_{j=1}^{n} [\mathbf{E}_\mathrm{N}\mathbf{A}_j]^t Z_j [\mathbf{E}_\mathrm{N}\mathbf{A}_j], \tag{29}$$

where $\mathbf{A}_j$ is the direction cosine matrix of the coordinate system associated with the jth set of barriers. As with channels, removal of a set of barriers is carried out by subtraction.

Channels and barriers that are parallel can be added and subtracted to a background rock in any order because they are representable by group elements. However, the medium equivalent to a background with intersecting sets of channels and barriers depends on the order in which they are introduced. To see this, consider only changes in permeability due to inclusion of channels or barriers. The change due to the introduction of channels at an orientation defined by $\mathbf{A}_c$ is, from (20), $\Delta\mathbf{K}_c = [\mathbf{E}_\mathrm{T}\mathbf{A}_c]^t\mathbf{Y}'[\mathbf{E}_\mathrm{T}\mathbf{A}_c]$. The change of permeability due to the introduction of barriers at an orientation defined by $\mathbf{A}_r$ is, from (28), $\Delta\mathbf{K}_r = [\mathbf{L} + [\mathbf{E}_\mathrm{N}\mathbf{A}_r]^t Z'[\mathbf{E}_\mathrm{N}\mathbf{A}_r]]^{-1} - \mathbf{L}^{-1}$, a matrix function, not only of Z' and $\mathbf{A}_r$, but also of the background, call it $\Delta\mathbf{K}_r(\mathbf{K})$. Thus the total change due to the introduction of channels and then barriers is $\Delta\mathbf{K}_c + \Delta\mathbf{K}_r(\mathbf{K} + \Delta\mathbf{K}_c)$, and this is not equal to the total change due to the introduction of barriers and then channels, $\Delta\mathbf{K}_r(\mathbf{K}) + \Delta\mathbf{K}_c$, except when the channels and barriers are parallel. This corresponds to the physical notion that if there are channels in a rock mass and subsequently the rock develops barriers intersecting the channels, the barriers will block the channels, and this is different from the case when there are barriers in a rock mass and subsequently the rock develops channels intersecting the barriers which allow flow through the barriers.

To illustrate the difference between (a) fracturing a rock with barriers, and (b) developing barriers in an already fractured rock, consider the following simple example. Let the background permeability be given by $K\mathbf{I}_3$, let the fractures be vertical in the yz-plane and have an excess permeability matrix $\tilde{\mathbf{L}}'^{-1}_{\mathrm{TT}} = Y\mathbf{I}_2$, and let the barriers be horizontal with flow impedance $\tilde{K}^{-1}_{\mathrm{NN}} = Z$. Then for (a), the background rock with barriers has a diagonal impermeability matrix, $\mathbf{diag}[K^{-1}, K^{-1}, K^{-1} + Z]$ and the addition of the vertical fractures gives the diagonal permeability matrix, $\mathbf{diag}[K, K + Y, K/(1 + KZ) + Y]$. For (b), the background rock with vertical fractures has a diagonal permeability matrix, $\mathbf{diag}[K, K + Y, K + Y]$. The addition of the horizontal barriers gives a diagonal impermeability matrix, $\mathbf{diag}[1/K, 1/(K + Y), 1/(K + Y) + Z]$, and thus a diagonal permeability matrix $\mathbf{diag}[K, K + Y, (K + Y)/[1 + (K + Y)Z]]$. Horizontal permeabilities are the same in the two cases. The permeability in the z-direction is bigger for case (a) (fractured barriers) than for case (b) (blocked fractures) by the ratio

$$\left[1 + \frac{YZ}{1 + Y/K}\right]\left[1 + \frac{YZ}{1 + KZ}\right],$$

showing that, if the background permeability K becomes bery small or very large compared to Y and $1/Z$, the ratio approaches $1 + YZ$. For large Z, fracturing the barriers changes the vertical permeability from almost zero to Y and increases the permeability horizontally along the fractures an amount Y. Once the barriers are fractured, their presence does not have a big effect. For large Y, blocking the fractures with even a small Z (the order of Y^{-1}) changes the vertical permeability considerably.

To illustrate this example numerically, select a background permeability $K = 0.1$, with vertical channel $Y = 5K$ and horizontal barrier $Z = 0.25K^{-1}$. The horizontal permeabilities in the x- and y-directions are 0.1 and 0.6, respectively. The vertical permeability for (a) (fractured barriers) is 0.58 showing that the presence of the fractured barriers has but a small effect because they only serve to decrease the already small contribution of the background permeability to the overall vertical permeability. For (b) (blocked fractures), the vertical permeability is 0.24 and both the barriers and the blocked fractures have a big effect.

Minimal representations

Minimal representations are useful to visualize flow in the quite complex permeable structures discussed above. Since the permeability tensor is a relatively simple mathematical object compared, for example, to an elastic modulus tensor which is fourth rank, there are some simple physical representations for the most general anisotropic permeability. In a coordinate system along the principal directions, the permeability matrix is diagonal and may be written $\mathbf{diag}[K_{xx}, K_{yy}, K_{zz}]$. Assume $K_{zz} \le K_{yy} \le K_{xx}$. Clearly, from (17), this permeability is equivalent to an isotropic background of permeability K_{zz} with channels perpendicular to the z-axis specified by excess permeability matrix $\mathbf{Y} = \mathbf{diag}[K_{xx} - K_{zz}, K_{yy} - K_{zz}]$.

The impermeability matrix is $\mathbf{diag}[K_{xx}^{-1}, K_{yy}^{-1}, K_{zz}^{-1}]$ with $K_{xx}^{-1} \le K_{yy}^{-1} \le K_{zz}^{-1}$. From (25), introducing, into an isotropic medium of impermeability K_{xx}^{-1}, barriers perpendicular to the z-axis with flow impedance $Z_z = K_{zz}^{-1} - K_{xx}^{-1}$ and barriers perpendicular to the y-axis with flow impedance $Z_y = K_{yy}^{-1} - K_{xx}^{-1}$ gives a medium having the desired impermeability matrix.

These are minimal representations for a general anisotropic permeable medium in terms of, first, an isotropic medium with a single set of channels with non-axial symmetry, and second, an isotropic medium with two perpendicular sets of flow barriers (each of which has axial symmetry by definition).

SUMMARY

The overall anisotropic permeability of a layered medium is easily determined knowing the anisotropic permeability of its constituent layers. The insertion or removal of a constituent can be accomplished by simple addition or subtraction of a group element consisting of the cumulative thickness of the constituent, and the matrix constructed from thickness times its hybrid submatrices. The result when many different constituents are added or subtracted is independent of the order in which these operations are carried out. The analysis involved in trying to find a model of a layered permeable reservoir that agrees with data and is in accord with some *a priori* information on the nature of the constituent layers becomes very straightforward.

In the domain of group elements, sets of parallel channels and sets of parallel flow barriers have convenient representations, and this implies that when such elements are parallel they can also be inserted in any order. A set of channels is characterized by its 2×2 excess permeability matrix. A set of barriers is characterized by its flow impedance, a scalar. In addition, because the changes in permeability due to the addition of a set of channels at any orientation is independent of the background permeability, successive sets of non-parallel channels can be introduced, in any order without ambiguity, to any background medium. Similarly, since changes in impermeability due to the addition of barriers at any orientation is independent of the background impermeability, successive sets of non-parallel barriers can also be introduced, in any order without ambiguity, to any background medium.

However, intersecting sets of channels *and* barriers can not be introduced without specifying whether the channels, or the barriers, are to be introduced first. This has been shown algebraically and by a simple example, and corresponds to the physical notion that if there are channels in a rock mass and subsequently the rock develops barriers intersecting the channels, the barriers will block the channels, and this is different from the case when there are barriers in a rock mass and subsequently the rock develops channels intersecting the barriers which allow flow through the barriers.

These concepts have their exact analogue in the case of electrical or heat conductors, or dielectrics. For example, channels would correspond to highly-conductive very thin layers (relative to the background), whereas barriers correspond to insulating thin layers.

Acknowledgements

I thank Robert Burridge, Adrian de Hoop, Tarak Habashy, Stefan Luthi and Francis Muir for valuable discussions (and S.L. for the contribution of Fig. 1), and Phil Christie, Klaus Helbig and Pabitra Sen for their critical reading of the manuscript.

Appendix

Matrix partitioning and hybrid coefficient matrices

Consider a linear constitutive relation of the form $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $m \times m$ positive definite symmetric matrix relating field variables which are components of the vectors $\mathbf{x}$ and $\mathbf{y}$ of length m. Positive definiteness is equivalent to $\mathbf{x}^t\mathbf{y} > 0$ for all non-trivial $\mathbf{x}$, $\mathbf{y}$ satisfying $\mathbf{y} = \mathbf{A}\mathbf{x}$. A partition of the vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix}, \tag{A1}$$

with p the length of the vectors with subscript T and q the length of the vectors with subscript N, $p + q = m$, implies a partitioning of matrix $\mathbf{A}$ into submatrices so that

$$\begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix}. \tag{A2}$$

The $p \times p$ $\mathbf{A}_{TT}$ and $q \times q$ $\mathbf{A}_{NN}$ are themselves symmetric and positive definite, and $\mathbf{A}_{NT} = \mathbf{A}^t_{TN}$. Subscript reversal always denotes a matrix transpose. Note that $\mathbf{B} \equiv \mathbf{A}^{-1}$ is itself symmetric and positive definite.

Solving for $\mathbf{y}_T$, $\mathbf{x}_N$ in terms of $\mathbf{x}_T$, $\mathbf{y}_N$ by solving the second of (A2) for $\mathbf{x}_N$, substituting the result into the first of $\mathbf{x}_N$ and collecting terms, gives

$$\begin{bmatrix} \mathbf{y}_T \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT} & \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1} \\ -\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT} & \mathbf{A}_{NN}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix} \equiv \begin{bmatrix} \mathbf{\Gamma}_{TT} & \mathbf{\Gamma}_{TN} \\ -\mathbf{\Gamma}_{NT} & \mathbf{\Gamma}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix}. \tag{A3}$$

The vectors $[\mathbf{y}_T, \mathbf{x}_N]^t$ and $[\mathbf{x}_T, \mathbf{y}_N]^t$ are hybrid vectors which are linearly related by the hybrid matrix $\mathbf{\Gamma}$. Backus (1990) suggested that the hybrid moduli can themselves be thought of as moduli of the medium, because they can be found from $\mathbf{A}$, and vice versa. Useful as the hybrid moduli will be seen to be in simplifying the derivation of the equivalent medium properties of a stratified medium, they are strange quantities because the dimensions of the various submatrices are different, and because, even if $\mathbf{A}$ is a tensor, the hybrid matrix $\mathbf{\Gamma}$ is not. However, note that

$$\begin{aligned} \mathbf{x}^t\mathbf{y} = \mathbf{x}_T^t\mathbf{y}_T + \mathbf{x}_N^t\mathbf{y}_N &= \mathbf{x}_T^t(\mathbf{\Gamma}_{TT}\mathbf{x}_T + \mathbf{\Gamma}_{TN}\mathbf{y}_N) + (-\mathbf{\Gamma}_{NT}\mathbf{x}_T + \mathbf{\Gamma}_{NN}\mathbf{y}_N)^t\mathbf{y}_N \\ &= \mathbf{x}_T^t\mathbf{\Gamma}_{TT}\mathbf{x}_T + \mathbf{y}_N^t\mathbf{\Gamma}_{NN}\mathbf{y}_N > 0, \end{aligned} \tag{A4}$$

implying that $\mathbf{\Gamma}_{NN}$ and $\mathbf{\Gamma}_{TT} = \mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}$ are positive definite. Similarly, (A2) could have been solved for $\mathbf{y}_T$ and $\mathbf{x}_N$ in terms of $\mathbf{x}_T$ and $\mathbf{y}_N$ and then $\mathbf{x}^t\mathbf{y} > 0$ would imply that $\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}$ is also positive definite. In addition, by inspection of (A3), the submatrices of $\mathbf{A}$, in terms of the submatrices of $\mathbf{\Gamma}$, are

$$\begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_{TT} + \mathbf{\Gamma}_{TN}\mathbf{\Gamma}_{NN}^{-1}\mathbf{\Gamma}_{NT} & \mathbf{\Gamma}_{TN}\mathbf{\Gamma}_{NN}^{-1} \\ \mathbf{\Gamma}_{NN}^{-1}\mathbf{\Gamma}_{NT} & \mathbf{\Gamma}_{NN}^{-1} \end{bmatrix}. \tag{A5}$$

