REVIEW OF THE FINITE ELEMENT METHOD (1)

1-D

Consider the Problem

$$(P1)\begin{cases} -\left(a(x)\,y'\right)' + c(x)\,y = f, & 0 < x < 1 \\ y(0) = y(1) = 0. & I = (0,1) \end{cases}$$

Assume $a(x)$, $c(x)$ are smooth functions on I and

$$0 < a_0 \leq a(x) \leq a_1 < \infty$$

$$c_1 \geq c_1(x) \geq 0 \; -$$

Sobolev Spaces (dimension 1)

Let (a,b) an interval not neccesarily bounded — and $m \geq 0$ —

$$H^m(a,b) = \left\{ f : (a,b) \longrightarrow R : \; f^{(k)}(t) \in L^2(a,b) \right\}_{k \leq m}$$

$$\|f\|_{m,(a,b)} = \left[\sum_{k=0}^{m} \int_a^b \left| f^{(k)}(t) \right|^2 dt \right]^{1/2} -$$

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Theorem 1.1 $H^m(a,b)$ is a Banach space (normed, complete) (See Adams, p. 45)

Theorem 1.2 $C_0^\infty(R)/_{(a,b)} = C^\infty([a,b])$ is dense in $H^m(a,b)$ $\left[C_0^\infty(a,b) = \{ v \in C^\infty(a,b) : \operatorname{supp} v \text{ is compact in } (a,b) \} \right.$

Proof: See Adams, p. 54.

$$W^{m,\infty}(a,b) = \{ f : (a,b) \longrightarrow R : f^{(k)}(t) \in L^\infty(a,b),\ k \le m \}$$

$$\|f\|_{m,\infty} = \max_{k \le m} \| f^{(k)} \|_{L^\infty(a,b)}.$$

Sobolev Imbedding Theorem 1.3 ($n=1$)

$$H^1(a,b) \overset{i}{\hookrightarrow} C_B^0(a,b)$$

$C_B^0(a,b) = \{ f(a,b) \longrightarrow R$ bdd and continuous$\}$.

Proof: Later.

Theorem 1.4. If $f \in H^K(I)$, $K \geq 0$, $\rightarrow$ $y \in H^{K+2}(I)$ and

$$\|y\|_{K+2} \leq C \|f\|_K \ -$$

Also, $\exists!$ solution of (P1) for every $f \in L^2(I)$ (shift theorem) –

Proof: Later –

Def. $H_0^1(I) = \overline{C_0^\infty(I)}^{\| \ \|_1}$

Note that, for any $v \in C_0^\infty(I)$

$$-\int_0^1 (ay')' v \, dx = \underbrace{-vay'\Big|_0^1}_{=0} + \int_0^1 a y' v' \, dx$$

Then,

$$\int_0^1 (a y' v' + c y v) \, dx = \int_0^1 f v \, dx = (f, v)$$

Thus,

$$(ay', v') + (cy, v) = (f, v), \quad v \in C_0^\infty(I)$$

Next, let $v \in H_0^1(I)$. Then $\exists \, (v_m)_{m \geq 1} \subset C_0^\infty(I)$ / $\|v_m - v\|_1 \xrightarrow[m \to \infty]{} 0$

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(4)

In particular,

$$(ay', v_n') \xrightarrow[n\to\infty]{} (ay', v')$$

$$(cy, v_n) \longrightarrow (cy, v)$$

$$(f, v_n) \longrightarrow (f, v)$$

But we know that

$$(ay', v_n') + (cy, v_n) = (f, v_n) \quad \forall n$$

Thus, taking limit in n

$$(ay', v') + (cy, v) = (f, v), \quad \forall v \in H_0^1(I).$$

Let

$$B(y,v) \equiv (ay', v') + (cy, v).$$

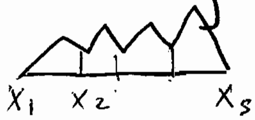
Thus, if y is solution of (P1) $\longrightarrow$

$$\text{(P2)} \qquad B(y,v) = (f,v), \quad \forall v \in H_0^1(I)$$

We will use (P2) to obtain an approximate solution to (P1) by a GALERKIN PROCEDURE —

Let $\delta = \{ \underset{0}{\overset{\|}{x_0}}, x_1, \ldots \underset{1}{\overset{\|}{x_m}} \} \quad x_k = kh, \quad h = 1/m.$

$$\mathcal{M}_h = \mathcal{M}^0_0(1,\delta) = \Big\{ v \in C^0(\bar{I}) : v \in P_1(x_{j-1}, x_j), \; V(0) = V(1) = 0 \Big\}$$



P_1 : polyn. of degree ≤ 1 in 1 variable –

Let $y_h \in \mathcal{M}_h$ defined by the relation

(P3). $\quad B(y_h, v) = (f, v), \quad v \in \mathcal{M}_h$.

