An Inverse Scattering Problem for Seismic While-drilling Technology

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The Forward Model

Let

$$\Omega = \{(r,\theta,z),\ 0 \le r \le R_0,\ 0 \le \theta < 2\pi,\ 0 \le z \le z_0\}$$

be a 3D layered and radially symmetric domain with boundary $\partial\Omega$.

Let $\Gamma^T = \partial\Omega \cap \{z = 0\}, \quad \Gamma = \partial\Omega \setminus \Gamma^T$.

Equation for the propagation of compressional waves in Ω with a point source of time shape $S(t)$ located on the symmetry axis $r = 0$ at depth $z = z_s$:

$$\begin{aligned} \rho \vec{u}_{tt}(A,t) - \nabla\big(A\nabla\cdot\vec{u}(A,t)\big) &= \delta_{r,z-z_s} S(t), \\ (r,\theta,z) \in \Omega, &\quad t \in (0,T) \end{aligned} \tag{1}$$

$\vec{u}(r,z,t) = \vec{u}(A,r,z,t) = (u_r, 0, u_z)$: displacement vector.

$A(r,z)$: compressional bulk modulus, $\quad c = (\frac{A}{\rho})^{\frac{1}{2}}$: wave speed

$\rho(r,z)$: density (assumed to be known).

Initial conditions (rest):

$$(2) \qquad \vec{u}(A, r, z, t = 0) = \vec{u}_t(A, r, z, t = 0) = 0, \quad (r, \theta, z) \in \Omega.$$

Boundary conditions:

$$(3) \qquad -A\nabla \cdot \vec{u} = (\rho A)^{\frac{1}{2}} \vec{u_t} \cdot \vec{\nu}, \quad (r, \theta, z) \in \Gamma, \quad t \in (0, T),$$

$$(4) \qquad -A\nabla \cdot \vec{u} = 0, \qquad (r, \theta, z) \in \Gamma^T, \quad t \in (0, T).$$

$\vec{\nu}$: unit outer normal to $\partial\Omega$

Remark: Equation (3) is an absorbing boundary condition which makes the artificial boundary Γ transparent to outward going waves arriving normally to Γ. Equation (4) represents the free–surface condition on Γ^T.

At time $t = 0$ the medium is excited with an appropriate choice of the source function $S(t)$. The values

$$\vec{u}^{obs}(r, z, t) = \left(u_r^{obs}(r, z, t), u_z^{obs}(r, z, t)\right)$$

are recorded at points $r = r_i, z = z^*, i = 1, \cdots, N_r$ inside Ω for all $t \in (0, T)$.

Assumption: $A(r,z)$ has a constant value on each layer Ω_j

$$\Omega_j = \{r : 0 \le r \le R_0\} \times \{z : z_{j-1} \le z < z_j\} \quad , j = 1, \cdots, N_z.$$

Thus,

$$A(r,z) = \sum_{j=1}^{N_z} A_j \chi_{\Omega_j}(z),$$

$$\chi_{\Omega_j}(z) = \begin{cases} 1 & z \in [z_{j-1}, z_j) \\ 0 & \text{otherwise} \end{cases}$$

The Inverse Problem:
Let $\vec{A} = (A_j)$, $j = 1, \cdots, N_z$. The objective is to infer from the measurements $\vec{u}^{obs}(r,z,t)$ the actual value of the parameter vector $\vec{A}$. Thus we minimize the (regularized) functional

$$\begin{aligned} J^\beta(\vec{A}) = \frac{1}{2} \sum_{i=1}^{N_r} \int_0^T & \Big(\big[u_r(\vec{A}, r_i, z^*, s) - u_r^{obs}(r_i, z^*, s)\big]^2 \\ & + \big[u_z(\vec{A}, r_i, z^*, s) - u_z^{obs}(r_i, z^*, s)\big]^2 \Big)\, ds \\ & + \frac{1}{2}\beta \|\vec{A}\|^2_{L^2(\Omega)} \end{aligned}$$

The Iterative Quasilinearization Procedure

Let $\vec{A}^0$ be an initial guess for $\vec{A}$. Then

$$\vec{A}^{i+1} = \vec{A}^i + \delta\vec{A}^i \tag{5}$$

where $\delta\vec{A}^i$ is the solution of the equations

$$M^\beta(\vec{A}^i)\delta\vec{A}^i = H^\beta(\vec{A}^i). \tag{6}$$

The matrix $M^\beta(\vec{A}^i) \in R^{N_z \times N_z}$ and the vector $H^\beta(\vec{A}^i) \in R^{N_z}$ are defined in terms of the (Frechét) derivatives $\vec{D}_A u$ of $\vec{u}(\vec{A})$ with respect to the parameter $\vec{A}$ as follows:

(7)

$$\left(M^\beta(\vec{A})\right)_{l,j} = \int_0^T \sum_{i=1}^{N_r} \Big([(D_A u)(s)]_r]^t_{l,i}[[(D_A u)(s)]_r]_{i,j} + [(D_A u)(s)]_z]^t_{l,i}[[(D_A u)(s)]_z]_{i,j} \Big) ds + \beta\Lambda, 1 \le l,j \le N_z$$

(8)

$$\left(H^\beta(\vec{A})\right)_j = \int_0^T \Bigg[\sum_{i=1}^{N_r} [(D_A u)(s)]_r]^t_{j,i} \Big(u_r^{obs}(r_i, z^*, s) - u_r(\vec{A}, r_i, z^*, s) \Big) + \sum_{i=1}^{N_r} [(D_A u)(s)]_z]^t_{j,i} \Big(u_z^{obs}(r_i, z^*, s) - u_z(\vec{A}, r_i, z^*, s) \Big) \Bigg] ds, \quad 1 \le j \le N_z.$$

$\beta,\ \Lambda$: regularization parameters.