Matrix inversion using submatrices

Solve (A3) for $\mathbf{x}_T$ and $\mathbf{x}_N$ in terms of $\mathbf{y}_T$ and $\mathbf{y}_N$, by solving the first of (A3) for $\mathbf{x}_T$, substituting the result in the second of (A3) and collecting terms, giving

$$\begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_{TT}^{-1} & -\mathbf{\Gamma}_{TT}^{-1}\mathbf{\Gamma}_{TN} \\ -\mathbf{\Gamma}_{NT}\mathbf{\Gamma}_{TT}^{-1} & \mathbf{\Gamma}_{NN} + \mathbf{\Gamma}_{NT}\mathbf{\Gamma}_{TT}^{-1}\mathbf{\Gamma}_{TN} \end{bmatrix} \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix} \equiv \begin{bmatrix} \mathbf{B}_{TT} & \mathbf{B}_{TN} \\ \mathbf{B}_{NT} & \mathbf{B}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix}, \tag{A6}$$

where $\mathbf{B} \equiv \mathbf{A}^{-1}$. By inspection of (A6), the submatrices of $\mathbf{\Gamma}$ can be expressed in terms of the submatrices of $\mathbf{B}$ giving

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{\Gamma}_{TT} & \mathbf{\Gamma}_{TN} \\ -\mathbf{\Gamma}_{NT} & \mathbf{\Gamma}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{TT}^{-1} & -\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN} \\ \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1} & \mathbf{B}_{NN} + \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN} \end{bmatrix}. \tag{A7}$$

It only remains to eliminate the submatrices of $\mathbf{\Gamma}$ from (A6) and (A3) giving the submatrices of $\mathbf{B}$ in terms of those of $\mathbf{A}$, thereby completing the inversion in terms of submatrices. Substitution of (A3) into (A6) yields

$$\begin{aligned}
\mathbf{B}_{TT} &= (\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}, \\
\mathbf{B}_{TN} &= -(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}, \\
\mathbf{B}_{NN} &= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1} \\
&\equiv (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}.
\end{aligned} \tag{A8}$$

This last identity can be proved as follows. Substitution of

$$(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1} \equiv \mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}(\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}, \tag{A9}$$

(which is seen to be an identity by post-multiplying by $\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}$ and premultiplying by $\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}$) into the first expression for $\mathbf{B}_{NN}$ in the third of (A8) gives

$$\begin{aligned}
\mathbf{B}_{NN} &= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}(\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1} \\
&= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}[\mathbf{A}_{NN} - (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})](\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1} \\
&= (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}.
\end{aligned} \tag{A10}$$

Similarly, substitute (A7) into (A5) and use identity (A9) but with $\mathbf{B}$ instead of $\mathbf{A}$ to give the analogous expressions for the submatrices of $\mathbf{A}$ in terms of those of $\mathbf{B}$,

$$\begin{aligned}
\mathbf{A}_{NN} &= (\mathbf{B}_{NN} - \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN})^{-1}, \\
\mathbf{A}_{TN} &= -\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN}(\mathbf{B}_{NN} - \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN})^{-1}, \\
\mathbf{A}_{TT} &= (\mathbf{B}_{TT} - \mathbf{B}_{TN}\mathbf{B}_{NN}^{-1}\mathbf{B}_{NT})^{-1}.
\end{aligned} \tag{A11}$$

Equivalent medium moduli

With these tools in hand, the procedure for finding the moduli of the homogeneous medium equivalent to a stationary finely-layered medium is straightforward. In such a medium, of stationarity thickness ℓ, assume $\mathbf{y}_N$ and $\mathbf{x}_T$ consist of field variables that are 'constant' over a thickness much larger than ℓ, and that $\mathbf{y}_T$ and $\mathbf{x}_N$ consist of variables that change markedly from layer to layer. Further assume that for a homogeneous medium to be equivalent to the finely-layered medium, the integrals of $\mathbf{y}_T$ and $\mathbf{x}_N$ over any depth range larger than ℓ must be the same in the layered medium and in the equivalent homogeneous medium. Then the equivalent medium properties are found by thickness-weighted averaging of the constitutive relations, $\mathbf{y} = \mathbf{A}\mathbf{x}$. However, to do the averaging, those relations must be rearranged so that $\mathbf{y}_T$ and $\mathbf{x}_N$ are isolated on one side of the equal sign. This is because products of changing field variables (the unknowns of the problem) and changing moduli (which are known) cannot be thickness-averaged, while products of constant field variables (also unknowns) and changing moduli can be averaged because the average of a constant times a variable modulus is merely the constant times the average of the modulus. This rearrangement to isolate the changing variables $\mathbf{y}_T$ and $\mathbf{x}_N$ is shown in (A3), and the hybrid moduli $\boldsymbol{\Gamma}$ are given in terms of $\mathbf{A}$ and $\mathbf{B} = \mathbf{A}^{-1}$ in (A3) and (A7), respectively. Averaging (A3) gives

$$\begin{bmatrix} \langle \mathbf{y}_T \rangle \\ \langle \mathbf{x}_N \rangle \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle & \langle \boldsymbol{\Gamma}_{TN} \rangle \\ -\langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{NN} \rangle \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix}, \tag{A12}$$

the hybrid moduli of the equivalent media, which are merely the thickness-weighted averages of the hybrid moduli of the individual constituents present in the finely-layered medium.

Matrices $\mathbf{A}_{eq}$ and/or $\mathbf{B}_{eq}$ are returned by applying (A5) and/or (A6), respectively, to $\langle \boldsymbol{\Gamma} \rangle$. The results of these operations for $\mathbf{A}_{eq}$ are,

$$\begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix}_{eq} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle + \langle \boldsymbol{\Gamma}_{TN} \rangle \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{TN} \rangle \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \\ \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \begin{matrix} \langle \mathbf{A}_{TT} \rangle - \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle \\ + \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \rangle \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \langle \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle \end{matrix} & \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \rangle \langle \mathbf{A}_{NN}^{-1}]^{-1} \\ \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \langle \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle & \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \end{bmatrix}; \tag{A13}$$

for $\mathbf{B}_{eq}$, they are

$$\begin{bmatrix} \mathbf{B}_{TT} & \mathbf{B}_{TN} \\ \mathbf{B}_{NT} & \mathbf{B}_{NN} \end{bmatrix}_{eq} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} & -\langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} \langle \boldsymbol{\Gamma}_{TN} \rangle \\ -\langle \boldsymbol{\Gamma}_{NT} \rangle \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} & \langle \boldsymbol{\Gamma}_{NN} \rangle + \langle \boldsymbol{\Gamma}_{NT} \rangle \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} \langle \boldsymbol{\Gamma}_{TN} \rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} & -\langle \mathbf{B}_{TT}^{-1} \rangle^{-1} \langle \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \\ -\langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \rangle \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} & \begin{matrix} \langle \mathbf{B}_{NN} \rangle - \langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \\ + \langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \rangle \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} \langle \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \end{matrix} \end{bmatrix}. \tag{A14}$$

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Geophysical Prospecting 39, 219–240, 1991

LAYERED PERMEABLE SYSTEMS[1]

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ABSTRACT

SCHOENBERG, M. 1991. Layered permeable systems. *Geophysical Prospecting* 39, 219–240.

Permeability is a second rank tensor relating flow rate to pressure gradient in a porous medium. If the permeability is a constant times the identity tensor the permeable medium is isotropic; otherwise it is anisotropic. A formalism is presented for the simple calculation of the permeability tensor of a heterogeneous layered system composed of interleaved thin layers of several permeable constituent porous media in the static limit. Corresponding to any cumulative thickness H of a constituent is an element consisting of scalar H and a matrix which is H times a hybrid matrix function of permeability. The calculation of the properties of a medium equivalent to the combination of permeable constituents may then be accomplished by simple addition of the corresponding scalar/matrix elements. Subtraction of an element removes a permeable constituent, providing the means to decompose a permeable medium into many possible sets of permeable constituents, all of which have the same flow properties. A set of layers of a constituent medium in the heterogeneous layered system with permeability of the order of $1/h$ as $h \to 0$, where h is that constituent's concentration, acts as a set of infinitely thin channels and is a model for a set of parallel cracks or fractures. Conversely, a set of layers of a given constituent with permeability of the order of h as $h \to 0$ acts as a set of parallel flow barriers and models a set of parallel, relatively impermeable, interfaces, such as shale stringers or some faults. Both sets of channels and sets of barriers are defined explicitly by scalar/matrix elements for which the scalar and three of the four submatrices vanish. Further, non-parallel sets of channels *or* barriers can be 'added' and 'subtracted' from a background homogeneous anisotropic medium commutatively and associatively, but not non-parallel sets of channels *and* barriers reflecting the physical reality that fractures that penetrate barriers will give a different flow behaviour from barriers that block channels. This analysis of layered media, and the representations of the phenomena that can occur as the thickness of a constituent is allowed to approach zero, are applicable directly to layered heat conductors, layered electrostatic conductors and layered dielectrics.

INTRODUCTION

Permeability is a second rank tensor relating the fluid flow rate vector in a porous solid to the macroscopic pressure gradient in the medium. It is a fundamental

[1] Based on a paper read at the 51st EAEG meeting, Berlin, May–June 1989; received September 1989, revision accepted August 1990.

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property of a porous medium, indicating how easily fluids move, for example, through rock in a hydrocarbon reservoir. However, permeability in the earth is almost everywhere anisotropic, sometimes by an order of magnitude or more. Figure 1 shows a piece of Navajo sandstone for which the horizontal permeability (parallel to the thin bands) is greater than 250 times the vertical permeability (perpendicular to the bands). The dark bands are layers with much finer grains and narrower pore throats than the lighter layers. The layers exhibit very little textural variation.

Basically, measurements are often made over distances large with respect to the width of individual layers in a finely-layered region, so the permeability observed is an average of the permeabilities of the individual constituent media (hereafter called constituents). Each layer is one of those constituents, and one must envisage perhaps many layers with only several constituents (see Fig. 2). Typically an alternating sequence of layers (not necessarily periodic) consists of many layers of only two constituents. Generally, each constituent may itself be anisotropic.

This situation was considered by Schoenberg and Muir (1989) with reference to elastic stiffness moduli. They constructed a calculus to deal efficiently with the calculation of stiffness moduli and thus plane wave phase velocities of a medium equivalent to the layered medium in the long wavelength limit. In addition, they showed how the calculus could be used to decompose an equivalent medium into possible constituents, and to handle in a coherent manner certain constituents, such as parallel fractures, that were limiting cases of layers of a given constituent.

The basic ideas of calculating the properties of equivalent media in the static limit are applied here to the less complicated situation of analysing the permeability tensor of a stratified medium under constant or slowly varying pressure gradients.