We will show that

1) $\exists!\; y_h \in \mathcal{M}_h$ satisfying (P3) ✓

2)
$$\|y - y_h\|_1 \le \inf_{\chi \in \mathcal{M}_h} \|y - \chi\|_1 \le c \|y\|_2 \, h.$$

3) $\|y - y_h\|_0 \le c h \|y - y_h\|_1 \le c h^2 \|y\|_2$

4) (P3) Represents an simple algebraic problem with a tridiagonal matrix –

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Lemma 1.5 There $\exists$ a unique sol'n of (P3)
(proof of statement 1))

Proof: Let y_h, z_h be solutions of (P3) and let $w = y_h - z_h$. Then,

$$(*) \qquad B(w, v) = 0 \qquad \forall v \in \mathcal{M}_h.$$

The choice $v = w$ in $(*)$ yields

$$B(w,w) = \int_0^1 \left(a(w')^2 + c w^2\right) dx = 0$$

Since $a \geq a_0 > 0$, $c \geq 0$,

$$(**) \qquad \int_0^1 (w')^2 \, dx = 0 \rightarrow \| w' \|_{L^2(I)} = 0$$

But note that if $\tilde{w} \in C_0^\infty(I) \rightarrow$

$$\tilde{w}(x) = \underset{\substack{\| \\ 0}}{\tilde{w}(0)} + \int_0^x \tilde{w}'(\xi)\, d\xi = \int_0^x \tilde{w}'(\xi)\, d\xi$$

$\rightarrow$

$$|\tilde{\omega}(x)|^2 = \left| \int_0^x 1 \cdot \tilde{\omega}'(\xi)\, d\xi \right|^2$$

$$\leq \int_0^x |\tilde{\omega}'(\xi)|^2 d\xi \cdot \int_0^x 1 \cdot d\xi$$

$$\leq \int_0^1 |\tilde{\omega}'(\xi)|^2 d\xi \cdot \int_0^1 1\, d\xi$$

$\to$ integration in x yields

$$\int_0^1 |\tilde{\omega}(x)|^2 dx \leq \int_0^1 dx \left(\int_0^1 |\tilde{\omega}'(\xi)|^2 d\xi \right)$$

so that

$$\| \tilde{\omega}(x) \|_{L^2(I)} \leq \| \tilde{\omega}' \|_{L^2(I)} \quad \forall\ \tilde{\omega} \in C_0^\infty(I)$$

By a density argument, the inequality above is valid for any $\tilde{\omega} \in H_0^1(I)$

(POINCARÉ INEQUALITY) —

Thus, from $((*)(*))$ we see that

$$\| \omega(x) \|_{L^2(I)} \leq \| \omega' \|_{L^2(I)} = 0$$

Hence $w(x) = 0$ almost every where.

But w is a continuous function (w is a polynomial).

Then

$$w \equiv 0$$

This completes the proof. —

Proof of Sobolev Imbedding Theorem 1.3

Assume we have shown that $\forall\, \phi \in C^\infty([a\,b])$

(A) $$\sup_{x \in [a\,b]} |\phi(x)| \leq K \|\phi\|_{1,(a,b)}$$

Let $\phi \in H^1(a,b)$. By theorem 1.2 $\exists$

$$(\phi_n)_{n \geq 1} \subset C^\infty([a\,b]) \;/\; \|\phi_n - \phi\|_1 \xrightarrow[n\to\infty]{} 0$$

(In particular, $\|\phi_n\|_1 \to \|\phi\|_1$)

Then, since $\phi_n(x)$ is a Cauchy sequence in H^1,

$$|\phi_n(x) - \phi_p(x)| \leq K \|\phi_n - \phi_p\|_1 \leq \varepsilon \,, \; n, p \geq n_0(\varepsilon)$$

(uniform convergence)