The matrices $[(D_A u)(s)]_r] \in R^{N_r \times N_z}$ and $[(D_A u)(s)]_z] \in R^{N_r \times N_z}$ in (7) and (8) are related to the derivatives of the solution $\vec{u}$ with respect to the vector parameter $\vec{A}$ and are defined as follows:

$$\Big([(D_A u)(s)]_r\Big)_{ij} = ([D_{A_j} u)(s)]_r(r_i, z^*),$$
$$\Big([(D_A u)(s)]_z\Big)_{ij} = ([D_{A_j} u)(s)]_z(r_i, z^*),$$
$$1 \le i \le N_r, 1 \le j \le N_z$$

Partial differential equations satisfied by the derivatives $\vec{D}_{A_j} u$:

$$\rho(\vec{D}_{A_j} u)_{tt}(t) - \nabla\big(A \nabla \cdot (\vec{D}_{A_j} u)(t)\big) = \nabla(\chi_{\Omega_j} \nabla \cdot \vec{u}(A))$$
$$(r, \theta, z) \in \Omega, \quad t \in (0, T),$$

Initial conditions :

$$\vec{D}_{A_j} u(r, z, t = 0) = (\vec{D}_{A_j} u)_t(r, z, t = 0) = 0, \quad (r, \theta, z) \in \Omega$$

Boundary conditions :

$$-A\nabla\cdot\vec{D}_{A_j}u = (\rho A)^{\frac{1}{2}}\vec{(}D_{A_j}u)_t\cdot\vec{\nu} - \frac{1}{2}(\frac{\rho}{A})^{\frac{1}{2}}(\vec{u}(A))_t\cdot\nu,$$

$$(r,\theta,z)\in\Gamma, \quad t\in(0,T),$$

$$-A\nabla\cdot\vec{D}_{A_j}u = 0, \qquad (r,\theta,z)\in\Gamma^T, \quad t\in(0,T),$$

Variational formulation:

$$V = \widehat{H}(div,\Omega) = \left\{\vec{v} = (v_r,v_\theta,v_z)\in[L^2(\Omega)]^3 : v_\theta = 0, \nabla\cdot\vec{v}\in L^2(\Omega)\right\}$$

$$(\phi,\psi) = \int_\Omega \phi(r,\theta,z)\psi(r,\theta,z)rdrd\theta dz : \quad \text{inner product in} \quad L^2(\Omega)$$

$$\langle\phi,\psi\rangle = \int_\Gamma \phi\psi d\sigma : \quad \text{inner product in} \quad L^2(\Gamma)$$

$d\sigma$: surface measure on Γ.

Forward problem: find $\vec{u} \in V$ such that

$$
\begin{aligned}
&(\rho\vec{u}_{tt}, \vec{v}) + (A\nabla \cdot \vec{u}, \nabla \cdot \vec{v}) + \langle (\rho A)^{\frac{1}{2}} \vec{u}_t \cdot \vec{\nu}, \vec{v} \cdot \nu \rangle_\Gamma \\
&= (\delta_{r,z-z_s} S(t), \vec{v}), \quad \vec{v} \in V, t \in (0,T)
\end{aligned}
$$

Derivative $\vec{D}_{A_j} u = ([D_A u)]_r, [D_A u)]_z)$: find $(\vec{D}_{A_j} u) \in V$ such that

$$
\begin{aligned}
&(\rho(\vec{D}_{A_j} u)_{tt}, \vec{v}) + (A\nabla \cdot (\vec{D}_{A_j} u), \nabla \cdot \vec{v}) + \langle (\rho A)^{\frac{1}{2}} (\vec{D}_{A_j} u)_t \cdot \vec{\nu}, \vec{v} \cdot \nu \rangle_\Gamma \\
&= -(\chi_{\Omega_j} \nabla \cdot \vec{u}, \nabla \cdot \vec{v}) - \langle \frac{1}{2} (\frac{\rho}{A})^{\frac{1}{2}} \chi_{\Omega_j} \vec{u}_t \cdot \vec{\nu}, \vec{v} \cdot \vec{\nu} \rangle_\Gamma, \\
&\qquad\qquad\qquad\qquad\qquad \vec{v} \in V, t \in (0,T)
\end{aligned}
$$

Discrete identification procedure: We employ an explicit finite element procedure to compute the solution of the forward problem and the Frechét derivatives in order to obtain a discrete version of our identification algorithm (5)–(6).

τ^h: quasiregular partition of Ω into elements generated by the rotation around the z-axis of rectangles in the (r, z)-variables of diameter bounded by h.

$V^h \subset V$: finite element space associated with τ^h.

Away from $r = 0$ the elements in V^h are locally of the form

$$(ar^{-1} + br, c + dz).$$

The innermost elements near $r = 0$ have the form

$$(br, c + dz).$$

Also require that the normal components of elements in V^h be continuos across the interelement boundaries.

Natural degrees of freedom for $\vec{v} = (v_r, 0, v_z) \in V^h$:

values of $\vec{v} \cdot \vec{\nu}$ at the mid points of the edges of the rectangles in τ^h.

Time discretization:
L : a positive integer, $\Delta t = T/L$, $f^n = f(n\Delta t)$.

$$\partial U^n = (U^{n+1} - U^{n-1}/2\Delta t,$$

$$\partial^2 U^n = (U^{n+1} - 2U^n - U^{n-1})/(\Delta t)^2.$$

To obtain an explicit finite element procedure, all integrals involving time derivatives are computed using a quadrature rule as follows. For a rectangle Q of side lengths h_r and h_z and mid points $a_i, i = 1, 2, 3, 4$,

$$\int_Q f(r,z) r dr d\theta dz \approx \frac{2\pi}{4} h_r h_z [f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4],$$

where f_i denotes the value of f at the node a_i.

For the boundary integrals we employ the mid-point quadrature rule.

$[f, g]$ and $\langle\langle f, g\rangle\rangle$: inner products (f, g) and $\langle f, g\rangle$ computed approximately using the indicated quadrature rules.