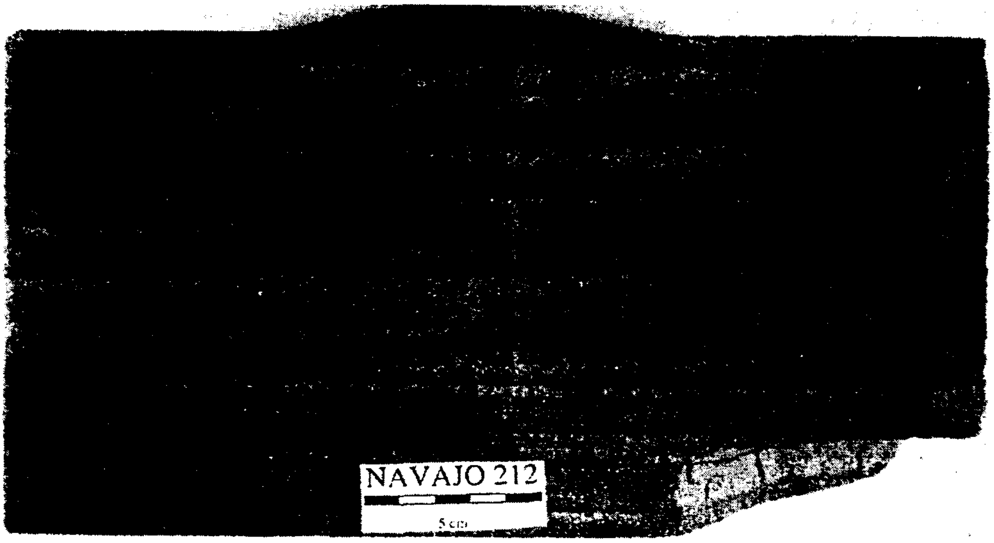

FIG. 1. The Navajo sandstone shown here exhibits highly anisotropic permeability. The permeability anisotropy is thought to be caused by the thin dark bands which consist of much smaller grains. The markings on the specimen denote the location of the cores which were used in the permeability experiments. Photo courtesy of Stefan M. Luthi, Schlumberger-Doll Research.

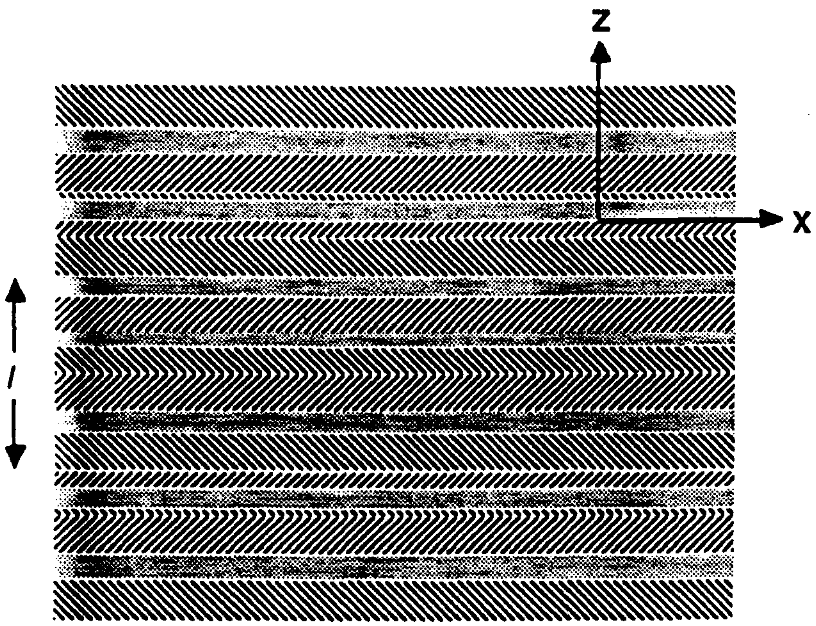

FIG. 2. A stack of permeable layers, in this case consisting of three constituents. Each constituent may be anisotropic. In any interval of thickness ℓ or larger, where ℓ is much smaller than a wavelength, the percentage of each constituent is assumed to be stationary with respect to the vertical coordinate z.

The purpose is to show how the permeability can be analysed in a layered porous reservoir and to expose the relevant parameters needed to specify flow channels and flow barriers. A secondary purpose is to show that this approach to layered media is useful in considering a broad class of linear constitutive relations, and the Appendix contains results for linear relations of arbitrary dimension. The particular example of the constitutive relation of a permeable solid, where a 3D vector field is linearly related to another vector field by a real symmetric second rank tensor is only one example of a class of problems including those of: (1) heat conducting solids where the heat flux vector is related to the temperature gradient by the heat conductivity tensor; (2) electrical conductors in the static limit where the conduction current density is related to the gradient of the potential (which is the electric field) by a real conductivity tensor; and (3) dielectrics in the static limit where the charge displacement vector is related to the electric field vector by the permittivity matrix. In general, heat conductivity, electrical conductivity and electrical permittivity tensors are anisotropic. All the ideas developed with the use of the calculus for permeable layered media have their exact analogue in the areas of heat conductivity and static electrical properties of layered media. In addition, a set of parallel flow channels in a rock mass, which may be modelled as a set of very thin layers of high permeability can perhaps be identified with very thin layers of high conductivity if the fluid flowing in the channels is an electrolyte, and with long parallel fractures or microcracks if they are open enough to change substantially the overall elastic compliance of the medium (Crampin 1984).

A porous medium is obviously extremely inhomogeneous at a level of the grain and pore size. However, for a porous medium homogeneous down to a scale covering many grains, the generalized Darcy's law states that the macroscopic pressure gradient ∇p and $\nu\rho\mathbf{q}$ are linearly related by a second rank permeability tensor $\mathbf{K}$ of dimension *length*^{2}. ν is the kinematic viscosity; ρ is fluid density; $\mathbf{q}$ is the flow rate of dimension *velocity* defined so that $\rho\mathbf{q}$ is the volume integral of the point-wise momentum of the fluid over the pore space in a volume divided by the volume. Thus $\rho\mathbf{q}$ is porosity times the volume average, in the Biot sense, of $\rho\mathbf{v}$ over the pore space, where $\mathbf{v}$ is the point-wise fluid velocity. Therefore the generalized Darcy's law may be written as

$$\mathbf{q} = -\frac{1}{\nu\rho}\mathbf{K}\cdot\nabla p, \tag{1}$$

or in matrix notation,

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = -\frac{1}{\nu\rho}\begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}\begin{bmatrix} p_{,x} \\ p_{,y} \\ p_{,z} \end{bmatrix},$$

with the comma [,] denoting partial differentiation.

In the very long wavelength, low-frequency range (quasi-steady state), the assumptions that the permeability matrix $\mathbf{K}$ be real and that the fluid be incompressible, i.e. $\nabla\cdot\mathbf{q} = 0$, are very good approximations even for gas-saturated media (Biot 1956; Schoenberg and Sen 1987). The condition that $-\nabla p\cdot\mathbf{q}$ be positive (thus assuming there is always some flow given sufficient pressure) implies $\mathbf{K}$ is positive definite. I further assume that reciprocity holds, which is equivalent to $\mathbf{K}$ being symmetric. Under these conditions, there is always a rectangular coordinate system in which $\mathbf{K}$ is diagonal, the diagonal elements being the eigenvalues which are real and positive. In general all three eigenvalues are different. The two more restrictive cases are when two eigenvalues are the same and when all three are the same, the isotropic case.

Note that the pressure gradient ∇p can be expressed in terms of $\mathbf{q}$ using the inverse of the permeability matrix, $\mathbf{L} = \mathbf{K}^{-1}$. $\mathbf{L}$ is the flow resistivity matrix, or the impermeability matrix, or simply the impermeability. As $\mathbf{K}$ has dimension *length*^{2}, $\mathbf{L}$ has dimension *length*$^{-2}$, and it too is symmetric, positive definite with eigenvalues equal to the inverses of those of $\mathbf{K}$. Equation (1) inverted is

$$\nabla p = -\nu\rho\mathbf{L}\cdot\mathbf{q}. \tag{2}$$

Its use greatly simplifies the insertion and removal of flow barriers, while fractures are easier to handle permeability. This is analogous to the fact that the elastic effects of fractures are much easier to analyse using elastic compliance instead of elastic stiffness moduli, a fact that was not appreciated in the original Schoenberg-Muir paper, but that has been used subsequently by Nichols, Muir and Schoenberg (1989) for elastic layers.

In addition to a constitutive relation, interface conditions on the field variables between homogeneous regions must be posited. Perfect contact at an interface $z = 0$

is defined as: (a) pressure p is continuous across $z = 0$ implying that any tangential derivative of p is continuous, or in vector form $\nabla_T p(0^-) = \nabla_T p(0^+)$; and (b) there are no sources or sinks for fluid in the interface between the different media so that the normal component of $\mathbf{q}$ is continuous across the interface, i.e. $q_z(0^-) = q_z(0^+)$. Interface condition (a) holds for temperature in the heat conduction problem and for the electric potential in the electrical conductivity problem, while (b) holds for the heat flux vector and the conduction current density vector. In all cases with constitutive relations and interface conditions of this same form, the analysis below applies.

In Section 1, the properties of a homogeneous medium equivalent to a layered permeable medium are formulated using submatrices of the permeability and impermeability matrices following the approach used in the Appendix of Helbig and Schoenberg (1987) which was for elastic equivalent medium properties. For any set of n constituent media, there exists a homogeneous anisotropic medium that behaves, in the quasi-static limit, exactly as does the finely-layered medium consisting of many layers, each layer being one of the n constituents. This means, in this case, that on a scale much larger than the scale of the layering, the equivalent medium flows exactly as does the layered medium under the same applied pressure gradients. The derivation and the appearance of the formulae for the equivalent medium properties are not dependent on the number of variables in the constitutive relation or the sizes of the submatrices. The approach, applicable to a broad range of problems of arbitrary dimension, is presented in the Appendix, which also includes a discussion of matrix inversion using submatrices and general equivalent media formulae.

In Section 2 the ideas of the Schoenberg–Muir calculus (1989) are applied, developed for elastic layers, to the problem of permeable layers under consideration here. Essentially, one mirrors the physical construction of a section of a given thickness of a layered medium composed of several constituents by associating with each constituent an element consisting of the cumulative thickness of the constituent and that thickness times the hybrid matrix function, its permeability. Then as one constructs the physical model by interleaving thin layers of each of the constituents, mathematically all one does is simply add these elements, giving a new element corresponding to the total thickness of the homogeneous medium equivalent, in the static limit, to the section of layered media just constructed. The order or way in which the constituents are inserted does not affect the result. The advantage of this approach is that removal of an amount of a given constituent is mathematically equivalent to subtraction of the element corresponding to that amount of the constituent, thereby providing the means to decompose a section of a permeable medium into a set of permeable constituents and their thicknesses. As each of the elements is merely a scalar and a matrix with certain specifiable properties, the set of all such elements is a commutative group under addition, called G, formalizing the operations that are allowed, both mathematically and physically.

The constituent properties which always carry over to the equivalent medium properties, i.e. for which properties is the set of all elements corresponding to layers with those properties a subgroup of G, is discussed in Section 3. Special attention is devoted to symmetry properties of the permeability tensor.

In Section 4, parallel cracks or fractures are characterized as infinitesimally thin, but free flowing, channels, while conversely, thin but highly impermeable layers, such as shale stringers or faults at which the pores are misaligned and clogged, are characterized as planar barriers to flow. Both these phenomena have simple explicit representations in the group domain. The insertion or removal of channels or barriers at any orientation becomes a simple arithmetic order-independent operation. Only when there are intersecting sets of channels and barriers, does the order in which they were introduced influence the properties of the equivalent medium. In addition, an arbitrary anisotropic permeable medium is shown to be equivalent to an isotropic background with a single set of flow channels, or, to an isotropic medium with two sets of flow barriers intersecting at right angles. In some sense these are minimal representations for an arbitrary medium.

1. Systems of Anisotropic Permeable Layers

Consider a region of porous, homogeneous (over a scale much larger than pore or grain size), but in general anisotropic layers, composed of n constituents, each with concentration h_i, so that $\sum_{i=1}^{n} h_i = 1$, and permeability $\mathbf{K}_i$, saturated with the same single fluid in all layers. Set Cartesian coordinates so that the z-axis is perpendicular to the layering, and the x- and y-axes lie in the plane of the layering (Fig. 2).

