MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

(1)

Ω : open polygonal domain in R^2

$$(1.a)\quad Lp = -\nabla\cdot(a\nabla p + \underset{\sim}{b}\, p) + cp = f, \quad \Omega,$$

$$(1.b)\quad p = -g, \quad \partial\Omega.$$

Assumptions : a) $\exists\ L^{-1} : L^2(\Omega)\times H^{3/2}(\partial\Omega) \longrightarrow H^2(\Omega)$

(i.e., (1.) is solvable for any f, g.)

b) $\quad \|p\|_2 \le C\left\{\|f\|_0 + |g|_{3/2,\partial\Omega}\right\}.$

Set
$$\underset{\sim}{u} = -(a\nabla p + \underset{\sim}{b}\, p)$$

$$\frac{1}{a}\underset{\sim}{u} = -\nabla p - \frac{\underset{\sim}{b}}{a} p$$

$$\alpha = \frac{1}{a}, \qquad \underset{\sim}{\beta} = \underset{\sim}{b}\,\alpha = \underset{\sim}{b}/a$$

Then,
$$\alpha\underset{\sim}{u} + \nabla p + \beta p = 0$$

(assume $0 < a_* \le a(x) \le a^* < \infty$)

Teoreme [Girault - Raviart p. 13]
FEM for Navier Stokes eq's.

$[C^\infty(\bar\Omega)]^n$ is dense in $H(\mathrm{div},\Omega)$ ②

Lemma

$$(\nabla\cdot q, v) + (q, \nabla v) = \langle q\cdot\nu, v\rangle,$$

$\forall\ q \in H(\mathrm{div},\Omega),\ v \in H^1(\Omega)$

Notation:
$$\langle f,g\rangle = \int_{\partial\Omega} f\bar g\, d\sigma$$
$$(f,g) = \int_\Omega f\bar g\, dx$$

Dem:

$$\int_\Omega \nabla\cdot F\, dx = \int_{\partial\Omega} F\cdot\nu\, d\sigma \quad \text{(Gauss)}$$

Note that

$$\nabla\cdot(\bar v q) = \bar v\, \nabla\cdot q + \nabla\bar v\cdot q$$

indicates conjugate

Then, for $q \in [C^\infty(\bar\Omega)]^n$, $v \in C^\infty(\bar\Omega)$,

$$\int \nabla\cdot q\, \bar v\, dx + \int q\cdot\nabla\bar v\, dx$$

$$= \int \nabla\cdot q\bar v\, dx \overset{\text{Gauss}}{=} \int_{\partial\Omega} q\bar v\cdot\nu\, d\sigma$$

$$\rightarrow (\nabla\cdot q, v) + (q, \nabla v) = \langle q\cdot\nu, v\rangle.$$

Formula above can be extended to $q \in H(\mathrm{div},\Omega)$, $v \in H^1(\Omega)$.

Lemme: $|V|_{L^2(\partial\Omega)} \leq C \|V\|_0^{1/2} \|V\|_1^{1/2}$ (3)

Proof.

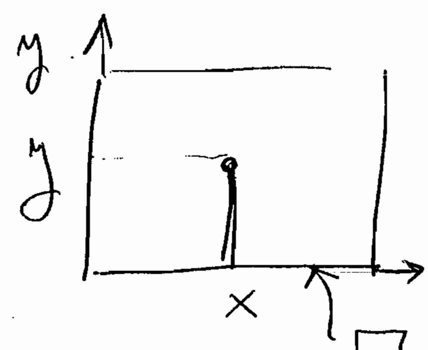
$$f^2(x,0) = -\int_0^y \frac{\partial f^2}{\partial y}(x,y)\, dy + f^2(x,y)$$

$$\int_0^1 f^2(x,0)\,dx = -\int_0^1\int_0^y 2 f(x,y) \frac{\partial f}{\partial y}(x,y)\,dx\,dy + \int_0^1 f^2(x,y)\,dy$$

$$\int_0^1 dy \left| \int_0^1 f^2(x,0)\,dx \right| \leq \int_0^1 dy \int\int_0^1 2|f|\left|\frac{\partial f}{\partial y}\right| dx\,dy + \int_0^1 dy \int_0^1 |f^2(x,y)|\,dy.$$

$$\leq C\|f\|_0 \|f\|_1 + \|f\|_0^2$$

$$\leq C \|f\|_0 \|f\|_1$$



$$\rightarrow \|f\|_{L^2(\Gamma)} \leq C\|f\|_0^{1/2} \|f\|_1^{1/2}$$

Thus problem (1) can be stated as a first order system in the form

$$\begin{cases} \alpha \underset{\sim}{u} + \nabla p + \underset{\sim}{\beta}\, p = 0 & , \Omega \quad (2.a) \\ \nabla \cdot \underset{\sim}{u} + c\, p = f & , \Omega \quad (2.b) \\ p = -g & , \partial\Omega \quad (2.c) \end{cases}$$

Weak form associated with (2):

Let $V = H(\mathrm{div}, \Omega) = \{ \underset{\sim}{v} \in [L^2(\Omega)]^2 : \nabla \cdot v \in L^2(\Omega)$

$W = L^2(\Omega)$.

