Introduction to the Theory of Poroelasticity

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CHAPTER 1
REVIEW OF THE THEORY OF ELASTICITY

This chapter gives a brief review of the classical theory of elasticity. For a more detailed treatment of the subject, we refer to [7].

The Stress Tensor

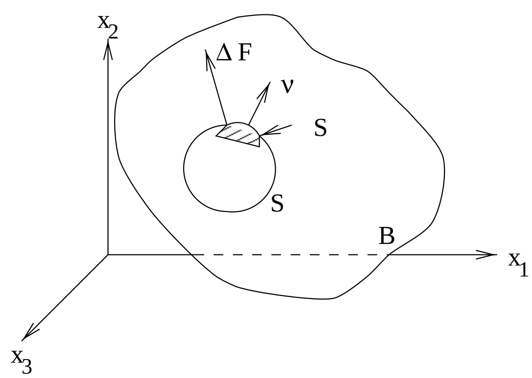


FIGURE 1. $\boldsymbol{\nu}$ = UNIT OUTWARD NORMAL TO ΔS, B = THREE–DIMENSIONAL BODY, AND S = CLOSED SURFACE WITHIN B.

Consider the part of material lying on the positive side of $\boldsymbol{\nu}$. That part of the material exerts a force ΔF on the other part situated in the $-\boldsymbol{\nu}$ direction. (ΔF is a function of ΔS and the orientation of the surface.)

Assumption: There exist

$$\lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} = T^{\nu} = \frac{dF}{dS}.$$

T^{ν} = stress vector or traction; it represents the force per unit area acting on the surface. Also, assume that the moment of the forces acting on ΔS about any point within ΔS vanishes in the limit. (Stress Principle of Euler–Cauchy). Now consider a special case in which S represents the face of a unit cube.

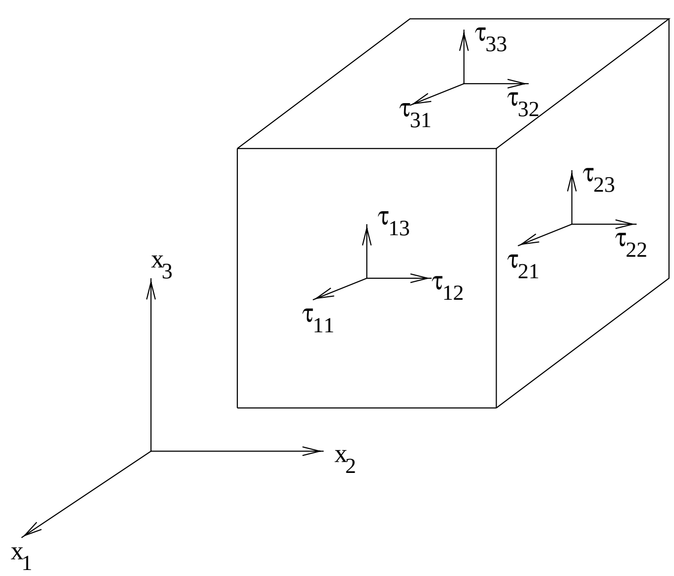


FIGURE 2.

Let $\overset{k}{T} = (\overset{k}{T}_1, \overset{k}{T}_2, \overset{k}{T}_3)$ be the stress vector acting on ΔS_k. Let *the normal* to ΔS_k be in the positive direction of the x_k–axis. Set

$$\overset{k}{T}_1 = \tau_{k1}, \qquad \overset{k}{T}_2 = \tau_{k2}, \qquad \overset{k}{T}_3 = \tau_{k3}.$$

τ_{ii}: normal stresses
τ_{ij}, $i \neq j$: shearing stresses.
τ_{ij}: force in the j–direction acting on the plane $x_i =$ constant.

Laws of Motion. (Euler's equations).

Let (x_1, x_2, x_3) be an inertial frame of reference and let $B(t)$ be the space occupied by a body at time t.

Let

$$\begin{aligned} \mathbf{r} &= \text{position vector of a particle with respect to the} \\ &\quad\ \text{origin of the coordinate system, and} \\ \mathbf{v} &= \text{particle velocity at } (x_1, x_2, x_3). \end{aligned}$$

Then, define

$$P = \int_{B(t)} \mathbf{v}\rho,\, dv, \qquad \text{(linear momentum)}, \tag{1.1}$$

$$H = \int_{B(t)} \mathbf{r} \times \mathbf{v}\rho\, dv, \quad \text{(moment of momentum)}, \tag{1.2}$$

where ρ is the material density.

Newton's Laws.

$$\dot{P} = \mathcal{F}, \tag{1.3}$$

(rate of change of linear momentum = total applied force $\mathcal{F}$);

$$\dot{H} = \mathcal{L}, \tag{1.4}$$

(rate of change of moment of momentum = total applied torque $\mathcal{L}$).

The torque $\mathcal{L}$ is taken with relation to the same point as the origin of the position vector r.

External forces:

1) body forces,
2) surface forces on stresses.

Examples:

1) gravitational forces,
2) pressure due to mechanical contact of two bodies.

Representation of body forces: $\int X\,dv$.

$X = (X_1, X_2, X_3) =$ dimensions of the force per unit volume (in gravitational forces $X_i = \rho g_i$). Then,

$$\mathcal{F} = \text{ total force } = \oint \overset{\nu}{T}\,ds + \int_B X\,dv \tag{1.5}$$

Likewise, the torque S about the origin is

$$\mathcal{L} = \int_S \mathbf{r} \times \overset{\nu}{T}\,ds + \int_B \mathbf{r} \times X\,dv. \tag{1.6}$$

Next, consider a small area ΔS as in Figure 3 below (small box).

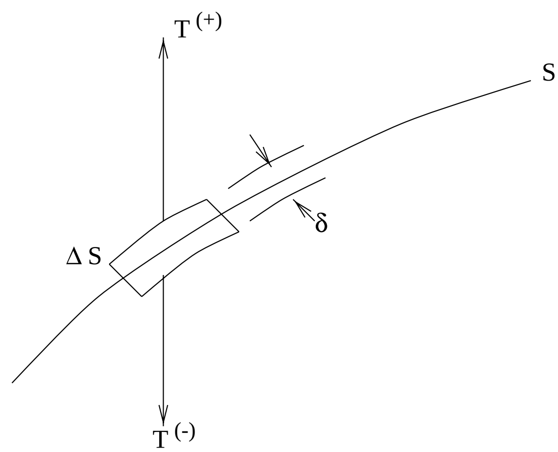


FIGURE 3.

Assume that $\delta \to 0$ while ΔS remains small but finite. Then, $\int_B X\,dv \to 0$ ($B = \Delta S \times \delta$). Also,

$$\oint_S \overset{\nu}{T}\,dx \to 0$$

on the lateral surfaces of the small box $\Delta S \times \delta$.

Also,

$$P = \int_{B\times\delta} \mathbf{v}\rho\,dv \underset{\delta\to 0}{\longrightarrow} 0, \qquad \dot{P} \underset{\delta\to 0}{\longrightarrow} 0.$$

Thus, it follows from (1.3) that

$$0 = \oint \overset{\nu}{T} ds \approx \Delta S(T^{(+)} + T^{(-)})$$

so that

$$\boxed{T^{(-)} = -T^{(+)}}. \tag{1.7}$$

Next, we want to show that if we know the components τ_{ij}, we can write the stress vector $\overset{\nu}{T}$ acting on any surface with outer normal $\boldsymbol{\nu}$ in the form

$$\overset{\nu}{T}_i = \nu_j \tau_{ji}.$$

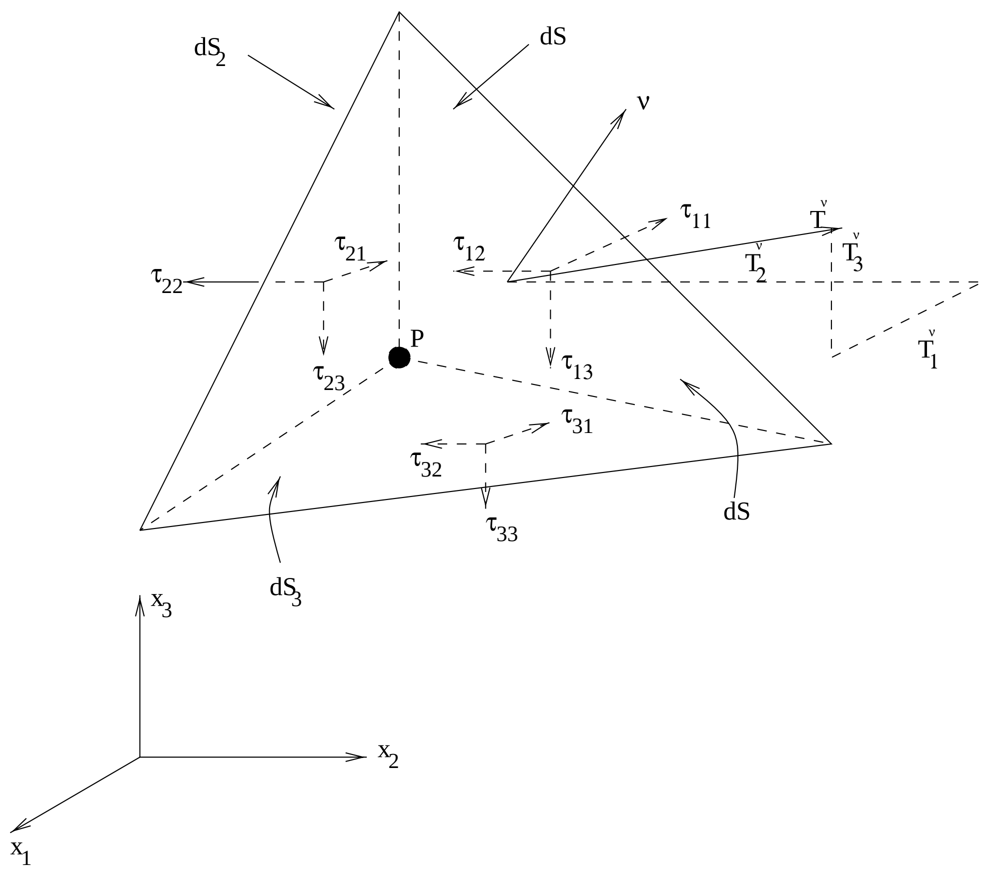


FIGURE 4.

Note that

$$\begin{aligned} dS_1 &= dS\cos(\nu, x_1) = \nu_1 dS \\ &= \text{ area of surface parallel to the } x_2x_3\text{–plane}, \\ dS_2 &= \nu_2 dS, \\ dS_3 &= \nu_3 dS. \end{aligned}$$

The tetrahedron volume is

$$dV = \frac{1}{3} h\, dS, \qquad h = \text{dist(vector } P, \text{ surface base } dS).$$

The forces acting in the positive x_1–direction on the three coordinate surfaces ($dS_1, dS_2,$ and dS_3) can be written as

$$(-\tau_{11} + \varepsilon_1) dS_1, \qquad (-\tau_{21} + \varepsilon_2) dS_2, \qquad (-\tau_{31} + \varepsilon_3) dS_3.$$

The negative sign in τ_{11} is taken because the outer normal ν_1 is opposite in sense with the x_1–axis, likewise, for x_2, x_3. If the stress field is continuous, the ε_i's are infinitesimal. The ε_i's are added because the tractions are acting at a point slightly different from P. Also, the force acting on the triangle normal to ν has a component $(\overset{\nu}{T}_1+\varepsilon)ds$ in the x_1–direction, the body force has an x_1–component $(X_1+\varepsilon')dv$, and the rate of change of linear motion has a component $\rho\dot{v}_1\,dv$ (ε', ε are infinitesimal and T_1^ν, X_1 refer to the point P). The first equation of motion is

$$(-\tau_{11}+\varepsilon_1)\nu_1 dS + (-\tau_{21}+\varepsilon_2)\nu_2 dS + (-\tau_{31}+\varepsilon_3)\nu_3 dS$$
$$+ (\overset{\nu}{T}_1+\varepsilon)dS + (x_1+\varepsilon')\frac{1}{3}h\,dS = \rho\dot{v}_1\frac{1}{3}h\,dS.$$

Dividing by dS and taking the limit $h\to 0$, we obtain

$$\overset{\nu}{T}_1 = \tau_{11}\nu_1 + \tau_{21}\nu_2 + \tau_3, \nu_3.$$

In general,

$$\overset{\nu}{T}_i = \nu_j\tau_{ij}. \tag{1.8}$$

Since (1.8) is valid for any vector ν, it follows from the quotient rule that τ_{ij} is a tensor called the stress tensor.

Equations of Equilibrium

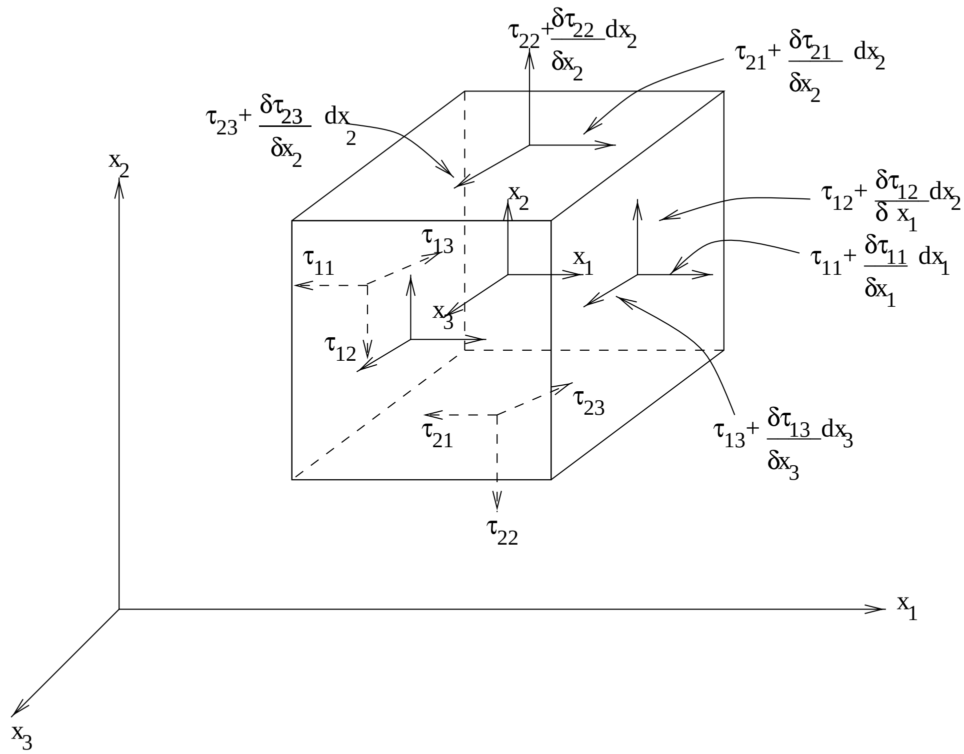


Figure 5.

Consider the static equilibrium of an infinitesimal parallelepiped with surfaces parallel to the coordinate planes (see Figure 5) ($dx_1 dx_2 dx_3 =$ volume of the parallellepiped). On the face $x_1 = 0$, we have the force $\tau_{11} dx_2 dx_3$, while on the face $x_1 = dx_1$, we have the force

$$\left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1\right) dx_2 dx_3, \text{ etc.}$$

The body force is $X_i dx_1 dx_2 dx_3$, $i = 1, 2, 3$. For equilibrium, the resulting force must vanish. Considering the forces in the x_1–direction, we have that

$$\left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \tau_{11} dx_2 dx_3 + \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 - \tau_{21} dx_1 dx_3$$
$$+ \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3\right) dx_2 dx_1 - \tau_{31} dx_2 dx_1 + X_1 dx_1 dx_2 dx_3 = 0.$$

Dividing by $dx_1 dx_2 dx_3$, we obtain

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + X_1 = 0.$$

Similarly for x_2, x_3 directions, then,

$$\boxed{\frac{\partial \tau_{ij}}{\partial x_j} + X_i = 0.} \tag{1.9}$$

Using moments we can see that

$$\tau_{ij} = \tau_{ji}. \tag{1.10}$$

Principal Stresses

A plane defined by a vector ν such that the stress vector is normal to the surface and the shearing stresses vanish is called a *principal plane,* and the value of the normal stress acting on the principal plane is a *principle stress.*

Let ν be a principal axis and σ the principal stress. Then,

$$\overset{\nu}{T} = \sigma \nu_i = \sigma \delta_{ji} \nu_j.$$

Also,

$$\overset{\nu}{T} = \tau_{ji} \nu_j;$$

thus,

$$\tau_{ji} \nu_j = \sigma \delta_{ji} \nu_j, \qquad (\tau_{ji} - \sigma \delta_{ji}) \nu_j = 0, \qquad i = 1, 2, 3.$$

We must find nonzero solutions such that

$$|\boldsymbol{\nu}| = 1.$$

The equation

$$\det(\tau_{ij} - \sigma\delta_{ij}) = 0 \tag{1.11}$$

gives a cubic equation for σ, the principal stress. Expanding (1.11) we have:

$$(\tau_{ij} - \sigma\delta_{ij}) = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0, \tag{1.12}$$

where

$$I_1 = T_s(\tau) = \tau_{11} + \tau_{22} + \tau_{33}, \qquad \text{(linear invariant)},$$

$$I_2 = \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix}, \qquad \text{(quadratic invariant)},$$

$$I_3 = \det\tau = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}, \qquad \text{(cubic invariant)}.$$

Let σ_1, σ_2, σ_3 be the roots of (1.11). Then,

$$(\sigma - \sigma_1)(\sigma - s_2)(\sigma - \sigma_3) = 0,$$
$$-\sigma^3 + (\sigma_1 + \sigma_2 + \sigma_3)\sigma^2 - \sigma(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) + \sigma_1\sigma_2\sigma_3 = 0,$$

so that

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \qquad I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3, \qquad I_3 = \sigma_1\sigma_2\sigma_3. \tag{1.13}$$

The principal stresses characterize the physical state of stress at a point and, consequently, must be independent of the coordinate system. Then, (1.12) and the coefficients I_1, I_2, I_3 are invariant with respect to the coordinate transformation. I_1, I_2, I_3 are the *invariant* of the stress tensor τ_{ij}. In the principal axis,

$$\tau_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}.$$

Strain Tensor

Solid bodies are deformed when subjected to forces, and the distances between material points vary. Consider a point of a solid body, represented by its position vector $\mathbf{x} = (x_1, x_2, x_3)$. After deformation, $\mathbf{x}$ becomes the point $\mathbf{x}'$. Displacement during deformation is then characterized by the vector

$$\mathbf{u} = \mathbf{x} - \mathbf{x}'.$$

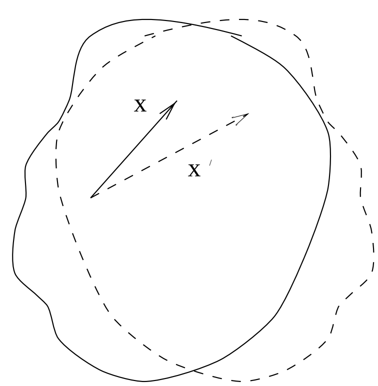


FIGURE 6.

The distance between two infinitely close points *before* deformation was

$$d\ell = (dx_1^2 + dx_2^2 + dx_3^2)^{1/2}$$

and, after deformation, it becomes

$$\begin{aligned} d\ell' &= (dx_1')^2 + (dx_2')^2 + (dx_3')^2 \\ &= \left((dx_1 + du_1)^2 + (dx_2 + du_2)^2 + (dx_3 + du_3)^2\right)^{1/2}. \end{aligned}$$

Now, use that

$$du_i = \frac{\partial u_i}{\partial x_k} dx_k$$

(sum over repeated indices from now on). Then,

$$\begin{aligned} (d\ell')^2 &= (dx_i + du_i)(dx_i + du_i) = dx_i dx_i + 2du_i dx_i + du_i du_i \\ &= dx_i dx_i + 2u_{i,k} dx_k dx_i + u_{i,k} dx_k u_{i,\ell} dx_\ell. \end{aligned}$$

Next, use that

$$\begin{aligned} 2u_{i,k} dx_k dx_i &= u_{i,k} dx_i dx_k + u_{i,k} dx_i dx_k \quad \begin{matrix} i \to k \\ k \to \ell \end{matrix} \\ &= u_{i,k} dx_i dx_k + u_{k,\ell} dx_\ell dx_k \quad \ell \to i \\ &= u_{i,k} dx_i dx_k + u_{k,i} dx_i dx_k = (u_{i,k} + u_{k,i}) dx_i dx_k. \end{aligned}$$

Next,

$$\begin{aligned} u_{i,k} u_{i,\ell} dx_k dx_\ell &= u_{j,k} u_{j,i} dx_k dx_i \quad \begin{matrix} \ell \to i \\ i \to j \end{matrix} \quad j \to \ell, \\ &= u_{\ell,k} u_{\ell,i} dx_k dx_i. \end{aligned}$$

Then,

$$\begin{aligned} (d\ell')^2 &= (d\ell)^2 + \Big[(u_{i,k} + u_{k,i}) + u_{\ell,k} u_{\ell,i}\Big] dx_i dx_k \\ &= (d\ell)^2 + 2\varepsilon_{ik} dx_i dx_k, \end{aligned}$$

where

$$\varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i} + u_{\ell,i} u_{\ell,k}) = \text{ the strain tensor.} \tag{1.14}$$

For small deformations, the variations in distances between material points and, hence, variation in displacements, are small. Then, we can discard the products $u_{\ell,i} u_{\ell,k}$. The *linearized* strain tensor is given by

$$\varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}). \tag{1.15}$$

Physical Interpretation

Consider a variation in length in the x_1–direction. Then,

$$(d\ell')^2 = (dx_1')^2, \qquad (d\ell)^2 = dx_1^2, \qquad dx_k = 0, \qquad k \neq 1,$$
$$d\mathbf{x}_1 = (dx_1, 0, 0), \qquad d\mathbf{x}_1' = (dx_1', 0, 0),$$

thus,

$$(dx_1')^2 = (dx_1)^2(1 + 2\varepsilon_{11}). \tag{1.16}$$

Set

$$\delta x_1 = \frac{\|d\mathbf{x}_1'\| - \|d\mathbf{x}_1\|}{\|d\mathbf{x}_1\|}.$$

Then,

$$\|d\mathbf{x}_1'\| = \|d\mathbf{x}_1\|(1 + \delta x_1),$$

so that

$$(dx_1')^2 = (dx_1)^2(1 + \delta x_1)^2. \tag{1.17}$$

From (1.16) and (1.17),

$$(1 + \delta x_1)^2 = 1 + 2\varepsilon_{11}.$$

Since

$$(1 + \delta x_1)^2 = 1 + 2\delta x_1 + (\delta x_1)^2 \approx 1 + 2\delta x_1$$

for small δx_1, we have

$$\varepsilon_{11} = \delta(x_1), \tag{1.18}$$

and from (1.17),

$$|dx_1'| = (1 + \varepsilon_{11})|dx_1|.$$

Then, the ε_{ii}'s correspond to linear dilation in the x_i–direction.

