1D WAVE PROPAGATION

BIOT MODEL -

BIOT : 1 FLUID PHASE

$$(1)\ -\omega^2 \rho\, u_s - \omega^2 \rho_f\, u_f - \nabla\cdot\tau = f^{(1)}$$

$$2)\ -\omega^2 \rho_f\, u_s - \omega^2 g(\omega) u_f + \nabla p + i\omega b(\omega) = f^{(2)}$$

$\rho^{(b)} = (1-\phi)\rho_s + \phi\rho_f$ $\qquad \phi$ = porosity –

$$(3)\ \tau_{ij} = 2\mu\, \varepsilon_{ij}(u_s) + \delta_{ij}(\lambda_c\, e - B\,\xi)$$

$$(4)\ p_f = -B e + M \xi$$

$$(5)\ \xi = -\nabla\cdot u_f$$

μ = shear modulus –

B-C : (2 D)

$$(6)\ (-\tau\nu\nu, -\tau\nu\chi, p_f) = i\omega D(u_s\cdot\nu, u_s\cdot\chi, u_f\cdot\nu), \ \partial\Omega$$

D = positive definite

$$(7)\ D = A^{1/2} S^{1/2} A^{1/2}$$

$$A = \begin{bmatrix} e^{(b)} & 0 & e_f \\ 0 & \hat{e} & 0 \\ e_f & 0 & g \end{bmatrix}$$

$$E = \begin{bmatrix} \lambda_c + 2\mu & 0 & B \\ 0 & \mu & 0 \\ B & 0 & M \end{bmatrix}$$

$$(8)\quad S = A^{-1/2}\, E\, A^{-1/2}$$

$$\hat{e} = e - (e_f)^2 / g(\omega)$$

$$9)\quad g(\omega) = \frac{e_f}{\phi}\left(S + \frac{F_i(\theta)}{\omega} \frac{\eta}{\mathbb{K}} \frac{\phi}{e_f} \right)$$

$$10)\quad b(\omega) = \frac{\eta}{\mathbb{K}} F_r(\theta)$$

$\mathbb{K}$ = permeability
S = structure factor

$$(11)\quad \theta(\omega) = a_p \left(\frac{\omega e_f}{\eta} \right)^{1/2}, \quad a_p = 2\left(\frac{A_0 K}{\phi} \right)^{1/2}$$

A_0 = Kozeny Carman constant $(= 5)$

$$F(\theta) = F_r(\theta) + i F_i(\theta)$$

$$\lambda_c + 2\mu = K_c + \frac{4}{3}\mu \quad \text{in 3D}$$

K_c = bulk modulus of saturated rock

K_c, μ = complex and frequency dependent in the general case

Weak formulation

From (1)

$$-\omega^2 (\rho^{(b)} u_s, v^1) - \omega^2 (\rho_f u_f, v^1)$$

$$+ (\tau_{ij}(u_s, u_f), \varepsilon_{ij}(v^1))$$

$$- \langle \tau \cdot \nu, v^1 \rangle_{\partial\Omega} = (f^{(1)}, v^1), \quad v^1 \in (H^1(\Omega))^2 \tag{12}$$

From (2)

$$-\omega^2 (\rho_f u_s, v^2) - \omega^2 (g\, u_f, v^2)$$

$$- (p, \nabla \cdot v^2) + \langle p, v^2 \cdot \nu \rangle_{\partial\Omega} = (f^{(2)}, v^2) \tag{13}$$

$$+ i\omega (b(\omega) u_f, v^2) \qquad v^2 \in H(\mathrm{div}, \Omega)$$

Adding (12) and (13)

$$-\omega^2(\rho^{(b)} u_s, v^1) - \omega^2(\rho_f u_f, v^1)$$

$$-\omega^2(\rho_f u_s, v^2) - \omega^2(g\, u_f, v^2)$$

$$(14) \quad + i\omega(b\, u_f, v^2) + (\tau_{ij}, \varepsilon_{ij}(v^1)) - (p_f, \nabla\cdot v^2)$$

$$-\langle \tau\cdot\nu, v^1\rangle + \langle p_f, v^2\cdot\nu\rangle = (f^{(1)}, v^1) + (f^{(2)}, v^2)$$

Boundary terms:

$$-\langle \tau\cdot\nu, v^1\rangle + \langle p_f\, v^2\cdot\nu\rangle$$

$$= -\langle \tau\nu\nu, \tau\nu\chi, v^1\cdot\nu, v^1\cdot\chi\rangle$$

$$+\langle p_f\, v^2\cdot\nu\rangle$$

$$= \langle(-\tau\nu\nu, -\tau\nu\chi, p_f), (v^1\cdot\nu, v^1\cdot\chi, v^2\cdot\nu)\rangle$$

$$= i\omega\langle D(u_s\cdot\nu, u_s\cdot\chi, u_f\cdot\nu), (v^1\cdot\nu, v^1\cdot\chi, v^2\cdot\nu)\rangle$$

Then (14) becomes

$$
\begin{aligned}
&-\omega^2\left(\overset{(b)}{\rho} u_s, v^1\right) - \omega^2\left(\rho_f u_f, v^1\right) \\
&-\omega^2\left(\rho_f u_s, v^2\right) - \omega^2\left(g\, u_f, v^2\right) \\
&+ i\omega\left(b\, u_f, v^2\right) + \left(\tau_{ij}, \varepsilon_{ij}(v^1)\right) - \left(p_f, \nabla\cdot v^2\right) \\
&+ i\omega \left\langle \mathcal{D}\left(u_s\cdot\nu,\ u_s\cdot\chi,\ u_f\cdot\nu\right), \left(v^1\cdot\nu, v^1\cdot\chi, v^2\cdot\nu\right)\right\rangle \\
&= \left(f^{(1)}, v^{(1)}\right) + \left(f^{(2)}, v^{(2)}\right), \qquad (15) \\
&\qquad \left(v^{(1)}, v^{(2)}\right) \in \left(H^1(\Omega)\right)^2 \times H(\mathrm{div}, \Omega)
\end{aligned}
$$

Reduction to 1-D Case

Delete rows and columns 2 in the definition of matrices A, E —

Remove $u_s\cdot\chi$, $v^1\cdot\chi$ in (15).