Assumption (1) is that the concentrations h_i of the finely-layered constituents are approximately the same in any interval in z of width ℓ or larger, i.e. the layered medium is stationary down to length scale ℓ, the stationarity length.
Assumption (2), that of slow variation over a length scale $L \gg \ell$, is that all layers of the same constituent encounter the same environment and thus have the same values of the field variables, ∇p and $\mathbf{q}$.

At any boundary between layers, which must be a plane of constant z, q_z (expressing the flow per unit area across the boundary) must be continuous, and the pressure p, and hence the derivatives of p parallel to the layering, must be continuous. Thus q_z, $p_{,x}$ and $p_{,y}$ are constant throughout the region while the components of $\mathbf{q}$ parallel to the layering and the derivative of p perpendicular to the layering depend on the local properties of the layered permeable system. Equation (1) in the ith constituent can be rewritten to separate field variables that are constant over long distances from those that vary with i as

$$\begin{aligned} \mathbf{q}_{\mathrm{T}_i} &= -\frac{1}{\nu\rho}\,[\mathbf{K}_{\mathrm{TT}_i}\nabla_{\mathrm{T}}\,p + \mathbf{k}_{\mathrm{TN}_i}\,p_{,z_i}], \\ q_z &= -\frac{1}{\nu\rho}\,[\mathbf{k}_{\mathrm{NT}_i}\nabla_{\mathrm{T}}\,p + K_{\mathrm{NN}_i}\,p_{,z_i}], \end{aligned} \tag{3}$$

where

$$\mathbf{q}_{\mathrm{T}_i} = \begin{bmatrix} q_{x_i} \\ q_{y_i} \end{bmatrix}, \qquad \nabla_{\mathrm{T}}\,p = \begin{bmatrix} p_{,x} \\ p_{,y} \end{bmatrix},$$

and

$$K_{\mathrm{NN}_i} = K_{zz_i}, \qquad \mathbf{k}_{\mathrm{TN}_i} = \begin{bmatrix} K_{xz_i} \\ K_{yz_i} \end{bmatrix}, \qquad \mathbf{k}_{\mathrm{NT}_i} = \mathbf{k}^{\mathrm{t}}_{\mathrm{TN}_i}, \qquad \mathbf{K}_{\mathrm{TT}_i} = \begin{bmatrix} K_{xx_i} & K_{xy_i} \\ K_{xy_i} & K_{yy_i} \end{bmatrix}.$$

$\mathbf{q}_{\mathrm{T}_i}$ is the velocity tangent to the layering and $\nabla_{\mathrm{T}} p$ is the tangential gradient of the pressure. Superscript t denotes the transpose. The convention used is that simple italics denote a scalar or a 1×1 submatrix, bold face, lower case denotes a vector or a 1×2 or 2×1 submatrix, and boldface capital denotes a matrix or a 2×2 submatrix.

Before these equations can be averaged to find an equivalent permeable medium, one has to solve for the variables that vary from layer to layer. Solving the second of (3) for $p_{,z_i}$ and substituting into the first of (3) gives

$$\begin{aligned} -\nu\rho\mathbf{q}_{\mathrm{T}_i} &= (\mathbf{K}_{\mathrm{TT}_i} - \mathbf{k}_{\mathrm{TN}_i} K^{-1}_{\mathrm{NN}_i} \mathbf{k}_{\mathrm{NT}_i})\nabla_{\mathrm{T}} p + \mathbf{k}_{\mathrm{TN}_i} K^{-1}_{\mathrm{NN}_i}(-\nu\rho q_z), \\ p_{,z_i} &= -K^{-1}_{\mathrm{NN}_i} \mathbf{k}_{\mathrm{NT}_i} \nabla_{\mathrm{T}} p + K^{-1}_{\mathrm{NN}_i}(-\nu\rho q_z). \end{aligned} \tag{4}$$

This is the hybrid form of the flow–pressure gradient relation, and the coefficients of $\nabla_{\mathrm{T}} p$ and $(-\nu\rho q_z)$ on the right-hand side of (4) are the submatrices of the hybrid modulus matrix (for short, the hybrid submatrices) of the ith permeable medium in terms of the submatrices of the permeability matrix.

Due to assumptions (a) and (b), the pressure drop for the equivalent homogeneous medium in the z-direction over any width $H \geq \ell$ but smaller than L must equal the sum over all the constituents of their z derivatives of pressure times their respective cumulative thickness in the width H. Dividing such a sum by H implies that $p_{,z}$ is given by the thickness-weighted average of the $p_{,z_i}$, i.e. $\sum_{i=1}^{n} h_i p_{,z_i} \equiv \langle p_{,z}\rangle$. A similar argument about the horizontal flow through a vertical section of width H implies that $\mathbf{q}_{\mathrm{T}}$ is given by the thickness-weighted average of the $\mathbf{q}_{\mathrm{T}_i}$, i.e. $\sum_{i=1}^{n} h_i \mathbf{q}_{\mathrm{T}_i} \equiv \langle \mathbf{q}_{\mathrm{T}}\rangle$. Thus thickness-weighted averaging of (4) gives, at length scales at least of the order of ℓ,

$$\begin{aligned} -\nu\rho\langle\mathbf{q}_{\mathrm{T}}\rangle &= (\langle\mathbf{K}_{\mathrm{TT}}\rangle - \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle)\nabla_{\mathrm{T}} p + \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle(-\nu\rho q_z), \\ \langle p_{,z}\rangle &= -\langle K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle \nabla_{\mathrm{T}} p + \langle K^{-1}_{\mathrm{NN}}\rangle(-\nu\rho q_z). \end{aligned} \tag{5}$$

This is the hybrid form of the averaged anisotropic flow–pressure gradient relation in the layered medium, which is precisely the flow–pressure gradient relation of the homogeneous medium that is equivalent to the heterogeneous layered medium at length scales of the order of ℓ or larger. The coefficients on the right-hand side of (5) are the hybrid submatrices of the equivalent medium, which are only the thickness-weighted averages of the hybrid submatrices of the constituents. From the hybrid submatrices, the permeability matrix of the equivalent medium is returned by solving (5) for $\langle\mathbf{q}_{\mathrm{T}}\rangle$ and q_z and identifying the coefficients of $\nabla_{\mathrm{T}} p$ and $\langle p_{,z}\rangle$ with the appropriate permeability submatrices, yielding

$$\begin{bmatrix} \mathbf{K}_{\mathrm{TT}} & \mathbf{k}_{\mathrm{TN}} \\ \mathbf{k}_{\mathrm{NT}} & K_{\mathrm{NN}} \end{bmatrix} = \begin{bmatrix} \begin{array}{c} \langle\mathbf{K}_{\mathrm{TT}}\rangle - \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle \\ + \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle\langle K^{-1}_{\mathrm{NN}}\rangle^{-1}\langle K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle \end{array} & \langle\mathbf{k}_{\mathrm{TN}} K^{-1}_{\mathrm{NN}}\rangle\langle K^{-1}_{\mathrm{NN}}\rangle^{-1} \\ \langle K^{-1}_{\mathrm{NN}}\rangle^{-1}\langle K^{-1}_{\mathrm{NN}} \mathbf{k}_{\mathrm{NT}}\rangle & \langle K^{-1}_{\mathrm{NN}}\rangle^{-1} \end{bmatrix}. \tag{6}$$

Equation (6) is the full anisotropic equivalent of the fact that, when layers are stacked together, permeability normal to the layering is the harmonic average of the constituent permeabilities (connection of conductors in series) while permeability tangential to the layering is the arithmetic average (connection of conductors in parallel).

The above analysis using (3) to (6) can be carried out in the same way using the impermeability rather than the permeability. Analogous to (3), **L** can be broken into submatrices for the purpose of separating variables which change from layer to layer from those that are constant over may layers, yielding for the ith constituent,

$$\begin{aligned} \nabla_T p &= -\nu\rho[\mathbf{L}_{TT_i} q_{T_i} + \mathbf{l}_{TN_i} q_z], \\ p_{,z_i} &= -\nu\rho[\mathbf{l}_{NT_i} \mathbf{q}_{T_i} + L_{NN_i} q_z]. \end{aligned} \tag{7}$$

The definitions of the submatrices of **L** are analogous to those of **K** of (3). Now solving (7) for the quantities which vary with i yields

$$\begin{aligned} -\nu\rho\mathbf{q}_{T_i} &= \mathbf{L}_{TT_i}^{-1} \nabla_T p - \mathbf{L}_{TT_i}^{-1} \mathbf{l}_{TN_i}(-\nu\rho q_z), \\ p_{,z_i} &= \mathbf{l}_{NT_i} \mathbf{L}_{TT_i}^{-1} \nabla_T p + (L_{NN_i} - \mathbf{l}_{NT_i} \mathbf{L}_{TT_i}^{-1} \mathbf{l}_{TN_i})(-\nu\rho q_z). \end{aligned} \tag{8}$$

This is the hybrid form of the flow–pressure gradient relation as is (4), but here the hybrid submatrices of the ith permeable medium are expressed in terms of the submatrices of the impermeability matrix. As above, the hybrid submatrices of the equivalent medium are the average of the hybrid submatrices of the constituents.

Then solving for $\nabla_T p$ and $\langle p_{,z} \rangle$ and identifying the coefficients of $\langle \mathbf{q}_T \rangle$ and q_z with the appropriate impermeability submatrices, yields for the impermeability matrix of the equivalent homogeneous medium, in submatrix form,

$$\begin{bmatrix} \mathbf{L}_{TT} & \mathbf{l}_{TN} \\ \mathbf{l}_{NT} & L_{NN} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{L}_{TT}^{-1} \rangle^{-1} & \langle \mathbf{L}_{TT}^{-1} \rangle^{-1} \langle \mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} \rangle \\ \langle \mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} \rangle \langle \mathbf{L}_{TT}^{-1} \rangle^{-1} & \begin{array}{c} \langle L_{NN} \rangle - \langle \mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} \rangle \\ + \langle \mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} \rangle \langle \mathbf{L}_{TT} \rangle^{-1} \langle \mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} \rangle \end{array} \end{bmatrix}, \tag{9}$$

the impermeability of the equivalent medium in terms of the impermeabilities of the constituent media. It is just as easy to find the permeability of the equivalent medium in terms of the impermeabilities of the constituent media, and the impermeability of the equivalent medium in terms of the permeabilities of the constituent media.

Since the hybrid submatrices are expressed both in terms of permeability and impermeability, they are a convenient point to derive a matrix inversion in terms of submatrices, which is shown in the Appendix for an m-dimensional linear constitutive relation in terms of p- and q-dimensional submatrices, where $p + q = m$.