Now consider two vectors $d\mathbf{x}$, $d\mathbf{y}$ that, after transformation, become $d\mathbf{x}'$, $d\mathbf{y}'$. Recall that

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}(x), \qquad \mathbf{y}' = \mathbf{y} + \mathbf{u}(y).$$

Then,

$$\begin{aligned}
d\mathbf{x}' \cdot d\mathbf{y}' &= (d\mathbf{x} + d\mathbf{u}) \cdot (d\mathbf{y} + d\mathbf{u}) \\
&= (dx_i + u_{i,k}dx_k)(dy_i + u_{i,\ell}dy_\ell) \\
&\cong dx_i dy_i + u_{i,k}dx_k dy_i + u_{i,\ell}dy_\ell dx_i \qquad \begin{matrix} \ell \to k \\ i \to \ell \end{matrix} \\
&= d\mathbf{x} \cdot d\mathbf{y} + u_{i,k}dx_k dy_i + u_{\ell,k}dy_k dx_\ell \qquad \ell \to i \\
&= d\mathbf{x} \cdot d\mathbf{y} + (u_{i,k} + u_{k,i})dx_k dy_i \\
&= d\mathbf{x} \cdot d\mathbf{y} + 2\underset{\downarrow}{\varepsilon_{ik}} dx_k dy_i.
\end{aligned}$$

$i \to k \quad k \to i$ does not change ε

Then,

$$dx' \cdot dy' = dx \cdot dy + 2\varepsilon_{ik} dx_i dy_k.$$

Assume that

$$d\mathbf{x} = d\mathbf{x}_1 = (dx_1, 0, 0), \qquad d\mathbf{y} = d\mathbf{x}_2 = (0, dx_2, 0), \qquad dx_1 > 0, \qquad dx_2 > 0,$$

are initially orthogonal. Recall that

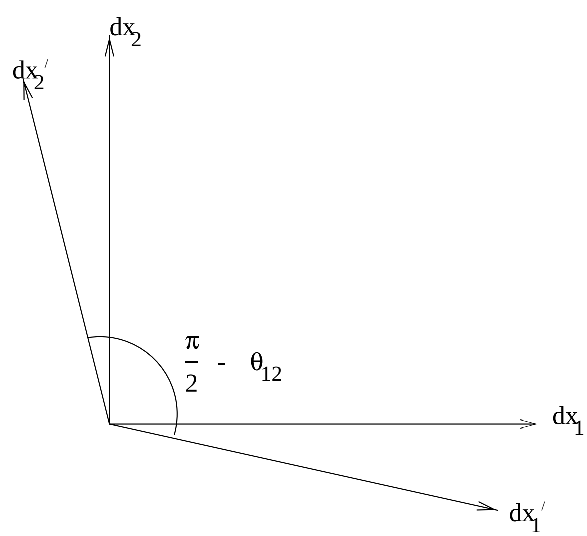
$$A \cdot B = |A|\,|B| \cos(A, B).$$

From (1.18),

$$\begin{aligned} d\mathbf{x}_1' \cdot d\mathbf{x}_2' &\simeq |d\mathbf{x}_1'|\,|d\mathbf{x}_2'| \cos(d\mathbf{x}_1', d\mathbf{x}_2') \\ &= dx_1 dx_2 (1+\varepsilon_{11})(1+\varepsilon_{22}) \cdot \cos(d\mathbf{x}_1', d\mathbf{x}_2') \\ &= 2\varepsilon_{12} dx_1 dx_2 + d\mathbf{x}_1 \cdot d\mathbf{x}_2 = 2\varepsilon_{12} dx_1 dx_2, \end{aligned} \tag{1.19}$$

but note that

$$(1+\varepsilon_{11})(1+\varepsilon_{22}) = 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{11}\varepsilon_{22} + \cdots \approx 1.$$



Also,

$$\cos(d\mathbf{x}_1' d\mathbf{x}_2') = \cos\left(\frac{\pi}{2} - \theta_{12}\right) = \cos\frac{\pi}{2}\cos\theta_{12} + \sin\frac{\pi}{2}\sin\theta_{12} \simeq \theta_{12} \tag{1.20}$$

for θ_{12} small. Then, from (1.19) and (1.20),

$$\theta_{12} = 2\varepsilon_{12}. \tag{1.21}$$

Thus, the nondiagonal elements characterize the change in angle between two basic vectors. The strain tensor has real eigenvectors, whose directions are called (orthogonal) principal strain directions. Let ε_I, ε_{II}, and ε_{III} be the diagonal strain tensor in such a reference frame. They are the principal strains. The elementary volume dv, built on the principal directions, is

$$dV = dx_I dx_{II} dx_{III}.$$

When transformed, $dx'_1 = (1+\varepsilon_I)dx_I$, etc. Then,

$$dV' = (1+\varepsilon_I)(1+\varepsilon_{II})(1+\varepsilon_{III})dV \approx dV + (\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})dV.$$

Up to the first order, the volumetric strain is now

$$dV' - dV \equiv \Delta V = \theta dV,$$

so that

$$\frac{dV'-dV}{dV} = \frac{\Delta V}{dV} = \theta \quad \text{(volume change/unit volume)}, \tag{1.22}$$
$$\theta = tr\,\varepsilon = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III} \quad \text{(tensorial invariant)},$$
$$\theta = \nabla \cdot u \quad \text{(displacement divergence)}.$$

The Equations of Motion

A body is composed of particles. To label the particles of the body, we choose a Cartesian frame of reference and identify the particles by its coordinates (a_1, a_2, a_3) at $t = 0$. At a later time, the particle has moved to another point of coordinates (x_1, x_2, x_3) *in the same* coordinate system. The relation

$$x_i = \widehat{x}_i(a_1, a_2, a_3, t), \qquad i = 1, 2, 3, \tag{1.23}$$

links the configurations of the body at different times t. The functions $\widehat{x}_i$ are single–valued, and the Jacobian is not zero. (See Figure 7.) Assume that there exists a positive quantity ρ (density), such that

$$\rho(\mathbf{x}) = \lim_{r_k \to 0} \frac{\text{mass of } B(\mathbf{x}, r_k)}{\text{volume of } B(\mathbf{x}, r_k)}, \tag{1.24}$$

where $B(\mathbf{x}, r_k)$ denotes the ball of radius r_k centered at $\mathbf{x}$.

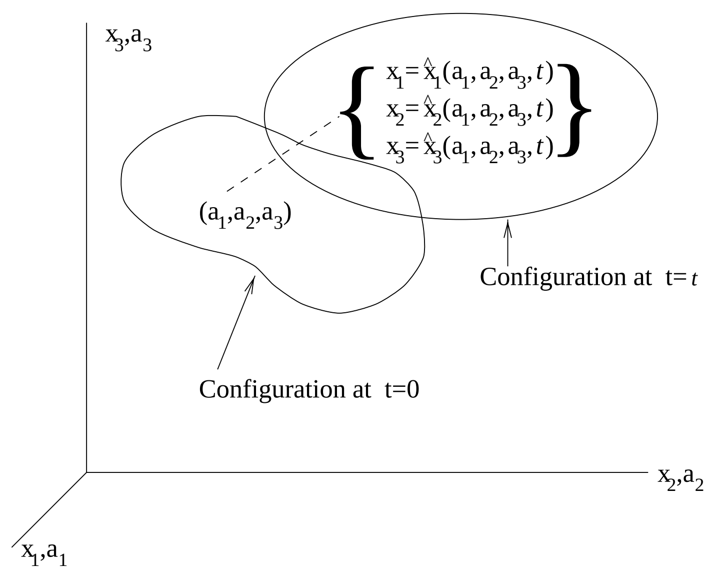


FIGURE 7

Assume that mass is conserved at all times. At $t = 0$, let $\rho_0(\mathbf{a})$ be the density at the point (a_1, a_2, a_3). Conservation of mass states that

$$\int \rho(\mathbf{x}) d\mathbf{x} = \int \rho_0(\mathbf{a}) d\mathbf{a},$$

where the integration is over the same set of particles. Since

$$\int \rho(\mathbf{x}) d\mathbf{x} = \int \rho(x) \left| \frac{\partial x_i}{\partial a_j} \right| d\mathbf{a},$$

we have that

$$\rho_0(\mathbf{a}) = \rho(\mathbf{x}) \left| \frac{\partial x_i}{\partial a_j} \right|, \qquad \rho(\mathbf{x}) = \rho_0(\mathbf{a}) \left| \frac{\partial a_i}{\partial x_j} \right|, \tag{1.25}$$

since the relation holds for all bodies. Here, $\left| \frac{\partial a_i}{\partial x_j} \right|$ denotes the determinant of the matrix $\left\{ \frac{\partial a_i}{\partial x_j} \right\}$. For a particle (a_1, a_2, a_3), whose trajectory is described by

$$x_i(a, t) = \widehat{x}_i(a_1, a_2, a_3, t),$$

the velocity is

$$v_i(\mathbf{a}, t) = \frac{\partial}{\partial t} x_i(\mathbf{a}, t),$$

and the acceleration is

$$\dot{v}_i(\mathbf{a}, t) = \frac{\partial^2}{\partial t^2} x_i(\mathbf{a}, t)$$

($\mathbf{a}$ is held constant). A description of mechanical evolution that uses (a_1, a_2, a_3, t) as independent variables is called *material description.* In hydrodynamics, usually the *spatial description* is chosen where the location is (x_1, x_2, x_3), and time t is used as independent variables. This is convenient because measurements in many materials are better interpreted in terms of what happens at a certain place, rather than following the particles. These two methods are called Lagrangian (material description) or Eulerian (spatial description):

$$\begin{aligned} a_1, a_2, a_3, t: &\quad \text{Lagrangian variables,} \\ (x_1, x_2, x_3, t): &\quad \text{Eulerian variables.} \end{aligned}$$

They are related by

$$x_i(\mathbf{a}, t) = \widehat{x}_i(\mathbf{a}, t). \tag{1.26}$$

In the spatial description $(\mathbf{x}, t)$ the instantaneous motion is described by the vector field $v_i(\mathbf{x}, t)$ associated with the instantaneous location of the particle (x_1, x_2, x_3) at time x. Then note that

$$\begin{aligned} \dot{v}_i(x, t) dt &= v_i(x_j + v_j dt, t + dt) - v_i(x, t) \\ &= v_i + \frac{\partial v_i}{\partial t} dt + \frac{\partial v_i(x, t)}{\partial x_j} v_j dt - v_i, \end{aligned}$$

so that

$$\dot{v}_i(x,t) = \frac{\partial v_i}{\partial t}(x,t) + v_j \frac{\partial v_i}{\partial x_j}$$

or

$$\dot{\mathbf{v}} = \underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{local part}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{\text{convective part}} \quad . \tag{1.27}$$

Considering any function $F(x_1, x_2, x_3, t)$, we can define the *material derivative*

$$\dot{F} \equiv \frac{DF}{Dt} \equiv \left.\frac{\partial F}{\partial t}\right|_{\mathbf{x}=\text{const}} + v_j \frac{\partial F}{\partial x_j} \equiv \left(\frac{\partial F}{\partial t}\right)_{\mathbf{a}=\text{const}}, \tag{1.28}$$

where $\mathbf{a} = (a_1, a_2, a_3) \equiv$ Lagrangian coordinate of the particle which is located at $\mathbf{x}$ at time t.

Material Derivative of a Volume Integral

We have that

$$I = \int_V A(x,t)dv, \tag{1.29}$$

where $A(x,t)$: a property of the continuum (example: mass).

We wish to know how fast the body itself sees the value of I changing. The particle x_i at time t will have coordinate $x_i + v\,dt = x_i'$ at time $t + dt$. The boundary S at time t will have moved to S' at $t + dt$, which bounds a domain V'. The material derivative of I is defined as

$$\frac{DI}{Dt} = \lim_{dt \to 0} \frac{1}{dt} \left[\int_{V'} A(x', t + dt)dv' - \int_V A(x,t)dv \right]. \tag{1.30}$$

There are two contributions to the right hand side of (1.30). One is over the region $V_0 = V \cap V'$. Such part contributes with

$$\lim_{dt \to 0} \frac{1}{dt} \int_{V_0} [A(x, t + dt) - A(x,t)]dx = \int_V \frac{\partial A}{\partial t} dv.$$

The other contribution comes from the value of A on S multiplied by the volume swept by the particles on the boundary in a time dt. Since the displacement of a particle on the boundary is $v_i dt$, the volume swept by particles occupying an area ds is $dV = \mathbf{v} \cdot \boldsymbol{\nu} dt\, ds$.

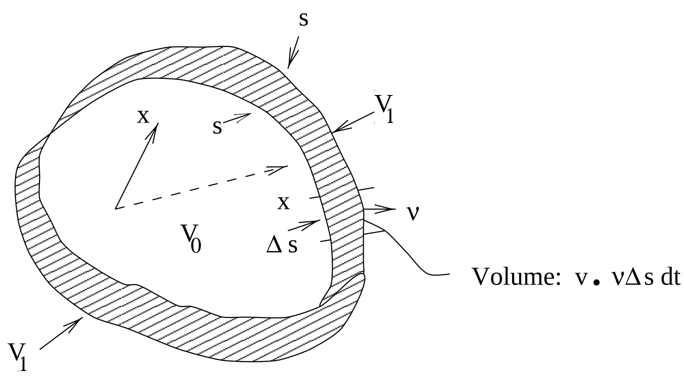


FIGURE 8

Then, the contribution of this part to $\dfrac{DI}{Dt}$ is $A\mathbf{v}\cdot\boldsymbol{\nu}ds$, and the total contribution is obtained integrating over S. Then,

$$\begin{aligned}\frac{D}{Dt}\int_V A(x,t)dv &= \int_V \frac{\partial A}{\partial t}dv + \int_S A\mathbf{v}\cdot\boldsymbol{\nu}ds\\ &= \int_V \left[\frac{\partial A}{\partial t} + \nabla\cdot(A\mathbf{v})\right]dv = \int_V \left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_j}v_j + A\frac{\partial v_j}{\partial x_j}\right)dv\\ &= \int_V \left(\frac{DA}{Dt} + A\nabla\cdot\mathbf{v}\right)dt = \frac{D}{Dt}\int_V A(x,t)dv.\end{aligned}$$

Thus,

$$\frac{D}{Dt}\int_V A(x,t)dv = \int_V \left(\frac{DA}{Dt} + A\nabla\cdot\mathbf{v}\right)dt = \int_V \left[\frac{\partial A}{\partial t} + \nabla\cdot(A\mathbf{v})\right]dv, \tag{1.31}$$

and also,

$$\frac{D}{Dt}\int_V A(x,t)dv = \int_V \frac{\partial A}{\partial t}dv + \int_S A\mathbf{v}\cdot\boldsymbol{\nu}ds. \tag{1.32}$$

Continuity Equation

Let

$$m = \int_V \rho dv$$

be the mass contained in a region at time t where $\rho = \rho(x,t)$ is the density. Conservation of mass implies that $\dfrac{Dm}{Dt} = 0$. From (1.31) and (1.32), for $A = \rho$ we get

$$\begin{aligned}&\text{i)} \quad \int_V \frac{\partial\rho}{\partial t}dv + \int_S \rho\mathbf{v}\cdot\nu ds = 0,\\ &\text{ii)} \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0,\\ &\text{iii)} \quad \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0.\end{aligned} \tag{1.33}$$

Equation of Motion

At an instant of time t, a region V of the space contains the linear momentum

$$P_i = \int_V \rho v_i dv.$$

If the body is subjected to surface tractions $\overset{\nu}{T}_i$ and body force/unit volume X_i, the resultant force is

$$\mathcal{F}_i = \int_S \overset{\nu}{T}_i ds + \int_V X_i dv.$$

We have seen that (c.f., 1.8) $\overset{\nu}{T}_i = \nu_j \tau_{ij}$,

$$\overset{\nu}{T}_i = \nu_j \tau_{ij}.$$

Thus,

$$\mathcal{F}_i = \int_S \tau_{ij}\nu_j ds + \int_V X_i dx = \int_V \left(\frac{\partial \tau_{ij}}{\partial x_j} + X_i\right) dv.$$

Then, using (1.31),

$$\begin{aligned}\frac{D}{Dt}\int_V \rho v_i dv &= \int_v \left[\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j)\right] dv \\ &= \int_V \left[v_i \frac{\partial \rho}{\partial t} + \frac{\rho \partial v_i}{\partial t} + v_i \frac{\partial(\rho v_j)}{\partial x_j} + \rho v_j \frac{\partial v_i}{\partial x_j}\right] dv \\ &= \int_V \left\{ v_i \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v})\right]}_{=0 \text{ by } (1.33)} + \underbrace{\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right]}_{\rho \frac{Dv_i}{Dt}} \right\} dv.\end{aligned}$$

Thus,

$$\int_V \rho \frac{Dv_i}{Dt} dv = \int_V \left(\frac{\partial \tau_{ij}}{\partial x_j} + X_i\right) dv.$$

Then, since V is arbitrary, we conclude that

$$\boxed{\rho \frac{Dv_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} + X_i} \qquad i = 1, 2, 3. \tag{1.34}$$

Now, (1.33) and (1.34) give us four equations for the ten unknowns: ρ, v_i (or u_i) and the six stress components τ_{ij}. We need additional relations and assumptions. One such restriction comes from a statement about the mechanical properties of the medium, in the form of a specification of the stress–strain relations (constitutive equations).

Generalized Hooke's Law

We have

$$\tau_{ij} = \tau^{ij} = C^{ijk\ell}\varepsilon_{k\ell}. \tag{1.35}$$

Since $\tau^{ij} = \tau^{ji}$, we must have

$$C^{ijk\ell} = C^{jik\ell}.$$

Next, since $\varepsilon_{k\ell} = \varepsilon_{\ell k}$, we have

$$\begin{aligned}
\tau^{ij} &= \frac{1}{2}C^{ijk\ell}\varepsilon_{k\ell} + \frac{1}{2}C^{ijk\ell}\varepsilon_{\ell k} \qquad k \to \ell, \qquad \ell \to s \\
&= \frac{1}{2}C^{ijk\ell}\varepsilon_{k\ell} + \frac{1}{2}C^{ij\ell s}\varepsilon_{s\ell} \qquad s \to k \\
&= \frac{1}{2}C^{ijk\ell}\varepsilon_{k\ell} + \frac{1}{2}C^{ij\ell k}\varepsilon_{k\ell} \\
&= \underbrace{\frac{1}{2}(C^{ijk\ell} + C^{ij\ell k})}_{\text{symmetrized } C^{ijk\ell} \text{ with respect to } k,\ell}\varepsilon_{k\ell}.
\end{aligned}$$

Then we can always assume that

$$C^{ijk\ell} = C^{ij\ell k}. \tag{1.36}$$

This give us 36 independent coefficients. Assume that there exists a strain energy function W in the form

$$W = \frac{1}{2}C^{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}, \tag{1.37}$$

with the property that

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ij}; \tag{1.38}$$

(i.e., W be an exact differential of the ε_{ij}'s). This will happen if and only if

$$\frac{\partial^2 W}{\partial \varepsilon_{ij}\partial \varepsilon_{k\ell}} = \frac{\partial^2 W}{\partial \varepsilon_{k\ell}\partial \varepsilon_{ij}}. \tag{1.39}$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon_{k\ell}}\left(\frac{\partial W}{\partial \varepsilon_{ij}}\right) &= \frac{\partial}{\partial \varepsilon_{k\ell}}\tau_{ij} = \frac{\partial}{\partial \varepsilon_{k\ell}}C^{ijk\ell}\varepsilon_{k\ell} = C^{ijk\ell}, \\
\frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial W}{\partial \varepsilon_{k\ell}}\right) &= \frac{\partial}{\partial \varepsilon_{ij}}\tau_{k\ell} = \frac{\partial}{\partial \varepsilon_{ij}}C^{k\ell ij}\varepsilon_{ij} = C^{k\ell ij}.
\end{aligned}$$

Thus, we have the conditions

$$C^{ijk\ell} = C^{k\ell ij}. \tag{1.40}$$

This additional condition (1.40) reduces the number of coefficients $C^{ijk\ell}$ to 21.

For isotropic materials (elastic solids),

$$\tau_{ij} = \lambda\delta_{ij}\nabla \cdot u + 2\mu\varepsilon_{ij}(u), \tag{1.41}$$

where λ and μ are called the Lamé constants. The constant μ is also called the shear modulus. To see the validity of (1.41), we proceed as follows (Love, p. 102). In an isotropic solid, (1.35) must be independent of direction. Also, in this case, the strain energy density W in (1.37) must be invariant under orthogonal transformations. Thus, W must be a function of the three invariants I_2, I_2, and I_3 of the strain tensor ε_{ij}; i.e.,

$$W = W(I_1, I_2, I_3). \tag{1.42}$$

Since we want to have a linear stress–strain relation, the I_3–term must be dropped and the strain energy density W becomes quadratic, including only I_1 and I_2. Thus, we use

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = e, \qquad I_2 = \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2.$$

It is convenient to use $-4I_2 = I_2'$; i.e.,

$$\begin{aligned} I_2' =& 4(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) - 4\varepsilon_{11}\varepsilon_{22} - 4\varepsilon_{22}\varepsilon_{33} - 4\varepsilon_{11}\varepsilon_{33} \\ &= 2(\varepsilon_{12}^2 + \varepsilon_{21}^2 + \varepsilon_{13}^2 + \varepsilon_{31}^2 + \varepsilon_{23}^2 + \varepsilon_{32}^2) - 4\varepsilon_{11}\varepsilon_{22} - 4\varepsilon_{22}\varepsilon_{33} - 4\varepsilon_{11}\varepsilon_{33}. \end{aligned}$$

Thus,

$$W = \frac{1}{2}(He^2 + \mu I_2'). \tag{1.43}$$

Now, using (1.38),

$$\begin{aligned} \frac{\partial W}{\partial \varepsilon_{11}} &= \tau_{11} = He + \mu(-2\varepsilon_{33} - 2\varepsilon_{22}), \\ \frac{\partial W}{\partial \varepsilon_{22}} &= \tau_{22} = He + \mu(-2\varepsilon_{11} - 2\varepsilon_{33}), \\ \frac{\partial W}{\partial \varepsilon_{33}} &= \tau_{33} = He + \mu(-\varepsilon_{11} - \varepsilon_{22}), \\ \frac{\partial W}{\partial \varepsilon_{12}} &= \tau_{12} = 2\mu\varepsilon_{12}, \quad \frac{\partial W}{\partial \varepsilon_{21}} = \tau_{21} = 2\mu\varepsilon_{21}, \\ \frac{\partial W}{\partial \varepsilon_{13}} &= \tau_{13} = 2\mu\varepsilon_{13}, \quad \frac{\partial W}{\partial \varepsilon_{23}} = \tau_{23} = 2\mu\varepsilon_{23}, \\ \frac{\partial W}{\partial \varepsilon_{31}} &= \tau_{31} = 2\mu\varepsilon_{31}, \quad \frac{\partial W}{\partial \varepsilon_{32}} = 2\mu\varepsilon_{32}. \end{aligned} \tag{1.44}$$

Set

$$H = \lambda + 2\mu.$$

Then,

$$\begin{aligned} \tau_{11} &= (\lambda + 2\mu)e + \mu(-2\varepsilon_{22} - 2\varepsilon_{33}) \\ &= \lambda e + 2\mu(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} - \varepsilon_{22} - \varepsilon_{33}) = \lambda e + 2\mu\varepsilon_{11}, \end{aligned}$$

$$\tau_{22} = \lambda e + 2\mu\varepsilon_{22}, \qquad \tau_{33} = \lambda e + 2\mu\varepsilon_{33}, \qquad \tau_{12} = 2\mu\varepsilon_{12}, \ldots, \text{ etc.}$$

In general,

$$\tau_{ij} = \lambda e \delta_{ij} + 2\mu\varepsilon_{ij},$$

and (1.41) is proved.