Also defining

$$\Lambda\left((u_s, u_f), (v^1, v^2)\right) = \left(\tau_{ij}, \varepsilon_{ij}(v^1)\right) - \left(p_f, \nabla\cdot v^2\right)$$

in the 1-D case Λ becomes

$$\Lambda\left((u_s, u_f), (v^1, v^2)\right) = \left(\tau_{11}, \varepsilon_{11}(v^1)\right) - \left(p_f, \nabla\cdot v^2\right)$$

(16) $$= \left(\lambda_c \frac{\partial u_s}{\partial x} + B\frac{\partial u_f}{\partial x} + 2\mu\frac{\partial u_s}{\partial x}, \frac{\partial v^1}{\partial x}\right) + \left(B\frac{\partial u^s}{\partial x} + M\frac{\partial u_f}{\partial x}, \frac{\partial v^2}{\partial x}\right)$$

Then we get the weak form (1D case)

(since $H(\mathrm{div}, \Omega) = H^1(\Omega)$ in 1-D)

$\Omega = (0,1)$.

Also set $\tilde{g}(\omega) = g(\omega) + \frac{b(\omega)}{i\omega} \rightarrow$

$= g(\omega) - i\, b(\omega)/\omega$

$$-\omega^2 (\rho^{(b)} u_s, v^1) - \omega^2 (\rho_f u_f, v^1)$$

$$-\omega^2 (\rho_f u_s, v^2) - \omega^2 (\tilde{g}(\omega) u_f, v^2)$$

$$+ \left((\lambda_c + 2\mu) \frac{\partial u_s}{\partial x}, \frac{\partial v^1}{\partial x} \right) + \left(B \frac{\partial u^f}{\partial x}, \frac{\partial v^1}{\partial x} \right)$$

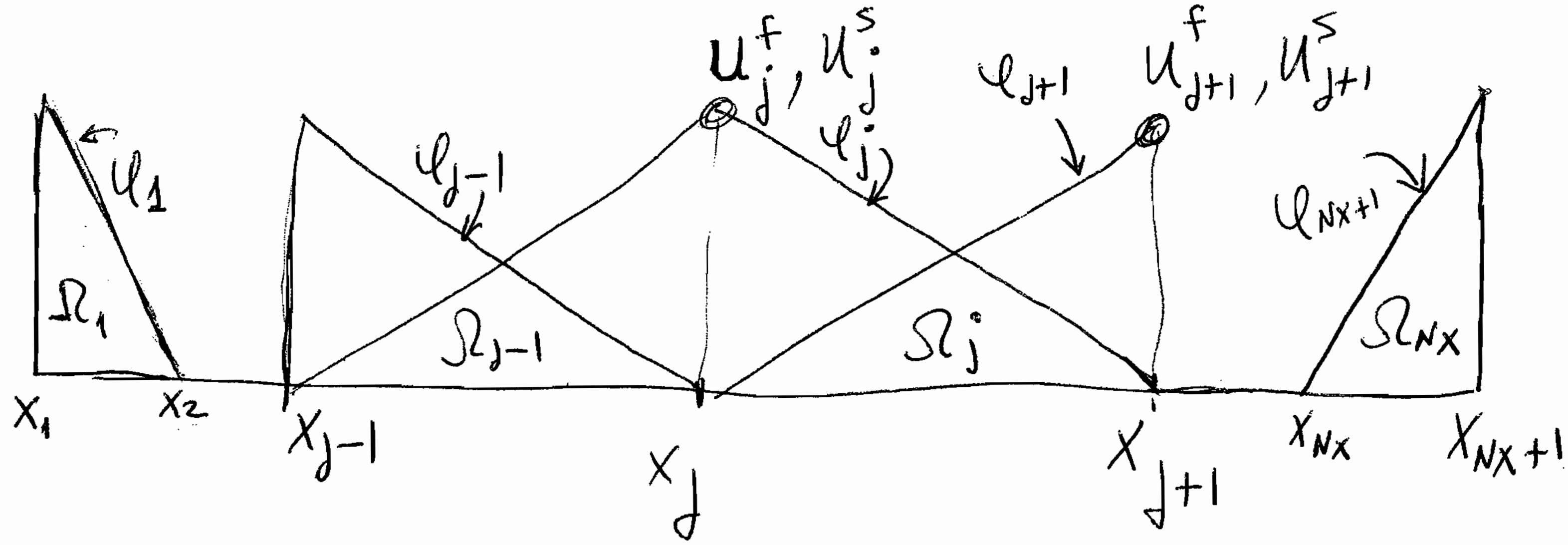
$$(17) \quad + \left(B \frac{\partial u^s}{\partial x} + M \frac{\partial u_f}{\partial x}, \frac{\partial v^2}{\partial x} \right)$$

$$+ i\omega \langle D(u_s \cdot \nu, u_f \cdot \nu), (v^1 \cdot \nu, v^2 \cdot \nu) \rangle$$

$$= (f^{(1)}, v^1) + (f^{(2)}, v^2), \quad (v^1, v^2) \in \left(H^1(\Omega) \right)^2 -$$