Thus $\phi_n(x)$ converges to a continuous function $\tilde{\phi}$ and since $\forall\, x \in (a,b)$

$$|\tilde{\phi}(x)| = \lim_n |\phi_n(x)| \leq K \lim_n \|\phi_n\|_1 = K \|\phi\|_1$$

we have that $\tilde{\phi} \in C^0_B(a,b)$ —

Also since $\phi_n \to \phi$ in $L^2(a,b)$, in particular we can assume that $\phi_n \to \phi$ a.e. in (a,b)

Thus, $\phi = \tilde{\phi}$ a.e., i.e., in the class of ϕ there exists a continuous representative $\tilde{\phi}$ and (9)

$$\sup_{x \in (a,b)} |\tilde{\phi}| \le K \|\phi\|_{1,(a,b)}$$

This proofs that the embedding is continuous.

Then we prove (A):

$$\phi(x) = \int_{t}^{x} \phi'(x)dx + \phi(t), \quad t, x \in (a,b)$$

$$\rightarrow (\phi(x))^2 \le 2\left[\left(\int_t^x \phi'(x)\,dx\right)^2 + (\phi(t))^2\right]$$

Cauchy Ineq.! $$\le 2\left[\int_t^x |\phi'(x)|^2 dx \cdot (x-t) + (\phi(t))^2\right]$$

$$\le 2\left[\int_a^b |\phi'(x)|^2 dx \cdot (b-a) + (\phi(t))^2\right]$$

Integration in t yields

$$(b-a)\,|\phi(x)|^2 \le (b-a)\,2\int_a^b (\phi'(x))^2\,dx + 2\int_a^b (\phi(t))^2\,dt$$

$$\rightarrow \quad |\phi(x)| \le \left(2\int_a^b (\phi'(x))^2\,dx + \frac{2}{(b-a)}\int_a^b (\phi(t))^2\,dt\right)^{1/2}$$

$$\le K\,\|\phi\|_{1,(a,b)} \quad \forall\ x \in (a,b)$$

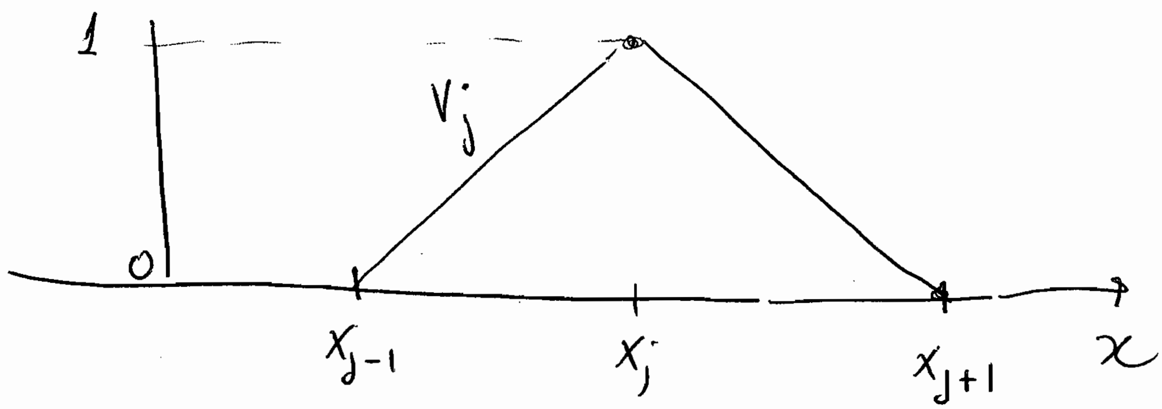
This completes the proof. ~~[illegible]~~

Proof of Statement 4. Lemma 1.6

(P3 represents a simple algebraic problem with a tridiagonal matrix) to determine the coefficients (c_j) in the equation

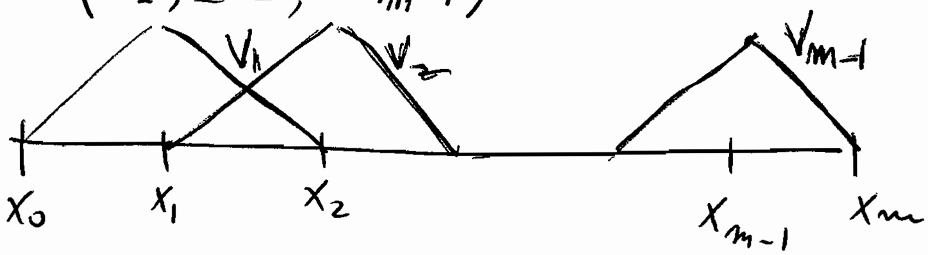
$$y_h = \sum_{j=1}^{m-1} c_j V_j(x)$$

where



Proof: Note that

$M_h = \text{span}(V_1, \ldots, V_{m-1})$.