Let us analyze the condition on the Lamé coefficients λ and μ so that the quadratic form W is positive definite.

$$\begin{aligned}
2W &= (\lambda + 2\mu)e^2 + \mu I_2' \\
&= \lambda e^2 + 2\mu e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) - 4\mu\varepsilon_{11}\varepsilon_{22} - 4\mu\varepsilon_{22}\varepsilon_{33} - 4\mu\varepsilon_{11}\varepsilon_{33} \\
&= \lambda e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\
&\quad + \frac{2}{3}\mu e^2 + \frac{4}{3}\mu(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 - 4\mu\varepsilon_{11}\varepsilon_{22} - 4\mu\varepsilon_{22}\varepsilon_{33} - 4\mu\varepsilon_{11}\varepsilon_{33} \\
&= \left(\lambda + \frac{2}{3}\mu\right)e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\
&\quad + \frac{4}{3}\mu[(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 - 3\varepsilon_{11}\varepsilon_{22} - 3\varepsilon_{22}\varepsilon_{33} - 3\varepsilon_{11}\varepsilon_{33}]
\end{aligned}$$

$$\begin{aligned}
&= \left(\lambda + \frac{2}{3}\mu\right)e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\
&\quad + \frac{4}{3}\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 - \varepsilon_{11}\varepsilon_{22} - \varepsilon_{11}\varepsilon_{33} - \varepsilon_{22}\varepsilon_{33}) \\
&= \left(\lambda + \frac{2}{3}\mu\right)e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\
&\quad + \frac{2}{3}\mu((\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
2W = &\left(\lambda + \frac{2}{3}\mu\right)e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\
&+ \frac{2}{3}\mu((\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2).
\end{aligned} \tag{1.45}$$

Now for $\varepsilon_{ii} = \varepsilon_{11}$, $\varepsilon_{ij} = 0$, $i \neq j$, we see that

$$k = \lambda + \frac{2}{3}\mu > 0. \tag{1.46}$$

For $\varepsilon_{ii} = 0$, $\varepsilon_{ij} \neq 0$ must be

$$\mu > 0. \tag{1.47}$$

To analyze the physical significance of the constant k in (1.46), we consider an experiment in which a cube of solid material is subjected to a hydrostatic pressure Δp; i.e.,

$$\tau_{11} = \tau_{22} = \tau_{33} = -\Delta p, \qquad \tau_{ij} = 0, \quad i \neq 1.$$

From (1.41)

$$-\Delta p = \tau_{11} = \lambda e + 2\mu\varepsilon_{11}, \qquad -\Delta p = \tau_{22} = \lambda e + 2\mu\varepsilon_{22}, \qquad -\Delta p = \tau_{33} = \lambda e + 2\mu\varepsilon_{33},$$

or

$$-3\Delta p = (3\lambda + 2\mu)e.$$

Then,

$$e = -\frac{\Delta p}{(\lambda + \frac{2}{3}\mu)} = -\frac{\Delta p}{k} = \frac{\Delta V}{V}. \tag{1.48}$$

Hence, (recall (1.22))

$$k = \lambda + \frac{2}{3}\mu = \text{ bulk modulus of elastic solid.}$$

CHAPTER 2
GASSMANN THEORY FOR FLUID-SATURATED POROUS MEDIA

Following the ideas in [8], in this section we will determine the bulk modulus of a fluid–saturated porous solid. We will assume that porous medium: (1) consists in a solid matrix which behaves like an isotropic homogeneous elastic solid as a whole, (2) the solid grains in the matrix also behave as a homogeneous elastic solid (with a higher compressibility), (3) the poral space is filled with a fluid, (4) all the poral space is interconnected, and (5) the whole system is macroscopically isotropic and homogeneous; i.e., for a cube of bulk material of dimensions much bigger than the pore diameter, the mechanical properties of the cube are identical and independent of the direction.

Let

$V_b =$ volume of a (representative) part of bulk material, big enough with respect to the pore diameter, small enough to be homogeneous,
$V_s =$ volume of the solid part contained in V_b, and
$V_f =$ poral space volume.

Then,

$$V_b = V_s + V_f, \qquad \phi = \frac{V_f}{V_b} = \text{ porosity.} \tag{2.1}$$

Let

$m_b =$ total mass of bulk material contained in V_b,
$m_s =$ mass of the matrix contained in V_b, and
$m_f =$ fluid mass contained in V_b.

Then,

$$\begin{aligned} m_b &= m_s + m_f, \\ \rho_b &= \frac{m_b}{V_b} = \text{ density of the bulk material,} \\ \rho_s &= \frac{m_s}{V_s} = \text{ density of the solid grains building the skeleton,} \\ \rho_f &= \frac{m_f}{V_f} = \text{ fluid density,} \\ \widehat{\rho} &= \frac{m_s}{V_b} = \text{ density of the dry matrix} \qquad (\widehat{\rho} < \rho_s \text{ always}). \end{aligned} \tag{2.2}$$

Note that

$$\widehat{\rho} = \frac{m_s}{V_b} = \frac{m_s}{V_s}\frac{V_s}{V_b} = \frac{m_s}{V_s}\frac{(V_b - V_f)}{V_b} = \rho_s\left(1 - \frac{V_f}{V_b}\right) = \rho_s(1-\phi),$$

$$\widehat{\rho} = \rho_s(1-\phi). \tag{2.3}$$

Also,

$$\rho_b = \frac{m_b}{V_b} = \frac{m_s + m_f}{V_b} = \widehat{\rho} + \frac{m_s}{V_b} = \widehat{\rho} + \frac{m_f}{V_f}\frac{V_f}{V_b} = \widehat{\rho} + \rho_f\phi.$$

Thus,

$$\rho_b = \widehat{\rho} + \phi\rho_f. \tag{2.4}$$

Let us consider a prismatic element Q of bulk material of base area F_b and height h as in Figure 10 (volume $V_b = Fh$):

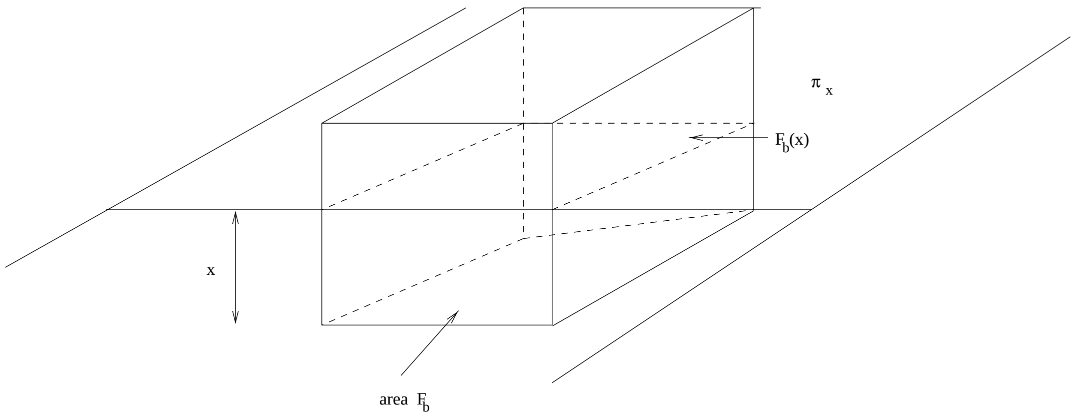


FIGURE 9

(assume F_b and h big compared to the pore diameter). Consider a plane $\pi_x \|$ to the base at a distance x. Let

$$F_b(x) = Q \cap \pi_x,$$
$$F_s(x) = F_b(x) \cap \text{ (solid part of } Q\text{), and}$$
$$F_f(x) = F_b(x) - F_s(x).$$

Then,

$$\frac{1}{h}\int_0^h F_f(x)dx = \frac{1}{h}V_f = \frac{1}{h}\phi V_b = \phi F_b.$$

We now know that the average of $F_f(x)$ over the cube is ϕF_b. If we *assume* that the variations of $F_f(x)$ are small, then

$$F_f = \phi F_b, \tag{2.5}$$

which in turn implies that

$$F_s = F_b - F_f = (1-\phi)F_b. \tag{2.6}$$

Next, consider a point A in the interior of the system and a neighborhood of A of a size much bigger than the pore diameter, but small enough to assume that the hydrostatic pressure in the fluid filling the poral space is uniform and equal to p_f. Now, consider a plane through A intersecting the neighborhood of A (see Figure 11) in a surface of measure F_b.

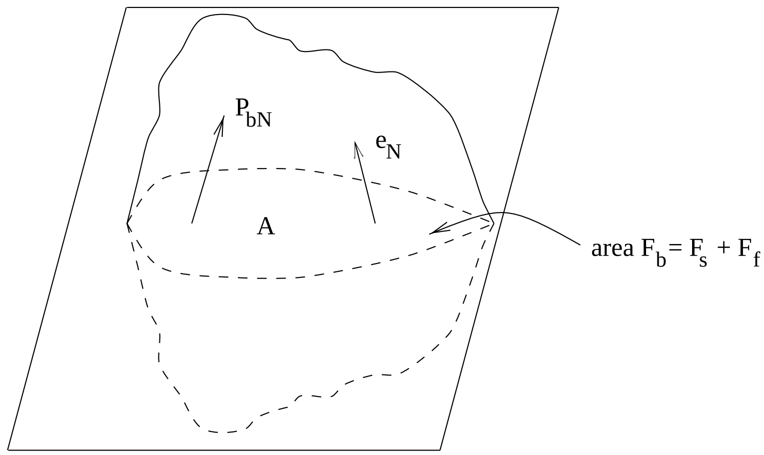


FIGURE 10

Let F_s and F_f be defined as before, and let e_N be a unit normal vector to the plane. Within the system around A, the forces are transmitted across surfaces. Let P_{bN} be the total force transmitted through F_b.

Set

$$\begin{aligned} P_{sN} &= \text{total force transmitted through the solid part of } F_b(F_s) \\ P_{fN} &= \text{total force transmitted through the fluid part of } F_b(F_f) \\ &= F_f p_f e_N \end{aligned}$$

(recall the pressure is uniform and equal to p_f in the neighborhood of A). Then, we have

$$P_{bN} = P_{sN} + F_f p_f e_N. \tag{2.7}$$

It is convenient to decompose P_{sN} into two parts: $P_{sN} = T_1 + \widehat{P}_N$. The term T_1 is chosen such that when added to the hydrostatic fluid pressure $F_f p_f e_N$ acting over the fluid part F_f, it gives the hydrostatic pressure $F_b p_f e_N$ over the whole surface area F_b; i.e.,

$$F_f p_f e_N + T_1 = F_b p_f e_N,$$

so that

$$T_1 = (F_b - F_f) p_f e_N = F_s p_f e_N$$

and

$$P_{sN} = F_s p_f e_N + \widehat{P}_N. \tag{2.8}$$

From (2.7) and (2.8), we see

$$P_{bN} = P_{sN} + F_f p_f e_N = F_s p_f e_N + \widehat{P}_N + F_f p_f e_N = F_b p_f e_N + \widehat{P}_N. \tag{2.9}$$

Set

$$p_N = \frac{P_{bN}}{F_b}, \qquad \widehat{p}_N = \frac{\widehat{P}_N}{F_b}, \tag{2.10}$$

(forces per unit area). Then, from (2.9),

$$p_N = \widehat{p}_N + p_f e_N. \tag{2.11}$$

Since the whole set of tensions P_{bN} and $P_{fN} e_N$ (for all possible directions defined by e_N) are each a tensor of rank 1 (a vector), from (2.11), the whole set of $\widehat{p}_N$ (for all possible directions defined by e_N) is also a tensor $\widehat{\mathbb{P}}$ of components $\widehat{p}_1, \dots, \widehat{p}_6$ (quotient rule) which will be called the *matrix residual stress tensor.*

Elasticity of the Open System.

Consider a part of bulk material of volume V_b with dimensions big enough with respect to the pore diameter. Assume a homogeneous tensional state $\mathbb{P}_b$ of a hydrostatic tension p_f and a residual stress tension $\widehat{\mathbb{P}}$ in the matrix. We will assume that in this tensional state the matrix behaves as elastic. This will allow us to determine the *elastic properties of the matrix* applying tensional changes Δp_f and $\Delta\widehat{\mathbb{P}}$, thereby allowing the fluid to respond to changes in the matrix. This shows why the poral space must be open to the exterior allowing the fluid to flow in or out. Also, the tensional changes must be done slowly to ignore friction effects of the fluid at the pore walls and also to ignore inertial effects. We will also assume that the fluid completely fills the poral space to avoid capillary effects, in which case p_f may not be uniform on V_f.

Since the solid grains building the matrix, the solid matrix and the fluid are assumed to be elastic, the idea is to divide a general change in tensional state $\Delta\mathbb{P}_b$ into parts Δp_f and $\Delta\widehat{\mathbb{P}}$; i.e.,

$$\Delta\mathbb{P}_b = \Delta p_f + \Delta\widehat{\mathbb{P}}, \tag{2.12}$$

and consider as separate cases,

$$\text{i)}\quad \Delta p_f \neq 0, \qquad \Delta\widehat{\mathbb{P}} = 0, \qquad \text{ii)}\quad \Delta p_f = 0, \qquad \Delta\widehat{\mathbb{P}} \neq 0.$$

Later, the general case will be obtained by superposition.

First, consider a tensional state such that

$$\Delta p_f \neq 0, \qquad \Delta\widehat{\mathbb{P}} = 0. \tag{2.13}$$

This corresponds to the so–called unjacketed–compressibility test in [5]. Since, from (2.8)

$$\Delta P_{sN} = F_s \Delta p_f e_N + \underset{\overset{|||}{0}}{\Delta\widehat{P}_N} = F_s \Delta p_f e_N, \tag{2.14}$$

we see that in this case, the matrix is subject to an additional hydrostatic pressure change Δp_f. Therefore, in this case, the matrix reduces its volume but keeps a similar shape; i.e.,

$$\frac{\Delta V_s}{V_s} = \frac{\Delta V_b}{V_b}. \tag{2.15}$$

Let k_s denote the bulk modulus of the solid grains composing the matrix. Then, by definition of k_s,

$$\frac{\Delta V_s}{V_s} = -\frac{\Delta p_f}{k_s}. \tag{2.16}$$

Thus, from (2.15),

$$\left.\frac{\Delta V_b}{V_b}\right|_{\substack{\Delta p_f \neq 0 \\ \Delta\widehat{\mathbb{P}}=0}} = -\frac{\Delta p_f}{k_s}. \tag{2.17}$$

Note that in this tensional state (i.e., in the case of (2.13)) we have that

$$\Delta\phi = 0. \tag{2.18}$$

In fact, using (2.15),

$$\Delta\phi = \Delta\left(\frac{V_f}{V_b}\right) = \Delta\left(\frac{V_b - V_s}{V_b}\right) = \frac{\Delta V_b V_s - V_b \Delta V_s}{V_b^2}$$
$$= \frac{\Delta V_b}{V_b}\frac{V_s}{V_b} - \frac{\Delta V_s}{V_b} = \frac{\Delta V_s}{V_s}\frac{V_s}{V_b} - \frac{\Delta V_s}{V_b} = 0.$$

Note that for a tensional change as in (2.13) it is also true that

$$\frac{\Delta V_f}{V_f} = \frac{\Delta V_b}{V_b}.$$

Next, let

$$\Delta\widehat{p} = \frac{1}{3}(\Delta\widehat{p}_1 + \Delta\widehat{p}_2 + \Delta\widehat{p}_3) \tag{2.19}$$

be the variation of the normal tension corresponding to $\Delta\widehat{\mathbb{P}}$.

Assume a tensional change such that

$$\Delta p_f = 0, \qquad \Delta\widehat{p} = 0, \qquad \Delta\widehat{\mathbb{P}} \neq 0, \tag{2.20}$$

so that only shear tensional changes are applied ($\Delta\widehat{p}_1 = \Delta\widehat{p}_2 = \Delta\widehat{p}_3 = 0$).

From (1.48) we see that in this case

$$e = \frac{\Delta V_b}{V_b} = 0,$$

so that we have matrix deformation without volume change. Such deformation is described with a constant $\widehat{\mu}$, the shear modulus of the matrix. For this tensional change we make the hypothesis that $\Delta\phi = 0$.

Now we consider a tensional change:

$$\Delta p_f = 0, \qquad \Delta\widehat{p} \neq 0. \tag{2.21}$$

If we measure the volume change ΔV_b, then the corresponding matrix bulk modulus $\widehat{k}$ is determined by the relation

$$\left.\frac{\Delta V_b}{V_b}\right|_{\substack{\Delta p_f = 0 \\ \Delta\widehat{p} \neq 0}} = -\frac{\Delta\widehat{p}}{\widehat{k}}. \tag{2.22}$$

Our next objective is to determine the porosity change $\Delta\phi$ for the tensional change (2.21). We use the following result of elasticity theory:

Let a homogeneous isotropic (differentiably) elastic body of arbitrary shape, volume V_s and compressional modulus k_s be compressed between parallel plates located at distance a, with a total pressure ΔH_i (i.e., $\Delta H_i =$ (Force/unit area) (plate area), see Figure 11).

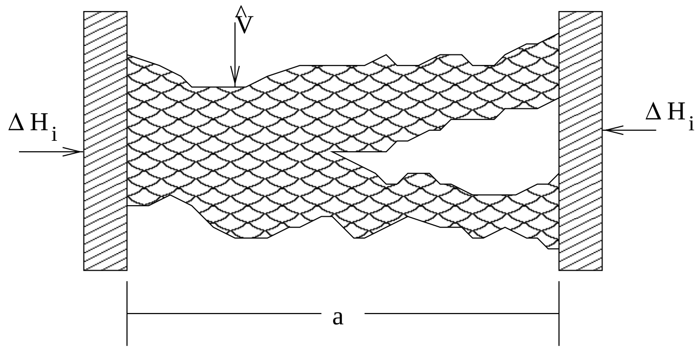


FIGURE 11

Then, the volumetric reduction ΔV_{s_i} is given by

$$\Delta V_{si} = -a\frac{\Delta H_i}{3k_s} \tag{2.23}$$

with a negative sign since the volume is reduced. The proof of (2.23) for a cube Q is as follows. Assume that the tensional state of V_s is described by (1.41) (this is our hypothesis about the elastic behavior of the body). Then consider a tensional state $\tau_{11} = -\Delta H_i$, $\tau_{22} = \tau_{33} = 0$, $\tau_{ij} = 0, \quad i \neq j$. Thus, for a unit cube Q,

$$\begin{aligned} -\Delta H_i &= \tau_{11} = \lambda e + 2\mu\varepsilon_{11}, \\ 0 &= \tau_{22} = \lambda e + 2\mu\varepsilon_{22}, \\ 0 &= \tau_{33} = \lambda e + 2\mu\varepsilon_{33}, \end{aligned}$$

$$\overline{}$$

$$-\Delta H_i = (3\lambda + 2\mu)e = 3\left(\lambda + \frac{2}{3}\mu\right)e = 3k_s e.$$

Since $e = -\frac{\Delta Q}{Q}$,

$$e = -\frac{\Delta Q_i}{Q} = -\frac{\Delta H_i}{3k_s}.$$

Now assume $V_s =$ cube of side a. In this case, $\tau_{11} = a^2 H_i$, $V_s = a^3$, so that

$$-a^2\Delta H_i = 3k_s e = 3k_s\frac{\Delta V_{s_i}}{V_s} = 3k_s\frac{\Delta V_{s_i}}{a^3}.$$

Hence,

$$\Delta H_i = -\frac{3k_s\Delta V_{s_i}}{a},$$

or

$$\Delta V_{s_i} = -\frac{a\Delta H_i}{3k_s},$$

as desired.

Now we apply (2.23) to our case. We consider a part of bulk material which is a cube of side a so that $V_b = a^3$ and let V_s be the matrix volume in the cube. For a tensional change of the form (2.21), the opposite faces of the cube are compressed by the forces

$$\Delta H_i = a^2 \Delta \widehat{p}_i, \qquad i = 1, 2, 3.$$

Thus,

$$\Delta V_s = \sum_{i=1}^{3} \Delta V_{s_i} = \sum_{i=1}^{3} -a \frac{\Delta H_i}{3k_s} = -\frac{a^3}{3k_s} \sum_{i=1}^{3} \Delta \widehat{p}_i = -\frac{V_b \Delta \widehat{p}}{k_s},$$

or

$$\frac{\Delta V_s}{V_b} = -\frac{\Delta \widehat{p}}{k_s}. \tag{2.24}$$

Next, since $V_s = (1 - \phi) V_b$,

$$\Delta V_s = (1 - \phi) \Delta V_b - V_b \Delta \phi,$$

or equivalently

$$\frac{\Delta V_s}{V_b} = (1 - \phi) \frac{\Delta V_b}{V_b} - \Delta \phi.$$

Using (2.22),

$$\frac{\Delta V_s}{V_b} = (1 - \phi) \left(-\frac{\Delta \widehat{p}}{\widehat{k}} \right) - \Delta \phi. \tag{2.25}$$

Now from (2.24) and (2.25),

$$-\frac{\Delta \widehat{p}}{k_s} = -(1 - \phi) \frac{\Delta \widehat{p}}{\widehat{k}} - \Delta \phi.$$

Therefore,

$$\Delta \phi = \left(\frac{1}{k_s} - \frac{(1 - \phi)}{\widehat{k}} \right) \Delta \widehat{p}. \tag{2.26}$$

Elasticity of the Closed System.

We want to apply tensional changes to a part V_b of the bulk material but avoid flow in or out of V_b, which we call a closed system.

We first consider a tensional change of the form

$$\Delta p_f = 0, \qquad \Delta \widehat{p} = 0, \qquad \Delta \widehat{\mathbb{P}} \neq 0. \tag{2.27}$$

As before, $\Delta V_b = 0$, and it is reasonable to assume that the shear modulus characterizing the deformation is independent of whether the system is open or closed, so that $\mu = \widehat{\mu}$.