<u>Remark</u>: <u>Fredholm</u> Alternative gives existence + uniqueness for (17) for $\omega > 0$ —

SPATIAL DISCRETIZATION



N_x = # subintervals

N_x+1 = # of nodes

$h_j = x_{j+1} - x_j$

$j = 1 - N_x$

(18) $$u_s = \sum_{j=1}^{N_x+1} u_j^s \varphi_j(x)$$

$$\Omega = \bigcup_{j=1}^{Nx} \Omega_j$$

(19) $$u_f = \sum_{j=1}^{Nx+1} u_j^f \varphi_j(x)$$

$$\varphi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h_{k-1}}, & x_{k-1} \le x \le x_k \\[2ex] 1 - \dfrac{x - x_k}{h_k}, & x_k \le x \le x_{k+1} \end{cases}$$

Take the test function $V^1 = \varphi_K$, $V^2 = 0$,

$K = 1 - NX+1$.

in (17). Assume piecewise constant coefficients. Then

$\rho = \rho_K$, $g = g_K$, $\lambda_c + 2\mu = \lambda_{c_K} + 2\mu_K$

$B = B_K$ $M = M_K$ on $(x_K, x_{K+1}) = \Omega_K$

Let us compute each term in (17):

$$(\rho u_s, \varphi_K)$$

$$= \left(\rho \sum_{j=K-1}^{K+1} u_j^s \varphi_j, \varphi_K\right)$$

$$(\varphi_{K-1}, \varphi_K)_{\Omega_{K-1}} \overset{(K \neq 1)}{=} \int_{x_{K-1}}^{x_K} \left(\frac{x_K - x}{h_{K-1}}\right)\left(\frac{x - x_{K-1}}{h_{K-1}}\right) dx$$

$$= \int_{x_{K-1}}^{x_K} \left(1 - \frac{x - x_{K-1}}{h_{K-1}}\right)\underbrace{\left(\frac{x - x_{K-1}}{h_{K-1}}\right)}_{u} dx \qquad du = \frac{1}{h_{K-1}} dx \to dx = h_{K-1} du$$

$$= h_{K-1}\int_0^1 (1-u)\, u\, du = h_{K-1}\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Big|_0^1 = \frac{h_{K-1}}{6}$$

Then,

$$(e\, u^s_{k-1}\, \varphi_{k-1}, \varphi_k) \overset{(k\neq 1)}{=} e_{k-1} \frac{h_{k-1} u^s_{k-1}}{6} (1-\delta_{k1})$$

$$(\varphi_k, \varphi_k)_{\Omega_{k-1}} = \int_{x_{k-1}}^{x_k} \Big(\underbrace{\frac{x - x_{k-1}}{h_{k-1}}}_{u}\Big)^2 dx$$

$$= h_{k-1} \int_0^1 u^2\, du = \frac{h_{k-1}}{3} (1 - \delta_{k1})$$

$$(\varphi_k, \varphi_k)_{\Omega_k} \overset{k\neq Nx+1}{=} \int_{x_k}^{x_{k+1}} \Big(1 - \underbrace{\frac{x - x_k}{h_k}}_{u}\Big)^2 dx$$

$$= h_k \int_0^1 (1-u)^2 du = h_k \left(u - u^2 + \frac{u^3}{3}\right)\Big|_0^1 =$$

$$= \frac{h_k}{3} (1 - \delta_{k, Nx+1})$$

Then,

$$u^s_k(e^{(b)} \varphi_k, \varphi_k) = u^s_k(e^{(b)} \varphi_k, \varphi_k)^{(1-\delta_{k1})}_{\Omega_{k-1}} + u^s_k(e^{(b)} \varphi_k, \varphi_k)^{(1-\delta_{k,Nx+1})}_{\Omega_k}$$

$$= \left[e^{(b)}_{k-1} h_{k-1} (1 - \delta_{k1}) + h_k e^{(b)}_k (1 - \delta_{k,Nx+1})\right] \frac{1}{3} u^s_k.$$

$$(\varphi_{K+1}, \varphi_K) \overset{K \neq NX+1}{=} \int_{x_K}^{x_{K+1}} \underbrace{\left(\frac{x - x_K}{h_K}\right)}_{u} \left(1 - \frac{x - x_K}{h_K}\right) dx$$

$$= h_K \int_0^1 u(1-u)\, du = \frac{h_K}{6}\left(1 - \delta_{K,NX+1}\right)$$

Then

$$\left(e^{(b)} u^s_{K+1} \varphi_{K+1}, \varphi_K\right) = e^{(b)}_K u^s_{K+1} \frac{h_K}{6}\left(1 - \delta_{K,NX+1}\right)$$

The terms $(e_f u_f, v')$ are computed in identical fashion. $\left(\cancel{e_f \equiv \text{constant}}\right)$

Calculation of the terms

$$T_1 = \left((\lambda_c + 2\mu) \sum_{j=K-1}^{K+1} u^s_j \frac{\partial \varphi_j}{\partial x}, \frac{\partial \varphi_K}{\partial x} \right)$$

$$\left(\frac{\partial \varphi_{K-1}}{\partial x}, \frac{\partial \varphi_K}{\partial x} \right) \overset{K \neq 1}{=} \int_{x_{K-1}}^{x_K} \left(-\frac{1}{h_{K-1}}\right) \frac{1}{h_{K-1}} dx = -\frac{1}{h_{K-1}}\left(1 - \delta_{K1}\right)$$

$$\left(\frac{\partial \varphi_k}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)_{\Omega_{k-1}} = \int_{x_{k-1}}^{x_k} \left(\frac{1}{h_{k-1}}\right)^2 dx = \frac{1}{h_{k-1}}(1-\delta_{k1})$$

$$\left(\frac{\partial \varphi_k}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)_{\Omega_{k+1}} = \int_{x_k}^{x_{k+1}} \left(\frac{1}{h_k}\right)^2 dx = \frac{1}{h_k}(1-\delta_{k,Nx+1})$$

$$\left(\frac{\partial \varphi_{k+1}}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)_{\Omega_{k+1}} = \int_{x_k}^{x_{k+1}} \frac{1}{h_k}\left(-\frac{1}{h_k}\right) dx = -\frac{1}{h_k}(1-\delta_{k,Nx+1})$$