2. Model Building by Addition and Subtraction

The Schoenberg–Muir calculus is a formal way of examining the elastic quantities that are unchanged in the replacement of a section of thickness H of one stratified medium by a section of another stratified medium of the same thickness when both stratified media are equivalent in the long wavelength limit. Applied to permeability, two stratified media equivalent means that in the static limit, they both have the same tangential flow, $\sum \mathbf{q}_{\mathrm{T}_i}$, and the same normal pressure drop, $\sum p_{,z_i}$, when q_z and $\nabla_{\mathrm{T}} p$ are the same across both sections. From (4) or (8), it is clear that, in addition to thickness being the same to preserve the geometry, the sums of thickness times hybrid submatrices must be the same. Thus associated with a homogeneous section of thickness H and permeability $\mathbf{K}$ will be an element consisting of a scalar and a matrix

$$\mathbf{G} = \left\{H, H\begin{bmatrix} \mathbf{K}_{\mathrm{TT}} - \mathbf{k}_{\mathrm{TN}} K_{\mathrm{NN}}^{-1} \mathbf{k}_{\mathrm{NT}} & \mathbf{k}_{\mathrm{TN}} K_{\mathrm{NN}}^{-1} \\ -K_{\mathrm{NN}}^{-1} \mathbf{k}_{\mathrm{NT}} & K_{\mathrm{NN}}^{-1} \end{bmatrix}\right\}$$

$$= \left\{H, H\begin{bmatrix} \mathbf{L}_{\mathrm{TT}}^{-1} & -\mathbf{L}_{\mathrm{TT}}^{-1} \mathbf{l}_{\mathrm{TN}} \\ \mathbf{l}_{\mathrm{NT}} \mathbf{L}_{\mathrm{TT}}^{-1} & L_{\mathrm{NN}} - \mathbf{l}_{\mathrm{NT}} \mathbf{L}_{\mathrm{TT}}^{-1} \mathbf{l}_{\mathrm{TN}} \end{bmatrix}\right\}. \tag{10}$$

If that homogeneous section is to be equivalent to a stratified section composed of n constituents, each of cumulative thickness H_i,

$$\mathbf{G} = \sum_{i=1}^{n} \mathbf{G}_i, \tag{11}$$

where $\mathbf{G}_i$ is the element corresponding to the ith constituent. If an element $\mathbf{G}_a$ is subtracted from $\mathbf{G}$, a new element is generated

$$\mathbf{G}_b = \mathbf{G} - \mathbf{G}_a, \tag{12}$$

corresponding to the section $\mathbf{G}$ with the thickness H_a of constituent a removed. The set of all such elements, i.e. the set of all elements consisting of a scalar of dimension *length*, a 3×3 matrix with its upper left (UL) 2×2 submatrix symmetric and of dimension *length*^{3}, its lower right (LR) 1×1 submatrix of dimension *length*$^{-1}$, its upper right (UR) 1×2 submatrix of dimension *length*, and its lower left (LL) 2×1 submatrix equal to the negative transpose of the UR submatrix, is closed under addition, and is a commutative group, called G; and hence, the elements are called group elements. This means nothing more than that the addition is associative and commutative; there exists a zero element $\{0, \mathbf{0}\}$; and every element $\mathbf{G}$ has its inverse $-\mathbf{G}$.

Clearly, dividing the 3×3 matrix by the constant gives the hybrid submatrices from which either the permeability, following the general derivation (A5), or the impermeability, following (A6), is easily found. The components $g(1)$ to $g(5)$ used by Schoenberg and Muir (1989) were the thickness, the mass (thickness times density), and the thickness times the 3×3 LR, 3×3 UR and 3×3 UL hybrid submatrices, respectively, the hybrid submatrices having been derived from the 6×6 elastic modulus matrix.

Summarizing: to combine sections with one another, add their group elements; to remove a given thickness of a constituent from a section, subtract its group element from that of the section. It is proved (see (A4)) that positive definite $\mathbf{K}$ is equivalent to positive definite UL and LR hybrid submatrices so, for a group element to correspond to a positive thickness of a realizable permeable medium, the scalar and the 1×1 LR submatrix must be positive, and the UL submatrix must be positive definite. Thus, combining constituents always yields a realizable medium, as can be seen both from physical grounds and mathematically, as the sum of positive numbers is positive and the sum of positive definite matrices is positive definite. If after removal (subtraction) of a constituent medium, one or more of these conditions is violated, the remaining group element does not correspond to a positive thickness of a realizable medium.

3. Subgroups of G

A subset of G may be defined by a set of distinguishing properties. When such a subset is closed, i.e. a subgroup, the sum of any elements in the subgroup retains the distinguishing properties of the subgroup. These distinguishing properties are reducible to the vanishing of a set of linear combinations of quantities in the group element, i.e. the leading scalar and the matrix components. But what quantities exactly? From dimensional considerations, any linear constraint in the specification of a subgroup must be either:

(a) a linear combination of H and the elements of the UR submatrix vanishing, i.e.

$$c_1 H K_{\mathrm{TN}_1} K_{\mathrm{NN}}^{-1} + c_2 H K_{\mathrm{TN}_2} K_{\mathrm{NN}}^{-1} + c_3 H$$
$$= c_1 H K_{xz}/K_{zz} + c_2 H K_{yz}/K_{zz} + c_3 H = 0;$$

(b) a linear combination of the elements of the UL submatrix vanishing, i.e.

$$d_1 H(K_{\mathrm{TT}_{11}} - K_{\mathrm{TN}_1}^2 K_{\mathrm{NN}}^{-1}) + d_2 H(K_{\mathrm{TT}_{22}} - K_{\mathrm{TN}_2}^2 K_{\mathrm{NN}}^{-1})$$
$$+ d_3 H(K_{\mathrm{TT}_{12}} - K_{\mathrm{TN}_1} K_{\mathrm{TN}_2} K_{\mathrm{NN}}^{-1})$$
$$= d_1 H(K_{xx} - K_{xz}^2/K_{zz}) + d_2 H(K_{yy} - K_{yz}^2/K_{zz}) + d_3 H(K_{xy} - K_{xz} K_{yz}/K_{zz}) = 0;$$

or (c) the 1×1 LR submatrix vanishing, i.e. $H K_{\mathrm{NN}}^{-1} = 0$.

All such constraints can equally well be written in terms of components of $\mathbf{L}$. Four subgroups corresponding to permeable layers with certain symmetry properties are:

(i) The group elements corresponding to constituents that have one of their eigenvectors of the permeability tensor in the z-direction normal to the layering. The permeability of these constituents is symmetric with respect to the plane of the layering. For this to occur, $\mathbf{k}_{\mathrm{TN}}$ must vanish leaving four independent permeability components instead of six for the general permeable medium. The vanishing of $\mathbf{k}_{\mathrm{TN}}$ can be written as two linear homogeneous linear constraints of type (a), $HK_{xz}/K_{zz} = 0$ and $HK_{yz}/K_{zz} = 0$, proving that such constituents form a subgroup, call it $EZ \subset G$ (EZ for eigenvector in z-direction) and if all constituents in a section have one of their eigenvectors normal to the layering, the equivalent medium does too. This is easily seen also from (6), which reduces, for constituents in subgroup EZ, to

$$\begin{bmatrix} \mathbf{K}_{TT} & \mathbf{k}_{TN} \\ \mathbf{k}_{NT} & K_{NN} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{K}_{TT} \rangle & \mathbf{0} \\ \mathbf{0} & \langle K_{NN}^{-1} \rangle^{-1} \end{bmatrix}. \tag{13}$$

Since $\mathbf{k}_{TN} = \mathbf{0}$, there is no coupling between the normal and tangential permeability, and the normal permeability is given, as always, by the harmonic average of the normal permeabilities whereas the tangential permeability components are given by the arithmetic averages of their corresponding components.

(ii) The group elements corresponding to constituents that not only have one of their eigenvectors of the permeability tensor in the z-direction, and thus are already in EZ, but have their xy-plane eigenvectors in fixed directions, with no loss of generality, say the x- and y-directions. Then not only must $\mathbf{k}_{TN}$ vanish but $\mathbf{K}_{TT}$ must be diagonal leaving only three independent permeability components. Constraints of type (b) for elements in EZ reduce to $d_1 HK_{xx} + d_2 HK_{yy} + d_3 HK_{xy} = 0$ and $\mathbf{K}_{TT}$ diagonal can be written as a constraint of type (b), $HK_{xy} = 0$, proving such elements form a subgroup, call it $OR \subset EZ$ (OR for 'orthorhombic'). If each constituent of a section has one of its eigenvectors perpendicular to the layering and the other two parallel to some fixed in-plane directions, then so does the equivalent medium.

(iii) The group elements corresponding to constituents that not only have one of their eigenvectors of the permeability tensor in the z-direction but have equal in-plane (xy-plane) eigenvectors. For this to occur, not only must $\mathbf{k}_{NT}$ vanish and $\mathbf{K}_{TT}$ be diagonal but $\mathbf{K}_{TT}$ must be proportional to $\mathbf{I}_2$, the 2×2 identity matrix, leaving two independent permeability components. This additional condition can be written as a constraint of type (b), $HK_{xx} - HK_{yy} = 0$, proving that such elements form a subgroup, denoted by $TI \subset OR$ (TI for 'transversely isotropic'). The permeability is rotationally symmetric about the z-axis.

(iv) The group elements corresponding to constituents that have an eigenvector in a fixed direction in the plane of the layering. Without loss of generality, let that direction be the y-axis. This requires that $K_{TN_2} = 0$ and $\mathbf{K}_{TT}$ be diagonal leaving four independent permeability components. These can be written as two linear homogeneous linear constraints, one of type (a), $HK_{yz}/K_{zz} = 0$, and one of type (b), $H(K_{xy} - K_{xz}K_{yz}/K_{zz}) = 0$, which, due to the first constraint, becomes $HK_{xy} = 0$, again proving that such constituents form a subgroup, call it $EY \subset G$ (for eigenvector in y-direction).

These relationships between these subgroups connected to the symmetry of the permeability matrix can be written as

$$TI \subset OR \subset \begin{Bmatrix} EZ \\ EY \end{Bmatrix} \subset G, \qquad OR \equiv EY \cap EZ. \tag{14}$$

If a constituent is isotropic, $\mathbf{K}$ is proportional to $\mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 identity matrix. Thus any isotropic constituent belongs to subgroup TI. From (13), noting the difference between arithmetic and harmonic means, it is clearly seen that even if *all* constituents in a section are isotropic, the equivalent medium is *not* isotropic but transversely isotropic. This is of course reinforced by the fact that the isotropy constraint (beyond those required for a constituent to be a member of TI) is that $K_{xx} - K_{zz} = 0$ which is not a constraint of either type (a), (b) or (c), and thus the set of elements corresponding to isotropic constituents does not form a subgroup of TI. It can also be shown that elements corresponding to constituents with a principal direction parallel to the layering but arbitrarily aligned in that plane also do not form a subgroup. The coefficients in the linear constraints specifying a given direction depend on the direction so different layers would satisfy different constraints.

4. Flow Channels and Flow Barriers

Parallel fractures as sets of flow channels

Consider a particular constituent denoted by subscript c, of a layered region of total thickness H, that has cumulative thickness H_c and hence relative thickness $h_c \equiv H_c/H$ and let the permeability become large while the thickness of this constituent, H_c, becomes small, allowing an infinitesimally small thickness of this constituent to flow a finite amount of fluid. The layers of this constituent can be thought of as a set of parallel free-flowing interfaces or open long planar cracks, called here 'channels'. Across each layer, q_z and $\nabla_T p$ are continuous, as across any single layer in the long wavelength limit. It is clear from the first of (4) that at least some components of $\mathbf{q}_{T_c}$ will become infinite but the total flow along the channels, $h_c H\mathbf{q}_{T_c}$, will remain finite. Parallel fractures in a rock mass, a set of open parallel faults, a set of aligned microcracks, or the set of fractures generated by a 'hydrofrac' could all be modelled by a set of such channels.

Instead of letting the permeability of the c medium become large, it is more convenient to let the impermeability, or in particular the TT submatrix of the impermeability, tend to zero as $h_c \to 0$, i.e. $\mathbf{L}_{TT_c}$ must be order h_c as $h_c \to 0$ and the positive definiteness of $\mathbf{L}$ requires that $\mathbf{l}_{TN_c}$ be of order h_c also. Define $\mathbf{L}_{TT_c} \equiv h_c\tilde{\mathbf{L}}_{TT}$ and $\mathbf{l}_{TN_c} \equiv h_c\tilde{\mathbf{l}}_{TN}$. Clearly, $\mathbf{L}_{TT_c}^{-1} = (1/h_c)\tilde{\mathbf{L}}_{TT}^{-1}$. The group element of such channels, i.e. the thickness and the thickness times the matrix of the hybrid submatrices, is, from (10),

$$\mathbf{G}_c = \lim_{h_c \to 0}\left\{h_c H, \begin{bmatrix} H\tilde{\mathbf{L}}_{TT}^{-1} & -h_c H\tilde{\mathbf{L}}_{TT}^{-1}\tilde{\mathbf{l}}_{TN} \\ h_c H\tilde{\mathbf{l}}_{NT}\tilde{\mathbf{L}}_{TT}^{-1} & h_c H[L_{NN_c} - h_c\tilde{\mathbf{l}}_{NT}\tilde{\mathbf{L}}_{TT}^{-1}\tilde{\mathbf{l}}_{TN}]\end{bmatrix}\right\}$$
$$\equiv \left\{0, \begin{bmatrix} H\mathbf{Y} & \mathbf{0} \\ \mathbf{0} & 0\end{bmatrix}\right\}, \tag{15}$$

where $\mathbf{Y} \equiv \tilde{\mathbf{L}}_{TT}^{-1}$, a 2×2 symmetric submatrix characterizing the set of channels and which I call the 'excess permeability matrix'. All group elements of the form (15) are a subgroup of G, implying that combining parallel sets of channels yields a set of channels.