Approximate solutions for the forward problem and the (Frechét) derivatives:
Approximate Forward problem: find $\vec{u}_h \in V^h$ such that

$$[\rho\partial^2\vec{u}_h^n, \vec{v}] + (A\nabla\cdot\vec{u}_h^n, \nabla\cdot\vec{v}) + \langle\langle(\rho A)^{\frac{1}{2}}\partial\vec{u}_h^n\cdot\vec{\nu}, \vec{v}\cdot\nu\rangle\rangle_\Gamma$$

$$= (\delta_{r,z-z_s}S^n, \vec{v}), \quad \vec{v}\in V^h,\ 1\le n\le L-1.$$

Approximate Derivative $(\vec{D}_{A_j}u)_h = ([D_A u)]_{h,r}, [D_A u)]_{h,z})$: find $(\vec{D}_{A_j}u)_h^n \in V^h$ such that

$$[\rho\partial^2(\vec{D}_{A_j}u)_h^n, \vec{v}] + (A\nabla\cdot(\vec{D}_{A_j}u)_h^n, \nabla\cdot\vec{v}) +$$

$$\langle\langle(\rho A)^{\frac{1}{2}}\partial(\vec{D}_{A_j}u)_h^n\cdot\vec{\nu}, \vec{v}\cdot\nu\rangle\rangle_\Gamma$$
$$= -(\chi_{\Omega_j}\nabla\cdot\vec{u}^n, \nabla\cdot\vec{v}) - \langle\langle\frac{1}{2}(\frac{\rho}{A})^{\frac{1}{2}}\chi_{\Omega_j}\partial\vec{u}_h^n\cdot\vec{\nu}, \vec{v}\cdot\vec{\nu}\rangle\rangle_\Gamma,$$

$$\vec{v}\in V^h,\ 1\le n\le L-1.$$

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$$\vec{A}^{i+1} = \vec{A}^i + \delta\vec{A}^i$$

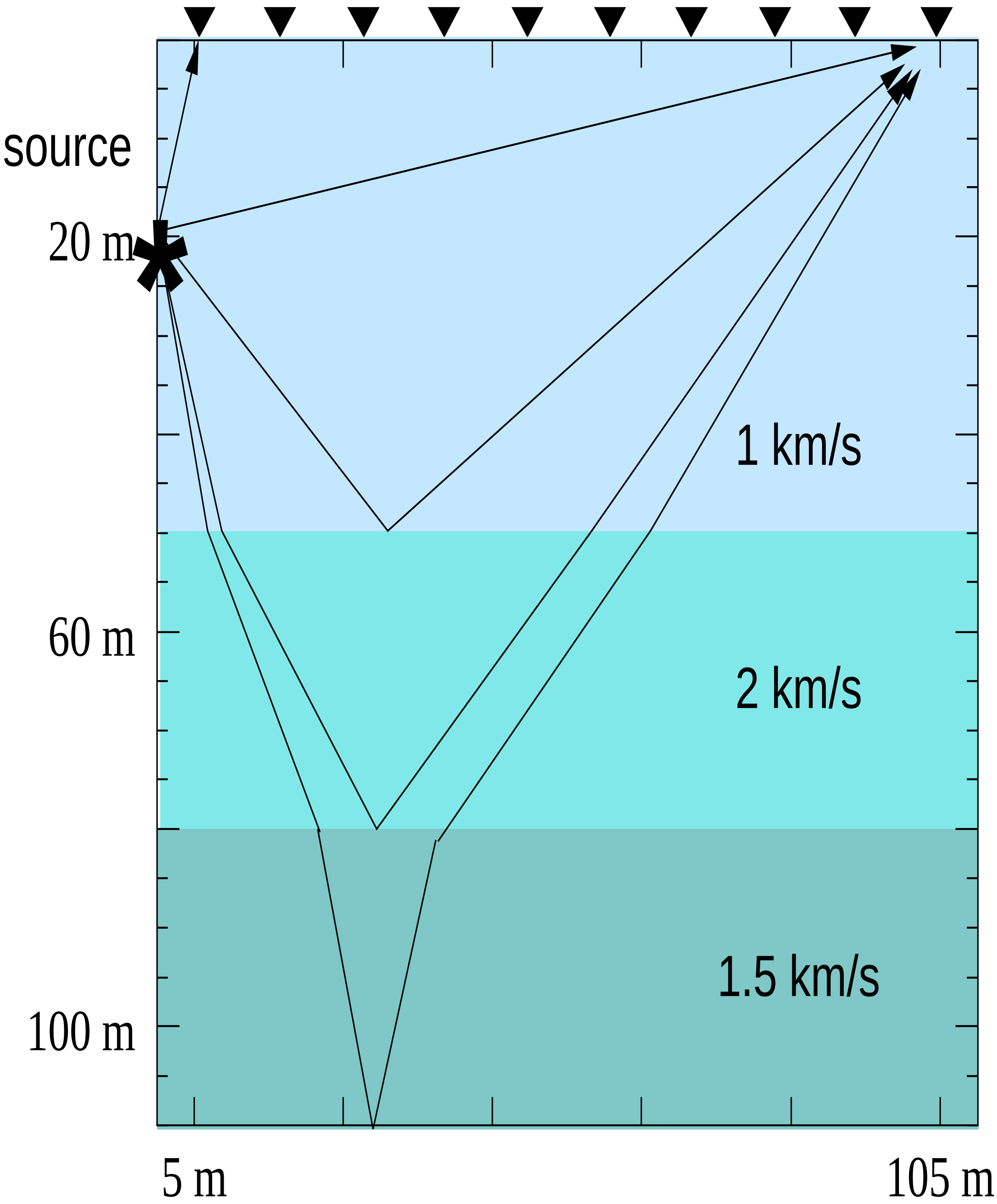
where $\delta\vec{A}^i$ is the solution of the equations