(3) $(\alpha \underset{\sim}{u}, \underset{\sim}{v}) + (\nabla p, \underset{\sim}{v}) + (\underset{\sim}{\beta}\, p, \underset{\sim}{v}) = 0, \quad \underset{\sim}{v} \in H(\mathrm{div}, \Omega)$

$[\ p \in H^2(\Omega) \rightarrow \quad \nabla p \in (H^1(\Omega))^2$

$\underset{\sim}{\beta}\, p \in (H^2(\Omega))^2 \rightarrow \quad \underset{\sim}{u} = -\frac{1}{\alpha}(\nabla p + \underset{\sim}{\beta}\, p) \in (L^2(\Omega))^2$

$\nabla \cdot u = f - c\, p \in L^2(\Omega) \longrightarrow \underset{\sim}{u} \in H(\mathrm{div}, \Omega)$

This implies that we can test (2.a) against $v \in V$. $(\alpha u + \nabla p + \beta p \in V)$.

Next

$$(\nabla \cdot v, p) + (v, \nabla p) = \langle v \cdot \nu, p \rangle$$
$$= -\langle g, v \cdot \nu \rangle$$

Then (3) becomes

$$(4) \quad (\alpha \underset{\sim}{u}, v) - (\nabla \cdot \underset{\sim}{v}, p) + (\beta p, \underset{\sim}{v}) = \langle g, v \cdot \nu \rangle$$

We can also test (2.b) against $w \in W$.

Then the weak form associated with (2) is find $(\underline{u}, p) \in V \times W$:

$$(5) \quad (\alpha \underline{u}, \underline{v}) - (\nabla \cdot \underline{v}, p) + (\beta p, \underset{\sim}{v}) = \langle \underline{g}, \underset{\sim}{v} \cdot \nu \rangle, \quad \underline{v} \in V,$$

$$(6) \quad (\nabla \cdot u, w) + (c p, w) = (f, w), \quad w \in W.$$

Lemma 1: Assume that problem (1) is uniquely solvable. Then (1) and (5)-(6) are equivalent

Proof: We have just shown that if p is solution of (1) and

$$\underline{u} = -\left(a \nabla p + \underline{b}\, p\right), \quad \alpha = a^{-1},$$

then $(\underline{u}, p)$ is solution of (5)-(6)

Also, since L^{-1} exists, given f, g, p exists and defining $\underline{u} = -\left(a \nabla p + \underline{b}\, p\right)$, $(\underline{u}, p)$ solves (5)-(6), so that there exists solution for (5)-(6).

Thus to show the equivalence of both formulations we only need to show that uniqueness holds for (5)-(6), which is left to the reader.

Let us go back to the mixed weak formulation of the original problem: find $(\underline{u}, p) \in V \times W$:

$$(20) \quad (\alpha \underline{u}, \underline{v}) - (\nabla \cdot \underline{v}, p) + (\beta p, v) = \langle g, \underline{v} \cdot \nu \rangle \quad v \in V,$$

$$(21) \quad (\nabla \cdot u, w) + (c p, w) = (f, w), \quad w \in W.$$

Let τ^h: quasi-regular partition of Ω into triangles or rectangles (or parallelopeans) of diameter bounded by h. such that every angle of each triangle is bounded below by a positive constant.

Let $V^h \subset V$, $W^h \subset W$ be finite element spaces associated with τ^h. Then the discrete form of (20)-(21) is as follows:

Find $(\underline{u}^h, p^h) \in V^h \times W^h$ such that

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(22) $(\alpha \underline{u}^h, \underline{v}) - (\nabla \cdot \underline{v}, p^h) + (\beta p^h, \underline{v}) = \langle g, \underline{v} \cdot \underline{\nu} \rangle$, $v \in V^h$,

(23) $(\nabla \cdot \underline{u}^h, w) + (c p^h, w) = (f, w)$, $w \in W$

We do not get a free choice for V^h and W^h.

The Raviart-Thomas Spaces (in R^2)

If R is a rectangle of $\mathcal{T}^h$ and $k \geq 0$ let

$$P_{k,l} = \left\{ \sum_{i=0}^{k} \sum_{j=0}^{l} c_{ij} x^i y^j \,;\, c_{ij} \in R \right\}$$

$$Q_k = P_{k+1,k} \times P_{k,k+1}$$

Note that if $\underline{q} = (q_1, q_2) \in Q_k$, then

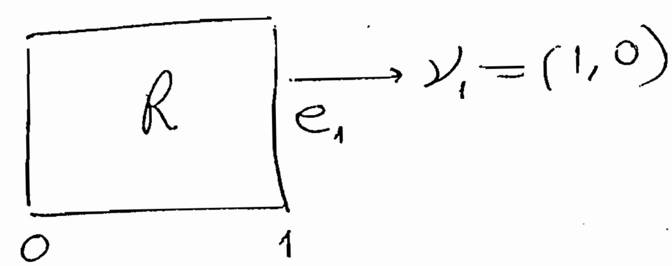
$$\frac{\partial q_1}{\partial x} \in P_{k,k}, \quad \frac{\partial q_2}{\partial y} \in P_{k,k},$$

so that

(24) $\quad \nabla \cdot \underline{q} \in P_{k,k}$

Also

(25) $\underline{q} \cdot \underline{\nu} \in P_k(e)$ for any edge e of R



In fact: on e_1

$$\underline{q} \cdot \underline{\nu}_1 = q_1(1, y) \in P_k(y)$$

and similarly on the other edges of R.