Then,

$$T_1 = -\frac{1}{h_{k-1}}(1-\delta_{k1})(\lambda_{c_{k-1}}+2\mu_{k-1})\, u^s_{k-1}$$

$$+ \left[(\lambda_{c_{k-1}}+2\mu_{k-1})\frac{1}{h_{k-1}}(1-\delta_{k1}) + \frac{1}{h_k}(\lambda_{c_k}+2\mu_k)(1-\delta_{kNx+1})\right]$$

$$\cdot\, u^s_k$$

$$-\frac{1}{h_k}(1-\delta_{kNx+1})\,(\lambda_{c_k}+2\mu_k)\,(1-\delta_{k,Nx+1})\, u^s_{k+1}$$

Similarly for $\tilde{T}_1 = \left(B\frac{\partial u^f}{\partial x}, \frac{\partial \varphi_k}{\partial x}\right)$ ✓

$$T_2 = \langle D(u_s \cdot \nu,\ u_f \cdot \nu),\ (\varphi_k \cdot \nu,\ 0) \rangle$$

$$= \langle D(u_s \cdot \nu,\ u_f \cdot \nu),\ (\varphi_k \cdot \nu, 0) \rangle \big|_{x=0}$$

$$+ \langle D(u_s \cdot \nu,\ u_f \cdot \nu),\ (\varphi_k \cdot \nu, 0) \rangle \big|_{x=1}$$

$$= \langle D(u_s \cdot \nu,\ u_f \cdot \nu),\ (\varphi_1 \cdot \nu, 0) \rangle \quad \delta_{k1}$$

$$+ \langle D(u_s \cdot \nu,\ u_f \cdot \nu),\ (\varphi_{N_x+1} \cdot \nu, 0) \rangle\ \delta_{k,Nx+1}$$

$$\varphi_1 \big|_{x=0} = 1 \quad , \quad \varphi_{Nx+1} \big|_{x=1} = 1$$

$\nu = -1$, on $x = 0$, $\nu = 1$, on $x = 1$.

Then,

$$T_2 = \langle D(-u_s, -u_f),\ (-1, 0) \rangle_{x=0}\ \delta_{k1}$$

$$+ \langle D(u_s, u_f),\ (1, 0) \rangle_{x=1}\ \delta_{k,Nx+1}$$

$$= \langle D(u_s, u_f),\ (1, 0) \rangle_{x=0}\ \delta_{k1}$$

$$+ \langle D(u_s, u_f),\ (1, 0) \rangle_{x=1}\ \delta_{k,Nx+1}$$

Now (see (18)-(19))

$$u_s\big|_{x=0} = u_1^s \qquad u_s\big|_{x=1} = u^s_{Nx+1}$$

$$u_f\big|_{x=0} = u_1^f \qquad u_f\big|_{x=1} = u^f_{Nx+1}$$

Then,

$$T_2 = \langle D^L(u_K^s, u_K^f), (1,0)\rangle \delta_{K1}$$

$$+ \langle D^R(u_K^s, u_K^f), (1,0)\rangle \delta_{K,Nx+1}$$

$$= \left(D_{11}^L u_K^s + D_{12}^L u_K^f\right)\delta_{K1}$$

$$+ \left(D_{11}^R u_K^s + D_{12}^R u_K^f\right)\delta_{K,Nx+1}$$

Then we collect all terms and get

[Definition $\langle u,v\rangle = u(0)\overline{v(0)} + u(1)\overline{v(1)}$]

$$-\omega^2 \rho^{(b)}_{k-1} \frac{h_{k-1}}{6}(1-\delta_{k1})\, u^s_{k-1}$$

$$-\omega^2 \left[\rho^{(b)}_{k-1} (1-\delta_{k1}) h_{k-1} + \rho^{(b)}_{k} (1-\delta_{k,Nx+1}) h_k \right] \frac{1}{3} u^s_k$$

$$-\omega^2 \rho^{(b)}_{k} \frac{h_k}{6}(1-\delta_{k,Nx+1})\, u^s_{k+1}$$

$$-\omega^2 \rho_{f,k-1} \frac{h_{k-1}}{6}(1-\delta_{k1})\, u^f_{k-1}$$

$$-\omega^2 \left[h_{k-1}\rho_{f_{k-1}}(1-\delta_{k1}) + h_k \rho_{f_k}(1-\delta_{k,Nx+1}) \right] \frac{1}{3} \quad u^f_k$$

$$-\omega^2 \rho_{f_k} \frac{h_k}{6}(1-\delta_{k,Nx+1})\, u^f_{k+1}$$

$$-\frac{1}{h_{k-1}}(\lambda_{c\,k-1} + 2\mu_{k-1})(1-\delta_{k1})\, u^s_{k-1}$$

$$+ \left[(\lambda_{c\,k-1} + 2\mu_{k-1})\frac{1}{h_{k-1}}(1-\delta_{k1}) + \frac{1}{h_k}(\lambda_{c_k} + 2\mu_k)(1-\delta_{k,Nx+1}) \right] u^s_k$$