$$M^{\beta,h}(\vec{A}^i)\delta\vec{A}^i = H^{\beta,h}(\vec{A}^i),$$

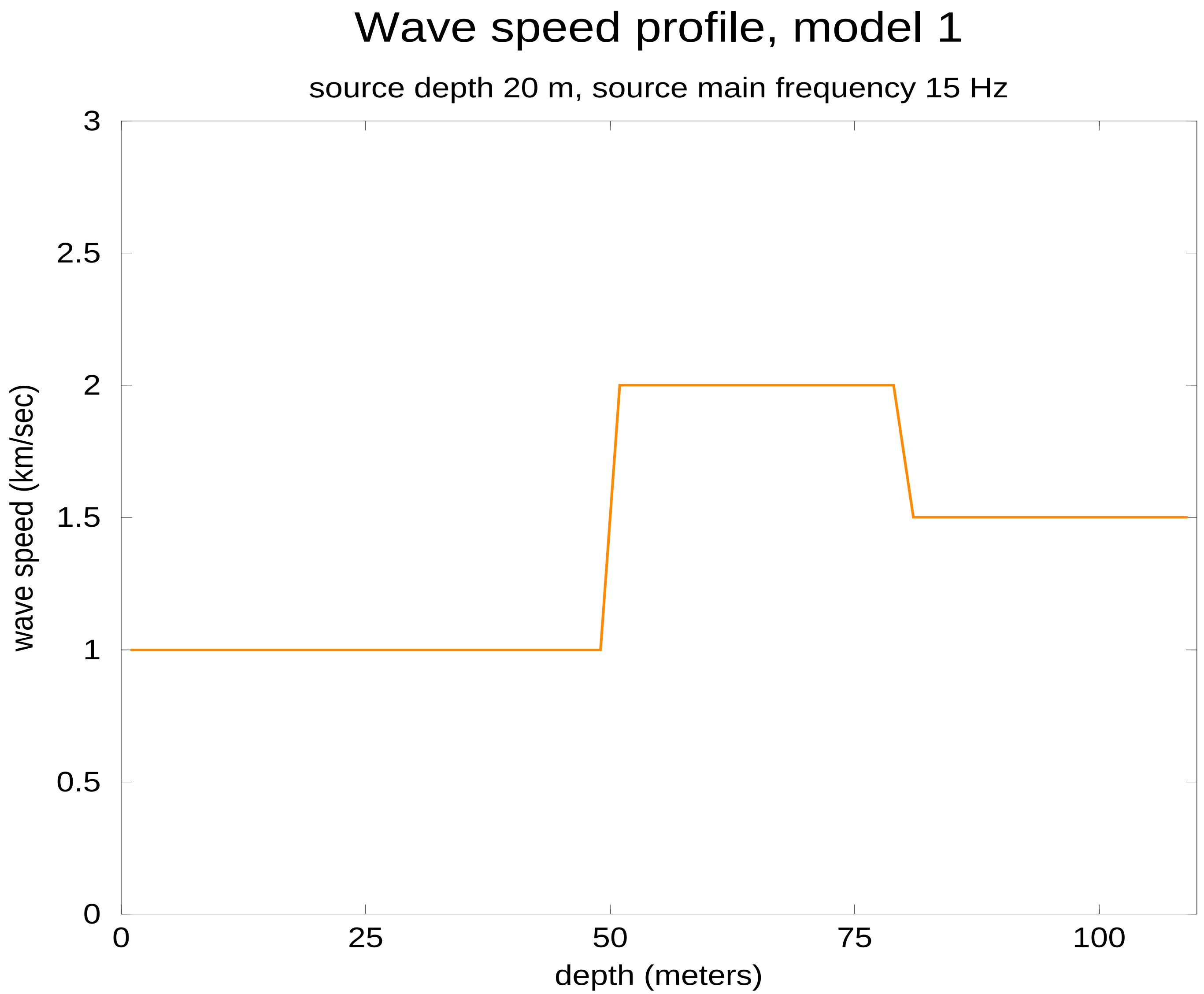
where

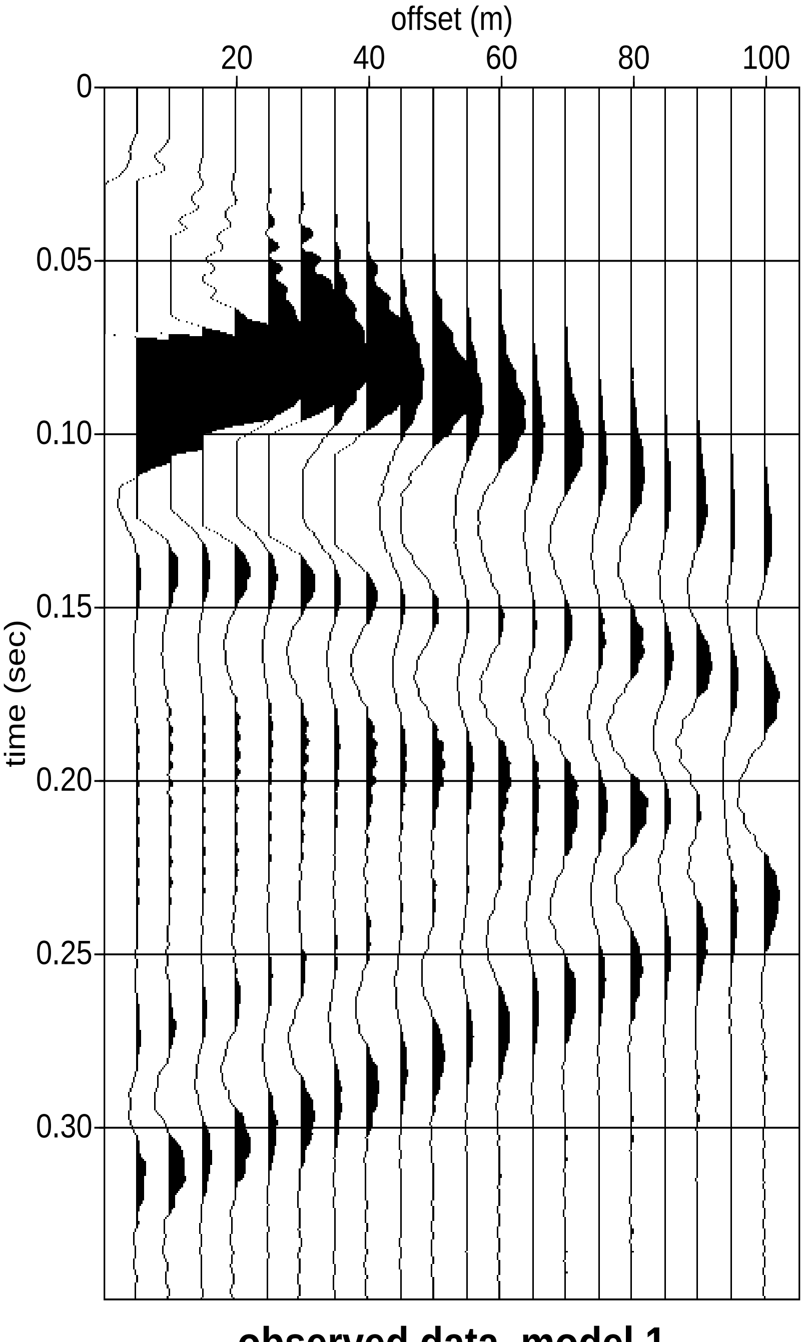
$$\begin{aligned}\left(M^{\beta,h}(\vec{A})\right)_{l,j} =& \sum_{n=1}^{L}\sum_{i}^{N_r}[(D_A u)(s)]^n_{h,r}]^t_{l,i}[[(D_A u)(s)]^n_{h,r}]_{i,j}+ \\ & [(D_A u)(s)]^n_{h,z}]^t_{l,i}[[(D_A u)(s)]^n_{h,z}]_{i,j}\Delta t + \beta\Lambda, \\ & 1 \le l,j \le N_z\end{aligned}$$