To introduce a set of channels with flow behaviour specified by excess permeability matrix $\mathbf{Y}$, into a background rock, with group element $\mathbf{G}_b$, of thickness H and permeability $\mathbf{K}$, add group elements $\mathbf{G}_b$ and $\mathbf{G}_c$, yielding the group element of the model of the cracked rock,

$$\mathbf{G}_b + \mathbf{G}_c = \left\{H, \begin{bmatrix} H(\mathbf{K}_{TT} - \mathbf{k}_{TN} K_{NN}^{-1} \mathbf{k}_{NT} + \mathbf{Y}) & H\mathbf{k}_{TN} K_{NN}^{-1} \\ -HK_{NN}^{-1}\mathbf{k}_{NT} & HK_{NN}^{-1} \end{bmatrix}\right\}. \tag{16}$$

Then, from (A5), given the hybrid matrix, the permeability of the cracked rock is given by

$$\begin{bmatrix} \mathbf{K}_{TT} + \mathbf{Y} & \mathbf{k}_{TN} \\ \mathbf{k}_{NT} & K_{NN} \end{bmatrix}. \tag{17}$$

Only the TT submatrix of the permeability is changed due to the addition of the channels, and the change is independent of the background. The change of the full permeability matrix due to the channels may be written

$$\Delta\mathbf{K} = \mathbf{E}_T^t \mathbf{Y} \mathbf{E}_T, \quad \text{where} \quad \mathbf{E}_T \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{18}$$

Note that the change in impermeability due to the channels is $\Delta\mathbf{L} = (\mathbf{K} + \Delta\mathbf{K})^{-1} - \mathbf{K}^{-1}$ which is strongly dependent on the background permeability and in general has no vanishing submatrices, which is why it is preferable to work with permeability when adding or subtracting channels.

Flow channels in other directions can easily be added by rotating the coordinate system to a primed coordinate system, with the z'-axis normal to the channels. In this coordinate system, the background permeability tensor is given by $\mathbf{AKA}^t$, where $\mathbf{A}$ is the direction cosine matrix of the primed coordinates relative to the original unprimed coordinates. Thus, assuming the channels have excess permeability matrix $\mathbf{Y}'$, the permeability of the rock with channels in the primed coordinates is, according to (18),

$$\mathbf{AKA}^t + \mathbf{E}_T^t \mathbf{Y}' \mathbf{E}_T, \tag{19}$$

and rotating back to the original coordinates gives the permeability as

$$\mathbf{A}^t(\mathbf{AKA}^t + \mathbf{E}_T^t \mathbf{Y}'\mathbf{E}_T)\mathbf{A} = \mathbf{K} + [\mathbf{E}_T \mathbf{A}]^t \mathbf{Y}' [\mathbf{E}_T \mathbf{A}] = \mathbf{K} + \mathbf{A}^t \begin{bmatrix} \mathbf{Y}' & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \mathbf{A}. \tag{20}$$

From (20), it is clear that the change in permeability due to channels at any orientation is independent of the properties of the background. Thus any number of sets of channels with arbitrary non-parallel orientations can be inserted into (by addition) or removed from (by subtraction) any background and in any order without ambiguity. The permeability of a medium with n sets of intersecting channels is given by

$$\mathbf{K} + \sum_{j=1}^{n} [\mathbf{E}_T \mathbf{A}_j]^t \mathbf{Y}_j [\mathbf{E}_T \mathbf{A}_j], \tag{21}$$

where $\mathbf{A}_j$ is the direction cosine matrix of the coordinate system associated with the jth set of channels.

Figure 3 is a representation of two intersecting sets of channels in an otherwise isotropic medium of permeability $K\mathbf{I}_3$. The thick channels are parallel to the

xy-plane and are assumed to have excess permeability matrix $\tilde{\mathbf{L}}_{\mathrm{TT}}^{-1} = Y_1 \mathbf{I}_2$. The thin channels are at angle θ to the thick channels and are assumed to have excess permeability matrix $\tilde{\mathbf{L}}_{\mathrm{TT}}'^{-1} = Y_2 \mathbf{I}_2$. The permeability of this system, provided flow disruption at the intersections can be neglected, is

$$K\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + Y_1\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

$$+ Y_2\begin{bmatrix}\cos\theta&0\\0&1\\\sin\theta&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\cos\theta&0&\sin\theta\\0&0&0\end{bmatrix}$$

$$= K\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + Y_1\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} + Y_2\begin{bmatrix}\cos^2\theta&0&\cos\theta\sin\theta\\0&1&0\\\cos\theta\sin\theta&0&\sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}K+Y_1+Y_2\cos^2\theta&0&Y_2\cos\theta\sin\theta\\0&K+Y_1+Y_2&0\\Y_2\cos\theta\sin\theta&0&K+Y_2\sin^2\theta\end{bmatrix}. \tag{22}$$

Because of the thin channels, $\mathbf{k}_{\mathrm{TN}} \neq 0$ unless $\theta = 0, \pi/2$. The group element of this medium is in subgroup EY. To provide numbers for this example, let the background permeability $K = 0.1$, the thick channel $Y_1 = 0.6$ and the thin channel $Y_2 = 0.3$. Then, referring to (22), the permeability has an eigenvalue equal to 1.0 with a corresponding eigenvector in the y-direction for all θ. The larger eigenvalue in the xz-plane, the ratio of the larger to smaller of the two xz-plane eigenvalues, and the angle between the principal direction corresponding to the larger eigenvalue and the x-axis are shown as a function of θ in Table 1 for this example. The maximum value of the angle for this example is 15° when $\theta = 60°$.

TABLE 1. Flow in xz-plane as a function of θ, corresponding to Fig. 3.

θ°	0	10	20	30	40	50	60	70	80	90
Larger eigenvalue	1.00	0.99	0.98	0.95	0.91	0.86	0.81	0.76	0.72	0.70
Ratio larger/smaller	10.00	9.37	7.87	6.18	4.73	3.61	2.79	2.22	1.87	1.75
Angle ° from x-axis	0.0	3.3	6.5	9.6	12.3	14.2	15.0	13.8	8.9	0.0

Note that the vanishing of the determinant of the excess permeability matrix $\mathbf{Y}$ implies the channels can flow in one direction only, thus simulating the flow behaviour of a set of parallel needle-like cracks, or flow tubes.

Planar flow-impeding interfaces as sets of flow barriers

Consider a particular permeable constituent, denoted by subscript r (for resistance to flow), of a layered region of total thickness H having a cumulative thickness H_r and hence relative thickness $h_r \equiv H_r/H$. Let the permeability become small as

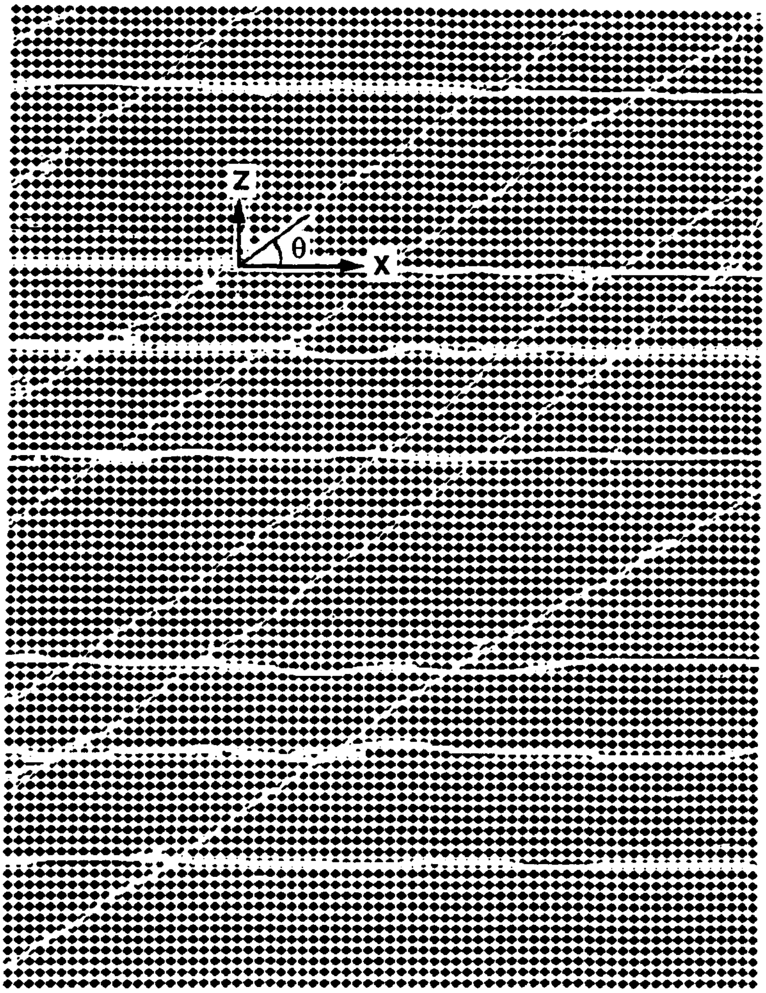

FIG. 3. A representation of a set of thick channels roughly parallel to the xy-plane (y-axis out of the paper), intersected by a thinner set of channels at an angle θ.

the total thickness of this constituent becomes small, so that a smaller and smaller thickness can still impede flow normal to the layering. Such impermeable planar interfaces can be thought of as relatively impermeable membranes or barriers to flow, called here simply 'barriers'. From the second of (4), any flow across a barrier must be accompanied by a pressure jump, i.e. $p_{,z_r}$ approaches infinity, but $h_r H p_{,z_r}$ will remain finite. A set of closed parallel faults across which tangential slip has taken place in an otherwise homogeneous porous medium may act as flow barriers due to pore misalignment or pore clogging that occurred at the time of slip. Thin parallel shale stringers can also be modelled as a set of flow barriers. Barriers occur commonly in tidal flats. Due to tidal deposition, thin sheets of fine clay are embedded in the sand often separated by as little as several centimetres. Such structures can continue to depths of tens of metres.

To model a set of barriers, let the 1×1 submatrix K_{nn} of the permeability tensor of this constituent be of order h_r as $h_r \to 0$. Positive definiteness requires that $\mathbf{k}_{TN_r}$ be also of order h_r. Define $K_{NN_r} \equiv h_r \tilde{K}_{NN}$ and $\mathbf{k}_{TN_r} \equiv h_r \tilde{\mathbf{k}}_{TN}$. The group element of the barriers are, from (10),

$$\mathbf{G}_r = \lim_{h_r \to 0} \left\{ h_r H, \begin{bmatrix} h_r H[\mathbf{K}_{TT_r} - h_r \tilde{\mathbf{k}}_{TN} \tilde{K}_{NN}^{-1} \tilde{\mathbf{k}}_{NT}] & h_r H \tilde{\mathbf{k}}_{TN} \tilde{K}_{NN}^{-1} \\ h_r H \tilde{K}_{NN}^{-1} \tilde{\mathbf{k}}_{NT} & H \tilde{K}_{NN}^{-1} \end{bmatrix} \right\}$$

$$\equiv \left\{ 0, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & HZ \end{bmatrix} \right\}, \quad (23)$$

where $Z \equiv \tilde{K}_{NN}^{-1}$ is a scalar characterizing the set of barriers, and which I call the 'flow impedance'. All group elements of this form are a subgroup of G.