We now need to determine the bulk modulus k_c of the saturated rock. For this purpose we apply a tensional change Δp_b. As before, we can decompose the tensional change in the form

$$\Delta p_b = \Delta p_f + \Delta \widehat{p}. \tag{2.28}$$

The desired modulus of the closed system is defined by

$$\frac{\Delta V_b}{V_b} = -\frac{\Delta p_b}{k_c}. \tag{2.29}$$

The fluid bulk modulus k_f is defined by

$$\frac{\Delta V_f}{V_f} = -\frac{\Delta p_f}{k_f}. \tag{2.30}$$

Now, using linear superposition and (2.17), (2.22),

$$\frac{\Delta V_b}{V_b} = \frac{\Delta V_b}{V_b}\bigg|_{\substack{\Delta p_f \neq 0 \\ \Delta \widehat{\mathbb{P}}=0}} + \frac{\Delta V_b}{V_b}\bigg|_{\substack{\Delta p_f = 0 \\ \Delta \widehat{p}\neq 0}} = -\frac{\Delta p_f}{k_s} - \frac{\Delta \widehat{p}}{\widehat{k}}. \tag{2.31}$$

Next, using (2.26) and (2.31),

$$\begin{aligned} \frac{\Delta V_s}{V_b} &= \frac{\Delta((1-\phi)V_b)}{V_b} = \frac{-\Delta\phi V_b + (1-\phi)\Delta V_b}{V_b} \\ &= -\Delta\phi + (1-\phi)\frac{\Delta V_b}{V_b} \\ &= \left(\frac{(1-\phi)}{\widehat{k}} - \frac{1}{k_s}\right)\Delta\widehat{p} + (1-\phi)\left(-\frac{\Delta p_f}{k_s} - \frac{\Delta\widehat{p}}{\widehat{k}}\right) \\ &= -(1-\phi)\frac{\Delta p_f}{k_s} - \frac{\Delta\widehat{p}}{k_s}. \end{aligned}$$

Then,

$$\frac{\Delta V_s}{V_b} = -(1-\phi)\frac{\Delta p_f}{k_s} - \frac{\Delta\widehat{p}}{k_s}. \tag{2.32}$$

Now, from (2.29) and (2.31),

$$\begin{aligned} -\frac{\Delta p_f}{k_c} &= -\frac{\Delta p_f}{k_s} - \frac{\Delta\widehat{p}}{\widehat{k}} \\ &= -\frac{(\Delta p_f - \Delta\widehat{p})}{k_s} - \frac{\Delta\widehat{p}}{\widehat{k}} = -\frac{\Delta p_f}{k_s} + \Delta\widehat{p}\left(\frac{1}{k_s} - \frac{1}{\widehat{k}}\right), \end{aligned}$$

so that

$$-\Delta p_f\left(\frac{1}{k_c} - \frac{1}{k_s}\right) = \Delta\widehat{p}\left(\frac{1}{k_s} - \frac{1}{\widehat{k}}\right). \tag{2.33}$$

Next, from (2.30) and (2.31),

$$\begin{aligned}\frac{\Delta V_s}{V_b} &= \frac{\Delta(V-V_f)}{V_b} = \frac{\Delta V_b}{V_b} - \frac{\Delta V_f}{V_b} \\ &= -\frac{\Delta p_f}{k_s} - \frac{\Delta \widehat{p}}{\widehat{k}} - \frac{V_f}{V_b}\frac{\Delta V_f}{V_f} = -\frac{\Delta p_f}{k_s} - \frac{\Delta \widehat{p}}{\widehat{k}} + \phi\frac{\Delta p_f}{k_f}.\end{aligned} \tag{2.34}$$

Now, from (2.34) and (2.32),

$$\begin{aligned}&-\frac{\Delta p_f}{k_s} - \frac{\Delta \widehat{p}}{\widehat{k}} + \phi\frac{\Delta p_f}{k_s} = -(1-\phi)\frac{\Delta p_f}{k_s} - \frac{\Delta \widehat{p}}{k_s}, \\ &\Delta p_f\left(-\frac{1}{k_s} + \frac{\phi}{k_s}\right) = -\frac{(1-\phi)}{k_s}\Delta p_f - \frac{\Delta \widehat{p}}{k_s} + \frac{\Delta \widehat{p}}{\widehat{k}}.\end{aligned}$$

Then,

$$\left(-\frac{1}{k_s} + \frac{\phi}{k_s}\right)(\Delta p_b - \Delta \widehat{p}) = -\frac{(1-\phi)}{k_s}(\Delta p_b - \Delta \widehat{p}) - \frac{\Delta \widehat{p}}{k_s} + \frac{\Delta \widehat{p}}{\widehat{k}}.$$

Dividing by $\Delta \widehat{p}$,

$$\left(-\frac{1}{k_s} + \frac{\phi}{k_f}\right)\left(\frac{\Delta p_b}{\Delta \widehat{p}} - 1\right) = -\frac{(1-\phi)}{k_s}\left(\frac{\Delta p_b}{\Delta \widehat{p}} - 1\right) - \frac{1}{k_s} + \frac{1}{\widehat{k}}. \tag{2.35}$$

Now from (2.33),

$$\Delta p_b\left(\frac{k_s - k_c}{k_c\, k_s}\right) = \Delta \widehat{p}\left(\frac{k_s - \widehat{k}}{k_s\, \widehat{k}}\right)$$

so that

$$\frac{\Delta p_b}{\Delta \widehat{p}} = \left(\frac{k_s - \widehat{k}}{k_s\, \widehat{k}}\right)\left(\frac{k_c\, k_s}{k_s - k_c}\right) = \left(\frac{\widehat{k} - k_s}{k_c - k_s}\right)\frac{k_c}{\widehat{k}}. \tag{2.36}$$

From (2.35) and (2.36),

$$\left(\frac{1}{k_s} - \frac{\phi}{k_f}\right) - \frac{(1-\phi)}{k_s} + \frac{1}{k_s} - \frac{1}{\widehat{k}} = \left(\frac{1}{k_s} - \frac{\phi}{k_f} - \frac{(1-\phi)}{k_s}\right)\frac{\Delta p_b}{\Delta \widehat{p}}$$

or

$$\begin{aligned}&\frac{(k_f - \phi k_s)\widehat{k} - (1-\phi)k_f\,\widehat{k} + (\widehat{k} - k_s)k_f}{k_s\, k_f\, \widehat{k}} \\ &= \frac{\Delta p_f}{\Delta \widehat{p}}\left(\frac{k_f - \phi k_s - (1-\phi)k_f}{k_s\, k_f}\right) \\ &= \left(\frac{\widehat{k} - k_s}{k_c - k_s}\right)\frac{k_c}{\widehat{k}}\left(\frac{k_f - \phi k_s - (1-\phi)k_f}{k_s\, k_f}\right) \\ &= \frac{(\widehat{k} - k_s)}{1 - \frac{k_s}{k_c}}(k_f - \phi k_s - (1-\phi)k_f).\end{aligned}$$

Then,

$$(k_f - \phi k_s)\widehat{k} - (1-\phi)k_f\,\widehat{k} + (\widehat{k} - k_s)k_f = \frac{(\widehat{k} - k_s)}{(1 - \frac{k_s}{k_c})}(k_f - \phi k_s - (1-\phi)k_f). \tag{2.37}$$

From (2.37),

$$\begin{aligned} 1 - \frac{k_s}{k_c} &= \frac{(\widehat{k} - k_s)(k_f - \phi k_s - (1-\phi)k_f)}{(k_f - \phi k_s)\widehat{k} - (1-\phi)k_f\,\widehat{k} + (\widehat{k} - k_s)k_f} \\ &= \frac{(\widehat{k} - k_s)(k_f - \phi k_s - k_f + \phi k_f)}{k_f\,\widehat{k} - \phi k_s\,\widehat{k} - k_f\,\widehat{k} + \phi k_f\,\widehat{k} + \widehat{k}\,k_f - k_s\,k_f} \\ &= \frac{\phi(\widehat{k} - k_s)(k_f - k_s)}{k_f(\widehat{k} - k_s) + \phi\widehat{k}(k_f - k_s)} \equiv \frac{a}{b}. \end{aligned} \tag{2.38}$$

Then,

$$\frac{k_s}{k_c} = 1 - \frac{a}{b} = \frac{b-a}{b}, \qquad k_c = k_s\frac{b}{b-a}. \tag{2.39}$$

Let us compute $b - a$ as follows:

$$\begin{aligned} b - a &= k_f(\widehat{k} - k_s) + \phi\widehat{k}(k_f - k_s) - \phi(\widehat{k} - k_s)(k_f - k_s) \\ &= k_f(\widehat{k} - k_s) + \phi\widehat{k}(k_f - k_s) - \phi\widehat{k}(k_f - k_s) + \phi k_s(k_f - k_s) \\ &= k_f(\widehat{k} - k_s) + \phi k_s(k_f - k_s). \end{aligned} \tag{2.40}$$

Then, from (2.39) and (2.40),

$$k_c = k_s\frac{k_f(\widehat{k} - k_s) + \phi\widehat{k}(k_f - k_s)}{k_f(\widehat{k} - k_s) + \phi k_s(k_f - k_s)}, \tag{2.41}$$

or,

$$k_c = k_s\frac{\widehat{k} + \dfrac{k_f(\widehat{k} - k_s)}{\phi(k_f - k_s)}}{k_s + \dfrac{k_f(\widehat{k} - k_s)}{\phi(k_f - k_s)}}.$$

Set

$$Q = \frac{k_f(k_s - \widehat{k})}{\phi(k_s - k_f)}. \tag{2.42}$$

Then,

$$k_c = k_f\frac{\widehat{k} + Q}{k_s + Q}. \tag{2.43}$$

Note that if k_f is very small, $Q = 0$, and $k_c = \widehat{k}$. Also, if $\phi = 0$, from (2.41) we see that

$$k_c = k_s.$$

Next, if $k_s \to \infty$, the incompressible solid, then

$$Q = \frac{k_f(1 - \frac{\widehat{k}}{k_s})}{\phi(1 - \frac{k_f}{k_s})} \to \frac{k_f}{\phi},$$

and

$$k_c = \frac{\widehat{k} + Q}{1 + \frac{Q}{k_s}} \to \widehat{k} + \frac{\widetilde{k}}{\phi}.$$

Finally, if $k_f \simeq k_s$, from (2.41) we have that

$$k_c \simeq k_s \simeq k_f.$$

CHAPTER 3
THEOREM OF MINIMUM POTENTIAL FOR ELASTIC BODIES

Assume a body V in static equilibrium under the action of specified body and surface forces; i.e.,

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \text{ in } V, \qquad \tau_{ij}\nu_j = \overset{v}{T}_i = \overset{v}{T}{}^*_i \text{ on } S. \tag{3.1}$$

Assume there exists a system of displacements u_i such that they satisfy (3.1). Consider a class or arbitrary displacements $u_i + \delta u_i$ with δu_i small so that the material remains elastic. The arbitrary displacements δu_i are called *virtual displacements.*

We assume that the body remains in static equilibrium and let us compute the "virtual work" done by the body forces F_i and surface forces per unit area (tractions) $\overset{\nu}{T}{}^*_i$ as:

$$\int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}{}^*_i \delta u_i ds. \tag{3.2}$$

Since we know that

$$\overset{\nu}{T}{}^*_i = \tau_{ij}\nu_j,$$

the second term in (3.2) is

$$\begin{aligned} \int_S \overset{\nu}{T}{}^*_i \delta u_i ds &= \int_S \tau_{ij}\delta u_i \nu_j ds = \int_V (\tau_{ij}\delta u_i)_{,j}\, dv \\ &= \int_V \tau_{ij,j}\delta u_i dv + \int_V \tau_{ij}\delta u_{i,j} dv = -\int_V F_i \delta u_i dv + \int_V \tau_{ij}\delta u_{i,j} dv. \end{aligned} \tag{3.3}$$

Since τ_{ij} is symmetric, we see that

$$\tau_{ij}\delta u_{i,j} = \frac{1}{2}\tau_{ij}\delta u_{i,j} + \frac{1}{2}\tau_{ji}\delta u_{i,j} = \frac{1}{2}\tau_{ij}\delta u_{i,j} + \frac{1}{2}\tau_{ji}\delta u_{j,i} = \tau_{ij}\delta\varepsilon_{ij}.$$

Then, (3.3) becomes the *virtual work principle:*

$$\int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}{}^*_i \delta u_i ds = \int_V \tau_{ij}\delta\varepsilon_{ij}. \tag{3.4}$$

Assume that there exists a strain–energy function $W(\varepsilon_{ij})$ such that

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ij}.$$

Then, in this case,

$$\int_V \tau_{ij}\delta\varepsilon_{ij} = \int_V \frac{\partial W}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} = \delta \int_V W\, dv.$$

The principle of virtual work can be stated here as

$$\delta \int_V W\, dv - \int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}{}^*_i \delta u_i ds \equiv \delta\mathcal{V} = 0, \tag{3.5}$$

where

$$\mathcal{V} = \int_V W\, dv - \int_V F_i u_i dv - \int_S \overset{\nu}{T}{}_i^* u_i ds \tag{3.6}$$

is called the potential energy of the system. Equation (3.5) shows that of all displacements satisfying the given boundary conditions, those that satisfy the equilibrium equations are distinguished by a stationary (extreme) value of the potential energy. Since linearity has not been used, this principle is valid for linear and nonlinear stress–strain laws.

Now we show that the principle of virtual work (3.5) yields (3.1). In fact,

$$\begin{aligned}\delta\mathcal{V} = 0 &= \int_V \frac{\partial W}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} dv - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}{}_i^* \delta u_i ds \\ &= \int_V \tau_{ij}\delta\varepsilon_{ij} dv - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}{}_i^* \delta u_i ds \\ &= \frac{1}{2}\int_V \tau_{ij}(\delta u_{i,j} + \delta u_{j,i}) dv - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}{}_i^* \delta u_i ds.\end{aligned}$$

But

$$\frac{1}{2}\int_V \tau_{ij}\delta u_{i,j} dv = -\frac{1}{2}\int_V \tau_{ij,j}\delta u_i dv + \frac{1}{2}\int_S \tau_{ij} v_j \delta u_i ds$$

and

$$\begin{aligned}\frac{1}{2}\int_V \tau_{ij}\delta u_{j,i} dv &= \frac{1}{2}\int_V \tau_{ji}\delta u_{j,i} dv \\ &= -\frac{1}{2}\int_V (\tau_{ji,i})\delta u_j dv + \frac{1}{2}\int_S \tau_{ji}\nu_i \delta u_j ds \qquad j \to i, \quad i \to k, \\ &= -\frac{1}{2}\int_V \tau_{ik,k}\delta u_i dv + \frac{1}{2}\int_S \tau_{ik}\nu_k \delta u_i ds \\ &= -\frac{1}{2}\int_V \tau_{ij,j}\delta u_i dv + \frac{1}{2}\int_S \tau_{ij}\nu_j \delta u_i ds.\end{aligned}$$

Therefore,

$$\delta\mathcal{V} = 0 = -\int_V \tau_{ij,j}\delta u_i dv + \int_S \tau_{ij}\nu_j\delta u_i ds - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}{}_i^* \delta u_i ds. \tag{3.7}$$

Since δu_i is arbitrary, from (3.7) we conclude that

$$\tau_{ij,j} + F_i = 0 \text{ in } V, \qquad \tau_{ij}\nu_j = \overset{\nu}{T}{}_i^* \text{ on } S,$$

so that (3.1) is satisfied. If we assume that $W(\varepsilon_{ij})$ is positive definite in a neighborhood of the original equilibrium state, then it can be shown that for any other system of displacements $u_i + \delta u_i$ with potential energy $\mathcal{V}'$ we have

$$\mathcal{V}' - \mathcal{V} \geq 0. \tag{3.8}$$

In fact,

$$\mathcal{V}' = \int_V W(\varepsilon_{ij} + \delta\varepsilon_{ij})dv - \int_V F_i(u_i + \delta u_i)dv - \int_S \overset{\nu}{T}{}_i^*(u_i + \delta u_i)ds,$$
$$\mathcal{V} = \int_V W(\varepsilon_{ij})dv - \int_V F_i u_i dv - \int_S \overset{\nu}{T}{}_i^* u_i ds.$$

Then,

$$\mathcal{V}' - \mathcal{V} = \int_V [W(\varepsilon_{ij} + \delta\varepsilon_{ij}) - W(\varepsilon_{ij})]dv - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}{}_i^* \delta u_i ds.$$

Expanding $W(\varepsilon_{ij} + \delta\varepsilon_{ij})$ we have

$$W(\varepsilon_{ij} + \delta\varepsilon_{ij}) = W(\varepsilon_{ij}) + \frac{\partial W}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{1}{2}\frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\delta\varepsilon_{ij}\delta\varepsilon_{k\ell} + \dots. \tag{3.9}$$

Then, up to second order terms we have

$$V' - V = \int_V \frac{\partial W}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij}dv - \int_V F_i\delta u_i dv - \int_S \overset{\nu}{T}{}_i^*\delta u_i ds + \frac{1}{2}\int_V \frac{\partial^2 W}{\partial\varepsilon_{ij}\delta\varepsilon_{k\ell}}\delta\varepsilon_{ij}\delta\varepsilon_{k\ell}. \tag{3.10}$$

It follows from (3.5), that the sum of the first three terms in the right hand side of (3.10) vanishes, so that

$$\mathcal{V}' - \mathcal{V} = \frac{1}{2}\int_V \frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\delta\varepsilon_{ij}\delta\varepsilon_{k\ell}. \tag{3.11}$$

Now set $\varepsilon_{ij} \equiv 0$ in (3.9). The constant term $W(\varepsilon_{ij})$ can be taken to be zero since it denotes the strain energy in the natural unstrained state. Also, since $\frac{\partial W}{\partial\varepsilon_{ij}} = \tau_{ij}$ and $\tau_{ij} = 0$ for $\varepsilon_{ij} = 0$, we have that up to second order terms,

$$\frac{1}{2}\frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\delta\varepsilon_{ij}\delta\varepsilon_{k\ell} = W(\delta\varepsilon_{ij}). \tag{3.12}$$

Then, if W is positive definite, from (3.11) we get

$$\mathcal{V}' - \mathcal{V} = \int_V W(\delta\varepsilon_{ij})dv \geq 0,$$

and (3.8) is demonstrated.

Review of Calculus of Variations.

Consider the functional

$$J(\boldsymbol{\varphi}) = \int_{t_0}^{t_1} L(\varphi_1, \dots, \varphi_n, \dot{\varphi}_1, \dots, \dot{\varphi}_n, t)dt \tag{3.13}$$

where $\varphi_j(t_0) = \varphi_j^0$, $\varphi_j(t_1) = \varphi_j^1$, $1 \leq j \leq n$, are fixed given values. Assume that $\mathbf{u} = (u_1, \dots, u_n)$ is such that

$$J(\mathbf{u}) = \underset{\boldsymbol{\varphi}}{\text{minimum}}\ J(\boldsymbol{\varphi}), \tag{3.14}$$

and let us find the equation to be satisfied by J (Euler Equations). Let us perturb u_j in $\varepsilon_j \eta_j(t)$, with ε_j independent of t and $\eta_j(t_0) = \eta_j(t_1) = 0$, $j = 1, \dots, n$, as follows:

$$\begin{aligned}
\delta L &= L(u_1 + \varepsilon_1 \eta_1, u_2 + \varepsilon_2 \eta_2, \dots, u_n + \varepsilon_n \eta_n, \dot{u}_1 + \varepsilon_1 \dot{\eta}_1, \dot{u}_2 + \varepsilon_2 \dot{\eta}_2, \dots, \dot{u}_n + \varepsilon_n \dot{\eta}_n, t) \\
&\quad - L(u_1, u_2, \dots, u_n, \dot{u}_1, \dot{u}_2, \dots, \dot{u}_n, t) \\
&= \frac{\partial L}{\partial u_1} \varepsilon_1 \eta_1 + \frac{\partial L}{\partial \dot{u}_1} \varepsilon_1 \dot{\eta}_1 + \cdots + \frac{\partial L}{\partial u_n} \varepsilon_n \eta_n + \cdots + \frac{\partial L}{\partial \dot{u}_n} \varepsilon_n \dot{\eta}_n \\
&= \sum_{j=1}^{n} \varepsilon_j \left(\frac{\partial L}{\partial u_j} \eta_j + \frac{\partial L}{\partial \dot{u}_j} \dot{\eta}_j \right).
\end{aligned}$$

Then, since $\delta J(u) = \int_{t_0}^{t_1} \delta L(\mathbf{u}, \dot{\mathbf{u}}, t) dt = 0$, we have

$$0 = \int_{t_0}^{t_1} \sum_{j=1}^{n} \varepsilon_j \left(\frac{\partial L}{\partial u_j} \eta_j + \frac{\partial L}{\partial \dot{u}_j} \dot{\eta}_j \right) dt.$$

Next, using integration by parts,

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{u}_j} \dot{\eta}_j dt = \underbrace{\frac{\partial L}{\partial \dot{u}_j} \eta_j \Big|_{t_0}^{t^1}}_{\equiv 0} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_j} \right) \eta_j dt.$$

Then,

$$\delta J = 0 = \int_{t_0}^{t_1} \sum_{j=1}^{n} \varepsilon_j \left(\frac{\partial L}{\partial u_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_j} \right) \right) \eta_j dt.$$

Choosing $\varepsilon_1 \neq 0$ and small, $\varepsilon_2 = \cdots = \varepsilon_n = 0$, and since $\eta_1(t)$ is arbitrary, we conclude that

$$\frac{\partial L}{\partial u_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_1} \right) = 0.$$

Similarly, for the other u_j's, so that we have the n Euler equations,

$$\frac{\partial L}{\partial u_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_j} \right) = 0, \qquad j = 1, \dots, n. \tag{3.15}$$

Hamilton Principle.