Set

$$\mathbb{V}_k(R) = Q_k(R) = (P_{k+1,k}\,,\, P_{k,k+1})$$

$$W_k(R) = P_{k,k}(R)$$

$$\dim \mathbb{V}_k(R) = 2 \dim P_{k+1,k} = 2(k+2)(k+1)$$

Degrees of freedom

Lemma 1

Any $\underline{v} \in \mathbb{V}_K(R)$ is uniquely determined by

i) $\underline{v} \cdot \underline{\nu}$ at $(K+1)$-points on each edge of R

or

(26) i') $\langle \underline{v} \cdot \underline{\nu}, \varphi \rangle_e$, $\varphi \in P_K(e)$ for any edge e of R

and

ii) $(\underline{v}, \underline{q})$, $\underline{q} \in P_{K-1,K} \times P_{K,K-1}$

Proof:

Obviously i) and i') are equivalent.

Next, the number of conditions in (26) is:

$4(K+1)$ from i) or i')

$2 \dim P_{K-1,K}$ from ii)

Then, the total number of conditions is

$$4(K+1) + 2K(K+1) = (K+1)(2K+4) = 2(K+1)(K+2)$$

Also,

$$\dim \mathbb{V}_K(R) = 2 \dim P_{K+1,K} = 2(K+1)(K+2)$$

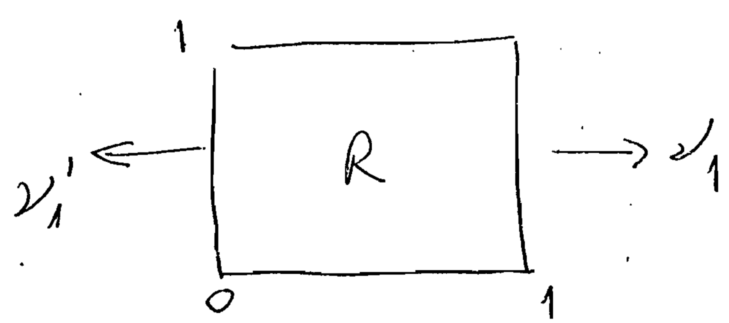
= # of conditions.

Then we only need to show uniqueness for (26). Thus assume that

$\underline{v} \in \mathbb{V}_K(R)$ and

i) $\langle \underline{v}\cdot\underline{\nu}, \varphi \rangle_e = 0$ $\quad \varphi \in P_K(e)$ on each side e of R

ii) $(\underline{v}, \underline{\Gamma}) = 0, \quad \underline{\Gamma} \in P_{K-1,K} \times P_{K,K-1}$



$$\underline{v}\cdot\underline{\nu}_1 = v_1\big|_{x=1} = 0$$

$$v\cdot\nu_1' = -v_1\big|_{x=0} = 0$$

Thus, x and $1-x$ divide q_1

or equivalently, 0 and 1 are roots of V_1. Thus, since $\underline{V}_1 \in P_{k+1,k}$

$$V_1 = x(1-x)\, r_1(x,y) \qquad r_1 \in P_{k-1,k}$$

Now $(r_1, 0)$ is a test function in ii)

$$0 = (\underline{V}, (r_1, 0)) = (x(1-x)\, r_1, r_1)$$

$$= \int_R \underbrace{x(1-x)}_{\geq 0}\, r_1^2 \, dx\, dy$$

Thus $r_1 = 0$ a.e and since r_1 is continuous $r_1 \equiv 0$ so that

$$V_1 \equiv 0 .$$

Similarly we can show that $V_2 \equiv 0$

This completes the proof.

Lemma 2: Let $\hat{R} = [0\,1]^2$ and let

$$\Pi_{\hat{R}}^{K} : H(\mathrm{div}, \Omega) \longrightarrow Q_K(\hat{R}) = \mathbb{V}_K(\hat{R})$$

defined by

(27)
i) $\langle (\Pi_{\hat{R}}^{K}\underline{q} - \underline{q}) \cdot \nu, \varphi \rangle_e = 0 \quad \forall \varphi \in P_K(e)$ on each side e of $\hat{R}$

ii) $(\Pi_{\hat{R}}^{K}\underline{q} - \underline{q}, \varphi) = 0, \quad \varphi \in P_{K-1,K} \times P_{K,K-1}$.

Then

$$\| \Pi_{\hat{R}}^{K}\underline{q} - \underline{q} \|_{0,\hat{R}} \le C\, |\underline{q}|_{K+1,\hat{R}} \tag{28}$$

$$\| \nabla \cdot (\Pi_{\hat{R}}^{K}\underline{q} - \underline{q}) \|_{0,\hat{R}} \le C\, \| \nabla \cdot \underline{q} \|_{K+1,\hat{R}} \tag{29}$$

Proof: Omitted.

Degrees of freedom in terms of values at nodal points.

Let $\delta_2 = \{ y_1, y_2, \dots, y_{K+1} \}$ the $(K+1)$-Gauss-points in $[0,1]$

$\delta_1 = \{ x_0 = 0, x_1, \dots, x_K, x_{K+1} = 1 \}$ the $(K+2)$-Lobatto-points in $[0,1]$.