$$-\frac{1}{h_k}(\lambda_{c_k} + 2\mu_k)(1-\delta_{k,Nx+1})\, u^s_{k+1}$$

$$+ i\omega(D^L_{11}\, u^s_k + D^L_{12}\, u^f_k)\,\delta_{k1} + i\omega(D^R_{11}\, u^s_k + D^R_{12}\, u^f_k)\,\delta_{k,Nx+1}$$

$$- \frac{1}{h_{k-1}} B_{k-1} (1 - \delta_{k1}) u^f_{k-1}$$

$$+ \left[\frac{1}{h_{k-1}} B_{k-1} (1 - \delta_{k1}) + \frac{1}{h_k} B_k (1 - \delta_{k,Nx+1}) \right] u^f_k$$

$$- \frac{1}{h_k} B_k (1 - \delta_{k,Nx+1}) u^f_{k+1}$$

$$= (f^{(1)}, \varphi_k) \quad , \quad k = 1 \text{ --- } Nx+1 \text{ --}$$

Then,

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$$-\left[\omega^2 \rho^{(b)}_{k-1} \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}}\left(\lambda_{c_{k-1}} + 2\mu_{k-1}\right)\right](1-\delta_{k1})\, u^{s}_{k-1}$$

$$+\left\{\left[-\omega^2 \rho^{(b)}_{k-1} \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}}\left(\lambda_{c_{k-1}} + 2\mu_{k-1}\right)\right](1-\delta_{k1}) + i\omega D^{L}_{11}\, \delta_{k1}\right.$$

$$\left.+\left[-\omega^2 \rho^{(b)}_{k} \frac{h_k}{3} + \frac{1}{h_k}\left(\lambda_{c_k} + 2\mu_k\right)\right](1-\delta_{k,Nx+1}) + i\omega D^{R}_{11}\, \delta_{k,Nx+1}\right\} u^{s}_{k}$$

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$$+\left[-\omega^2 \rho^{(b)}_{k} \frac{h_k}{6} - \frac{1}{h_k}\left(\lambda_{c_k} + 2\mu_k\right)\right](1-\delta_{k,Nx+1})\, u^{s}_{k+1}$$

$$+\left[-\omega^2 \rho_f \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}} B_{k-1}\right](1-\delta_{k1})\, u^{f}_{k-1}$$

$$+\left\{\left[-\omega^2 \rho_f \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}} B_{k-1}\right](1-\delta_{k1}) + i\omega D^{L}_{12}\, \delta_{k1}\right.$$

$$\left.+\left[-\omega^2 \rho_f \frac{h_k}{3} + \frac{1}{h_k} B_k\right](1-\delta_{k,Nx+1}) + i\omega D^{R}_{12}\, \delta_{k,Nx+1}\right\} u^{f}_{k}$$

$$+\left[-\omega^2 \rho_f \frac{h_k}{6} - \frac{1}{h_k} B_k\right](1-\delta_{k,Nx+1})\, u^{f}_{k+1}$$

$$= \left(f^{(1)}, \varphi_k\right) \qquad , k = 1 \text{ — } Nx+1$$

Next we take the test function

$V^1 = 0$, $V^2 = \varphi_K$, $K = 1 — NX+1$ in (17).

The quadratures for the mass terms are identical —

New terms:

$$\left(B \sum_{j=K-1}^{K+1} u_j^s \frac{\partial \varphi_j}{\partial x}, \frac{\partial \varphi_K}{\partial x} \right)$$

is identical to the T_1 — term changing $\lambda_c + 2\mu$ by B —

$$\left(M \sum_{j=K-1}^{K+1} u_j^f \frac{\partial \varphi_j}{\partial x}, \frac{\partial \varphi_K}{\partial x} \right) : \text{idem}$$

(changing u_j^s by u_j^f)

Boundary term:

$$T_3 = \langle D(u^s \circ \nu,\ u^f \circ \nu), (0, \varphi_k \circ \nu) \rangle$$

$$= \langle D(u^s \circ \nu, u^f \circ \nu), (0, \varphi_1 \circ \nu) \rangle_{x=0} \delta_{k1}$$

$$+ \langle D(u^s \circ \nu, u^f \circ \nu), (0, \varphi_{Nx+1} \circ \nu) \rangle_{x=1} \delta_{k,Nx+1}$$

$$= \langle D(-u^s, -u^f), (0, -1) \rangle_{x=0} \delta_{k1}$$

$$+ \langle D(u^s, u^f), (0, 1) \rangle_{x=1} \delta_{k,Nx+1}$$

$$= \langle D(u^s, u^f), (0, 1) \rangle_{x=0} \delta_{k1}$$

$$+ \langle D(u^s, u^f), (0, 1) \rangle_{x=1} \delta_{k,Nx+1}$$

$$= \langle D(u^s_k, u^f_k), (0, 1) \rangle \delta_{k1}$$

$$+ \langle D(u^s_k, u^f_k), (0, 1) \rangle \delta_{k,Nx+1}$$

$$= (D^L_{21} u^s_k + D^L_{22} u^f_k) \delta_{k1}$$

$$+ (D^R_{21} u^s_k + D^R_{22} u^f_k) \delta_{k\,Nx+1}$$

$$-\omega^2 \rho_f \frac{h_{k-1}}{6}(1-\delta_{k1})\, u^s_{k-1}$$

$$-\omega^2 \left[\rho_f (1-\delta_{k1})\frac{h_{k-1}}{3} + \rho_f (1-\delta_{k,nx+1}) \frac{h_k}{3}\right] u^s_k$$