Assume an oscillating body with displacements small enough so that the acceleration is given by $\frac{\partial^2 u_i}{\partial t^2}$ in Eulerian coordinates. Then the equation of motion is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} + F_i \quad \text{in } V. \tag{3.16}$$

Again, consider virtual displacements δu_i but, instead of the body in static equilibrium, we now have a vibrating body. Assume that we have prescribed surface tensions over $S = \partial V$; i.e.,

$$\tau_{ij} \nu_j = \overset{\nu}{T}{}_i^0, \qquad \text{on } S. \tag{3.17}$$

As before, the virtual work done by the body force and surface force is

$$\int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}{}_i^0 \delta u_i ds. \tag{3.18}$$

Consider the last integral:

$$\int_S \overset{\nu}{T}{}_i^0 \delta u_i ds = \int_S \tau_{ij}\nu_j \delta u_i ds = \int_V \frac{\partial(\tau_{ij}\delta u_i)}{\partial x_j} dv$$
$$= \int_V \frac{\partial \tau_{ij}}{\partial x_j}\delta u_i dv + \int_V \tau_{ij}\delta u_{i,j} dv = \int_V \left(\rho\frac{\partial^2 u_i}{\partial t^2} - F_i\right)\delta u_i dv + \int_V \tau_{ij}\delta\varepsilon_{ij} dv.$$

This yields the following variational equation of motion:

$$\int_V \tau_{ij}\delta\varepsilon_{ij} dv = \int_V \left(F_i - \rho\frac{\partial^2 u_i}{\partial t^2}\right)\delta u_i dv + \int_S \overset{\nu}{T}{}_i^0 \delta u_i ds. \tag{3.19}$$

If we assume that a strain energy density function exists, then (3.19) can be written as

$$\delta \int_V W\, dv = \int_V \left(F_i - \rho\frac{\partial^2 u_i}{\partial t^2}\right)\delta u_i dv + \int_S \overset{\nu}{T}{}_i^0 \delta u_i ds. \tag{3.20}$$

Let us integrate (3.20) between t_0 and t_1 under the assumption that the virtual displacements δu_i are functions of both space and time and that

$$\delta u_i(t_0) = \delta u_i(t_1) = 0. \tag{3.21}$$

Then,

$$\int_{t_0}^{t_1}\int_V \delta W\, dv = \int_{t_0}^{t_1} dt \int_V F_i\delta u_i dv + \int_{t_0}^{t_1}\int_S \overset{\nu}{T}{}_i^0 \delta u_i ds - \int_{t_0}^{t_1} dt\int_V \rho\frac{\partial^2 u_i}{\partial t^2}\delta u_i dv. \tag{3.22}$$

Let

$$I = -\int_V\int_{t_0}^{t_1} \rho\frac{\partial^2 u_i}{\partial t^2}\delta u_i dv\, dt.$$

Using integration by parts in time,

$$J = -\int_V \rho\frac{\partial u_i}{\partial t}\delta u_i dv\bigg|_{t_0}^{t_1} + \int_V\int_{t_0}^{t_1} dv\frac{\partial u_i}{\partial t}\left(\rho\frac{\partial\delta u_i}{\partial t} + \frac{\partial\rho}{\partial t}\delta u_i\right)dt.$$

From the continuity equation and for constant in space density ρ

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v}) = -\frac{\partial}{\partial x_i}\left(\rho\frac{\partial u_i}{\partial t}\right) = -\rho\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t}\right).$$

For small displacements, this term is small since we have assumed that velocities and gradients of velocities are small, and consequently

$$\rho\frac{Dv_i}{Dt} = \rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho\frac{\partial^2 u_i}{\partial t^2} + \rho\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial t}\right), \simeq \rho\frac{\partial^2 u_i}{\partial t^2}.$$

Then,

$$I = \int_{t_0}^{t_1} \int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial(\delta u_i)}{\partial t} dv\, dt = \int_{t_0}^{t_1} \int_V \rho \frac{\partial u_i}{\partial t} \delta \frac{\partial u_i}{\partial t} dv\, dt$$
$$= \int_{t_0}^{t_1} \delta \int_V \frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dv\, dt = \int_{t_0}^{t_1} \delta T(t) dt,$$

where

$$T(t) = \frac{1}{2} \int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dv \tag{3.23}$$

represents the kinetic energy of the body. Thus, under the assumption (3.21), (3.22) becomes

$$\int_{t_0}^{t_1} \delta(\mathcal{V} - T) dt = 0, \tag{3.24}$$

where, as in (3.6),

$$\mathcal{V} = \int_V W\, dv - \int_V F_i u_i dv - \int_S \overset{\nu}{T}{}_i^0 u_i ds, \tag{3.25}$$

represents the potential energy of the system. The function

$$L = \mathcal{V} - T \tag{3.26}$$

(or sometimes $-L$) is called the Lagrangian function, and (3.24) represents Hamilton's principle. In words, the Hamilton principle states that the system will move so that the time average of the difference between kinetic and potential energies will be an extremum (minimum). Using the Euler equations (3.15) we obtain the Lagrangian formulation of the equation of motion

$$\frac{\partial(T - \mathcal{V})}{\partial u_j} = \frac{d}{dt}\left(\frac{\partial(T - \mathcal{V})}{\partial \dot{u}_j}\right).$$

Since T is independent of u_j and V is independent of $\dot{u}_j$, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) = -\frac{\partial \mathcal{V}}{\partial u_j}, \qquad j = 1, 2, 3, \tag{3.27}$$

where we have chosen the displacements u_j as generalized coordinates. Now let

$$T_d = \frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} \tag{3.28}$$

be the kinetic energy density so that

$$T(t) = \int_V T_d(x, t) dv.$$

Also, transforming the surface integral (3.25) into a volume integral we obtain

$$\mathcal{V} = \int_V W\,dv - \int_V F_i u_i dv - \int_V \frac{\partial}{\partial x_j}(\tau_{ij} u_i) dv = \int_V \mathcal{V}_d dv,$$

where V_d represents the potential energy density of the system. Then, from (3.27)

$$\int_V \frac{d}{dt}\left(\frac{\partial T_d}{\partial \dot{u}_j}\right) dv = -\int_V \frac{\partial V_d}{\partial u_j} dv,$$

so that, since the above system holds for any V, then

$$\frac{d}{dt}\left(\frac{\partial T_d}{\partial \dot{u}_j}\right) = -\frac{\partial V_d}{\partial u_j}, \qquad j = 1, 2, 3. \tag{3.29}$$

We now compute the right hand side in (3.29) as follows:

$$\begin{aligned}
\delta\mathcal{V} &= \int_V \delta W\,dx - \int_V F_i \delta u_i dv - \int_S \overset{\nu}{T}_i \delta u_i dv \\
&= \int_\Omega \delta W\,dx - \int_V F_i \delta u_i - \int_S \tau_{ij} \delta u_i \nu_j ds \\
&= \int_\Omega \delta W\,dx - \int_V F_i \delta u_i - \int_V \frac{\partial}{\partial x_j} \tau_{ij} \delta u_i - \int_V \tau_{ij} \frac{\partial \delta u_i}{\partial x_j} dv \\
&= \int_\Omega \delta W\,dx - \int_V F_i \delta u_i - \int_V \frac{\partial \tau_{ij}}{\partial x_j} \delta u_i - \int_V \tau_{ij} \delta \varepsilon_{ij} dv \\
&= -\int_V \left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) \delta u_i dv = \int_\Omega \delta \mathcal{V}_d dv.
\end{aligned}$$

Thus,

$$\delta \mathcal{V}_d = -\left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) \delta u_i. \tag{3.30}$$

Assume that $\mathcal{V}_d$ is an exact differential of the variables u_i; i.e., that the system is conservative. Then,

$$\frac{\partial \mathcal{V}_d}{\partial u_i} = -\left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right). \tag{3.31}$$

Also, it follows from (3.28) that

$$\frac{\partial T_d}{\partial \dot{u}_i} = \rho \dot{u}_i.$$

Thus, (3.29) becomes

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \frac{\partial \tau_{ij}}{\partial x_j},$$

which are the original equations of motion in (3.16).

Let us check that the boundary conditions (3.17) are satisfied. We proceed as in the derivation of (3.7), where from the principle of virtual work (3.5) we derived the equilibrium equations (3.1). We start with the statement of Hamilton principle in the form of (3.22) and rewrite the left hand side of (3.22) in the form

(3.32)
$$\int_{t_0}^{t_1} dt \int_V \delta W\, dv = \int_{t_0}^{t_1} dt \int_V \tau_{ij}\delta\varepsilon_{ij} dv = \int_{t_0}^{t_1} dt \int_V \tau_{ij}\frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) dv$$
$$= -\int_{t_0}^{t_1} dt \int_V \frac{\partial \tau_{ij}}{\partial x_j}\delta u_i dv + \int_{t_0}^{t_1} \int_S \tau_{ij}\nu_j \delta u_i ds.$$

Using (3.32) in (3.22) we obtain

$$-\int_{t_0}^{t_1} dt \int_V \frac{\partial \tau_{ij}}{\partial x_j}\delta u_i dv + \int_{t_0}^{t_1} dt \int_S \tau_{ij}\nu_j \delta u_i ds$$
$$= \int_{t_0}^{t_1} dt \int_V F_i \delta u_i dv + \int_{t_0}^{t_1} \int_S \overset{\nu}{T}{}_i^0 \delta u_i ds - \int_{t_0}^{t_1} dt \int_V \rho \frac{\partial^2 u_i}{\partial t^2}\delta u_i dv,$$

so that

$$\int_{t_0}^{t_1} \left\{ \int_V \left[\frac{\partial \tau_{ij}}{\partial x_j} + F_i - \rho\frac{\partial^2 u_i}{\partial t^2} \right] \delta u_i dv + \int_S (\overset{\nu}{T}{}_i^0 - \tau_{ij}\nu_j)\delta u_i ds \right\} dt = 0. \tag{3.33}$$

Since the δu_i's are arbitrary, from (3.33) we see that (3.16) holds, and that

$$\tau_{ij}\nu_j = \overset{\nu}{T}{}_i^0 \text{ on } S,$$

so that the boundary conditions (3.17) are satisfied.

Summarizing from the Hamilton principle (3.24) we have obtained the Lagrange equations of motion (3.29). If the system is conservative in the sense of (3.31), then from the Hamilton principle we can recover the equations of motion (3.16)–(3.17).

In the next section, we will use these tools to derive the equation of motion for fluid–saturated porous media, also known as Biot media.

CHAPTER 4
WAVE PROPAGATION IN FLUID SATURATED POROUS MEDIA

1. The Stress-Strain Relations.

Let Ω be a porous medium saturated by a single–phase fluid, let $\phi(x)$ be the effective porosity, and let $\mathbf{u}^s$, $\widetilde{\mathbf{u}}^f$ be the locally averaged solid and water displacements in Ω. The physical meaning of $\widetilde{\mathbf{u}}^f$ is as follows: take a unit cube Q of bulk material. Then, for any face F of the cube, the quantity

$$\int_F \phi \widetilde{\mathbf{u}}^f \cdot \boldsymbol{\nu} d\sigma \tag{4.1}$$

represents the amount of fluid displaced through F, where $\boldsymbol{\nu}$ denotes the unit outward normal to F. Let τ_{ij} and σ_{ij} be the total stress tensor of the bulk material and the stress tensor in the solid part, respectively. Also, let p_f denote the fluid pressure and set

$$\sigma = -\phi p_f. \tag{4.2}$$

Then,

$$\tau_{ij} = \sigma_{ij} + \delta_{ij}\sigma = \sigma_{ij} - \phi p_f \delta_{ij}. \tag{4.3}$$

Assume that the domain Ω of bulk material is originally in static equilibrium and consider a system for surface forces g_i^s, g_i^f such that Ω remains in equilibrium under the action of such forces.

Since the system is in static equilibrium, the fluid pressure is constant on Ω so that

$$\nabla p_f = \frac{\partial p_f}{\partial x_i} = 0. \tag{4.4}$$

Also, since the total stress field is in equilibrium,

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0. \tag{4.5}$$

Let W denote the strain energy density for the fluid–solid system. Then, the virtual work principle states that the variation of strain energy in a body Ω is equal to the virtual work of the surface forces on $\partial\Omega$ (body forces such as gravity are neglected here); i.e.,

$$\int_\Omega \delta W \, d\Omega = \int_{\partial\Omega} (g_i^s \delta u_i^s + g_i^f \delta \widetilde{u}_i^f) d\sigma. \tag{4.6}$$

The forces g_i^s act on the solid part of $\partial\Omega$, while g_i^f acts on the fluid part of $\partial\Omega$. Here $\partial\Omega$ is not the physical termination of the body, but any closed surface within the body. In this way, we do not have to introduce the surface tension at the physical boundary.

Next, use that

$$g_i^s = \sigma_{ij}\nu_j, \qquad g_i^f = -\phi p_f \delta_{ij} \nu_j. \tag{4.7}$$

Also, from (4.3),

$$g_i^s = (\tau_{ij} + \delta_{ij}\phi p_f)\nu_j. \tag{4.8}$$

Using (4.7)–(4.8) in (4.6) we obtain

$$\begin{aligned}\int_\Omega \delta W \, d\Omega &= \int_{\partial\Omega} ((\tau_{ij} + \delta_{ij}\phi p_f)\delta u_i^s \nu_j - \phi p_f \delta_{ij} \delta \widetilde{u}_i^f \nu_j) d\sigma \\ &= \int_{\partial\Omega} [\tau_{ij}\delta u_i^s \nu_j - \phi p_f \delta_{ij}(\delta \widetilde{u}_i^f - \delta u_i^s)\nu_j] d\sigma.\end{aligned} \tag{4.9}$$

Set

$$u_i^f = \phi(\widetilde{u}_i^f - u_i^s), \tag{4.10}$$

which represents the displacement of the fluid relative to the solid but measured in terms of volume per unit area of bulk material. Then, (4.9) becomes

$$\int_\Omega \delta W \, d\Omega = \int_{\partial\Omega} (\tau_{ij}\delta u_i^s \nu_j - p_f \delta_{ij} \delta u_i^f \nu_j) d\sigma. \tag{4.11}$$

Next, using Gauss's theorem,

$$\int_\Omega \delta W \, d\Omega = \int_\Omega \frac{\partial}{\partial x_j}(\tau_{ij}\delta u_i^s) dv - \int_\Omega \frac{\partial}{\partial x_j}(p_f \delta_{ij} \delta u_i^f) dv. \tag{4.12}$$

Next, note that

$$\frac{\partial}{\partial x_j}(\tau_{ij}\delta u_i^s) = \frac{\partial}{\partial x_j}\tau_{ij}\delta u_i^s + \tau_{ij}\frac{\partial \delta u_i^s}{\partial x_j}, \qquad \frac{\partial}{\partial x_j}(p_f \delta_{ij}\delta u_i^f) = \frac{\partial p_f}{\partial x_j}\delta u_i^f + p_f \frac{\partial \delta u_i^f}{\partial x_i}. \tag{4.13}$$

Since the body is in equilibrium, (4.4) and (4.5) hold and consequently

$$\begin{aligned}\frac{\partial}{\partial x_j}(\tau_{ij}\delta u_i^s) &= \tau_{ij}\frac{\partial \delta u_i^s}{\partial x_j} = \frac{1}{2}\tau_{ij}\frac{\partial \delta u_i^s}{\partial x_j} + \frac{1}{2}\tau_{ji}\frac{\partial \delta u_i^s}{\partial x_j} \\ &= \frac{1}{2}\tau_{ij}\frac{\partial \delta u_i^s}{\partial x_j} + \frac{1}{2}\tau_{ij}\frac{\partial \delta u_j^s}{\partial x_i} = \tau_{ij}\frac{1}{2}\delta\left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i}\right) = \tau_{ij}\delta\varepsilon_{ij}(\mathbf{u}^s).\end{aligned} \tag{4.14}$$

Also,

$$\frac{\partial}{\partial x_j}(p_f \delta_{ij}\delta u_i^f) = p_f \delta \nabla \cdot \mathbf{u}^f. \tag{4.15}$$

Set

$$\xi = -\nabla \cdot \mathbf{u}^f. \tag{4.16}$$

Thus, (4.12) becomes

$$\int_\Omega \delta W \, d\Omega = \int_\Omega (\tau_{ij}\delta\varepsilon_{ij}(\mathbf{u}^s) + p_f \delta\xi) dv. \tag{4.17}$$

Since this is true for any Ω, we conclude that

$$\delta W = \tau_{ij}\delta\varepsilon_{ij}(\mathbf{u}^s) + p_f\delta\xi. \tag{4.18}$$

Next, since δW must be an exact differential of the strains $\varepsilon_{ij}(\mathbf{u}^s)$ and ξ, we have that W must satisfy the conditions

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ij}, \qquad \frac{\partial W}{\partial \xi} = p_f, \tag{4.19}$$

and

$$\frac{\partial^2 W}{\partial \varepsilon_{ij}\delta\xi} = \frac{\partial^2 W}{\partial\xi\partial\varepsilon_{ij}}, \qquad \frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}} = \frac{\partial^2 W}{\partial\varepsilon_{k\ell}\partial\varepsilon_{ij}}. \tag{4.20}$$

The Stress-Strain Relations.

In the isotropic case, W needs to be a function of the three invariants I_1, I_2, and I_3 of ε_{ij}, and the variable ξ; i.e.,

$$W = W(I_1, I_2, I_3, \xi).$$

To remain in the linear case, we must include only I_1, I_2, and ξ. As in the elastic solid case, it is more convenient to use $I_2' = -4I_2$ (c.f. 1.42). Thus,

$$W = \frac{1}{2}(He^2 + \mu I_2' - 2Be\xi + M\xi^2). \tag{4.21}$$

Using (4.19), we obtain

$$\begin{aligned}
\frac{\partial W}{\partial \varepsilon_{11}} &= \tau_{11} = He + \mu(-2\varepsilon_{33} - 2\varepsilon_{22}) - B\xi,\\
\frac{\partial W}{\partial \varepsilon_{22}} &= \tau_{22} = He + \mu(-2\varepsilon_{11} - 2\varepsilon_{33}) - B\xi,\\
\frac{\partial W}{\partial \varepsilon_{33}} &= \tau_{33} = He + \mu(-2\varepsilon_{22} - 2\varepsilon_{22}) - B\xi,\\
\frac{\partial W}{\partial \varepsilon_{ij}} &= \tau_{ij} = 2\mu\varepsilon_{ij}, \quad i \neq j, \quad \frac{\partial W}{\partial \xi} = p_f = -Be + M\xi.
\end{aligned} \tag{4.22}$$

Set

$$H = \lambda_c + 2\mu, \qquad B = \alpha M, \qquad \lambda_c = \lambda + \alpha^2 M. \tag{4.23}$$

Then, (4.22) becomes

$$\tau_{11} = \lambda_c e + 2\mu e - 2\mu(\varepsilon_{22} + \varepsilon_{33}) - \alpha M\xi = \lambda_c e + 2\mu\varepsilon_{11} - \alpha M\xi.$$

Similarly,

$$\begin{aligned}
\tau_{ii} &= \lambda_c e + 2\mu\varepsilon_{ii} - \alpha M\xi, \qquad i = 2, 3.\\
\tau_{ij} &= 2\mu\varepsilon_{ij}, \qquad i \neq j. \qquad p_f = -\alpha Me + M\xi.
\end{aligned}$$

In abbreviated form,

$$\tau_{ij} = (\lambda_c e - \alpha M\xi)\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad p_f = -\alpha Me + M\xi. \tag{4.24}$$

The inverse relation for (4.24) can be written in the form

$$\varepsilon_{ij} = \frac{1}{2\mu}\tau_{ij} + \delta_{ij}(D\tau - Fp_f), \qquad \xi = -F\tau + Hp_f, \tag{4.25}$$

where

$$\tau = \tau_{11} + \tau_{22} + \tau_{33} = T_r(\tau). \tag{4.26}$$

Physical significance of the variables e and ξ.

Let $\overline{V}_b$, $\overline{V}_s$, $\overline{V}_f$ be the bulk, solid, and fluid volumes, respectively, of a homogeneous part Ω of bulk material in the initial equilibrium state. Since $\mathbf{u}^s$ is the averaged solid displacement vector over the *whole* bulk material, e represents the change $\Delta V_b = V_b - \overline{V}_b$ in bulk volume per unit volume of bulk material; i.e.,

$$e = \frac{\Delta V_b}{\overline{V}_b}. \tag{4.27}$$

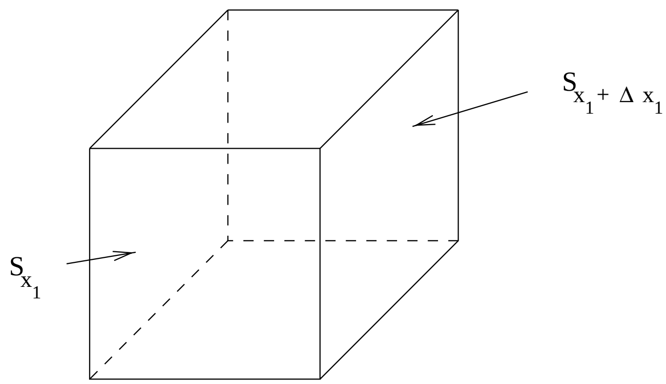


FIGURE 12

Next, consider a cube of bulk material of uniform porosity $\overline{\phi} = \overline{V}_f/\overline{V}_b$. The amount of fluid entering the face S_{x_1} is $\overline{\phi}(\widetilde{u}_1^f(x_1) - u_1^s(x_1))\Delta x_2 \Delta x_3$, and the amount of fluid leaving the face $S_{x_1+\Delta x_1}$ is $\overline{\phi}(\widetilde{u}_1^f(x_1+\Delta x_1) - u^s(x_1+\Delta x_1))\Delta x_2 \Delta x_3$. Then, the change in fluid content δF_c is given by

$$\delta F_c = \overline{\phi}\frac{[(\widetilde{u}_1^f(x_1+\Delta x_1) - u_1^s(x_1+\Delta x_1)) - (\widetilde{u}_1^f(x) - u_1^s(x_1))]}{\Delta x_1}\Delta x_1 \Delta x_2 \Delta x_3 \simeq \frac{\partial u_1^f}{\partial x_1}\overline{V}_b.$$

In general,

$$\frac{\delta F_c}{\overline{V}_b} = \nabla \cdot \mathbf{u}^f = -\xi. \tag{4.28}$$

Thus, ξ represents the change in fluid content per unit bulk volume.

Next, let us denote by ΔV_f^c the part of the total change $\Delta V_f = V_f - \overline{V}_f$ in fluid volume due to changes in fluid pressure. Then,

$$\frac{\Delta V_f^c}{V_f} = -\frac{p_f}{k_f}. \tag{4.29}$$

Now observe that the change in fluid content is the difference between ΔV_f and ΔV_f^c. Since ξ measures this difference per unit bulk volume, we see that

$$\xi = \frac{\Delta V_f - \Delta V_f^c}{\overline{V}_b} = \frac{\overline{V}_f}{\overline{V}_b}(\Delta V_f - \Delta V_f^c)\frac{1}{\overline{V}_f} = \overline{\phi}(\Delta V_f - \Delta V_f^c)/\overline{V}_f. \tag{4.30}$$

For the analysis, it is convenient to decompose τ_{ij} in the form

$$\tau_{ij} = -p_f \delta_{ij} + \widehat{\tau}_{ij}, \tag{4.31}$$

where $\widehat{\tau}_{ij}$ is the "residual" or "effective" stress of the material.