Then we choose as nodes $\delta_1 \otimes \delta_2$

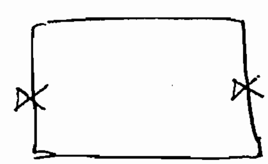
Example

a) $K = 0$

$\delta_2 = \{ y_1 \} = \{ \frac{1}{2} \}$, $\delta_1 = \{ 0, 1 \}$

Then,

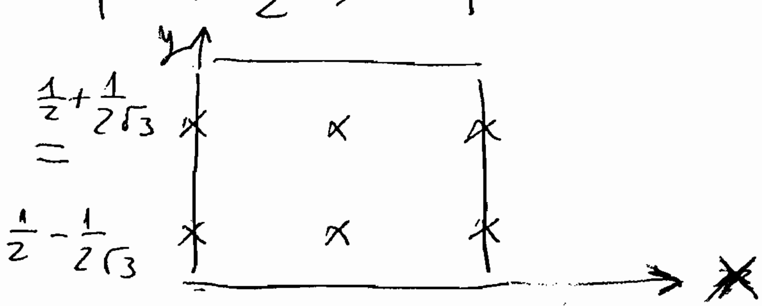
$\delta_1 \otimes \delta_2 =$



b) $K = 1$ $\quad \delta_2 = \{ y_1, y_2 \} = \{ \frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}} \}$

$\delta_1 = \{ 0, \frac{1}{2}, 1 \}$

$\delta_1 \times \delta_2 =$



The following result can be proved

Lemma: Any $\underline{v} \in \mathbb{V}_K(\hat{R})$ is determined by the values of v_1 on $\delta_1 \otimes \delta_2$ and of v_2 on $\delta_2 \otimes \delta_1$.

Proof:

Veremos que valen i) y ii) del lema 3.7

Veamos que vale i).

Si damos los valores de q_1 en $\delta_1 \otimes \delta_2$,

como $0, 1 \in \delta_1$, estamos dando los valores

de q_1 en $k+1$ puntos distintos (los puntos de

Gauss) sobre los ejes $x_1 = 0$, $x_1 = 1$.

Luego, como $q \cdot \nu = q_1$ en el eje $x_1 = 1$,

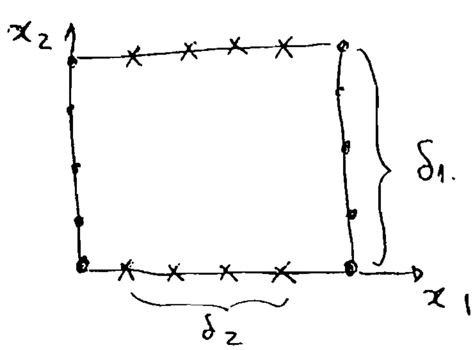
$q \cdot \nu = -q_1$ sobre el eje $x_1 = 0$, conocemos

$k+1$ valores distintos de $q \cdot \nu$ en los ejes $x_1 = 0$, $x_1 = 1$.

Además damos los valores de q_2 en $\delta_2 \otimes \delta_1$.

Como $0, 1 \in \delta_1$, conocemos q_2 en $k+1$ puntos

sobre los ejes $x_2 = 0$, $x_2 = 1$.



Luego conocemos $q \cdot \nu$ sobre los ejes $x_2 = 0$, $x_2 = 1$ en $k+1$ puntos distintos. Entonces vale i)

Veamos ahora que vale ii), o sea que conocemos los momentos (17)

$$(q, r) \quad , \quad r \in P_{k-1,k} \times P_{k,k-1} \,.$$

Como $q = (q_1, q_2)$, esto es equivalente a decir que conocemos

$$(q_1, r_1) \,. \quad , r_1 \in P_{k-1,k}$$

$$(q_2, r_2) \quad , r_2 \in P_{k,k-1} \,.$$

Recordemos algo sobre las cuadraturas de Gauss y de Gauss-Lobatto.

<u>Regla de Gauss</u>:

$$\int_0^1 g(x)\,dx = \sum_{j=1}^{m} w_j \, g(y_j) \quad , \text{ donde}$$

y_j = raíces del polinomio de Legendre P_m,

$$w_j = \int_0^1 l_j(x)\,dx \,,$$

$$l_j = \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)} \qquad 1 \leq j \leq m \,.$$

Esta regla es exacta para polinomios de grado $\leq 2n-1$.

Regla de Gauss – Lobatto

$$\int_0^1 g(x)\,dx = \sum_{j=1}^{n} \omega_j\, g(x_j) \quad , x_0 = 0 \, , x_1 = 1 ,$$

y $\{x_j\}_{j=2}^{n-1}$ son las raíces de $P'_{n-1}(x)$, donde P_{n-1} es el polinomio de Legendre de orden $n-1$. Los ω_j son los mismos de antes. Esta regla es exacta para polinomios de grado $\leq 2n-3$.

Corolario: Definimos en $[0\ 1]^2$ la regla Lobatto$_m$ x Gauss$_n$ en la forma natural

$$\iint_0^1 g(x_1, x_2)\, dx_1\, dx_2 = \sum_{j=1}^{m} \sum_{k=1}^{n} \omega_j\, \omega_k\, g(x_j, y_k)$$

Entonces esta regla es exacta para polinomios de grado $\leq 2m-3 \times 2n-1$.