$$-\omega^2 \rho_f \frac{h_k}{6}(1-\delta_{k,nx+1})\, u^s_{k+1}$$

$$-\omega^2 \tilde{g}_{k-1} \frac{h_{k-1}}{6}(1-\delta_{k1})\, u^f_{k-1}$$

$$-\omega^2 \left[\tilde{g}_{k-1} (1-\delta_{k1})\frac{h_{k-1}}{3} + \tilde{g}_k (1-\delta_{k,nx+1}) \frac{h_k}{3}\right] u^f_k$$

$$-\omega^2 \tilde{g}_k \frac{h_k}{6}(1-\delta_{k,nx+1})\, u^f_{k+1}$$

$$-\frac{1}{h_{k-1}} B_{k-1} (1-\delta_{k1})\, u^s_{k-1}$$

$$+ \left[\frac{1}{h_{k-1}} B_{k-1} (1-\delta_{k1}) + \frac{1}{h_k} B_k (1-\delta_{k,nx+1})\right] u^s_k$$

$$-\frac{1}{h_k} B_k (1-\delta_{k,nx+1})\, u^s_{k+1}$$

$$- \frac{1}{h_{k-1}} M_{k-1} \, (1-\delta_{k1}) \, u^{f}_{k-1}$$

$$+ \left[M_{k-1} \frac{1}{h_{k-1}} (1-\delta_{k1}) + \frac{1}{h_k} M_k \left(1-\delta_{k\,Nx+1}\right) \right] u^{f}_{k}$$

$$- \frac{1}{h_k} M_k \, (1-\delta_{k,Nx+1}) \, u^{f}_{k+1}$$

$$+ i\omega \left(D^{L}_{21} \, u^{s}_{k} + D^{L}_{22} \, u^{f}_{k} \right) \delta_{k1}$$

$$+ i\omega \left(D^{R}_{21} \, u^{s}_{k} + D^{R}_{22} \, u^{f}_{k} \right) \delta_{k\,Nx+1}$$

$$= \left(f^{(2)}, \varphi_k \right) \qquad , \; k = 1 - Nx+1$$

Rearranging terms:

$$\left[-\omega^2 \tilde{g}_{k-1} \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}} M_{k-1}\right](1-\delta_{k1})\, u^f_{k-1}$$

$$+ \Big\{ \left[-\omega^2 \tilde{g}_{k-1} \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}} M_{k-1}\right](1-\delta_{k1})$$

$$+ i\omega\, D^L_{22}\, \delta_{k1}$$

$$+ \left[-\omega^2 \tilde{g}_k \frac{h_k}{3} + \frac{1}{h_k} M_k\right](1-\delta_{k,NX+1})$$

$$+ i\omega\, D^R_{22}\, \delta_{k,NX+1} \Big\}\, u^f_k \tag{21}$$

$$+ \left[-\omega^2 \tilde{g}_k \frac{h_k}{6} - \frac{1}{h_k} M_k\right](1-\delta_{k,NX+1})\, u^f_{k+1}$$

$$\left[-\omega^2 e_f \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}} B_{k-1}\right](1-\delta_{k1})\, u^s_{k-1}$$

$$+ \Big\{ \left[-\omega^2 e_f \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}} B_{k-1}\right](1-\delta_{k1}) + i\omega\, D^L_{21}\, \delta_{k1}$$

$$+ \left[-\omega^2 e_f \frac{h_k}{3} + \frac{1}{h_k} B_k\right](1-\delta_{k,NX+1})$$

$$+ i\omega\, D^R_{21}\, \delta_{k,NX+1} \Big\}\, u^s_k$$

$$+ \left[-\omega^2 \rho_f \frac{h_k}{6} - \frac{1}{h_k} B_k\right](1-\delta_{k,Nx+1})\, u^s_{k+1}$$

(21)

$$= (f^{(2)}, \varphi_k)\,, \quad k = 1 \text{ — } Nx+1 \text{ —}$$

——— - ———

Global numbering of unknowns:

$$Z^s(2j-1) = u^s_j$$

$$Z^f(2j) = u^f_j \quad , \; j = 1 - Nx+1$$

Then, in equation (20) we have

unknowns:

$$u^s_{k-1} = Z^s_{2(k-1)-1} = Z^s_{2k-3}$$

$$u^s_k = Z^s_{2k-1} \, , \quad u^s_{k+1} = Z^s_{2(k+1)-1} = Z^s_{2k+1}$$

$$u^f_{k-1} = Z^f_{2(k-1)} = Z^f_{2k-2}$$

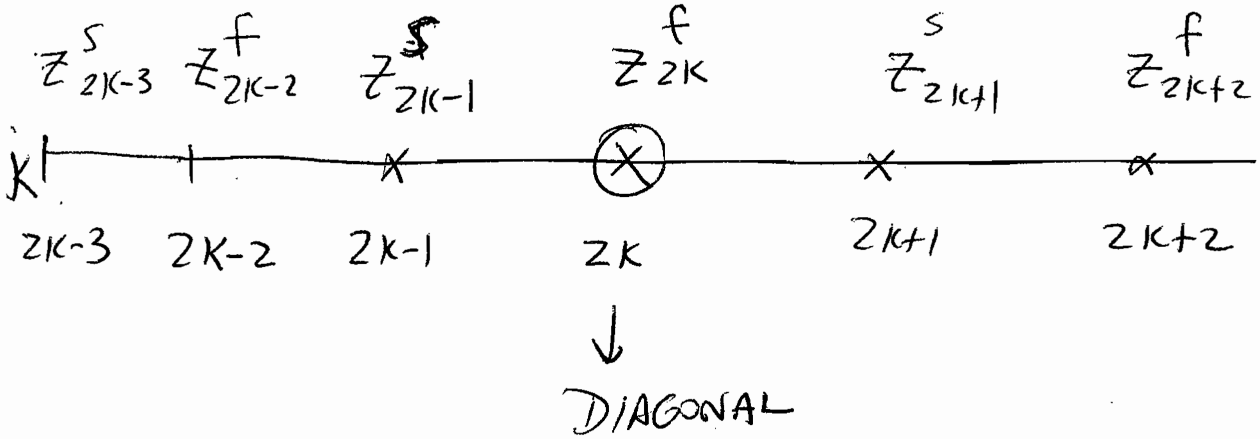
$$u^f_k = Z^f_{2k} \qquad u^f_{k+1} = Z_{2(k+1)} = Z_{2k+2}$$