Let us determine the elastic coefficients in the stress–strain relations (4.24):

$$\tau_{ij} = (\lambda_c e - B\xi)\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad p_f = -Be + M\xi.$$

Here we follow the argument given in [14]. First, since the fluid does not support any shear, μ is identical to the shear modulus of the solid matrix. For the other coefficients, it is sufficient to consider tensional changes τ_{ij} such that

$$\tau_{11} = \tau_{22} = \tau_{33} = \frac{1}{3}\Delta\tau = -\Delta p, \quad \Delta p > 0, \qquad \tau_{ij} = 0, \quad i \neq j.$$

Set

$$\frac{1}{3}\widehat{\tau} = \widehat{\tau}_{11} + \widehat{\tau}_{22} + \widehat{\tau}_{33} = -\Delta\widehat{p}.$$

Then the decomposition (4.31) becomes

$$\begin{aligned} &\tau_{11} = -p_f + \widehat{\tau}_{11}, \quad \text{(same for } \tau_{22},\, \tau_{33}\text{)} \\ &\frac{1}{3}\tau = -p_f + \frac{1}{3}\widehat{\tau}, \qquad -\Delta p = -p_f - \Delta\widehat{p}, \qquad \Delta p = p_f + \Delta\widehat{p}. \end{aligned} \tag{4.32}$$

Also, (4.24)–(4.25) reduce to

$$\begin{aligned} \frac{1}{3}\Delta\tau &= \tau_{11} = (\lambda_c e - B\xi) + 2\mu\varepsilon_{11}, \\ \frac{1}{3}\Delta\tau &= \tau_{22} = (\lambda_c e - B\xi) + 2\mu\varepsilon_{22}, \\ \frac{1}{3}\Delta\tau &= \tau_{33} = (\lambda_c e - B\xi) + 2\mu\varepsilon_{33}. \end{aligned}$$

Adding the last three equations, we obtain

$$\frac{1}{3}\Delta\tau = -\Delta p = \left(\lambda_c + \frac{2}{3}\mu\right)e - B\xi \equiv Ge - B\xi. \tag{4.33}$$

Also,

$$p_f = -Be + M\xi. \tag{4.34}$$

Now, from (4.25),

$$\varepsilon_{11} = \frac{1}{2\mu}\tau_{11} + (D\tau - Fp_f), \qquad \varepsilon_{22} = \frac{1}{2\mu}\tau_{22} + (D\tau - Fp_f), \qquad \varepsilon_{33} = \frac{1}{2\mu}\tau_{33} + (D\tau - Fp_f).$$

Adding the last three equations,

$$e = \frac{1}{2\mu}\tau + 3D\tau - 3Fp_f = \left(3D + \frac{1}{2\mu}\right)\tau - 3Fp_f, \tag{4.35}$$

and

$$\xi = -F\tau + Hp_f. \tag{4.36}$$

Consider the closed system, in which no fluid is allowed to flow in or out of the bulk material, and let k_c, the bulk modulus of the system, be defined by

$$e = -\frac{\Delta p}{k_c}. \tag{4.37}$$

This corresponds to a compressibility test in which a sample of bulk material is enclosed in an impermeable jacket and then subjected to an additional external pressure Δp.

Note that for a closed system $\xi = 0$. Then from (4.33),

$$e = -\Delta p / G. \tag{4.38}$$

Thus, from (4.37) and (4.38),

$$G = k_c. \tag{4.39}$$

To determine k_c, we first need to use (4.35) to derive expressions for $3D + \frac{1}{2\mu}$ and F using the jacketed compressibility test, which corresponds to a tensional state such that

$$p_f = 0, \quad e = -\frac{\Delta p}{k_m} = -\frac{\Delta \widehat{p}}{k_m}, \tag{4.40}$$

so that the fluid pressure is held constant and the external applied pressure

$$-\Delta p = -(p_f + \Delta \widehat{p}) = -\Delta \widehat{p} = \widehat{\tau}_{11} = \widehat{\tau}_{22} = \widehat{\tau}_{33} \tag{4.41}$$

is supported only by the solid matrix. Here k_m denotes the bulk modulus of the empty matrix (see Biot and Willis [5]).

Now, using (4.29) and (4.30),

$$\begin{aligned} \xi &= \frac{\Delta V_f - \Delta V_f^c}{\overline{V}_b} = \frac{\Delta V_f}{\overline{V}_b} = \frac{\Delta(\phi V_b)}{\overline{V}_b} = \frac{\Delta\phi V_b + \phi \Delta V_b}{\overline{V}_b} \\ &= \frac{\Delta\phi(\overline{V}_b + \Delta V_b) + (\overline{\phi} + \Delta\phi)\Delta V_b}{\overline{V}_b}. \end{aligned}$$

Then,

$$\xi \simeq \Delta\phi + \overline{\phi}\,\frac{\Delta V_b}{\overline{V}_b}. \tag{4.42}$$

Now, according to (2.26),

$$\Delta\phi = \left(\frac{1}{k_s} - \frac{(1-\overline{\phi})}{k_m}\right)\Delta\widehat{p}, \tag{4.43}$$

where k_s denotes the bulk modulus of the solid grains. Thus, using (4.40) and(4.42),

$$\xi = \left(\frac{1}{k_s} - \frac{1}{k_m}\right)\Delta\widehat{p} + \overline{\phi}\frac{\Delta\widehat{p}}{k_m} + \overline{\phi}\frac{\Delta V_b}{\overline{V}_b} = \left(\frac{1}{k_s} - \frac{1}{k_m}\right)\Delta\widehat{p}. \tag{4.44}$$

Now, using (4.44) and (4.40) in (4.35) we obtain

$$\begin{aligned} -\frac{\Delta\widehat{p}}{k_m} &= \left(3D + \frac{1}{2\mu}\right)\tau = \left(3D + \frac{1}{2\mu}\right)(\widehat{\tau}_{11} + \widehat{\tau}_{22} + \widehat{\tau}_{33}) \\ &= \left(3D + \frac{1}{2\mu}\right)(-3\Delta\widehat{p}). \end{aligned}$$

Therefore,

$$3D + \frac{1}{2\mu} = \frac{1}{3k_m}. \tag{4.45}$$

Also,

$$\left(\frac{1}{k_s} - \frac{1}{k_m}\right)\Delta\widehat{p} = -F\tau = F \cdot 3\Delta\widehat{p},$$

so that

$$F = \frac{1}{3}\left(\frac{1}{k_s} - \frac{1}{k_m}\right). \tag{4.46}$$

Now we will obtain an expression for k_c using (4.45) and (4.46).

Using (4.37), (4.45), and (4.46) in (4.35) we see that for the closed system,

$$e = -\frac{\Delta p}{k_c} = \frac{1}{3k_m}\tau - 3Fp_f = \frac{1}{3k_m}(-3\Delta p) - 3\frac{1}{3}\left(\frac{1}{k_s} - \frac{1}{k_m}\right)p_f.$$

Thus,

$$\frac{\Delta p}{k_c} = \frac{\Delta p}{k_m} + \left(\frac{1}{k_s} - \frac{1}{k_m}\right)p_f. \tag{4.47}$$

Next we will derive a relation between Δp and p_f, valid for the closed system.

First note that since, for the closed system $\xi = 0$, from (4.29) and (4.20) we have

$$0 = \overline{\phi}\left(\frac{\Delta V_f}{\overline{V}_f} - \frac{\Delta V_f^c}{\overline{V}_f}\right).$$

Then,

$$\frac{\Delta V_f}{\overline{V}_f} = \frac{\Delta V_f^c}{\overline{V}_f} = -\frac{p_f}{k_f}. \tag{4.48}$$

In (4.48), k_f denotes fluid bulk modulus. Next, using (4.43) up to first order terms, we have that

$$\begin{aligned}\frac{\Delta V_f}{\overline{V}_f} &= \frac{\Delta(\phi V_b)}{\overline{V}_f} = \phi\frac{\Delta V_b}{\overline{V}_f} + \frac{V_b}{\overline{V}_f}\Delta\phi = \frac{V_f}{V_b}\frac{\Delta V_b}{\overline{V}_f} + \frac{(\overline{V}_b + \Delta V_b)}{\overline{V}_f}\Delta\phi \\ &= \frac{(\overline{V}_f + \Delta V_f)\Delta V_b}{V_b\overline{V}_f} + \frac{(\overline{V}_b + \Delta V_b)\Delta\phi}{\overline{V}_f} \simeq \frac{\Delta V_b}{V_b} + \frac{\Delta\phi}{\overline{\phi}} \\ &= -\frac{\Delta p}{k_c} + \frac{1}{\overline{\phi}}\left(\frac{1}{k_s} - \left(\frac{1-\overline{\phi}}{k_m}\right)\right)\Delta\widehat{p}.\end{aligned} \tag{4.49}$$

Combining (4.48)–(4.49) and the decomposition (4.32) ($\Delta p = p_f + \Delta\widehat{p}$), we see that

$$-\frac{p_f}{k_f} = -\frac{\Delta p}{k_c} + \frac{1}{\overline{\phi}}\left(\frac{1}{k_s} - \frac{(1-\overline{\phi})}{k_m}\right)(\Delta p - p_f).$$

Thus,

$$\Delta p\left(-\frac{1}{k_c} + \frac{1}{\overline{\phi}}\left(\frac{1}{k_s} - \frac{(1-\overline{\phi})}{k_m}\right)\right) = p_f\left(-\frac{1}{k_f} + \frac{1}{\overline{\phi}}\left(\frac{1}{k_s} - \frac{(1-\overline{\phi})}{k_m}\right)\right).$$

Multiplying by $\overline{\phi}$,

$$\Delta p\left(\frac{1}{k_s} - \frac{1}{k_m} + \overline{\phi}\left(\frac{1}{k_m} - \frac{1}{k_c}\right)\right) = p_f\left(\frac{1}{k_s} - \frac{1}{k_m} + \overline{\phi}\left(\frac{1}{k_m} - \frac{1}{k_f}\right)\right),$$

so that

$$p_f = \frac{\frac{1}{k_s} - \frac{1}{k_m} + \overline{\phi}(\frac{1}{k_m} - \frac{1}{k_c})}{\frac{1}{k_s} - \frac{1}{k_m} + \overline{\phi}(\frac{1}{k_s} - \frac{1}{k_c})}\Delta p. \tag{4.50}$$

Using (4.50) in (4.47) we obtain the relation

$$\frac{1}{k_c} = \frac{1}{k_m} + \left(\frac{1}{k_s} - \frac{1}{k_m}\right)\frac{(\frac{1}{k_s} - \frac{1}{k_m}) + \overline{\phi}(\frac{1}{k_m} - \frac{1}{k_c})}{(\frac{1}{k_s} - \frac{1}{k_m}) + \overline{\phi}(\frac{1}{k_m} - \frac{1}{k_f})}. \tag{4.51}$$

From (4.49), a calculation yields

$$k_c = k_s \cdot \frac{k_m + Q}{k_s + Q}, \tag{4.52}$$

$$Q = \frac{k_f(k_m - k_s)}{\phi(k_s - k_f)}, \tag{4.53}$$

which coincides with the expression given in (2.42)–(2.43).

We need to compute the remaining coefficients B and M. They can be obtained from the jacketed compressibility test described by the tensional state (4.40); i.e.,

$$p_f = 0, \quad e = -\frac{\Delta p}{k_m} = -\frac{\Delta\widehat{p}}{k_m}.$$

From (4.33), (4.34), (4.39), and the expression for ξ in (4.44) we obtain

$$-\Delta p = -\Delta\widehat{p} = k_c\left(-\frac{\Delta\widehat{p}}{k_m}\right) - B\left(\frac{1}{k_s} - \frac{1}{k_m}\right)\Delta\widehat{p},$$
$$0 = -B\left(-\frac{\Delta\widehat{p}}{k_m}\right) + M\left(\frac{1}{k_s} - \frac{1}{k_m}\right)\Delta\widehat{p}.$$

Thus,

$$\text{(4.54)} \qquad \begin{aligned} &\text{i)} \quad 1 = \frac{k_c}{k_m} + B\left(\frac{1}{k_s} - \frac{1}{k_m}\right), \\ &\text{ii)} \quad 0 = \frac{B}{k_m} + M\left(\frac{1}{k_s} - \frac{1}{k_m}\right). \end{aligned}$$

Then,

$$B\left(\frac{k_m - k_s}{k_s k_m}\right) = 1 - \frac{k_c}{k_m} = 1 - \frac{k_s}{k_m}\frac{k_m + \frac{k_f(k_m-k_s)}{\phi(k_f-k_s)}}{k_s + \frac{k_f(k_m-k_s)}{\phi(k_f-k_s)}}$$

$$= 1 - \frac{k_s}{k_m}\frac{k_m\phi(k_f - k_s) + k_f(k_m - k_s)}{(k_s\phi(k_f - k_s) + k_f(k_m - k_s))}$$

$$= 1 - \frac{k_s\phi(k_f - k_s) + \frac{k_s k_f}{k_m}(k_m - k_s)}{k_s\phi(k_f - k_s) + k_f(k_m - k_s)}$$

$$= \frac{k_f(k_m - k_s) - \frac{k_s k_f}{k_m}(k_m - k_s)}{k_s\phi(k_f - k_s) + k_f(k_m - k_s)}$$

$$= \frac{k_f(k_m - k_s)(1 - \frac{k_s}{k_m})}{k_s\phi(k_f - k_s) + k_f(k_m - k_s)}.$$

Then,

$$B = \frac{k_s k_m}{(k_m - k_s)}\frac{k_f(k_m - k_s)(\frac{k_m - k_s}{k_m})}{(k_s\phi(k_f - k_s) + k_f(k_m - k_s)}$$

or

$$\text{(4.55)} \qquad B = \frac{k_s k_f(k_s - k_m)}{k_s\phi(k_s - k_f) + k_f(k_s - k_m)}.$$

Next, from (4.54.ii),

$$M\left(\frac{k_m - k_s}{k_s k_m}\right) = -\frac{B}{k_m}, \qquad M = \frac{k_s}{(k_s - k_m)} \cdot \frac{k_s k_f(k_s - k_m)}{k_s\phi(k_s - k_f) + k_f(k_s - k_m)}.$$

Thus,

$$M = \frac{k_s^2 k_f}{k_s\phi(k_s - k_f) + k_f(k_s - k_m)}. \tag{4.56}$$

Also, B and M may be determined using the unjacketed compressibility test ([5]) corresponding to a tensional state of the form

$$\Delta\widehat{p} = 0, \qquad \tau_{11} = \tau_{22} = \tau_{33} = -\Delta p = -p_f.$$

In this test, a sample of bulk material is immersed in a container with the same fluid as that inside the poral space and then subjected to a hydrostatic pressure change Δp.

Thus, in this case, the pressure change is supported by both the solid and fluid parts of the bulk material, and the residual stress, acting only on the matrix, is zero. Thus, according to (2.26) or (4.43),

$$\Delta\phi = 0. \tag{4.57}$$

Next, note that from (4.57),

$$\begin{aligned}
\frac{\Delta V_s}{\overline{V}_s} &= \frac{\Delta((1-\phi)V_b)}{\overline{V}_s} = \frac{(1-\phi)\Delta V_b - \Delta\phi V_b}{\overline{V}_s} \\
&= \frac{[1-(\overline{\phi}+\Delta\phi)]\Delta V_b}{(1-\overline{\phi})\overline{V}_b} \approx \frac{\Delta V_b}{\overline{V}_b}, \\
\frac{\Delta V_f}{\overline{V}_f} &= \frac{\Delta(\phi V_b)}{\overline{V}_f} = \frac{\phi\Delta V_b}{\overline{\phi}\,\overline{V}_b} \approx \frac{\Delta V_b}{\overline{V}_b}.
\end{aligned}$$

Thus,

$$\frac{\Delta V_f}{\overline{V}_f} = \frac{\Delta V_s}{\overline{V}_s} = \frac{\Delta V_b}{\overline{V}_b}. \tag{4.58}$$

Since

$$\frac{\Delta V_s}{\overline{V}_s} = -\frac{\Delta p}{k_s},$$

we conclude that

$$e = -\frac{\Delta p}{k_s}. \tag{4.59}$$

Also, using (4.29), (4.30), and (4.58),

$$\xi = \overline{\phi}\left(\frac{\Delta V_f}{\overline{V}_f} - \frac{\Delta V_f^c}{\overline{V}_f}\right) = \overline{\phi}\left(-\frac{\Delta p}{k_s} + \frac{p_f}{k_f}\right) = \overline{\phi}\left(\frac{1}{k_f} - \frac{1}{k_s}\right)\Delta p. \tag{4.60}$$

Now using (4.59) and (4.60) in (4.33)–(4.34), we obtain

$$\begin{aligned}
-\Delta p &= k_c\left(-\frac{\Delta p}{k_s}\right) - B\overline{\phi}\left(\frac{1}{k_f} - \frac{1}{k_s}\right)\Delta p, \\
-\Delta p &= p_f = -B\left(-\frac{\Delta p}{k_s}\right) + M\overline{\phi}\left(\frac{1}{k_f} - \frac{1}{k_s}\right)\Delta p
\end{aligned}$$

or

$$1 = \frac{k_c}{k_s} + B\overline{\phi}\left(\frac{1}{k_f} - \frac{1}{k_s}\right), \qquad 1 = \frac{B}{k_s} + M\overline{\phi}\left(\frac{1}{k_f} - \frac{1}{k_s}\right). \tag{4.61}$$

Now, (4.61) and an algebraic manipulation using the expression for k_c in (4.52)–(4.53) yields again the equations for B and M in (4.55) and (4.56).

We now examine the restrictions on the coefficients imposed by the nonnegative character of the strain energy W. Recall that (c.f. (4.21))

$$2W = He^2 + \mu I_2' - 2Be\xi + M\xi^2. \tag{4.62}$$

Using an argument similar to that leading to (1.45), we can write (4.62) in the equivalent form

$$\begin{aligned} 2W = &\left(\lambda_c + \frac{2}{3}\mu\right)e^2 + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) \\ &+ \frac{2}{3}\mu\Big((\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2\Big) - 2Be\xi + M\xi^2. \end{aligned} \tag{4.63}$$

Setting

$$e = \xi = 0,$$

we see that we must have

$$\mu \geq 0. \tag{4.64}$$

Next, setting

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}, \quad \varepsilon_{ij} = 0, \quad i \neq j,$$

we obtain

$$2W = k_c e^2 - 2Be\xi + M\xi^2 = [e, \xi]\begin{bmatrix} k_c & -B \\ -B & M \end{bmatrix} \begin{bmatrix} e \\ \xi \end{bmatrix} \equiv [e, \xi]E\begin{bmatrix} e \\ \xi \end{bmatrix}.$$

From $\det(E - rI) = 0$, we find

$$r = \frac{(k_c + M) \pm \sqrt{(k_c + M)^2 - 4(k_c M - C^2)}}{2}.$$

Thus, for the eigenvalues r to be nonnegative, we find the conditions

$$(k_c + M)^2 - 4k_c M + 4B^2 = (k_c - M)^2 + 4B^2 \geq 0 \text{ (always true)}, \tag{4.65}$$

$$k_c M - B^2 \geq 0, \quad k_c \geq 0, \quad M \geq 0. \tag{4.66}$$

Next, we observe that, since $B = \alpha M$ and $\lambda_c = \lambda + \alpha^2 M$ (c.f. (4.23)),

$$k_c M - B^2 = k_c M - \alpha^2 M^2 = (k_c - \alpha^2 M)M.$$

Next, set

$$k = \lambda + \frac{2}{3}\mu.$$

Then,

$$k_c - k = \lambda_c + \frac{2}{3}\mu - \lambda - \frac{2}{3}\mu = \lambda_c - \lambda = \alpha^2 M$$

so that

$$k_c M - B^2 = (k_c - \alpha^2 M)M = kM.$$

Therefore, for W to be nonnegative, we have the necessary and sufficient conditions

$$\mu \geq 0, \quad M \geq 0, \quad k = \lambda + \frac{2}{3}\mu \geq 0. \tag{4.67}$$

To interpret the condition $k = \lambda + \frac{2}{3}\mu \geq 0$, we proceed as follows.

We start with (4.24):

$$\begin{aligned} &\text{i)} \quad \tau_{ij} = (\lambda_c e - \alpha M \xi)\delta_{ij} + 2\mu\varepsilon_{ij} \\ &\text{ii)} \quad p_f = -\alpha M e + M\xi. \end{aligned} \tag{4.68}$$

Next, write ξ as function of p_f and e:

$$\xi = \frac{1}{M} p_f + \alpha e. \tag{4.69}$$

Using (4.69) in (4.68), we obtain

$$\tau_{ij} = \left[\lambda_c e - \alpha M \left(\frac{1}{M} p_f + \alpha e\right)\right]\delta_{ij} + 2\mu\varepsilon_{ij},$$

or

$$\tau_{ij} + \delta_{ij}\alpha p_f = 2\mu\varepsilon_{ij} + \delta_{ij}(\lambda_c - \alpha^2 M)e + 2\mu\varepsilon_{ij} + \delta_{ij}\lambda e. \tag{4.70}$$

Now use (4.31) to write τ_{ij} as a function of the residual stress $\widehat{\tau}_{ij}$; i.e.,

$$\tau_{ij} = -p_f \delta_{ij} + \widehat{\tau}_{ij}.$$

Then, (4.70) becomes

$$\widehat{\tau}_{ij} - (1 - \alpha)\delta_{ij} p_f = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda e. \tag{4.71}$$

Recall that the jacketed compressibility test is defined by the tensional state

$$p_f = 0, \quad \tau_{11} = \tau_{22} = \tau_{33} = \widehat{\tau}_{11} = \widehat{\tau}_{22} = \widehat{\tau}_{33} = -\Delta\widehat{p}.$$

Then, from (4.71) we obtain

$$-\Delta\widehat{p} = 2\mu\varepsilon_{ii} + \lambda e, \quad i = 1, 2, 3.$$

Adding the three equations, we get

$$-3\Delta\widehat{p} = (2\mu + 3\lambda)e$$

or

$$e = -\frac{\Delta\widehat{p}}{\lambda + \frac{2}{3}\mu} = -\frac{\Delta\widehat{p}}{k}.$$

Thus, the condition $k = \lambda + \frac{2}{3}\mu \geq 0$ simply states that, for the open system ($p_f = 0$), the coefficient $k = \lambda + \frac{2}{3}\mu$, which represents the inverse of the jacketed compressibility, be nonnegative.

The Equation of Motion.

Following the ideas given in Chapter 3, we will choose u_i^s and u_i^f, $1 \le i \le 3$ as generalized coordinates or state variables to describe the evolution of the fluid–solid system. As in (3.29), the Lagrange formulation of the equation of motion is given by

$$\text{(4.72)} \quad \begin{aligned} &\text{i)} \quad \frac{d}{dt}\left(\frac{\partial T_d}{\partial \dot{u}_i^s}\right) + \frac{\partial \mathcal{D}_d}{\partial \dot{u}_i^s} = -\frac{\partial V_d}{\partial u_i^s}, \\ &\text{ii)} \quad \frac{d}{dt}\left(\frac{\partial T_d}{\partial \dot{u}_i^f}\right) + \frac{\partial \mathcal{D}_d}{\partial \dot{u}_i^f} = -\frac{\partial V_d}{\partial u_i^f}, \quad 1 \le i \le 3. \end{aligned}$$

In (4.72), T_d, $\mathcal{D}_d$, and V_d are, respectively, the kinetic energy density, the dissipation energy density function, and the potential energy density of the system.