En efecto: basta verificar que es exacta para un polinomio de la forma

$$x^p y^q, \quad p \le 2m-3, \quad q \le 2n-1.$$

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Pero,

$$\iint_0^1 x^p y^q \, dx\, dy = \left(\int_0^1 x^p dx\right)\left(\int_0^1 y^q dy\right)$$

$$= \left(\sum_j w_j x_j^p\right)\left(\sum_K w_K y_K^q\right)$$

$$\quad p \le 2m-3 \qquad\qquad q \le 2n-1$$

$$= \sum_{jK} w_j w_K x_j^p y^q$$

Luego vale la afirmación.

Por lo tanto la regla $\text{Lobatto}_{K+2} \times \text{Gauss}_{K+1}$

es exacta para polinomios de grado $\le (2(K+2)-3) \times (2(K+1)-1)$

$= 2K+1 \times 2K+1$, o sea en $P_{2K+1,\, 2K+1}$.

Ahora, como $q_1 \in P_{K+1,\, K}$, $r_1 \in P_{K-1,\, K}$, entonces

$q_1 r_1 \in P_{2K,\, 2K}$. Luego $\text{Lob}_{K+2} \times \text{Gauss}_{K+1}$

es exacta para $q_1 r_1$. (8.15)

Por lo tanto, si q_1 se conoce en $S_1 \otimes S_2$

$= (K+2)$ - puntos de Lobatto $\times$ $(K+1)$ - puntos de Gauss,

o sea conocemos

$q_1(x_j, y_\rho)$, $j = 0, \dots, k+1$,
$\rho = 1, \dots, k+1$

Entonces

$$(q_1, r_1) = \int_{\hat{R}} q_1 r_1 \underset{\text{por } (8.15)}{=} \sum_{j=0}^{k+1} \sum_{\rho=1}^{k+1} \omega_j \omega_\rho q_1(x_j, y_\rho) r_1(x_j, y_\rho)$$

Eligiendo r_1 en una base de $P_{k-1,k}$ (o sea r_1 es conocido) resulta que el conocimiento de los valores $q_1(x_j, y_\rho)$ implica que conocemos

(q_1, r_1) $\forall\, r_1 \in P_{k-1,k}$.

De la misma forma se puede ver que conocemos

(q_2, r_2) , $r_2 \in P_{k,k-1}$.

Luego vale ii) y esto finaliza la demostración.

Let $R = [0, h_x] \times [0, h_y]$ any rectangle in the partition τ^h (recall that τ^h is quasiregular) — Let

$$\Pi_R^K : H(\mathrm{div}, R) \longrightarrow \mathbb{V}_K(R) = (P_{K+1,K}, P_{K,K+1})(R)$$

defined by

$$\text{i)} \quad \langle (\Pi_R^K \underline{v} - \underline{v}) \cdot \nu, \varphi \rangle_e = 0, \quad \varphi \in P_K(e) \quad \forall \text{ side } e \text{ of } R$$

(30)

$$\text{ii)} \quad (\Pi_R^K \underline{v} - \underline{v}, \underline{\Gamma}) = 0, \quad \Gamma \in P_{K-1,K} \times P_{K,K-1}$$

<u>Lemma 4</u> Let $h = \max_i \operatorname{diam} R_i$.

$$(31) \quad \| \Pi_R^K \underline{v} - \underline{v} \|_{0,R} \leq C(K, \sigma_0) h^{K+1} | \underline{v} |_{K+1,R}$$

$$(32) \quad \| \nabla \cdot (\Pi_R^K \underline{v} - \underline{v}) \|_{0,R} \leq C(K, \sigma_0) h^{K+1} | \nabla \cdot \underline{v} |_{K+1,R} -$$

where $\sigma_0 = \frac{1}{e}$ is the quasiregularity constant

<u>Proof</u>: Scaling argument.

THE RAVIART-THOMAS-NEDELEC SPACES OVER TRIANGLES

If T is a triangle, for $k \geq 0$ set

$P_k(T)$ = polynomials of degree $\leq k$ on T

$$\mathbb{P}_k(T) = (P_k(T), P_k(T))$$

$$\mathbb{V}_k(T) = \mathbb{P}_k(T) \oplus \text{Span}(\vec{x}\, P_k(T))$$

$$W_k(T) = P_k(T),$$

where

$$\vec{x}\, P_k(T) = (x_1 P_k(T),\ x_2 P_k(T))$$

Note that $\dim \mathbb{W}_K(T) = K^2+4K+3$

In fact: first recall that

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$$\dim P_K(T) = \frac{(K+1)(K+2)}{2}$$

Next, if $v \in \mathbb{W}(T)$,

$$v = (p_1, p_2) + (x_1 p_3, x_2 p_3), \quad p_i \in P_K(T)$$

If $\deg(p_3) \le K-1$, then $\deg(x_i p_3) \le K$ and $x_i p_3 \in P_K(T)$. Then the only new polynomials added by $\vec{x} P_K(T)$ are of the form

$(x_1 p_3, x_2 p_3)$, with

$$p_3 = x_1^{\alpha} x_2^{\beta} \qquad \alpha+\beta = K$$

i.e.