$$\begin{aligned}\left(H^{\beta,h}(\vec{A})\right)_{j} =& \sum_{n=1}^{L}\Bigg[\sum_{i}^{N_r}[(D_A u)(s)]_{h,r}]^t_{j,i} \\ & \left(u^{obs,n}(r_i,z^*) - u^n_{h,r}(\vec{A},r_i,z^*)\right)+ \\ & \sum_{i}^{N_r}[(D_A u)(s)]_{h,z}]^t_{j,i} \\ & \left(u_z^{obs,n}(r_i,z^*) - u^n_{h,z}(\vec{A},r_i,z^*)\right)\Bigg]\Delta t,\ 1 \le j \le N_z\end{aligned}$$

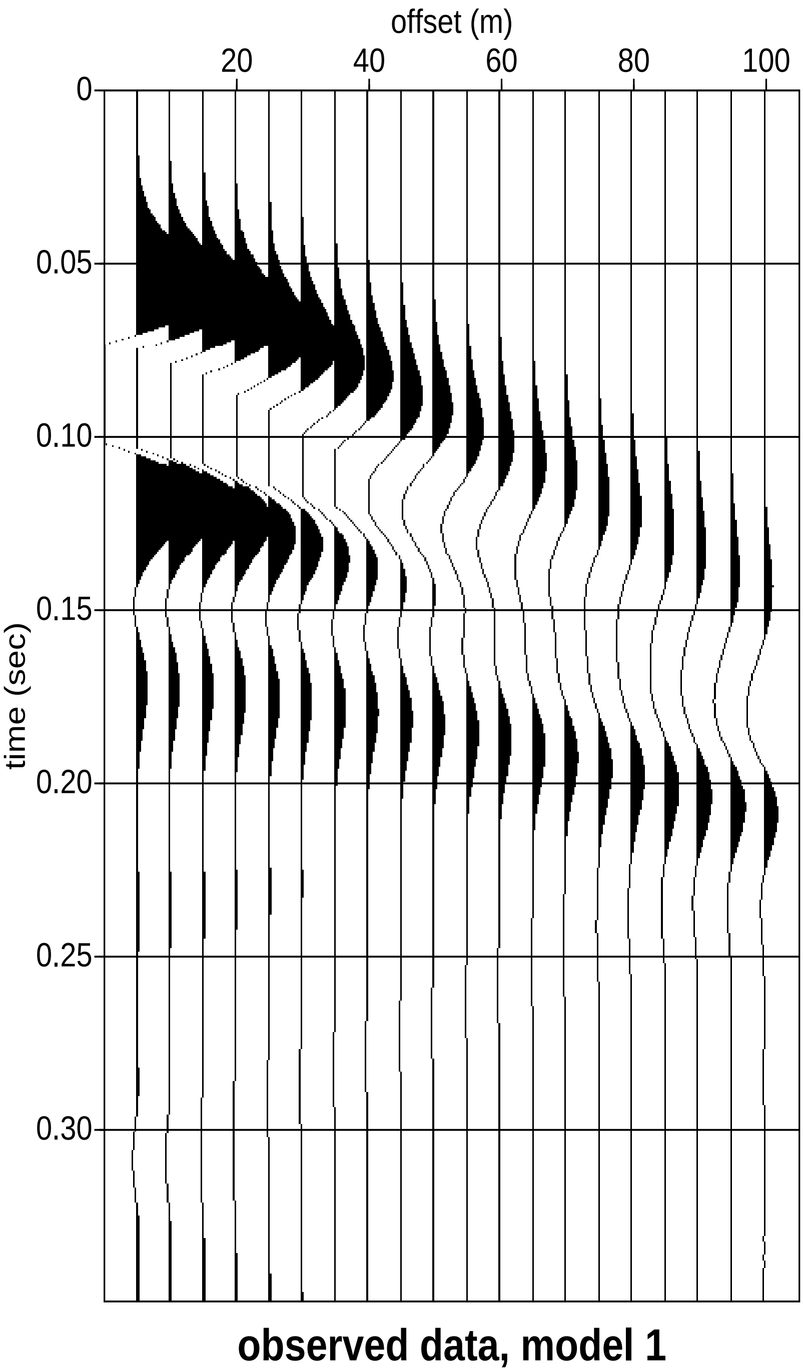


Model 1


Wave speed profile, model 1
source depth 20 m, source main frequency 15 Hz
wave speed (km/sec)
depth (meters)
0
0.5
1
1.5
2
2.5
3
0
25
50
75
100