To introduce barriers specified by flow impedance Z into a background rock, denoted by subscript b, of thickness H and impermeability matrix $\mathbf{L}$, add the group elements of the background and of the barriers, yielding the group element of the rock with barriers

$$\mathbf{G}_b + \mathbf{G}_r = \left\{ H, \begin{bmatrix} H\mathbf{L}_{TT}^{-1} & -H\mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} \\ H\mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} & H[L_{NN} - \mathbf{l}_{NT} \mathbf{L}_{TT}^{-1} \mathbf{l}_{TN} + Z] \end{bmatrix} \right\}, \quad (24)$$

and using (A6), from the hybrid matrix, $\mathbf{L}$ of the rock with barriers is

$$\begin{bmatrix} \mathbf{L}_{TT} & \mathbf{l}_{TN} \\ \mathbf{l}_{NT} & L_{NN} + Z \end{bmatrix}. \quad (25)$$

Only the NN submatrix of $\mathbf{L}$, a scalar, is changed by the addition of barriers, and the change is independent of the background. The total change in the impermeability due to the barriers may be written in matrix form,

$$\Delta\mathbf{L} = \mathbf{E}_N^t Z \mathbf{E}_N, \quad \text{where} \quad \mathbf{E}_N \equiv [0 \quad 0 \quad 1], \quad (26)$$

analogous to the changes of the permeability tensor due to the channels given in (18). I have used the impermeability here because the permeability change due to the barriers, $\Delta\mathbf{K} = (\mathbf{L} + \Delta\mathbf{L})^{-1} - \mathbf{L}^{-1}$, is again dependent on the properties of the background and in general has no vanishing submatrices.

As above with flow channels, a set of barriers with arbitrary orientation can be added to any background by rotating to a primed coordinate system with its z'-axis normal to the barriers to be added. Let the flow impedance of the barriers be Z'. Then the total flow impedance in the primed system is

$$\mathbf{ALA}^t + \mathbf{E}_N^t Z' \mathbf{E}_N \quad (27)$$

and rotating back to the original coordinates gives the flow impedance as

$$\mathbf{A}^t(\mathbf{ALA}^t + \mathbf{E}_N^t Z' \mathbf{E}_N)\mathbf{A} = \mathbf{L} + [\mathbf{E}_N \mathbf{A}]^t Z' [\mathbf{E}_N \mathbf{A}] = \mathbf{L} + \mathbf{A}^t \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z' \end{bmatrix} \mathbf{A}. \quad (28)$$

Note that $[\mathbf{E}_N \mathbf{A}]$ depends only on the direction cosines of the z'-axis. As with channels, the change in impermeability due to the barriers is independent of the properties of the background. Thus sets of barriers with arbitrary orientation can also be introduced to any background in any order. The impermeability of a medium with n sets of intersecting barriers is given by

$$\mathbf{L} + \sum_{j=1}^{n} [\mathbf{E}_N \mathbf{A}_j]^t Z_j [\mathbf{E}_N \mathbf{A}_j], \tag{29}$$

where $\mathbf{A}_j$ is the direction cosine matrix of the coordinate system associated with the jth set of barriers. As with channels, removal of a set of barriers is carried out by subtraction.

Channels and barriers that are parallel can be added and subtracted to a background rock in any order because they are representable by group elements. However, the medium equivalent to a background with intersecting sets of channels and barriers depends on the order in which they are introduced. To see this, consider only changes in permeability due to inclusion of channels or barriers. The change due to the introduction of channels at an orientation defined by $\mathbf{A}_c$ is, from (20), $\Delta\mathbf{K}_c = [\mathbf{E}_T \mathbf{A}_c]^t \mathbf{Y}' [\mathbf{E}_T \mathbf{A}_c]$. The change of permeability due to the introduction of barriers at an orientation defined by $\mathbf{A}_r$ is, from (28), $\Delta\mathbf{K}_r = [\mathbf{L} + [\mathbf{E}_N \mathbf{A}_r]^t Z' [\mathbf{E}_N \mathbf{A}_r]]^{-1} - \mathbf{L}^{-1}$, a matrix function, not only of Z' and $\mathbf{A}_r$, but also of the background, call it $\Delta\mathbf{K}_r(\mathbf{K})$. Thus the total change due to the introduction of channels and then barriers is $\Delta\mathbf{K}_c + \Delta\mathbf{K}_r(\mathbf{K} + \Delta\mathbf{K}_c)$, and this is not equal to the total change due to the introduction of barriers and then channels, $\Delta\mathbf{K}_r(\mathbf{K}) + \Delta\mathbf{K}_c$, except when the channels and barriers are parallel. This corresponds to the physical notion that if there are channels in a rock mass and subsequently the rock develops barriers intersecting the channels, the barriers will block the channels, and this is different from the case when there are barriers in a rock mass and subsequently the rock develops channels intersecting the barriers which allow flow through the barriers.

To illustrate the difference between (a) fracturing a rock with barriers, and (b) developing barriers in an already fractured rock, consider the following simple example. Let the background permeability be given by $K\mathbf{I}_3$, let the fractures be vertical in the yz-plane and have an excess permeability matrix $\tilde{\mathbf{L}}'^{-1}_{TT} = Y\mathbf{I}_2$, and let the barriers be horizontal with flow impedance $\tilde{K}^{-1}_{NN} = Z$. Then for (a), the background rock with barriers has a diagonal impermeability matrix, $\mathbf{diag}[K^{-1}, K^{-1}, K^{-1} + Z]$ and the addition of the vertical fractures gives the diagonal permeability matrix, $\mathbf{diag}[K, K + Y, K/(1 + KZ) + Y]$. For (b), the background rock with vertical fractures has a diagonal permeability matrix, $\mathbf{diag}[K, K + Y, K + Y]$. The addition of the horizontal barriers gives a diagonal impermeability matrix, $\mathbf{diag}[1/K, 1/(K + Y), 1/(K + Y) + Z]$, and thus a diagonal permeability matrix $\mathbf{diag}[K, K + Y, (K + Y)/[1 + (K + Y)Z]]$. Horizontal permeabilities are the same in the two cases. The permeability in the z-direction is bigger for case (a) (fractured barriers) than for case (b) (blocked fractures) by the ratio

$$\left[1 + \frac{YZ}{1 + Y/K}\right]\left[1 + \frac{YZ}{1 + KZ}\right],$$

showing that, if the background permeability K becomes bery small or very large compared to Y and $1/Z$, the ratio approaches $1 + YZ$. For large Z, fracturing the barriers changes the vertical permeability from almost zero to Y and increases the permeability horizontally along the fractures an amount Y. Once the barriers are fractured, their presence does not have a big effect. For large Y, blocking the fractures with even a small Z (the order of Y^{-1}) changes the vertical permeability considerably.

To illustrate this example numerically, select a background permeability $K = 0.1$, with vertical channel $Y = 5K$ and horizontal barrier $Z = 0.25K^{-1}$. The horizontal permeabilities in the x- and y-directions are 0.1 and 0.6, respectively. The vertical permeability for (a) (fractured barriers) is 0.58 showing that the presence of the fractured barriers has but a small effect because they only serve to decrease the already small contribution of the background permeability to the overall vertical permeability. For (b) (blocked fractures), the vertical permeability is 0.24 and both the barriers and the blocked fractures have a big effect.

Minimal representations

Minimal representations are useful to visualize flow in the quite complex permeable structures discussed above. Since the permeability tensor is a relatively simple mathematical object compared, for example, to an elastic modulus tensor which is fourth rank, there are some simple physical representations for the most general anisotropic permeability. In a coordinate system along the principal directions, the permeability matrix is diagonal and may be written $\mathbf{diag}[K_{xx}, K_{yy}, K_{zz}]$. Assume $K_{zz} \leq K_{yy} \leq K_{xx}$. Clearly, from (17), this permeability is equivalent to an isotropic background of permeability K_{zz} with channels perpendicular to the z-axis specified by excess permeability matrix $\mathbf{Y} = \mathbf{diag}[K_{xx} - K_{zz}, K_{yy} - K_{zz}]$.

The impermeability matrix is $\mathbf{diag}[K_{xx}^{-1}, K_{yy}^{-1}, K_{zz}^{-1}]$ with $K_{xx}^{-1} \leq K_{yy}^{-1} \leq K_{zz}^{-1}$. From (25), introducing, into an isotropic medium of impermeability K_{xx}^{-1}, barriers perpendicular to the z-axis with flow impedance $Z_z = K_{zz}^{-1} - K_{xx}^{-1}$ and barriers perpendicular to the y-axis with flow impedance $Z_y = K_{yy}^{-1} - K_{xx}^{-1}$ gives a medium having the desired impermeability matrix.

These are minimal representations for a general anisotropic permeable medium in terms of, first, an isotropic medium with a single set of channels with non-axial symmetry, and second, an isotropic medium with two perpendicular sets of flow barriers (each of which has axial symmetry by definition).

SUMMARY

The overall anisotropic permeability of a layered medium is easily determined knowing the anisotropic permeability of its constituent layers. The insertion or removal of a constituent can be accomplished by simple addition or subtraction of a group element consisting of the cumulative thickness of the constituent, and the matrix constructed from thickness times its hybrid submatrices. The result when many different constituents are added or subtracted is independent of the order in which these operations are carried out. The analysis involved in trying to find a model of a layered permeable reservoir that agrees with data and is in accord with

some *a priori* information on the nature of the constituent layers becomes very straightforward.

In the domain of group elements, sets of parallel channels and sets of parallel flow barriers have convenient representations, and this implies that when such elements are parallel they can also be inserted in any order. A set of channels is characterized by its 2×2 excess permeability matrix. A set of barriers is characterized by its flow impedance, a scalar. In addition, because the changes in permeability due to the addition of a set of channels at any orientation is independent of the background permeability, successive sets of non-parallel channels can be introduced, in any order without ambiguity, to any background medium. Similarly, since changes in impermeability due to the addition of barriers at any orientation is independent of the background impermeability, successive sets of non-parallel barriers can also be introduced, in any order without ambiguity, to any background medium.

However, intersecting sets of channels *and* barriers can not be introduced without specifying whether the channels, or the barriers, are to be introduced first. This has been shown algebraically and by a simple example, and corresponds to the physical notion that if there are channels in a rock mass and subsequently the rock develops barriers intersecting the channels, the barriers will block the channels, and this is different from the case when there are barriers in a rock mass and subsequently the rock develops channels intersecting the barriers which allow flow through the barriers.

These concepts have their exact analogue in the case of electrical or heat conductors, or dielectrics. For example, channels would correspond to highly-conductive very thin layers (relative to the background), whereas barriers correspond to insulating thin layers.

ACKNOWLEDGEMENTS

I thank Robert Burridge, Adrian de Hoop, Tarak Habashy, Stefan Luthi and Francis Muir for valuable discussions (and S.L. for the contribution of Fig. 1), and Phil Christie, Klaus Helbig and Pabitra Sen for their critical reading of the manuscript.

APPENDIX

Matrix partitioning and hybrid coefficient matrices

Consider a linear constitutive relation of the form $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $m \times m$ positive definite symmetric matrix relating field variables which are components of the vectors $\mathbf{x}$ and $\mathbf{y}$ of length m. Positive definiteness is equivalent to $\mathbf{x}^t\mathbf{y} > 0$ for all non-trivial $\mathbf{x}$, $\mathbf{y}$ satisfying $\mathbf{y} = \mathbf{A}\mathbf{x}$. A partition of the vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix}, \tag{A1}$$

with p the length of the vectors with subscript T and q the length of the vectors with subscript N, $p + q = m$, implies a partitioning of matrix $\mathbf{A}$ into submatrices so that

$$\begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix}. \tag{A2}$$

The $p \times p$ $\mathbf{A}_{TT}$ and $q \times q$ $\mathbf{A}_{NN}$ are themselves symmetric and positive definite, and $\mathbf{A}_{NT} = \mathbf{A}_{TN}^t$. Subscript reversal always denotes a matrix transpose. Note that $\mathbf{B} \equiv \mathbf{A}^{-1}$ is itself symmetric and positive definite.