$$p_3 = \{ x_1^0 x_2^K, x_1^1 x_2^{K-1}, \ldots x_1^K x_2^0 \}$$

$= K+1$ distinct polynomials

Thus,

$$\dim \mathbb{W}_K(T) = 2\frac{(K+1)(K+2)}{2} + K+1$$

$$= (K+1)(K+3) = K^2+4K+3.$$

Next note that if $\underline{v} \in \mathbb{V}_K(T)$

$$(v_1, v_2) = \underline{v} = (p_1, p_2) + (x_1 p_3, x_2 p_3), \ p_i \in P_K(T)$$

$$\rightarrow \ \frac{\partial v_1}{\partial x_1} = \frac{\partial p_1}{\partial x_1} + p_3 + x_1 \frac{\partial p_3}{\partial x_1} \in P_K(T)$$

$$\frac{\partial v_2}{\partial x_2} = \frac{\partial p_2}{\partial x_2} + p_3 + x_2 \frac{\partial p_3}{\partial x_2} \in P_K(T)$$

$$\rightarrow \ \boxed{\operatorname{div} \mathbb{V}_K(T) \subset W_K(T)}$$

Degrees of freedom:

Lemma 10 Any $\underline{v} \in \mathbb{V}_K(T)$ is uniquely determined by

i). $\underline{v} \cdot \underline{\nu}$ at $(K+1)$ - distinct points on each edge e of T

or

i') $\langle \vec{v} \cdot \vec{\nu}, \varphi \rangle_e$, $\varphi \in P_K(e)$

(33) and

ii) $(\underline{v}, \underline{q})$, $\underline{q} \in \mathbb{P}_{K-1}(T)$.

Proof: Note that the number of

conditions in (50) is

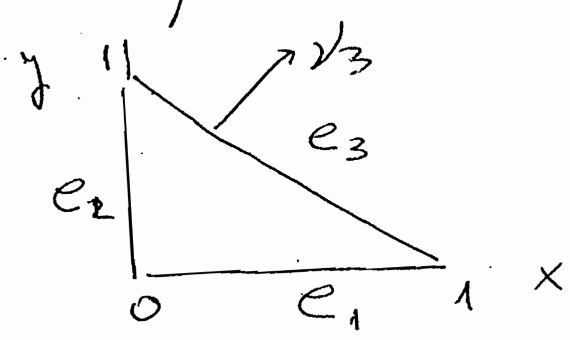
$$3(K+1) + 2\,\frac{K(K+1)}{2} = K^2 + 4K + 3 = \dim \mathbb{V}(T).$$

Then to proof the lemma we only need to show that if $\underline{v} \in \mathbb{V}(T)$ and $\underline{v}$ has homogeneous degrees of freedom, then $\underline{v} \equiv 0$.

First note that if $\underline{v} \in \mathbb{V}(T)$, then

$\underline{v} \cdot \nu \in P_K(e)$ for any edge e of T.

To see this is is sufficient to see this on a reference triangle $\hat{T}$



On e_1 $\quad \underline{v} \cdot \nu = v \cdot (0,-1) = -v_2\big|_{y=0}$

$$v_2\big|_{y=0} = p_2 + y\,p_3\big|_{y=0} = p_2\big|_{y=0} \in P_K(e)$$

Similarly on e_2.

On e_3, $x+y=1$ $\quad \nu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$v \cdot \nu_3 = (p_1, p_2) \cdot \nu_3 + p_3 (x, y) \cdot \left(\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{2}}\right)$$ 26

$$= (p_1, p_2) \cdot \nu_3 + p_3|_{e_3} \frac{1}{\sqrt{2}} \underbrace{(x+y)}_{=1} \in P_k(e_3)$$

$$\boxed{\text{Then} \quad v \cdot \nu \in P_k(e) \text{ for any edge } e \text{ of } T}$$

Example $\quad k=0$, $\quad \dim \mathbb{V}_k(T) = 3$

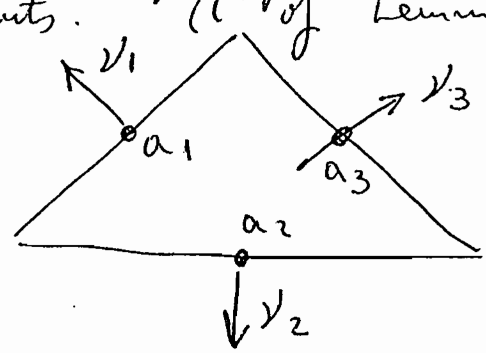
$$v(x,y) = (p_1, p_2) + (x p_3 ; y p_3) \quad p_i \in P_0(T)$$

$$= (a, b) + (cx, cy), \quad a, b, c \text{ constant}$$

$$v_1(x) = a + cx$$

$$v_2(x) = b + cy$$

Local degrees of freedom in terms of values at nodal points. (i of Lemma 10)



a_i = mid point of each edge

We may choose three basis functions $(\underline{\psi}_i)$ (
such that

$$\underline{\psi}_1 \cdot \underline{\nu}_1 = 1, \qquad \underline{\psi}_1 \cdot \nu_i = 0, \quad i \neq 1$$

$$\underline{\psi}_2 \cdot \nu_2 = 1, \qquad \underline{\psi}_2 \cdot \nu_i = 0, \quad i \neq 2$$

$$\psi_3 \cdot \nu_3 = 1, \qquad \psi_3 \cdot \nu_i = 0, \quad i \neq 3.$$

Then $\hat{v} \cdot \nu = 0$ on $\partial \hat{T}$

Let $\varphi \in P_K(\hat{T})$

$$(\hat{v}, \nabla \varphi) + (\nabla \cdot \hat{v}, \varphi) = \langle \underbrace{\hat{v} \cdot \nu}_{=0}, \varphi \rangle$$