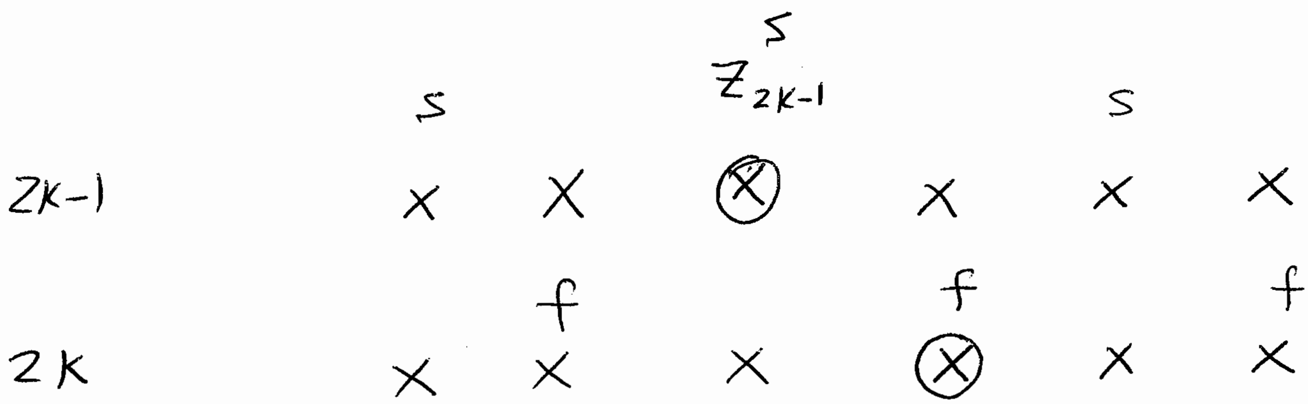
Eq'n (20)

	Z^s_{2k-3}	Z^f_{2k-2}	Z^s_{2k-1}	Z^f_{2k}	Z^s_{2k+1}	Z^f_{2k+2}
$2k$		$\times$	$\times$			
	$2k-3$	$2k-2$	$2k-1$	$2k$	$2k+1$	$2k+2$

$\downarrow$ diagonal

Eq'n (21)





Rectangular band matrix A:

$$\begin{array}{c|ccccccc|}
 & & u^s_{K-1} & u^f_{K-1} & \underset{u^s_K}{s} & u^f_K & u^s_{K+1} & u^f_{K+1} \\
2K-1 & 0 & \times & \times & \otimes & \times & \times & \times \\
 & u^s_{K-1} & u^f_{K-1} & u^s_K & f & u^s_{K+1} & u^f_{K+1} & \\
2K & \times & \times & \times & \otimes & \times & \times & 0 \\
 & & & & \uparrow u^f_K & & &
\end{array}$$

$$A(2k-1,1) = 0$$

$$A(2k-1,2) = \left[-\omega^2 e_{k-1} \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}}\left(\lambda_{c\,k-1} + 2\mu_{k-1}\right)\right](1-\delta_{k1})$$

$$A(2k-1,3) = -\omega^2 e_f \frac{h_{k-1}}{6}(1-\delta_{k1}) - \frac{1}{h_{k-1}} B_{k-1}(1-\delta_{k1})$$

$$A(2k-1,4) = \left[-\omega^2 e_{k-1} \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}}\left(\lambda_{c_{k-1}} + 2\mu_{k-1}\right)\right](1-\delta_{k1})$$

$$+ i\omega D^L_{11} \delta_{k1}$$

$$+ \left[-\omega^2 e_k \frac{h_k}{3} + \frac{1}{h_k}\left(\lambda_{c_k} + 2\mu_k\right)\right](1-\delta_{k,Nx+1})$$

$$+ i\omega D^R_{11} \delta_{k,Nx+1}$$

$$A(2k-1,5) = -\omega^2\left[e_f \frac{h_{k-1}}{3}(1-\delta_{k1}) + e_f \frac{h_k}{3}(1-\delta_{k,Nx+1})\right]$$

$$+ i\omega D^L_{12} \delta_{k1} + i\omega D^R_{12} \delta_{k,Nx+1}$$

$$+ \frac{1}{h_{k-1}} B_{k-1}(1-\delta_{k1}) + \frac{1}{h_k}(1-\delta_{k,Nx+1}) B_k$$

$$A(2k-1,6) = \left[-\omega^2 e_k \frac{h_k}{6} - \frac{1}{h_k}\left(\lambda_{c_k} + 2\mu_k\right)\right]\left(1-\delta_{k,Nx+1}\right)$$

$$A(2k-1,7) = -\omega^2 e_f \frac{h_k}{6}\left(1-\delta_{k,Nx+1}\right) - \frac{1}{h_k} B_k \left(1-\delta_{k,Nx+1}\right)$$

$$A(2k,1) = \left[-\omega^2 e_f \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}} B_{k-1}\right]\left(1-\delta_{k1}\right)$$

$$A(2k,2) = \left[-\omega^2 \tilde{g}_{k-1} \frac{h_{k-1}}{6} - \frac{1}{h_{k-1}} M_{k-1}\right]\left(1-\delta_{k1}\right)$$

