Appendix A Numerical fractional derivatives

The SEIR equations (1) are of the form

$$D^{\nu} f(t) = g[f(t)], \quad \text{with} \quad f(0) = f_0, \tag{A.1}$$

where f and g are functions of time and we omit the spatial variable.

A.1 Euler derivative

The most simple time approximation in fractional calculus is the Euler method,

$$f^{n+1} = f^0 + h^{\nu} \sum_{j=0}^{n} a_{j(n+1)} g(f^j), \tag{A.2}$$

where

$$a_{j(n+1)} = \frac{1}{\Gamma(1+\nu)} [(n-j+1)^{\nu} - (n-j)^{\nu}] \tag{A.3}$$

(e.g., Hassouna et al., 2018).

A.2 Grünwald-Letnikov derivative

A widely used time approximation in fractional calculus is the backward Grünwald-Letnikov (GL) derivative. The GL fractional derivative of a function f is

$$h^{\nu} D^{\nu} \sim \sum_{k=0}^{n+1} c_k f^{n+1-k} = f^{n+1} + \sum_{k=1}^{n+1} c_k f^{n+1-k} \qquad c_k = (-1)^k \binom{\nu}{k} \tag{A.4}$$

where h is the time step and $t = (n+1)h$. The derivation of this expression can be found, for instance, in Carcione et al. (2002). The binomial coefficients can be defined in terms of Euler's Gamma function as

$$\binom{\nu}{k} = \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu-k+1)}$$

and can be calculated by a simple recursion formula

$$\binom{\nu}{k} = \frac{\nu-k+1}{k} \binom{\nu}{k-1}, \qquad \binom{\nu}{0} = 1.$$

If ν is a natural number, we have the classical derivatives. The GL approximation is of order O(h). The fractional derivative of f at time t depends on all the previous values of f. This is the memory property of the fractional derivative. In our calculations we consider the whole memory history since for $\nu < 1$ it is not possible to use the short-memory principle, i.e., less terms in the sum of equation (A.1), as can be used in the simulation of wave propagation (Carcione et al., 2002). Waves "forget" the past but diffusion fields "remember" it.

A.3 CL method

A time discretization of equation (A.1) using the GL derivative is given in Murillo and Yuste (2009), called the CL method (Ciesielski and Leszczynski, 2003). Scherer et al. (2011) [Eq. (4.3)] re-propose this method. It has the form

$$f^{n+1} = -\sum_{k=1}^{n+1} c_k f^{n+1-k} + h^\nu [r_{n+1} f_0 + g(f^n)], \tag{A.5}$$

where

$$r_{n+1} = \frac{(n+1)^{-\nu}}{\Gamma(1-\nu)}. \tag{A.6}$$

We have solve the SEIR equations using this method, but it does not conserve the population at short times, even if $\mu = \alpha = 0$, i.e., it introduces "negative deaths".

A.4 GMMP method

Murillo and Yuste (2009) compare the CL method to the so-called GMMP method (Gorenflo et al., 2002). The algorithm is

$$f^{n+1} = -\sum_{k=1}^{n+1} c_k f^{n+1-k} + f_0 \sum_{k=0}^{n+1} c_k + h^\nu g(f^n), \tag{A.7}$$

This algorithm conserves the population and yields the same results of the most precise ABM method (see next section). The purpose for introducing several algorithms is proper testing of the solution and show that some methods, used in the literature without testing, do not work for the epidemic equations.