$\nabla \varphi \in P_{K-1}(T) \rightarrow (\hat{v}, \nabla \varphi) = 0 \rightarrow$

$$(\nabla \cdot \hat{v}, \varphi) = 0 \quad \forall \varphi \in P_K(T)$$

$\rightarrow$ taking $\varphi = \nabla \cdot \hat{v} \in P_K(T)$ we conclude that

$$\nabla \cdot \hat{v} = 0$$

$\rightarrow \exists \; \hat{w} \in P_{K+1}(T)$:

$$\hat{v} = \operatorname{curl} \hat{w} = \left(\frac{\partial \hat{w}}{\partial x_2}, - \frac{\partial \hat{w}}{\partial x_1} \right)$$

$$\frac{\partial \hat{w}}{\partial \tau} = \hat{v} \cdot \nu = 0 \quad \text{on } \partial \hat{T}$$

Thus $\hat{w}$ is constant on $\partial \hat{T}$ and we can take $\hat{w} \equiv 0$ on $\partial \hat{T}$

Then $\hat{\omega}$ can be written as

$$\hat{\omega} = \lambda_1 \lambda_2 \lambda_3 \, \hat{q} \qquad \hat{q} \in P_{K-2}(\hat{T})$$

Let $r \in \left(P_{K-1}(\hat{T})\right)^2 \longrightarrow$ proof later

$$\underset{\substack{\downarrow \\ \text{Zero degrees of freedom}}}{0 =} (\hat{V}, r) = \left(\frac{\partial \hat{w}}{\partial x_2}, r_1\right) - \left(\frac{\partial \hat{w}}{\partial x_1}, r_2\right)$$

$$= \left(\hat{\omega}, \frac{\partial r_2}{\partial x_1} - \frac{\partial r_1}{\partial x_2}\right) + \langle \underset{\substack{\| \\ 0 \text{ on } \partial\hat{T}}}{\hat{\omega}}, r \cdot \tau \rangle_{\partial \hat{T}}$$

Choose r such that $\frac{\partial r_2}{\partial x_1} - \frac{\partial r_1}{\partial x_2} = \hat{q}$

$\longrightarrow$

$$(\lambda_1 \lambda_2 \lambda_3 \hat{q}, \hat{q}) = 0 \longrightarrow$$

Since $\lambda_1 \lambda_2 \lambda_3 > 0 \qquad \hat{q} = 0 \longrightarrow \hat{w} = 0$

$\longrightarrow \hat{V} = 0$ ___

Definition of $\lambda_i(T)$:

$$\lambda_i(x,y) = \frac{1}{D}\left(\eta_i x - \xi_i y + \omega_i\right)$$

$\xi_1 = x_2 - x_3$, $\xi_2 = x_3 - x_1$, $\xi_3 = x_1 - x_2$,

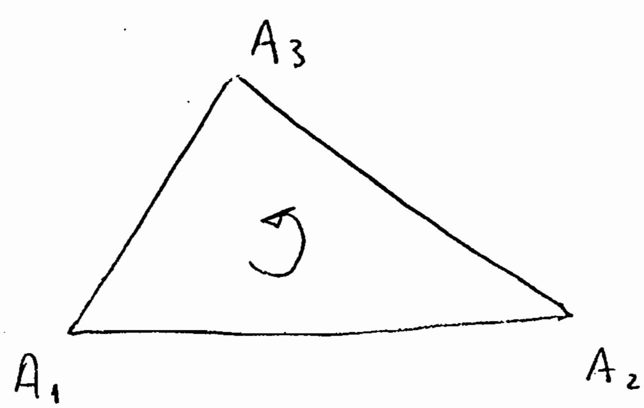
$\eta_1 = y_2 - y_3$, $\eta_2 = y_3 - y_1$, $\eta_3 = y_1 - y_2$

$\omega_1 = x_2 y_3 - x_3 y_2$

$\omega_2 = x_3 y_1 - x_1 y_3$

$\omega_3 = x_1 y_2 - x_2 y_1$

$$D = \omega_1 + \omega_2 + \omega_3 = \det \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



$A_i = (x_i, y_i)$

$i = 1, 2, 3$.