Solving for $\mathbf{y}_T$, $\mathbf{x}_N$ in terms of $\mathbf{x}_T$, $\mathbf{y}_N$ by solving the second of (A2) for $\mathbf{x}_N$, substituting the result into the first of $\mathbf{x}_N$ and collecting terms, gives

$$\begin{bmatrix} \mathbf{y}_T \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT} & \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1} \\ -\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT} & \mathbf{A}_{NN}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\Gamma}_{TT} & \boldsymbol{\Gamma}_{TN} \\ -\boldsymbol{\Gamma}_{NT} & \boldsymbol{\Gamma}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix}. \tag{A3}$$

The vectors $[\mathbf{y}_T, \mathbf{x}_N]^t$ and $[\mathbf{x}_T, \mathbf{y}_N]^t$ are hybrid vectors which are linearly related by the hybrid matrix $\boldsymbol{\Gamma}$. Backus (1990) suggested that the hybrid moduli can themselves be thought of as moduli of the medium, because they can be found from $\mathbf{A}$, and vice versa. Useful as the hybrid moduli will be seen to be in simplifying the derivation of the equivalent medium properties of a stratified medium, they are strange quantities because the dimensions of the various submatrices are different, and because, even if $\mathbf{A}$ is a tensor, the hybrid matrix $\boldsymbol{\Gamma}$ is not. However, note that

$$\begin{aligned} \mathbf{x}^t\mathbf{y} = \mathbf{x}_T^t\mathbf{y}_T + \mathbf{x}_N^t\mathbf{y}_N &= \mathbf{x}_T^t(\boldsymbol{\Gamma}_{TT}\mathbf{x}_T + \boldsymbol{\Gamma}_{TN}\mathbf{y}_N) + (-\boldsymbol{\Gamma}_{NT}\mathbf{x}_T + \boldsymbol{\Gamma}_{NN}\mathbf{y}_N)^t\mathbf{y}_N \\ &= \mathbf{x}_T^t\boldsymbol{\Gamma}_{TT}\mathbf{x}_T + \mathbf{y}_N^t\boldsymbol{\Gamma}_{NN}\mathbf{y}_N > 0, \end{aligned} \tag{A4}$$

implying that $\boldsymbol{\Gamma}_{NN}$ and $\boldsymbol{\Gamma}_{TT} = \mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}$ are positive definite. Similarly, (A2) could have been solved for $\mathbf{y}_T$ and $\mathbf{x}_N$ in terms of $\mathbf{x}_T$ and $\mathbf{y}_N$ and then $\mathbf{x}^t\mathbf{y} > 0$ would imply that $\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}$ is also positive definite. In addition, by inspection of (A3), the submatrices of $\mathbf{A}$, in terms of the submatrices of $\boldsymbol{\Gamma}$, are

$$\begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_{TT} + \boldsymbol{\Gamma}_{TN}\boldsymbol{\Gamma}_{NN}^{-1}\boldsymbol{\Gamma}_{NT} & \boldsymbol{\Gamma}_{TN}\boldsymbol{\Gamma}_{NN}^{-1} \\ \boldsymbol{\Gamma}_{NN}^{-1}\boldsymbol{\Gamma}_{NT} & \boldsymbol{\Gamma}_{NN}^{-1} \end{bmatrix}. \tag{A5}$$

Matrix inversion using submatrices

Solve (A3) for $\mathbf{x}_T$ and $\mathbf{x}_N$ in terms of $\mathbf{y}_T$ and $\mathbf{y}_N$, by solving the first of (A3) for $\mathbf{x}_T$, substituting the result in the second of (A3) and collecting terms, giving

$$\begin{bmatrix} \mathbf{x}_T \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_{TT}^{-1} & -\boldsymbol{\Gamma}_{TT}^{-1}\boldsymbol{\Gamma}_{TN} \\ -\boldsymbol{\Gamma}_{NT}\boldsymbol{\Gamma}_{TT}^{-1} & \boldsymbol{\Gamma}_{NN} + \boldsymbol{\Gamma}_{NT}\boldsymbol{\Gamma}_{TT}^{-1}\boldsymbol{\Gamma}_{TN} \end{bmatrix} \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix} \equiv \begin{bmatrix} \mathbf{B}_{TT} & \mathbf{B}_{TN} \\ \mathbf{B}_{NT} & \mathbf{B}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{y}_T \\ \mathbf{y}_N \end{bmatrix}, \tag{A6}$$

where $\mathbf{B} \equiv \mathbf{A}^{-1}$. By inspection of (A6), the submatrices of $\boldsymbol{\Gamma}$ can be expressed in terms of the submatrices of $\mathbf{B}$ giving

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{TT} & \boldsymbol{\Gamma}_{TN} \\ -\boldsymbol{\Gamma}_{NT} & \boldsymbol{\Gamma}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{TT}^{-1} & -\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN} \\ \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1} & \mathbf{B}_{NN} + \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN} \end{bmatrix}. \tag{A7}$$

It only remains to eliminate the submatrices of $\mathbf{\Gamma}$ from (A6) and (A3) giving the submatrices of $\mathbf{B}$ in terms of those of $\mathbf{A}$, thereby completing the inversion in terms of submatrices. Substitution of (A3) into (A6) yields

$$\begin{aligned}
\mathbf{B}_{TT} &= (\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1},\\
\mathbf{B}_{TN} &= -(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1},\\
\mathbf{B}_{NN} &= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\\
&\equiv (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}. \qquad \text{(A8)}
\end{aligned}$$

This last identity can be proved as follows. Substitution of

$$(\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT})^{-1}\mathbf{A}_{TN}\mathbf{A}_{NN}^{-1} \equiv \mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}(\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}, \qquad \text{(A9)}$$

(which is seen to be an identity by post-multiplying by $\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}$ and premultiplying by $\mathbf{A}_{TT} - \mathbf{A}_{TN}\mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}$) into the first expression for $\mathbf{B}_{NN}$ in the third of (A8) gives

$$\begin{aligned}
\mathbf{B}_{NN} &= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}\mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN}(\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}\\
&= \mathbf{A}_{NN}^{-1} + \mathbf{A}_{NN}^{-1}[\mathbf{A}_{NN} - (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})](\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}\\
&= (\mathbf{A}_{NN} - \mathbf{A}_{NT}\mathbf{A}_{TT}^{-1}\mathbf{A}_{TN})^{-1}. \qquad \text{(A10)}
\end{aligned}$$

Similarly, substitute (A7) into (A5) and use identity (A9) but with $\mathbf{B}$ instead of $\mathbf{A}$ to give the analogous expressions for the submatrices of $\mathbf{A}$ in terms of those of $\mathbf{B}$,

$$\begin{aligned}
\mathbf{A}_{NN} &= (\mathbf{B}_{NN} - \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN})^{-1},\\
\mathbf{A}_{TN} &= -\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN}(\mathbf{B}_{NN} - \mathbf{B}_{NT}\mathbf{B}_{TT}^{-1}\mathbf{B}_{TN})^{-1},\\
\mathbf{A}_{TT} &= (\mathbf{B}_{TT} - \mathbf{B}_{TN}\mathbf{B}_{NN}^{-1}\mathbf{B}_{NT})^{-1}. \qquad \text{(A11)}
\end{aligned}$$

Equivalent medium moduli

With these tools in hand, the procedure for finding the moduli of the homogeneous medium equivalent to a stationary finely-layered medium is straightforward. In such a medium, of stationarity thickness ℓ, assume $\mathbf{y}_N$ and $\mathbf{x}_T$ consist of field variables that are 'constant' over a thickness much larger than ℓ, and that $\mathbf{y}_T$ and $\mathbf{x}_N$ consist of variables that change markedly from layer to layer. Further assume that for a homogeneous medium to be equivalent to the finely-layered medium, the integrals of $\mathbf{y}_T$ and $\mathbf{x}_N$ over any depth range larger than ℓ must be the same in the layered medium and in the equivalent homogeneous medium. Then the equivalent medium properties are found by thickness-weighted averaging of the constitutive relations, $\mathbf{y} = \mathbf{A}\mathbf{x}$. However, to do the averaging, those relations must be rearranged so that $\mathbf{y}_T$ and $\mathbf{x}_N$ are isolated on one side of the equal sign. This is because products of changing field variables (the unknowns of the problem) and changing moduli (which are known) cannot be thickness-averaged, while products of constant field variables (also unknowns) and changing moduli can be averaged because the average of a constant times a variable modulus is merely the constant times the average of the modulus. This rearrangement to isolate the changing variables $\mathbf{y}_T$ and $\mathbf{x}_N$ is shown in (A3), and the hybrid moduli $\boldsymbol{\Gamma}$ are given in terms of $\mathbf{A}$ and $\mathbf{B} = \mathbf{A}^{-1}$ in (A3) and (A7), respectively. Averaging (A3) gives

$$\begin{bmatrix} \langle \mathbf{y}_T \rangle \\ \langle \mathbf{x}_N \rangle \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle & \langle \boldsymbol{\Gamma}_{TN} \rangle \\ -\langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{NN} \rangle \end{bmatrix} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{y}_N \end{bmatrix}, \tag{A12}$$

the hybrid moduli of the equivalent media, which are merely the thickness-weighted averages of the hybrid moduli of the individual constituents present in the finely-layered medium.

Matrices $\mathbf{A}_{eq}$ and/or $\mathbf{B}_{eq}$ are returned by applying (A5) and/or (A6), respectively, to $\langle \boldsymbol{\Gamma} \rangle$. The results of these operations for $\mathbf{A}_{eq}$ are,

$$\begin{bmatrix} \mathbf{A}_{TT} & \mathbf{A}_{TN} \\ \mathbf{A}_{NT} & \mathbf{A}_{NN} \end{bmatrix}_{eq} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle + \langle \boldsymbol{\Gamma}_{TN} \rangle \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{TN} \rangle \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \\ \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \langle \boldsymbol{\Gamma}_{NT} \rangle & \langle \boldsymbol{\Gamma}_{NN} \rangle^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \begin{matrix} \langle \mathbf{A}_{TT} \rangle - \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle \\ + \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \rangle \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \langle \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle \end{matrix} & \langle \mathbf{A}_{TN} \mathbf{A}_{NN}^{-1} \rangle \langle \mathbf{A}_{NN}^{-1}]^{-1} \\ \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \langle \mathbf{A}_{NN}^{-1} \mathbf{A}_{NT} \rangle & \langle \mathbf{A}_{NN}^{-1} \rangle^{-1} \end{bmatrix}; \tag{A13}$$

for $\mathbf{B}_{eq}$, they are

$$\begin{bmatrix} \mathbf{B}_{TT} & \mathbf{B}_{TN} \\ \mathbf{B}_{NT} & \mathbf{B}_{NN} \end{bmatrix}_{eq} = \begin{bmatrix} \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} & -\langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} \langle \boldsymbol{\Gamma}_{TN} \rangle \\ -\langle \boldsymbol{\Gamma}_{NT} \rangle \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} & \langle \boldsymbol{\Gamma}_{NN} \rangle + \langle \boldsymbol{\Gamma}_{NT} \rangle \langle \boldsymbol{\Gamma}_{TT} \rangle^{-1} \langle \boldsymbol{\Gamma}_{TN} \rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} & -\langle \mathbf{B}_{TT}^{-1} \rangle^{-1} \langle \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \\ -\langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \rangle \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} & \begin{matrix} \langle \mathbf{B}_{NN} \rangle - \langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \\ + \langle \mathbf{B}_{NT} \mathbf{B}_{TT}^{-1} \rangle \langle \mathbf{B}_{TT}^{-1} \rangle^{-1} \langle \mathbf{B}_{TT}^{-1} \mathbf{B}_{TN} \rangle \end{matrix} \end{bmatrix}. \tag{A14}$$

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