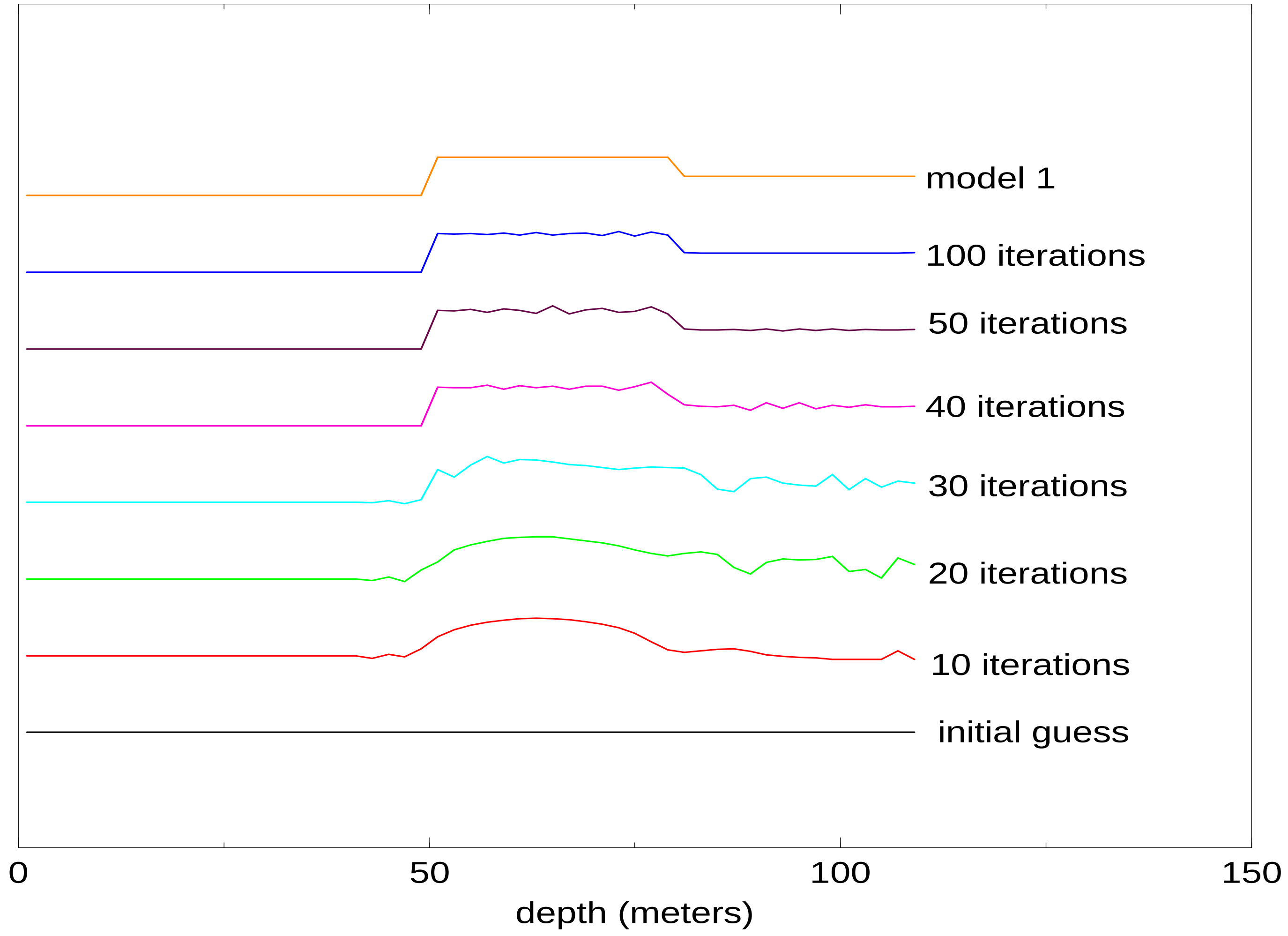
Traces of the r-component of the displacement vector