A.5 Adams-Bashforth-Moulton scheme

Baleanu et al. (2012) report the predictor-corrector Adams- Bashforth-Moulton scheme (Eqs. 2.3.7, 2.1.7 and 2.1.9) to solve equation (A.1). For $0 < \nu \leq 1$ and one corrector iteration, the method is

$$\begin{aligned} f^{np} &= f_0 + h^\nu \textstyle\sum_{j=0}^{n-1} a_{jn} g(f^j), && \text{predictor,} \\ f^{n} &= f_0 + h^\nu \textstyle\sum_{j=0}^{n-1} b_{jn} g(f^j) + h^\nu b_{nn} g(f^{np}), && \text{corrector,} \end{aligned} \tag{A.8}$$

where a_{jn} is given by equation (A.3), and

$$b_{jn} = \frac{1}{\Gamma(2+\nu)} \begin{cases} (n-1)^{1+\nu} - (n-\nu-1)n^\nu & j = 0, \\ (n-j+1)^{1+\nu} + (n-j-1)^{1+\nu} - 2(n-j)^{1+\nu} & 1 \leq j \leq n-1, \\ 1 & j = n. \end{cases} \tag{A.9}$$

Equation (A.2) is the predictor in the Adams-Bashforth-Moulton scheme. For instance, Abdullah et al. (2017) solve the SEIR model using this methodology.

A.6 Simple examples

A.6.1 Example 1

Let us consider the particular case

$$D^\nu f(t) = \alpha f(t), \quad \text{with} \quad f(0) = f_0, \tag{A.10}$$

whose exact solution is

$$f(t) = f_0 E_{\nu,1}(\alpha t^\nu) = f_0 E_\nu(\alpha t^\nu) \tag{A.11}$$

(Garra and Polito, 2010; Scherer et al., 2011), where E denotes the Mittag-Leffler function.

A.6.2 Example 2

We consider the following differential equation

$$D^{\nu} f(t) = \frac{\Gamma(6)t^{5-\nu}}{\Gamma(6-\nu)} - \frac{3\Gamma(5)t^{4-\nu}}{\Gamma(5-\nu)} + \frac{2\Gamma(4)t^{3-\nu}}{\Gamma(4-\nu)}. \tag{A.12}$$

The exact solution for $0 < \nu < 1$ and $f(0) = 0$ is

$$f(t) = t^5 - 2t^4 + 2t^3. \tag{A.13}$$

Appendix B SEIR semi-analytical solution

We consider the solution obtained by Abdullah et al. (2017), neglecting their metapopulation terms, spatial diffusion and natural births and deaths. Then, the governing differential equations (1) at $t = t_n$ become

$$\begin{aligned}
D^{\nu} S^n &= -\beta^{\nu} S^n \frac{I^{n-1}}{N}, \\
D^{\nu} E^n &= \beta^{\nu} S^n \frac{I^{n-1}}{N} - \epsilon^{\nu} E^n, \\
D^{\nu} I^n &= \epsilon^{\nu} E^n - \gamma^{\nu} I^n, \\
D^{\nu} R^n &= \gamma^{\nu} I^n,
\end{aligned} \tag{B.1}$$

whose solution is

$$\begin{aligned}
S^n &= S(0)[1 - \beta^{\nu} t_n^{\nu} I^{n-1} E_{\nu,\nu+1}(-t_n^{\nu}\beta^{\nu} I^{n-1})], \\
E^n &= \int_0^{t_n} \beta^{\nu} I^{n-1} S^n \tau^{\nu-1} E_{\nu,\nu} d\tau + E(0)[1 - \epsilon^{\nu} t_n^{\nu} E_{\nu,\nu+1}(-\epsilon^{\nu} t_n^{\nu})], \\
I^n &= \int_0^{t_n} \epsilon^{\nu} E^n \tau^{\nu-1} E_{\nu,\nu} d\tau + I(0)[1 - \gamma^{\nu} t_n^{\nu} E_{\nu,\nu+1}(-\gamma^{\nu} t_n^{\nu})], \\
R^n &= R(0) + \int_0^{t_n} \gamma^{\nu} I^n \tau^{\nu-1} E_{\nu,\nu} d\tau.
\end{aligned} \tag{B.2}$$

Equations (B.1) and (B.2) are particular case of equations (26)-(29) and (40)-(43) in Abdullah et al. (2017), respectively.

Fig. 1 SEIR model. The total population, N, is categorized in four classes, namely, susceptible, S, exposed E, infected I and recovered R (Chitnis et al., 2008). Λ and μ correspond to births and natural deaths independent of the disease.