In fact: we have m subintervals and in each ~~one~~ one of them the functions in M_h

have two parameters, since is linear –
Thus we have $\underline{2m}$ parameters –

But in $x_1, —, x_{m-1}$ we have $(m-1)$ continuity conditions and 2 additional conditions at x_0 and x_m of being zero –
Thus, we have

$$m-1+2 = m+1 \qquad \text{restrictions}$$

Hence,

$$\dim M_h = 2m - (m+1) = m-1 .$$

Since $v_1 —— v_{m-1}$ are obviously l. i.,
they are a basis for M_h – Let

$$y_h = \sum_{j=1}^{m-1} c_j v_j$$

Then $(P3)$ is equivalent to the equations

$$B(y_h, v_i) = (f, v_i) = \gamma_i \quad , \quad i = 1 — m-1 .$$

Thus,

$$\sum_{j=1}^{m-1} B(v_j, v_i)\, c_j = \gamma_i \quad , i=1 - m-1$$

But note that

$$B(v_j\, v_i) = \int_0^1 [a(x)\, v_j'\, v_i' + c(x)\, v_j\, v_i]\, dx$$

$$= \int_{(x_{j-1}, x_{j+1}) \cap (x_{i-1}, x_{i+1})} (a\, v_j' v_i' + c\, v_j\, v_i)\, dx = 0 \quad \text{if } |i-j| > 1.$$

Thus, the matrix

$$A = (a_{ij})_{1 \le i,j \le m-1} = \big(B(v_j\, v_i)\big)_{1 \le i,j \le m-1}$$

is a tridiagonal matrix and the algebraic system is written as follows:

$$A \underline{c} = \underline{\gamma} \qquad \underline{c} = \begin{pmatrix} c_1 \\ c_{m-1} \end{pmatrix},$$

$$\underline{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_{m-1} \end{pmatrix}.$$

Lemma 1.7 There exists a unique solution $\eta_h \in \mathcal{M}_h$ of (P3) —

Proof: In a linear system uniqueness and existence are equivalent. Then Lemma 1.5 implies that the matrix A is non-singular and consequently there exists a unique $\underline{c}$ such that

$$A\,\underline{c} = \underline{\gamma} \quad \text{for any } \underline{\gamma} \in \mathbb{R}^{m-1}.$$

Thus there exists a unique solution η_h of (P3) given by

$$\eta_h = \sum_{j=1}^{m-1} c_j v_j \quad — \checkmark$$

Let us see how to solve a tridiagonal system

$$A = L\,U$$

(example $m = 7$)

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & 0 & 0 \\ 0 & 0 & c_3 & a_4 & b_4 & 0 \\ 0 & 0 & 0 & c_4 & a_5 & b_5 \\ 0 & 0 & 0 & 0 & c_5 & a_6 \end{bmatrix} =$$

(3,4)

$$\begin{array}{c} \\ \\ \\ 4 \\ \\ \\ \end{array} \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_1 & d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_2 & d_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_3 & d_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_4 & d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_5 & d_6 \end{bmatrix} \begin{array}{c} \begin{array}{cccccc} & & 3 & & & \end{array} \\ \begin{bmatrix} 1 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & u_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & u_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & u_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(1,1)

$$\boxed{d_1 \cdot 1 = a_1}$$

(1,2)

$$d_1 u_1 = b_1 \rightarrow$$

$$\boxed{u_1 = b_1/d_1}$$

(2,1)

$$l_1 \cdot 1 = c_1$$

$$\boxed{l_1 = c_1}$$

(2,2)

$$l_1 u_1 + d_2 = a_2 \rightarrow \boxed{d_2 = a_2 - l_1 u_1}$$

(2,3)

$$l_1 \cdot 0 + d_2 u_2 = b_2 \rightarrow \boxed{u_2 = b_2/d_2}$$

(3,2)

$$0 \cdot u_1 + l_2 \cdot 1 = c_2 \rightarrow \boxed{l_2 = c_2}$$

$$\underset{}{\overset{(3,3)}{l_2 u_2 + d_3 = a_3}} \longrightarrow \boxed{d_3 = a_3 - l_2 u_2}$$

$$\overset{(3,4)}{d_3 u_3 = b_3} \longrightarrow \boxed{u_3 = b_3 / d_3}$$

$$\overset{(4,3)}{\boxed{l_3 = c_3}}$$

(Row 4 of L * column 3 of U)