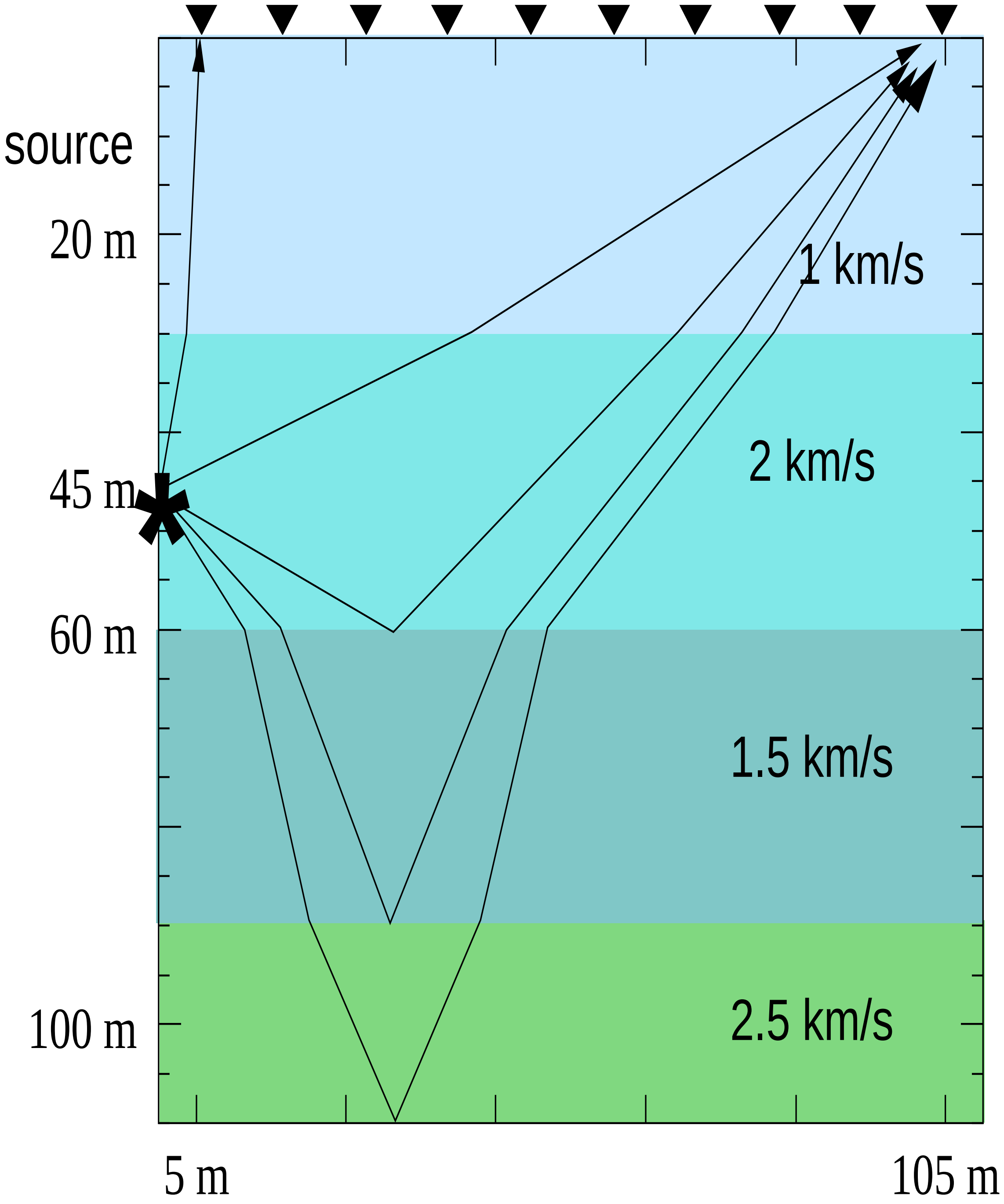


Traces of the z-component of the displacement vector

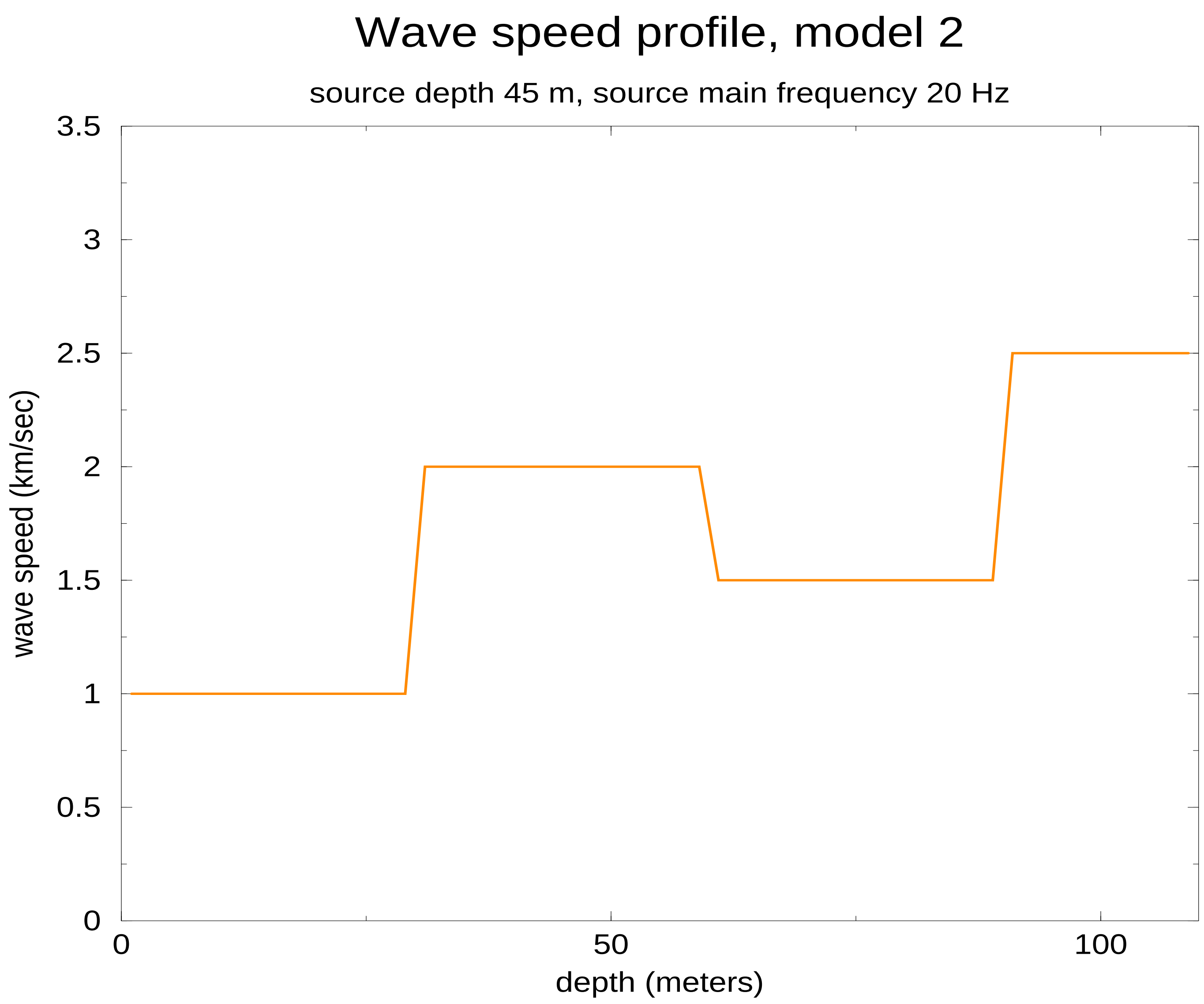
Model 1. Updated wave speed profiles

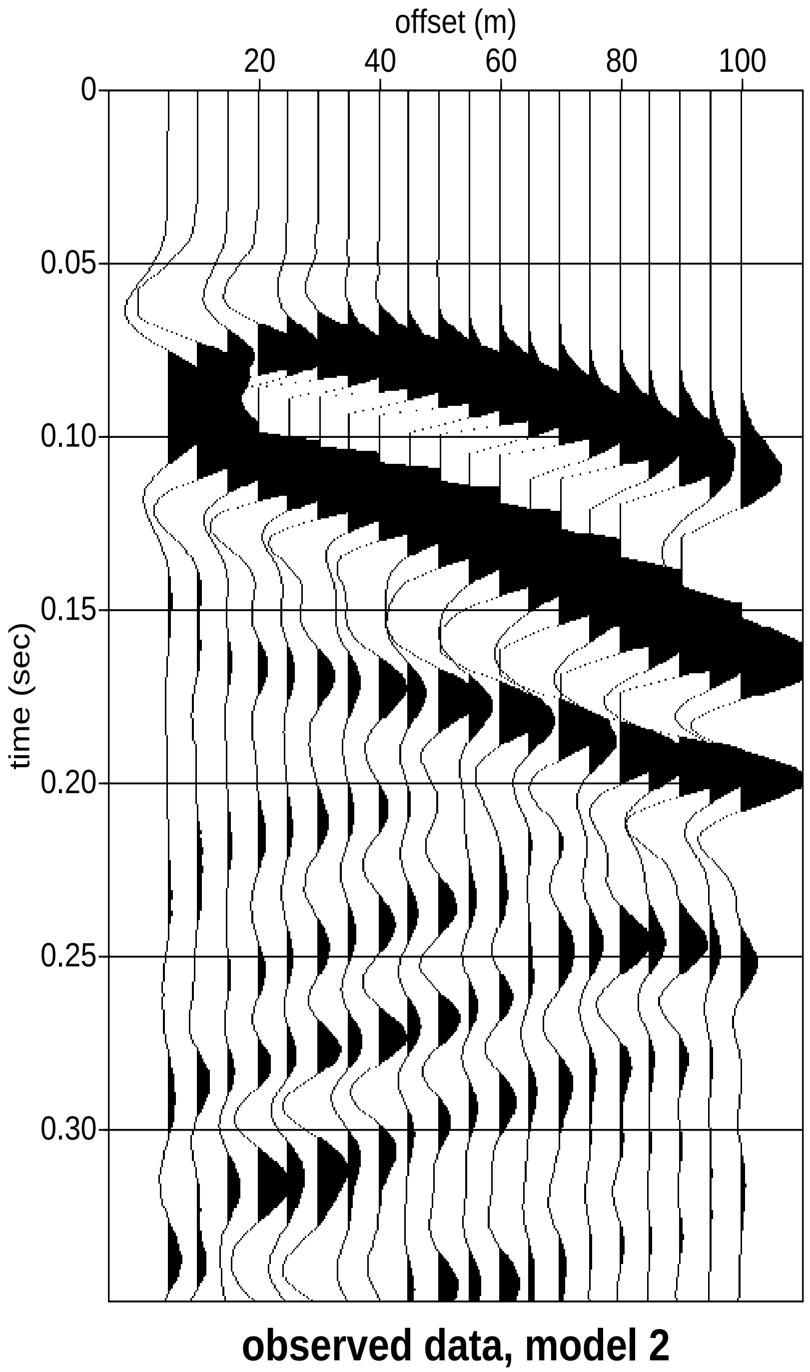
at several steps of the identification procedure



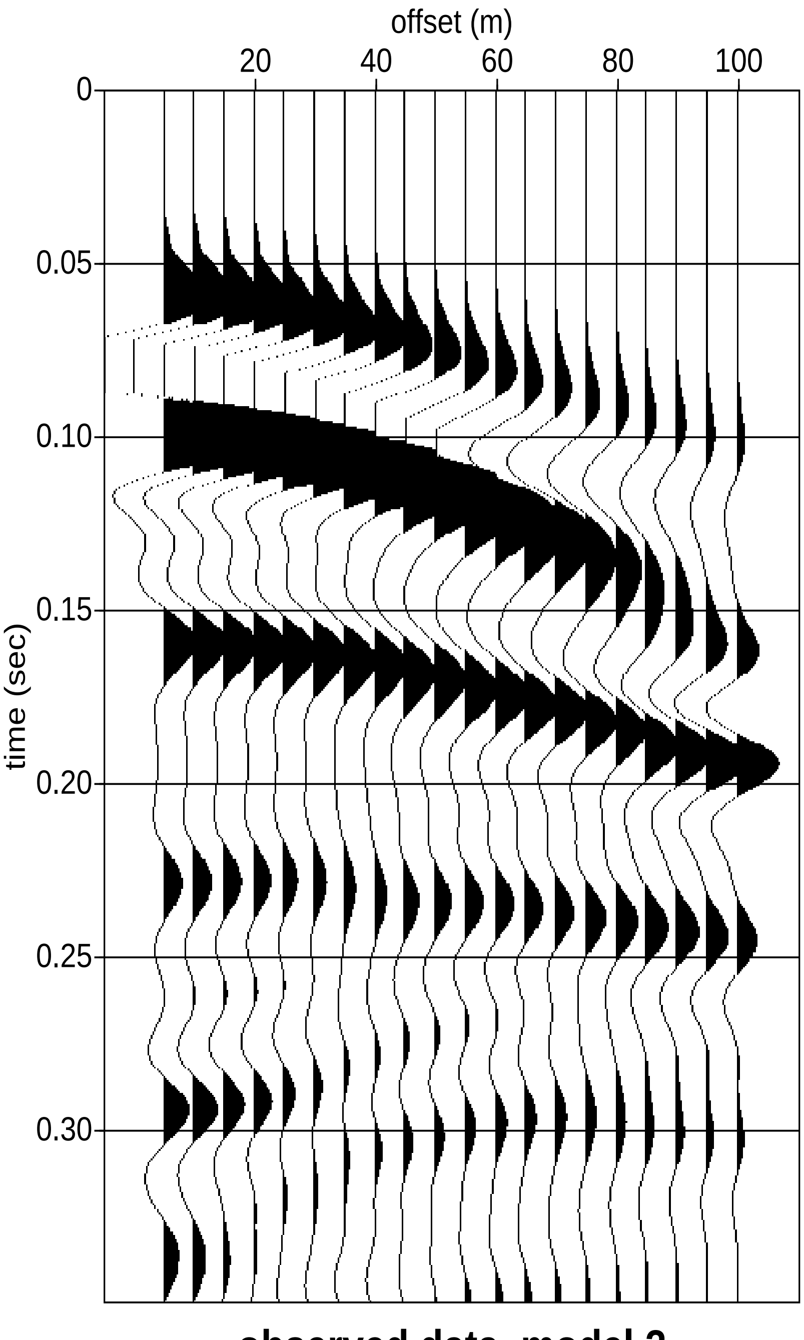


Model 2


Wave speed profile, model 2
source depth 45 m, source main frequency 20 Hz
wave speed (km/sec)
depth (meters)
0
0.5
1
1.5
2
2.5
3
3.5
0
50
100



Traces of the r-component of the displacement vector



Traces of the z-component of the displacement vector

Model 2. Updated wave speed profiles

at several steps of the identification procedure