$$A(2k,3) = \left[-\omega^2 e_f \frac{h_{k-1}}{3} + \frac{1}{h_{k-1}} B_{k-1}\right]\left(1-\delta_{k1}\right)$$

$$+ i\omega D_{21}^{L} \delta_{k1} + i\omega D_{21}^{R} \delta_{k,Nx+1}$$

$$+ \left[-\omega^2 e_f \frac{h_k}{3} + \frac{1}{h_k} B_k\right]\left(1-\delta_{k,Nx+1}\right)$$

$$A(2K,4) = \left[-\omega^2 \tilde{g}_{K-1} \frac{h_{K-1}}{3} + \frac{1}{h_{K-1}} M_{K-1}\right](1-\delta_{K1})$$

$$+ i\omega D_{22}^{L} \delta_{K1} + i\omega D_{22}^{R} \delta_{K,Nx+1}$$

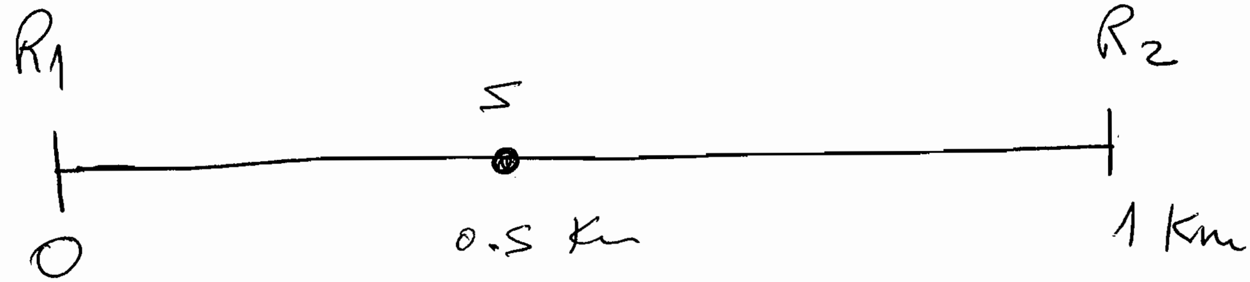
$$+ \left[-\omega^2 \tilde{g}_{K} \frac{h_{K}}{3} + \frac{1}{h_{K}} M_{K}\right](1-\delta_{K,Nx+1})$$

$$A(2K,5) = \left[-\omega^2 e_f \frac{h_K}{6} - \frac{1}{h_K} B_K\right](1-\delta_{K,Nx+1})$$

$$A(2K,6) = \left[-\omega^2 \tilde{g}_K \frac{h_K}{6} - \frac{1}{h_K} M_K\right](1-\delta_{K,Nx+1})$$

$$A(2K,7) = 0$$

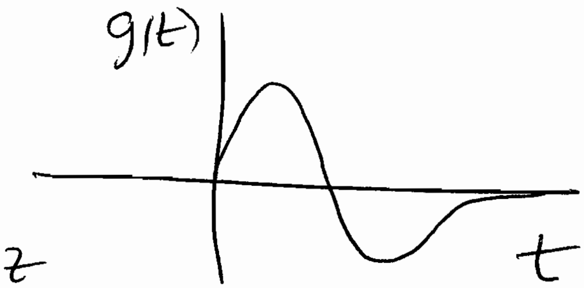
100 frequencies in $(0, 100\ Hz)$



Berez water

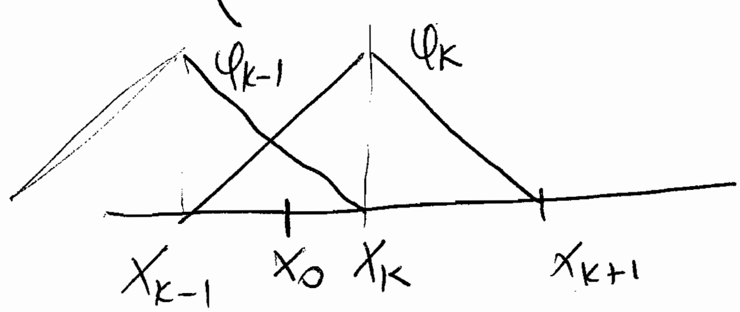
$$\tilde{f}^{(1)} = \frac{\partial}{\partial x} \delta(x - x_0)\, g(\omega)$$

$g(t)$ = waveform — main frequency 50 Hz



$$\left(f^{(1)}, \varphi_k\right) = \left(\frac{\partial}{\partial x}\delta(x-x_0) g(\omega), \varphi_j\right)$$

$$= -\left(\delta(x-x_0)\, g(\omega), \frac{\partial \varphi_j}{\partial x}\right)$$



$$= -g(\omega)\, \frac{\partial \varphi_j}{\partial x}(x_0)$$

$$\frac{\partial \varphi_j}{\partial x}(x_0) = 0 \quad \text{except for}$$

$j = k-1,\ k$

$j = k-1$ $$\frac{\partial \varphi_{k-1}}{\partial x}(x_0) = -\frac{1}{h_{k-1}}$$

$j = k$ $$\frac{\partial \varphi_k}{\partial x}(x_0) = \frac{1}{h_{k-1}}$$

$$g(t) = -2\xi(t-t_0)\, e^{-\xi(t-t_0)^2}$$

$$\xi = 8 f_0^2 \qquad t_0 = \frac{1.25}{f_0}$$

$f_0 =$ dominant frequency (50 Hz)

<u>Options</u>: $f_2 = 0$

$$f_1 = (1-\phi)\,\tilde{f}_1$$

$$f_2 = \phi\,\tilde{f}_1$$