First, let us compute the right hand side in (4.72). Following the definition in (3.6), let

$$\text{(4.73)} \quad \mathcal{V} = \int_\Omega W\, d\Omega - \int_{\partial\Omega} (g_i^s u_i^s + g_i^f \widetilde{u}_i^f) d\sigma$$

be the potential energy of the fluid–solid system (neglecting body forces) where

$$\text{(4.74)} \quad g_i^s = \sigma_{ij}\nu_j = (\tau_{ij} + \delta_{ij}\phi p_f)\nu_j, \qquad g_i^f = -\phi p_f \delta_{ij}\nu_j.$$

Also, recall that according to (4.18),

$$\text{(4.75)} \quad \delta W = \tau_{ij}\delta\varepsilon_{ij}(u^s) + p_f \delta\xi.$$

If the system is in static equilibrium, then

$$\text{(4.76)} \quad \delta\mathcal{V} = 0 = \int_\Omega \delta W\, d\Omega - \int_{\partial\Omega} (g_i^s \delta u_i^s + g_i^f \delta\widetilde{u}_i^f) d\sigma$$

states the virtual work principle for the fluid–solid system. Now we consider a perturbation of the system from the equilibrium state; i.e., we do not have any more of the conditions,

$$\tau_{ij,j} = 0, \quad \frac{\partial p_f}{\partial x_j} = 0.$$

Then,

$$\begin{aligned} \delta\mathcal{V} &= \int_\Omega \delta W\, d\Omega - \int_{\partial\Omega} (\tau_{ij} + \phi p_f \delta_{ij})\delta u_i^s \nu_j d\sigma + \int_{\partial\Omega} \phi p_f \delta_{ij} \delta\widetilde{u}_i^f \nu_j d\sigma \\ &= \int_\Omega \delta W\, d\Omega - \int_{\partial\Omega} (\tau_{ij}\delta u_i^s \nu_j - p_f \delta_{ij} \delta u_i^f \nu_j) d\sigma \\ &= \int_\Omega \delta W\, d\Omega - \int_\Omega \frac{\partial}{\partial x_j}(\tau_{ij}\delta u_i^s) d\Omega + \int_\Omega \frac{\partial}{\partial x_j}(p_f \delta_{ij} \delta u_i^f) d\Omega \\ &= \int_\Omega (\tau_{ij}\delta\varepsilon_{ij} + p_f \delta\xi) d\Omega - \int_\Omega \tau_{ij,j}\delta u_i^s d\Omega - \int_\Omega \tau_{ij}\delta\varepsilon_{ij} d\Omega \\ &\quad + \int_\Omega \frac{\partial p_f}{\partial x} \delta u_i^f d\Omega + \int_\Omega p_f(-\delta\xi) d\Omega. \end{aligned}$$

Hence,

$$\delta \mathcal{V} = -\int_{\Omega} \left(\tau_{ij,j} \delta u_i^s - \frac{\partial p_f}{\partial x} \delta u_i^f \right) d\Omega = \int_{\Omega} \delta V_d d\Omega. \tag{4.77}$$

Thus,

$$\delta V_d = -\frac{\partial \tau_{ij}}{\partial x_j} \delta u_i^s + \frac{\partial p_f}{\partial x_i} \delta u_i^f. \tag{4.78}$$

Assuming that $\mathcal{V}_d$ is an exact differential in the variables u_i^s and u_i^f, we see that

$$\frac{\partial V_d}{\partial u_i^s} = -\frac{\partial \tau_{ij}}{\partial x_j}, \qquad \frac{\partial V_d}{\partial u_i^f} = \frac{\partial p_f}{\partial x_i}, \quad i = 1, 2, 3. \tag{4.79}$$

Next, we will compute the kinetic energy density T_d for the fluid–solid system. Let us consider a unit cube Q of bulk material, and let Q_p denote the porous part of Q. Let ρ_f and ρ_s be the mass densities of the fluid and solid phases, respectively.

Let $(v_i)_{1 \le i \le 3}$ be the relative microvelocity field; i.e., the velocity of each fluid particle with respect to the solid frame. Assuming that the relative flow inside the poral space is of laminar type (i.e., we are in the low frequency range) we can write

$$v_i = a_{ij} u_j^f, \tag{4.80}$$

with the coefficients a_{ij} depending on the coordinates of the pores and the pore geometry. Let

$$\rho_1 = (1 - \phi) \rho_s$$

be the mass of solid per unit volume of bulk material. Then, on the solid part of Q (i.e., in $Q \setminus Q_p$), the kinetic energy is given by

$$\frac{1}{2} \int_{Q \setminus Q_p} \rho_s \dot{u}_i^s \dot{u}_i^s d(Q \setminus Q_p) = \frac{1}{2} |Q \setminus Q_p| \rho_s \dot{u}_i^s \dot{u}_i^s = \frac{1}{2} (1 - \phi) \rho_s \dot{u}_i^s \dot{u}_i^s = \frac{1}{2} \rho_1 \dot{u}_i^s \dot{u}_i^s. \tag{4.81}$$

Here we have used that since u_i^s is the average solid displacement over Q, u_i^s is constant over Q.

Next, on the porous part Q_p, the velocity of any given particle is the relative microvelocity plus the averaged solid velocity; i.e., $\dot{u}_i + v_i$. Then the kinetic energy in Q_p is obtained by integration of $(\dot{u}_i + v_i)(\dot{u}_i + v_i)$ over Q_p. Thus, the total kinetic energy per unit volume of bulk material is given by

$$T_d = \frac{1}{2} \rho_1 \dot{u}_i^s \dot{u}_i^s + \frac{1}{2} \rho_f \int_{Q_p} (\dot{u}_i^s + v_i)(\dot{u}_i^s + v_i) dQ_p. \tag{4.82}$$

Next, note that

$$\frac{1}{2} \rho_f \int_{Q_p} \dot{u}_i^s \dot{u}_i^s dQ_p = \frac{1}{2} \rho_f \phi \dot{u}_i^s \dot{u}_i^s \tag{4.83}$$

and that

$$\rho_f \int_{Q_p} \dot{u}_i^s v_i dQ_p = \rho_f \dot{u}_i^s \int_{Q_p} v_i dQ_p = \rho_f \dot{u}_i^s \dot{u}_i^f, \tag{4.84}$$

since the averaged relative fluid velocity is obtained by averaging the relative microvelocity field over Q_p.

Next, using (4.80),

$$\begin{aligned}\rho_f \int_{Q_p} v_k v_k dQ_p &= \rho_f \int_{Q_p} a_{ki}\dot{u}_i^f a_{kj}\dot{u}_j^f dQ_p \\ &= \left(\rho_f \int_{Q_p} a_{ki}a_{kj} dQ_p\right)\dot{u}_i^f \dot{u}_j^f = g_{ij}\dot{u}_i^f \dot{u}_j^f,\end{aligned} \tag{4.85}$$

where

$$g_{ij} = \rho_f \int_{Q_p} a_{ki}a_{kj} dQ_p. \tag{4.86}$$

Note that $g_{ij} = g_{ji}$.

Using (4.83), (4.84), and (4.85) in (4.82), we obtain

$$T_d = \frac{1}{2}\rho_1 \dot{u}_i^s \dot{u}_i^s + \frac{1}{2}\rho_f \phi \dot{u}_i^s \dot{u}_i^s + \rho_f \dot{u}_i^s \dot{u}_i^f + \frac{1}{2} g_{ij}\dot{u}_i^f \dot{u}_j^f.$$

Let

$$\rho = \rho_1 + \rho_f\phi = [(1-\phi)\rho_s + \phi\rho_f] = \text{ mass density of bulk material.} \tag{4.87}$$

Then,

$$T_d = \frac{1}{2}\rho \dot{u}_i^s \dot{u}_i^s + \rho_f \dot{u}_i^s \dot{u}_i^f + \frac{1}{2} g_{ij}\dot{u}_i^f \dot{u}_j^f. \tag{4.88}$$

Note that g_{ij} must be positive definite, otherwise, we may have, for $u_i^s \equiv 0$,

$$T = \frac{1}{2} g_{ij}\dot{u}_i^f \dot{u}_j^f = 0 \quad \text{for } u_i^f \neq 0.$$

For an isotropic microvelocity field, we have that

$$g_{ij} = g\delta_{ij}, \tag{4.89}$$

and (4.88) becomes

$$T_d = \frac{1}{2}\rho \dot{u}_i^s \dot{u}_i^s + \rho_f \dot{u}_i^s \dot{u}_i^f + \frac{1}{2} g \dot{u}_i^f \dot{u}_i^f. \tag{4.90}$$

Next, we will compute the form of the dissipation energy density function $\mathcal{D}_d$. Following [3], we will assume that dissipation depends only on the relative flow between the fluid and the solid. Assuming that the relative flow is of Poiseuille type, the microscopic flow pattern inside the pores is uniquely determined by the six generalized velocities $\dot{u}_i^s$, $\dot{\tilde{u}}_i^f$. The dissipation function vanishes when $u_s^i = \dot{\tilde{u}}^f$. Thus, we can write $\mathcal{D}_d$ in the form

$$\mathcal{D}_d = \frac{1}{2}\eta r_{ij}\dot{u}_i^f\dot{u}_j^f \tag{4.91}$$

where η is the fluid viscosity and r_{ij} is a symmetric positive definite matrix. Now, from (4.88) we have that

$$\begin{aligned} &\frac{\partial T_d}{\partial \dot{u}_k^s} = \rho\dot{u}_k^s + \rho_f\dot{u}_k^f, \qquad \frac{\partial T_d}{\partial \dot{u}_k^f} = \rho_f\dot{u}_k^s + g_{kj}\dot{u}_j^f,\\ &\frac{\partial \mathcal{D}_d}{\partial \dot{u}_k^s} = 0, \qquad \frac{\partial \mathcal{D}_d}{\partial \dot{u}_k^f} = \eta r_{kj}\dot{u}_j^f. \end{aligned} \tag{4.92}$$

Thus, combining (4.79) and (4.92) we see that the Lagrange equations (4.72) become

$$\begin{aligned} &\text{ii)} \quad \rho\ddot{u}_i^s + \rho_f\ddot{u}_i^f = \frac{\partial \tau_{ij}}{\partial x_j},\\ &\text{i)} \quad \rho_f\ddot{u}_i^s + g_{ij}\ddot{u}_i^f + \eta r_{ij}\dot{u}_j^f = -\frac{\partial p_f}{\partial x_i}, \end{aligned} \tag{4.93}$$

which are Biot's equation of motion for the fluid–solid system.

Note that in the case of steady flow rate ($\dot{u}_i^f = \text{const}$) from (4.93), we have that

$$\eta r_{ij}\dot{u}_j^f = \frac{\partial p_f}{\partial x_i}. \tag{4.94}$$

Let $\mathbb{K} = (k_{ij})$ be the inverse of the matrix $R = (r_{ij})$. Then, the equation above becomes

$$\eta\mathbb{K}^{-1}\dot{\mathbf{u}}^f = \nabla p \qquad \text{(Darcy's law)}, \tag{4.95}$$

so that $\mathbb{K}$ can be identified with the rock permeability.

Next, in the isotropic case,

$$r_{ij} = r\delta_{ij} = \mathbb{K}^{-1}\delta_{ij}.$$

Thus, in the isotropic case, (4.93) becomes

$$\begin{aligned} &\text{i)} \quad \rho\ddot{u}_i^s + \rho_f\ddot{u}_i^f = \frac{\partial \tau_{ij}}{\partial x_j},\\ &\text{ii)} \quad \rho_f\ddot{u}_i^s + g\ddot{u}_i^f + \eta\mathbb{K}^{-1}\dot{u}_i^f = -\frac{\partial p_f}{\partial x_i}, \qquad i = 1,2,3. \end{aligned} \tag{4.96}$$

Equations (4.96) together with the constitutive relations given in (4.24); i.e.,

$$\begin{aligned} &\text{i)} \quad \tau_{ij} = (\lambda_c e - B\xi)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}^s),\\ &\text{ii)} \quad p_f = -Be + M\xi, \end{aligned} \tag{4.97}$$

completely determines the dynamic behaviour of the solid–fluid system in the low–frequency range.

Note that for the kinetic energy to be positive, the conditions

$$\rho g - \rho_f^2 > 0, \quad g > 0, \quad (\rho > 0 \text{ always}), \tag{4.98}$$

must be satisfied.

In fact,

$$\det \begin{pmatrix} \rho - \beta & \rho_f \\ \rho_f & g - \lambda \end{pmatrix} = \beta^2 - \beta(\rho + g) + (\rho g - \rho_f^2) = 0,$$

$$\beta = \frac{\rho + g \pm \sqrt{(\rho + g)^2 - 4(\rho g - \rho_f^2)}}{2} = \frac{\rho + g \pm \sqrt{(\rho - g)^2 + 4\rho_f^2}}{2}.$$

Thus, for the eigenvalues β to be positive we need that

$$\rho g - \rho_f^2 > 0,$$

and since $\rho > 0$, for the condition above to be true, g must be positive. Thus, (4.98) is proved.

For the analysis that follows, it is convenient to write (4.96) in terms of u_i^s, $\widetilde{u}_i^f$ for the case of constant porosity ϕ. Since (c.f., 4.3)

$$\tau_{ij} = \sigma_{ij} + \delta_{ij}\sigma = \sigma_{ij} - \phi p_f \delta_{ij},$$

from (4.96.i) we have

$$\rho \ddot{u}_i^s + \rho_f \phi (\ddot{\widetilde{u}}_i^f - \ddot{u}_i^s) = \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma}{\partial x_i}. \tag{4.99}$$

Multiplying (4.96.ii) by ϕ we see that

$$\begin{aligned} \frac{\partial \sigma}{\partial x_i} = -\phi \frac{\partial p_f}{\partial x_i} &= \phi \rho_f \ddot{u}_i^s + \phi g \ddot{u}_i^f + \eta \mathbb{K}^{-1} \phi \dot{u}_i^f \\ &= \phi \rho_f \ddot{u}_i^s + \phi^2 g (\ddot{\widetilde{u}}_i^f - \ddot{u}_i^s) + \eta \mathbb{K}^{-1} \phi^2 (\dot{\widetilde{u}}_i^f - \dot{u}_i^s) \\ &= (\phi \rho_f - \phi^2 g) \ddot{u}_i^s + \phi^2 g \ddot{\widetilde{u}}_i^f + \eta \mathbb{K}^{-1} \phi^2 (\dot{\widetilde{u}}_i^f - \dot{u}_i^s). \end{aligned} \tag{4.100}$$

Using (4.100) in (4.99), we obtain

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} &= \rho \ddot{u}_i^s + \rho_f \phi \ddot{\widetilde{u}}_i^f - \rho_f \phi \ddot{u}_i^s - (\phi \rho_f - \phi^2 g) \ddot{u}_i^s - \phi^2 g \ddot{\widetilde{u}}_i^f - \eta \mathbb{K}^{-1} \phi^2 (\dot{\widetilde{u}}_i^f - \dot{u}_i^s) \\ &= (\rho - 2\phi \rho_f + \phi^2 g) \ddot{u}_i^s + (\phi \rho_f - \phi^2 g) \ddot{\widetilde{u}}_i^f + \phi^2 \eta \mathbb{K}^{-1} (\dot{u}_i^s - \widetilde{u}_i^f). \end{aligned} \tag{4.101}$$

Set

$$\begin{aligned} &\rho_{11} = \rho - 2\phi \rho_f + \phi^2 g, \qquad \rho_{12} = \phi \rho_f - \phi^2 g, \\ &\rho_{22} = \phi^2 g, \qquad b = \phi^2 \eta \mathbb{K}^{-1}. \end{aligned} \tag{4.102}$$

Then (4.99) and (4.101) become

$$\text{(4.103)} \qquad \begin{aligned} &\text{i)} \quad \rho_{11}\ddot{u}_i^s + \rho_{12}\ddot{\tilde{u}}_i^f - b(\dot{u}_i^f - \dot{\tilde{u}}_i^s) = \frac{\partial \sigma_{ij}}{\partial x_j}, \\ &\text{ii)} \quad \rho_{12}\ddot{u}_i^s + \rho_{22}\ddot{\tilde{u}}_i^f + b(\dot{u}_i^f - \dot{\tilde{u}}_i^s) = \frac{\partial \sigma}{\partial x_i}. \end{aligned}$$

Next we will give constitutive relations for σ_{ij}, the stress in the solid part of the bulk material, and $\sigma = -\phi p_f$ in terms of $\varepsilon_{ij}(\mathbf{u}^s)$, e, and $\theta = \nabla \cdot \tilde{\mathbf{u}}^f$. First, note that

$$\xi = -\nabla \cdot \mathbf{u}^f = -\nabla \cdot (\phi(\tilde{\mathbf{u}}^f - \mathbf{u}^s)).$$

Thus,

$$\xi = \phi(e - \theta).$$

Next, from (4.98.i), using that $B = \alpha M$,

$$\text{(4.104)} \qquad \sigma = -\phi p_f = \phi \alpha M e - \phi M \phi (e - \theta)$$

or

$$\sigma = \phi M(\alpha - \phi)e + \phi^2 M \theta.$$

Using (4.104), since $\lambda_c = \lambda + \alpha^2 M$, we obtain

$$\text{(4.105)} \qquad \begin{aligned} \sigma_{ij} &= \tau_{ij} - \delta_{ij}\sigma \\ &= [\lambda_c e - \alpha M \phi(e - \theta)]\delta_{ij} + 2\mu\varepsilon_{ij} - [\phi M \alpha e - \phi^2 M e + \phi^2 M \theta] \\ &= \Big[[\lambda_c - 2\alpha M \phi + \phi^2 M]e + \phi M(\alpha - \phi)\theta\Big]\delta_{ij} + 2\mu\varepsilon_{ij} \\ &= \Big[[\lambda + \alpha^2 M - 2\alpha M \phi + \phi^2 M]e + \phi M(\alpha - \phi)\theta\Big]\delta_{ij} + 2\mu\varepsilon_{ij} \\ &= \Big[[\lambda + M(\alpha - \phi)^2]e + \phi M(\alpha - \phi)\theta\Big]\delta_{ij} + 2\mu\varepsilon_{ij}. \end{aligned}$$

Setting

$$\text{(4.106)} \qquad A = \lambda + M(\alpha - \phi)^2, \qquad Q = \phi(\alpha - \phi)M, \qquad R = \phi^2 M,$$

we can rewrite (4.104) and (4.105) in the form

$$\text{(4.107)} \qquad \begin{aligned} &\text{i)} \quad \sigma_{ij} = (Ae + Q\theta)\delta_{ij} + 2\mu\varepsilon_{ij}, \\ &\text{ii)} \quad \sigma = Qe + R\theta. \end{aligned}$$

The coefficient α in (4.106) was shown to be in the range $\phi \le \alpha \le 1$ [5]. The equations of motion (4.103) together with the constitutive relations (4.107) are the original equations derived by Biot in [3].

Let us analyze the significance of the coefficients ρ_{11}, ρ_{12}, ρ_{22} in (4.103) following the ideas in [3]. In these arguments, we will ignore friction effects so that $b \equiv 0$. Let $\rho_1 = (1-\phi)\rho_s$, $\rho_2 = \phi\rho_f$ be the solid and fluid masses per unit volume of bulk material. Assume that $u_i^s = \widetilde{u}_i^f \equiv u_i$. In this case, adding (4.103.i) and (4.103.ii) we obtain

$$(\rho_{11} + 2\rho_{12} + \rho_{22})\ddot{u}_i = \frac{\partial}{\partial x_i}(\sigma_{ij} + \delta_{ij}\sigma) = \frac{\partial}{\partial x_i}\tau_{ij}$$

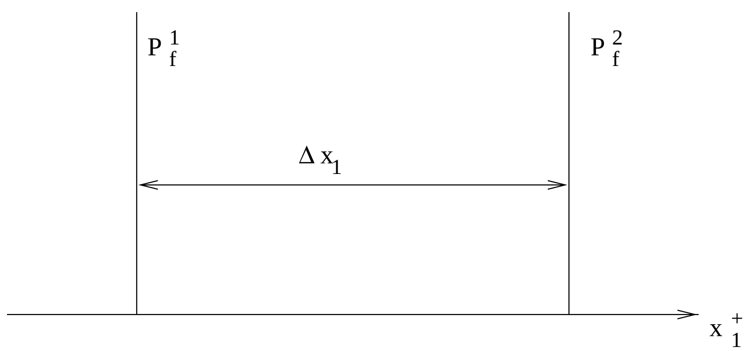
which shows that $\rho_{11} + 2\rho_{12} + \rho_{22}$ represents the total mass ρ of the solid–fluid system per unit volume. Thus,

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}. \tag{4.108}$$

On the other hand,

$$\rho = (1-\phi)\rho_s + \phi\rho_f. \tag{4.109}$$

Next, let us assume displacements only in the x_1–direction and that $u_1^s = u_1^f$.



Assume that $p_f^2 < p_f^1$ so that

$$\frac{\partial p_f}{\partial x_1} \approx \frac{p_f^2 - p_f^1}{\Delta x_1} < 0.$$

Since the fluid moves in the positive x_1–direction with positive acceleration, then

$$-\frac{\partial p_f}{\partial x_1} = \rho_f \ddot{u}_1^f = \rho_f \ddot{u}_1^s$$

or

$$-\phi\frac{\partial p_f}{\partial x_1} = \phi\rho_f\frac{\partial^2 u_1^s}{\partial t^2}. \tag{4.110}$$

On the other hand, from (4.103.ii) we have that

$$(\rho_{12} + \rho_{22})\ddot{u}_1^s = -\phi\frac{\partial p_f}{\partial x_1}. \tag{4.111}$$

Then, from (4.110) and (4.111), we see that

$$\rho_{12} + \rho_{22} = \phi\rho_f = \rho_2. \tag{4.112}$$

Also, using (4.112), (4.108), and (4.109) we see that

$$\rho = (1-\phi)\rho_s + \phi\rho_f = \rho_{11} + \rho_{12} + \rho_{12} + \rho_{22} = \rho_{11} + \rho_{12} + \phi\rho_f$$

so that

$$\rho_{11} + \rho_{12} = (1-\phi)\rho_s = \rho_1. \tag{4.113}$$

The coefficient ρ_{12} is a mass–coupling parameter between fluid and solid. To see this, we consider the special case in which $\widetilde{u}_i^f \equiv 0$. In this case and for displacements in the x_1–direction only, from (4.103) we have that

$$\begin{aligned} &\text{i)} \quad \rho_{11}\ddot{u}_1^s = \frac{\partial\sigma_{11}}{\partial x_1}, \\ &\text{ii)} \quad \rho_{12}\ddot{u}_1^s = -\phi\frac{\partial p_f}{\partial x_1}. \end{aligned} \tag{4.114}$$

The force $-\phi\frac{\partial p_f}{\partial x_1}$ is acting on the fluid to prevent its displacement. This force is in a direction opposite to the acceleration of the solid, and then from (4.114.ii) we see that

$$\rho_{12} < 0. \tag{4.115}$$

Also, from (4.114) we see that, since $\widetilde{u}_1^f = 0$, the solid is moving within the fluid with more inertia; i.e., moves slowly since the fluid is preventing the solid motion. The coefficient ρ_{11} in (4.114.i) represents the total effective mass of the solid moving within the fluid. Since the fluid is restricting the solid motion, the solid has an "apparent mass" ρ_{11} greater than its own mass ρ_1; i.e.,

$$\rho_{11} = \rho_1 + \rho_\alpha, \tag{4.116}$$

where ρ_α is an additional mass due to the fluid. Now from (4.113) and (4.116), we see that

$$\rho_{12} = -\rho_\alpha \tag{4.117}$$

so that ρ_{12} is the additional apparent mass with a change in sign. Thus, we can write the coefficients ρ_{11}, ρ_{12}, and ρ_{22} in the form

$$\rho_{11} = \rho_1 + \rho_\alpha, \qquad \rho_{22} = \rho_2 + \rho_\alpha, \qquad \rho_{12} = -\rho_\alpha. \tag{4.118}$$

The kinetic energy density function T_d associated with (4.103) is

$$T_d = \frac{1}{2}(\rho_{11}\dot{u}_i^s\dot{u}_i^s + 2\rho_{12}\dot{u}_i^s\dot{\widetilde{u}}_i^f + \rho_{22}\dot{\widetilde{u}}_i^f\dot{\widetilde{u}}_i^f). \tag{4.119}$$

In order that T_d be a positive quadratic form, we need the conditions

$$\begin{aligned} &\text{i)} \quad \rho_{11} > 0, \quad \rho_{22} > 0, \\ &\text{ii)} \quad \rho_{11}\rho_{22} - (\rho_{12})^2 > 0. \end{aligned} \tag{4.120}$$

If ρ_{11}, ρ_{12}, and ρ_{22} are given by (4.118), with ρ_1, ρ_2, and ρ_α positive by their physical nature, then it is obvious that (4.120.ii) is always satisfied.