$\lambda_i(x_j, y_j) = \delta_{ij}$

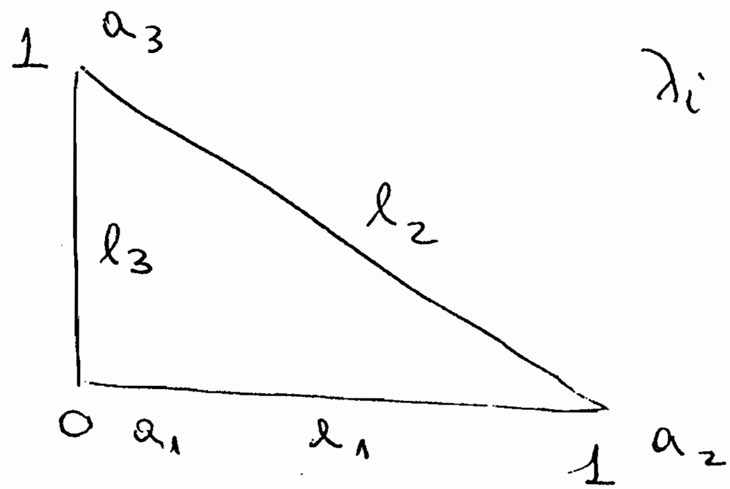
$\lambda_i(A_j) = \delta_{ij}$

Let us show that

$$B_{k+1}(T) = \{ p \in P_{k+1}(T) : p/_{\partial T} = 0 \}$$ (31)

$$= \lambda_1 \lambda_2 \lambda_3 \, P_{k-2}(T)$$

<u>Proof</u> Let $p \in \lambda_1 \lambda_2 \lambda_3 P_{k-2}(T)$



$\lambda_i(a_j) = \delta_{ij}$

$\lambda_1 = 1 - x - y$

$\lambda_2 = x$

$\lambda_3 = y$

$p = (1-x-y)\, x y \, p_1(x,y) \qquad p_1 \in P_{k-2}(T).$

$$\left.\begin{array}{l} \lambda_1/_{\ell_2} = 0 \Rightarrow p/_{\ell_2} = 0 \\ \lambda_2/_{\ell_3} = 0 \Rightarrow p/_{\ell_3} = 0 \\ \lambda_2/_{\ell_1} = 0 \rightarrow p/_{\ell_1} = 0 \end{array}\right\} \Rightarrow p/_{\partial T} = 0.$$

Also since $\deg(\lambda_1) = \deg(\lambda_2) = \deg(\lambda_3) = 1$,
$\deg p = 3 + \deg(p_1) = 3 + k - 2 = k+1 \Rightarrow$
$p \in P_{k+1}(T) \Rightarrow \quad p \in B_{k+1}(T)$

Reciprocally, assume $\quad p \in B_{k+1}(T)$.

Since $p(x,0) = p/_{y=0} = 0 \qquad p = y\, p_1(x,y) \quad p_1 \in P_{k}(T)$
(y divide to p).

Since $\quad p(0,y) = y\, p_1(0,y) = 0 \quad$ then

$$p_1(x,y) = x\, p_2(x,y)$$

$$p = x y\, p_2(x,y)$$

$$p(x,y)/_{\ell_2} = p(x, 1-x) = 0 \Rightarrow$$

$$p = x y (1-x-y)\, p_3(x,y)$$
$$= \lambda_1 \lambda_2 \lambda_3\, p_3(x,y) \quad p \in P_{k-2}(T).$$

Now Lemma 10 allows us to define the local projection

$\Pi_T^K : H(\mathrm{div}, T) \longrightarrow W_K(T)$ by

(34) i) $\langle (\Pi_T^K \underline{v} - \underline{v}) \cdot \nu, \varphi \rangle_e = 0$, $\varphi \in P_K(e)$ for each edge e of T

ii) $((\Pi_T^K \underline{v} - \underline{v}), \underline{\Gamma}) = 0$, $\underline{\Gamma} \in P_{K-1} \times P_{K-1}$.

Next, if $\varphi \in P_K(\hat{T}) = W_K(\hat{T})$, then $\nabla \varphi \in P_{K-1} \times P_{K-1}$ and

$$(\nabla \cdot (\Pi_T^K \underline{v} - \underline{v}), \varphi) = -(\Pi_T^K \underline{v} - v, \nabla\varphi) + \langle (\Pi_T^K \underline{v} - \underline{v}) \cdot \nu, \varphi \rangle = 0$$

by definition of Π_T^K.

Also, since $\Pi_T^K \underline{v} \in W_K(T)$, $\nabla \cdot \Pi_T^K \underline{v} \in W_K(T) = P_K(T)$

Then the equation

$$\boxed{(35)\quad (\nabla\cdot \Pi_T^K \underline{v} - \nabla\cdot \underline{v}, \varphi) = 0 \quad \forall \varphi \in P_K(\hat{T}) = W_K(\hat{T})}$$

implies that $\nabla\cdot \Pi_T^K \underline{v}$ is the L^2-projection of $\nabla\cdot \underline{v}$ onto $W_K(\hat{T}) = P_K(\hat{T})$.

Then, using Bramble-Hilbert,

$$\|\nabla\cdot(\Pi_T^K \underline{v} - \underline{v})\|_{0,\hat{T}} = \inf_{p\in P_K(\hat{T})} \|\nabla\cdot \underline{v} - p\|_{0,\hat{T}}$$

$$\leq C\, |\nabla\cdot v|_{K+1,\hat{T}}$$

and by scaling we can show that the estimates in Lemma 4 hold, i.e.,

$$\text{i)}\ \|\Pi_T^K \underline{v} - \underline{v}\|_{0,T} \leq C\, h^{K+1} |\underline{v}|_{K+1,R}$$

(36) $$\text{ii)}\ \|\nabla\cdot(\Pi_T^K \underline{v} - \underline{v})\|_{0,T} \leq C\, h^{K+1} |\nabla\cdot v|_{K+1,R}.$$