~~Assume that we use the vector [illegible] to store the vector~~

Assume that the matrix A is stored in the vectors a(·), b(·), c(·)

Assume that we use a(·) to construct the vector d(·), b(·) to construct u(·) and c(·) to construct l(·) -

Then the algorithm is

```
u(1) = u(1) / d(1)
Do i = 2, m-2
d(i) = d(i) - c(i-1) * u(i-1)
u(i) = u(i) / d(i)
enddo
d(m-1) = d(m-1) - c(m-2) * u(m-2)
```

At this step we have factored the matrix A. Now we need to solve

$$A\,x = LU\,x = z \searrow \text{data}$$

Let

$$U x = Y$$

Then first we solve

$$LY = z \rightarrow \text{data}$$

just computed

$$\begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ l_1 & d_2 & 0 & 0 & 0 & 0 \\ 0 & l_2 & d_3 & 0 & 0 & 0 \\ 0 & 0 & l_3 & d_4 & 0 & 0 \\ 0 & 0 & 0 & l_4 & d_5 & 0 \\ 0 & 0 & 0 & 0 & l_5 & d_6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

$$d_1 y_1 = z_1 \rightarrow y_1 = z_1/d_1$$

$$l_1 y_1 + d_2 y_2 = z_2 \rightarrow y_2 = (z_2 - l_1 y_1)/d_2$$

$$l_2 y_2 + d_3 y_3 = z_3 \rightarrow y_3 = (z_3 - l_2 y_2)/d_3$$

$$\vdots$$

Thus, the algorithm is (Forward Solution) (18)

$$Y(1) = z(1)/d(1)$$

Do $i = 2 - m-1$

$$y(i) = \left(z(i) - l(i-1) * y(i-1)\right)/d(i)$$

enddo

Next, we need to solve

$$U x = Y \leftarrow \text{just computed}$$

$$\begin{bmatrix} 1 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & u_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & u_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & u_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$x_6 = y_6$, $x_5 + u_5 x_6 = y_5 \rightarrow x_5 = y_5 - u_5 x_6$

$x_4 + u_4 x_5 = y_4 \rightarrow x_4 = y_4 - u_4 x_5$

$\vdots$

$x_1 + u_1 x_2 = y_1 \rightarrow x_1 = y_1 - u_1 x_2$.

Thus, (backward solution)

```
X(m-1) = Y(m-1)
DO i=1, m-2
K = m-2+i-1
X(K) = Y(K) - U(K)*X(K-1)
ENDDO
```

Note that if

$$E^h = \| y - y^h \|_0 = c\, h^\alpha$$

$$E^{\frac{h}{2}} = \| y - y^{h/2} \|_0 = c\left(\frac{h}{2}\right)^\alpha$$

Then

$$\frac{E^h}{E^{h/2}} = 2^\alpha \quad \text{, so that}$$

$$\alpha \ln 2 = \ln\left(\frac{E^h}{E^{h/2}}\right) \quad \text{or}$$

$$\alpha = \frac{\ln\left(E^h / E^{h/2}\right)}{\ln 2}$$

determines α

Now we go back to the proof of statement 2) (21)

<u>Lemma 1.12</u> Let y be the solution of (P2) and y_h the solution of (P3). Then,

$$\|y - y_h\|_1 \leq C \inf_{x \in M_h} \|y - x\|_1$$

<u>Proof</u>.
First note that it follows from (P2) and (P3) that

$$B(y - y_h, v) = 0, \quad v \in M_h$$

Choose $v = y - y_h + x - y, \quad x \in M_h$.
Then,

$$B(y - y_h, y - y_h + x - y) = 0,$$

or

$$B(y - y_h, y - y_h) = B(y - y_h, y - x), \quad x \in M_h$$

Next recall Poincare's inequality

$$\| w \|_{0,I} \leq \| w' \|_{0,I} \quad \forall w \in H_0^1(I)$$

(in general, for an interval (a,b) we would have

$$\| w \|_{0,(a,b)} \leq C(a,b) \| w' \|_{0,(a,b)} \quad .)$$

Hence,

$$\|w\|_1 = \left(\|w\|_0^2 + \|w'\|_0^2\right)^{1/2}$$
$$\leq \left(2\,\|w'\|_0^2\right)^{1/2}$$

Then,

$$\frac{1}{\sqrt{2}}\,\|w\|_{1,I} \leq \|w'\|_{0,I} \underset{\text{obvious}}{\leq} \|w\|_{1,I}$$

so that $\|w'\|_{0,I}$ and $\|w\|_{1,I}$ are equivalent norms in $H_0^1(I)$.

Then,

$$\frac{a_0}{2}\|y-y_h\|_{1,I}^2 \leq a_0\,\|(y-y_h)'\|_{0,I}^2$$

$$\leq B(y-y_h, y-y_h) = B(y-y_h, y-\chi)$$

$$= \int_0^1 \left[a\,(y-y_h)'(y-\chi)' + c\,(y-y_h)(y-\chi)\right]dx$$

$$\leq a_1\,\|(y-y_h)'\|_{0,I}\|(y-\chi)'\|_{0,I} + c_1\,\|y-y_h\|_{0,I}\,\|y-\chi\|_{0,I}$$

$$\leq (a_1+c_1)\,\|y-y_h\|_{1,I}\,\|y-\chi\|_{1,I}$$

Thus,

$$\|y-y_h\|_{1,I} \leq \frac{2(a_1+c_1)}{a_0}\|y-\chi\|_{1,I}, \quad \forall \chi \in \mathcal{M}_h.$$

Then,

$$\|y-y_h\|_{1,I} \leq C(a_0, a_1, c_1) \inf_{\chi \in \mathcal{M}_h} \|y-\chi\|_{1,I}.$$

This completes the proof —

<u>Corollary 1.13</u> $\|y - y_h\|_1 \leq c\, h \|y\|_2$.

<u>Proof</u>: Recall that the piecewise Lagrange interpolant $P_1 y$ of y satisfies the estimate
$(n=2)$

$$\|y - P_1 y\|_1 \leq c\, h \|y\|_2$$

Then, using theorem 1.12,

$$\|y - y_h\| \leq c \inf_{x \in M} \|y - x\|_1$$

$$\leq C \|y - P_1 y\|_1$$

$$\leq C h \|y\|_2 \, .$$

This completes the proof.

<u>Lemma 1.14</u>

$$\|y - y_h\|_0 \leq c\, h \|y - y_h\|_1 \, .$$

<u>Proof</u>: We will use the so-called "duality argument". Let $\xi = y - y_h$ and let φ be the solution of

$$L^*\varphi = L\varphi = -(a\varphi')' + c\varphi = \xi, \quad x \in I$$

$$\varphi(0) = \varphi(1) = 0$$

An elliptic regularity result (to be proved later) tells us that

$$(*) \qquad \|\varphi\|_2 \le C \|\xi\|_0 .$$

Hence,

$$\|\xi\|_0^2 = (\xi, -(a\varphi')' + c\varphi)$$

$$\left[\text{integration by parts}\right] \quad = B(\xi, \varphi)$$

$$= B(\xi, \varphi - \chi), \quad \chi \in \mathcal{M}.$$

$$\le (a_1 + c_1) \|\xi\|_1 \|\varphi - \chi\|_1 .$$

Then, if $P_1\varphi$ is the p.w. Lagrange interpolant of φ, using $(*)$

$$\|\xi\|_0^2 \le C \|\xi\|_1 \inf_{\chi \in \mathcal{M}} \|\varphi - \chi\|_1$$

$$\le C \|\xi\|_1 \|\varphi - P_1\varphi\|_1$$

$$\le C \|\xi\|_1 h \|\varphi\|_2$$

$$\le Ch \|\xi\|_1 \|\xi\|_0 . \longrightarrow$$

$\|\xi\|_0 \le ch \|\xi\|_1$. This completes the proof.

<u>Corollary</u> <u>1.15</u> $\|y - y_h\|_0 \leq c\, h^2 \|y\|_2$.

<u>Proof</u>: It follows immediately from Lemma 1.14 and Corollary 1.13 —