Note that using (4.103) and (4.120) we see immediately that $g > 0$ and that

$$\rho_{11}\rho_{22} - (\rho_{12})^2 = (\rho - 2\phi\rho_f + \phi^2 g)\phi^2 g - (\phi\rho_f - \phi^2 g)^2 = \phi^2[\rho g - \rho_f^2] > 0$$

so that if ρ_{11}, ρ_{12}, ρ_{22} are given in the form (4.118) then the conditions (4.98) are automatically satisfied.

Classification of the Waves in the Isotropic Case.

First note that in the constant coefficients case

$$\begin{aligned}\frac{\partial}{\partial x_j}\tau_{ij} &= \frac{\partial}{\partial x_j}\left(\lambda_c \frac{\partial u_k^2}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_j} B\xi\delta_{ij} \\ &= \lambda_c \frac{\partial}{\partial x_i}\frac{\partial u_k^s}{\partial x_k} + \mu\frac{\partial^2}{\partial x_j^2}u_i^s + \mu\frac{\partial}{\partial x_i}\frac{\partial u_j^s}{\partial x_j} - B\frac{\partial}{\partial x_i}\xi \\ &= (\lambda_c + \mu)\frac{\partial}{\partial x_i}e + \mu\Delta u_i^s - B\frac{\partial}{\partial x_i}\xi.\end{aligned}$$

Then, from (4.96) and (4.98) we obtain

$$\begin{aligned}\rho\ddot{u}_i^s + \rho_f\ddot{u}_i^f &= (\lambda_c + \mu)\frac{\partial e}{\partial x_i} - B\frac{\partial}{\partial x_i}\xi + \mu\Delta u_i^s, \\ \rho_f\ddot{u}_i^s + g\ddot{u}_i^f + \eta\mathbb{K}^{-1}\dot{u}_i^f &= -\frac{\partial}{\partial x_i}(-Be + M\xi).\end{aligned} \tag{4.121}$$

Using vector notation we can write (4.121) in the equivalent form

$$\begin{aligned}\rho\ddot{\mathbf{u}}^s + \rho_f\ddot{\mathbf{u}}^f &= (\lambda_c + \mu)\nabla e - B\nabla\xi + \mu\Delta\mathbf{u}^s, \\ \rho_f\ddot{\mathbf{u}}^s + g\ddot{\mathbf{u}}^f + \eta\mathbb{K}^{-1}\dot{\mathbf{u}}^f &= -\nabla[-Be + M\xi].\end{aligned} \tag{4.122}$$

Applying the divergence operator to (4.122) we obtain the equations governing the propagation of dilatational waves:

$$\rho\ddot{e} + \rho_f\ddot{\theta} = H\Delta e + B\Delta\theta, \qquad \rho_f\ddot{e} + g\ddot{\theta} + \eta\mathbb{K}^{-1}\dot{\theta} = B\Delta e + M\Delta\theta, \tag{4.123}$$

where $H = \lambda_c + 2\mu$ and $\theta = \nabla\cdot\mathbf{u}^f$. Now consider a plane compressional wave of angular frequency ω and wave number $\ell = \ell_r + i\ell_i$ travelling in the x_1–direction; i.e.,

$$\begin{aligned}e &= C_1^{(\ell)}e^{i(\ell x_1 - \omega t)} = C_1^{(\ell)}e^{-\ell_i x_1}e^{i\ell_r(x_1 - \frac{\omega}{\ell_r}t)}, \\ \theta &= C_2^{(\ell)}e^{i(\ell x_1 - \omega t)} = C_2^{(\ell)}e^{-\ell_i x_1}e^{i\ell_r(x_1 - \frac{\omega}{\ell_r}t)}.\end{aligned} \tag{4.124}$$

Thus, the wave has phase velocity $\omega/|\ell_r|$ and attenuation factor ℓ_i; this factor should be nonnegative to have a physically meaningful solution. Substitution of (4.124) in (4.123) yields

$$\begin{aligned}&-\omega^2[C_1\rho + C_2\rho_f] = -\ell^2[C_1 H + C_2 B], \\ &-\omega^2\left[C_1\rho_f + C_2 g - \frac{\eta\mathbb{K}^{-1}}{i\omega}C_2\right] = -\ell^2[C_1 B + C_2 M].\end{aligned} \tag{4.125}$$

Set

$$\gamma = \frac{\omega}{\ell} = \frac{\omega}{\ell_r + i\ell_i} = \frac{\omega(\ell_r - i\ell_i)}{\ell_r^2 + \ell_i^2} = \gamma_r + i\gamma_i.$$

Then,

$$
\begin{aligned}
C_1 H + C_2 B &= \gamma^2 [C_1 \rho + C_2 \rho_f], \\
C_1 B + C_2 M &= \gamma^2 \left[C_1 \rho_f + C_2 g + i \frac{\eta \mathbb{K}^{-1}}{\omega} C_2 \right].
\end{aligned} \tag{4.126}
$$

Set

$$
\begin{aligned}
\widetilde{A} &= \begin{pmatrix} \rho & \rho_f \\ \rho_f & g \end{pmatrix}, \qquad \widetilde{E} = \begin{pmatrix} H & B \\ B & M \end{pmatrix}, \\
\widetilde{\mathcal{C}} &= \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\eta K^{-1}}{\omega} \end{bmatrix}, \qquad \mathbb{C}^{(\gamma)} = \begin{pmatrix} C_1^{(\gamma)} \\ C_2^{(\gamma)} \end{pmatrix}.
\end{aligned}
$$

In matrix form, (4.125) now becomes the generalized eigenvalue problem

$$
\gamma^2 (\widetilde{A} + i \widetilde{\mathcal{C}}) \mathbb{C}^{(\gamma)} = \widetilde{E} \mathbb{C}^{(\gamma)}. \tag{4.127}
$$

Next note that since $\widetilde{E}$ and $\widetilde{A}$ are associated with the strain and kinetic energies, they are positive definite, and $\widetilde{\mathcal{C}}$ is nonnegative. Let $\lambda = \lambda_r + i\lambda_i$ and $\mathbf{x}$ be an eigenvalue and eigenvector of the generalized eigenvalue problem (4.127); i.e.,

$$
\widetilde{E} \mathbf{x} = \lambda (\widetilde{A} + i \widetilde{\mathcal{C}}) \mathbf{x}. \tag{4.128}
$$

Set

$$
r = (\widetilde{E}\mathbf{x}, \mathbf{x}) > 0, \quad p = (\widetilde{A}\mathbf{x}, \mathbf{x}) > 0, \quad q = (\widetilde{\mathcal{C}}\mathbf{x}, \mathbf{x}) \geq 0.
$$

Then, from (4.128),

$$
(\widetilde{E}\mathbf{x}, \mathbf{x}) = \lambda ((\widetilde{A}\mathbf{x}, \mathbf{x}) + i (\widetilde{\mathcal{C}}\mathbf{x}, \mathbf{x})) \tag{4.129}
$$

or

$$
r = (\lambda_r + i\lambda_i)(p + iq) = \lambda_r p - \lambda_i q + i(\lambda_r q + \lambda_i p).
$$

Then,

$$
\begin{aligned}
\text{i)} \quad r &= \lambda_r p - \lambda_i q, \\
\text{ii)} \quad 0 &= \lambda_r q + \lambda_i p.
\end{aligned} \tag{4.130}
$$

Multiplying (4.130.i) by p and (4.130.ii) by q and adding the resulting equations, we obtain

$$
\lambda_r (p^2 + q^2) = rp.
$$

Therefore,

$$
\begin{aligned}
\text{i)} \quad \lambda_r &= \frac{rp}{p^2 + q^2} > 0, \\
\text{ii)} \quad \lambda_i &= -\lambda_r \frac{q}{p} = -\frac{rq}{p^2 + q^2} \leq 0.
\end{aligned} \tag{4.131}
$$

Thus,

$$\gamma^2 = \lambda = \frac{r}{p^2+q^2}(p - iq) = \mathrm{Re}(\gamma^2) + i\,\mathrm{Im}(\gamma^2)$$

so that

$$\mathrm{Re}(\gamma^2) > 0, \quad \mathrm{Im}(\gamma^2) \leq 0. \tag{4.132}$$

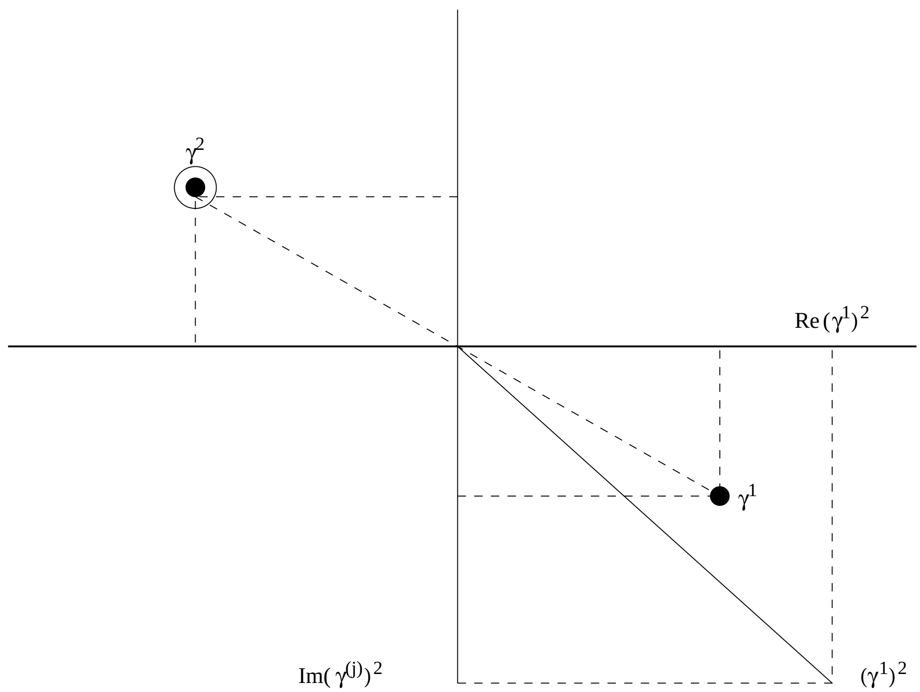


FIGURE 13

Let $(\gamma^{(j)})^2$, $j = 1, 2$ be the two solutions of the generalized eigenvalue problem (4.127). Since

$$\ell^{(j)} = \ell_r^{(j)} + i\ell_i^{(j)} = \frac{\omega}{\gamma^{(j)}} = \frac{\omega(\gamma_r^{(j)} - i\gamma_i^{(j)})}{(\gamma_r^{(j)})^2 + (\gamma_i^{(j)})^2}, \tag{4.133}$$

we choose $\gamma^{(j)}$ such that $\gamma_i^{(j)} \geq 0$ so that $\ell_i^{(j)} \leq 0$ in order to have physically meaningful solutions; i.e., the root circled in Figure 13.

The corresponding phase velocities are given by

$$\nu^{(j)} = \frac{\omega}{|\ell_r^{(j)}|}, \quad j = 1, 2, \tag{4.134}$$

corresponding to the type I and type II compressional waves, respectively. Instead of the attenuation coefficient ℓ_i in (4.124), it is convenient to use another attenuation coefficient defined as follows: At $x_1 = 0$, the original amplitude for e is $e_0 = C_1^{(\ell^{(j)})}$. Let

$$\chi^{(j)} = \text{ wavelength } = \frac{\text{phase velocity}}{\text{frequency}} \equiv \frac{\nu^{(j)}}{f}.$$

Then, at $x_1 = \chi^{(j)}$, the amplitude of e is

$$e(\chi^{(j)}) = C_1^{(\ell^{(j)})} e^{-\ell_i^{(j)} \frac{\nu^{(j)}}{f}} = e_0 e^{-\ell_i^{(j)} \frac{\nu^{(j)}}{f}}.$$

Then,

$$\log_{10}\left(\frac{e(\chi^{(j)})}{e_0}\right) = -\ell_i^{(j)} \frac{\nu^{(j)}}{f} \log_{10}(e).$$

We define the attenuation coefficient $b^{(j)}$ measured in DB/HZ-sec by the formula

$$\begin{aligned} b^{(j)} &= -20 \log_{10}\left(\frac{e(\chi^{(j)})}{e_0}\right) = 20 \log_{10}(e) \ell_i^{(j)} \frac{\nu^{(j)}}{f} \\ &= 20 \log_{10}(e) \ell_i^{(j)} \frac{\omega}{|\ell_f^{(j)}| \omega / 2\pi} = (2\pi)(8.685889) \ell_i^{(j)} / |\ell_r^{(j)}|. \end{aligned}$$

This coefficient measures the wave attenuation after travelling one wavelength. For example, an attenuation coefficient $b^{(1)}$ of 20 DB implies that after travelling one wavelength the wave has reduced ten times its original amplitude.

Let us consider the purely elastic case ($\eta = 0$). In this case, (4.127) becomes

$$\widetilde{E}\mathbb{C}^{(\gamma)} = \gamma^2 \widetilde{A}\mathbb{C}^{(\gamma)}. \tag{4.135}$$

Multiplying by $\widetilde{A}^{-\frac{1}{2}}$, we get the equation

$$\widetilde{A}^{-\frac{1}{2}} \widetilde{E} \widetilde{A}^{-\frac{1}{2}} \widetilde{A}^{\frac{1}{2}} \mathbb{C}^{(\gamma)} = \gamma^2 \widetilde{A}^{\frac{1}{2}} \mathbb{C}^{(\gamma)}. \tag{4.136}$$

Set

$$\begin{aligned} &\text{i)} \quad \mathbf{q}^{(\gamma)} = \widetilde{A}^{\frac{1}{2}} \mathbb{C}^{(\gamma)}, \\ &\text{ii)} \quad D = \widetilde{A}^{-\frac{1}{2}} \widetilde{E} \widetilde{A}^{-\frac{1}{2}}. \end{aligned} \tag{4.137}$$

Note that D is symmetric, positive definite. Then (4.136) becomes

$$D\mathbf{q}^{(\gamma)} = \gamma^2 \mathbf{q}^{(\gamma)}. \tag{4.138}$$

Let $q^{(\gamma_j)}$, $j = 1, 2$ be the orthonormal eigenvectors associated with D. Then,

$$[\mathbf{q}^{(\gamma_1)}, \mathbf{q}^{(\gamma_2)}] = [\widetilde{A}^{\frac{1}{2}} \mathbb{C}^{(\gamma_1)}, \widetilde{A}^{\frac{1}{2}} \mathbb{C}^{(\gamma_2)}] = [\widetilde{A}\mathbb{C}^{(\gamma_1)}, \mathbb{C}^{(\gamma_2)}] = 0, \tag{4.139}$$

which is analogous to the orthogonality relation derived by Biot in [3]. To interpret the orthogonality relation (4.139) it is convenient to rewrite it in terms of the absolute fluid displacement $\widetilde{\mathbf{u}}^f$ and $\mathbf{u}^s$. Since

$$\mathbf{u}^f = \phi(\widetilde{\mathbf{u}}^f - \mathbf{u}^s),$$

we see that

$$\widetilde{\theta} = \nabla \cdot \widetilde{\mathbf{u}}^f = \widetilde{C}_2 e^{i(\ell x_1 - \omega t)}. \tag{4.140}$$

Since

$$\theta = \phi(\widetilde{\theta} - e) = \phi(\widetilde{C}_2 - C_1)e^{i(\ell x_1 - \omega t)} = C_2 e^{i(\ell x_1 - \omega t)},$$

we conclude that the amplitude relation is

$$C_2 = \phi(\widetilde{C}_2 - C_1). \tag{4.141}$$

Set

$$\mathbb{C}^{(\gamma_1)} = (C_1^{(\gamma_1)}, C_2^{(\gamma_1)}) \equiv (x_1, C_2^{(\gamma_1)},), \mathbb{C}^{(\gamma_2)} = (C_1^{(\gamma_2)}, C_2^{(\gamma_2)}) \equiv (y_1, C_2^{(\gamma_2)}).$$

Then,

$$C_2^{(\gamma_1)} = \phi(\widetilde{C}_2^{(\gamma_1)} - x_1), \qquad C_2^{(\gamma_2)} = \phi(\widetilde{C}_2^{(\gamma_2)} - y_1).$$

To simplify this notation, set $x_2 = \widetilde{C}_2^{(\gamma_1)}$, $y_2 = \widetilde{C}_2^{(\gamma_2)}$, so that

$$\mathbb{C}^{(\gamma_1)} = (x_1, \phi(x_2 - x_1)), \mathbb{C}^{(\gamma_2)} = (y_1, \phi(y_2 - y_1)). \tag{4.142}$$

Using (4.142) in (4.139), we obtain

(4.143)

$$\begin{aligned}
[\widetilde{A}\mathbb{C}^{(\gamma_1)}, \mathbb{C}^{(\gamma_2)}] &= \left[\begin{pmatrix} \rho & \rho_f \\ \rho_f & g \end{pmatrix} \begin{pmatrix} x_1 \\ \phi(x_2 - x_1) \end{pmatrix}, \begin{pmatrix} y_1 \\ \phi(y_2 - y_1) \end{pmatrix} \right] \\
&= [\rho x_1 + \rho_f \phi(x_2 - x_1)] y_1 + [\rho_f x_1 + g\phi(x_2 - x_1)]\phi(y_2 - y_1) \\
&= (\rho - 2\rho_f \phi + g\phi^2) x_1 y_1 + (\rho_f \phi - g\phi^2)(x_2 y_1 + x_1 y_2) + g\phi^2 x_2 y_2 = 0.
\end{aligned}$$

Next, using the notation defined in (4.103) we can rewrite (4.143) in the form

$$\rho_{11} x_1 y_1 + \rho_{12}(x_2 y_1 + x_1 y_2) + \rho_{22} x_2 y_2 = 0. \tag{4.144}$$

Possible cases.

a) $x_1 > 0 \quad x_2 < 0$,
$y_1 > 0 \quad y_2 < 0$.
Then,

$$\rho_{11} x_1 y_1 > 0, \qquad \rho_{22} x_2 y_2 > 0, \qquad \rho_{12}(x_2 y_1 + x_1 y_2) > 0,$$

and (4.144) cannot be satisfied.
Thus, this case is *not possible.*

b) $x_1 > 0 \quad x_2 > 0$
$y_1 > 0 \quad y_2 > 0$.
In this case we can rewrite (4.114) in the form
$(\rho_{11} + \rho_{12}) y_1 (x_1 + x_2) + (\rho_{12} + \rho_{22}) y_2 (x_1 + x_2) = 0$,
or
$(1 - \phi)\rho_s y_1 (x_1 + x_2) + \phi \rho_f y_2 (x_1 + x_2) = 0$,
that cannot be satisfied since all terms are positive.
Then, this case is also *not possible*

c) $x_1 < 0, \quad x_2 < 0$,
$y_1 < 0, \quad y_2 < 0$.

is not possible for the same reason as case b).

d) Case $x_1 < 0$, $x_2 > 0$, $y_1 < 0$, $y_2 > 0$ is not possible because it is like case a). Then, *the only possible cases* are

$$\begin{aligned} &x_1 > 0, \quad x_2 > 0, \\ &y_1 < 0, \quad y_2 > 0 \quad (\text{or } y_1 > 0, y_2 < 0). \end{aligned} \tag{4.145}$$

This says that if the amplitudes are in phase for one velocity; i.e., same amplitude signs, then they are in opposite phase for the other. Thus, there is a wave in which the amplitudes are in phase and another in which they are in opposite phase.

Now we go back to (4.135) and take the inner product with $\mathbb{C}^{(\gamma)} = (C_1^{(\gamma)}, C_2^{(\gamma)})$:

$$[\widetilde{E}\mathbb{C}^{(\gamma)}, \mathbb{C}^{(\gamma)}] = \gamma^2 [\widetilde{A}\mathbb{C}^{(\gamma)}, \mathbb{C}^{(\gamma)}]. \tag{4.146}$$

Using the notation defined above, we write

$$\mathbb{C}^{(\gamma)} = (x_1, \phi(x_2 - x_1)) \tag{4.147}$$

and, consequently,

$$[\widetilde{A}\mathbb{C}^{(\gamma)}, \mathbb{C}^{(\gamma)}] = \rho_{11} x_1^2 + 2\rho_{12} x_1 x_2 + \rho_{22} x_2^2. \tag{4.148}$$

Next, using that $B = \alpha M$,

$$\begin{aligned} [\widetilde{E}\mathbb{C}^{(\gamma)}, \mathbb{C}^{(\gamma)}] &= \left[\begin{bmatrix} H & \alpha M \\ \alpha M & M \end{bmatrix} \begin{bmatrix} x_1 \\ \phi(x_2 - x_1) \end{bmatrix}, \begin{bmatrix} x_1 \\ \phi(x_2 - x_1) \end{bmatrix} \right] \\ &= [H x_1 + \alpha M \phi(x_2 - x_1)] x_1 + [\alpha M x_1 + M\phi(x_2 - x_1)]\phi(x_2 - x_1) \\ &= (H - 2\alpha M \phi + M\phi^2)(x_1)^2 + 2(\alpha M \phi - M\phi^2) x_1 x_2 + M\phi^2 (x_2)^2. \end{aligned}$$

Since $H = \lambda_c + 2\mu = \lambda + \alpha^2 M + 2\mu$, using (4.106) we see that

$$[\widetilde{E}C^{(\alpha)}, C^{(\alpha)}] = A(x_1)^2 + 2Q x_1 x_2 + R(x_2)^2. \tag{4.149}$$

Thus, using (4.148) and (4.149) in (4.146) we see that

$$\frac{\rho_{11}(x_1)^2 + 2\rho_{12} x_1 x_2 + \rho_{22}(x_2)^2}{A(x_1)^2 + 2Q x_1 x_2 + R(x_2)^2} = \frac{1}{\gamma^2}. \tag{4.150}$$

From (4.150) we see that, since the only negative coefficient in (4.150) is ρ_{12}, the minimum of $\frac{1}{\gamma^2}$ corresponds to the case in which x_1 and x_2 have the same sign; i.e., amplitudes in phase. Thus, the higher velocity ($\min \frac{1}{\gamma}$) has amplitudes in phase and the lower velocity has amplitudes in opposite phase.

Next we consider the rotational waves. Let

$$\boldsymbol{\kappa}^s = \operatorname{curl} \mathbf{u}^s, \qquad \boldsymbol{\kappa}^f = \operatorname{curl} \mathbf{u}^f. \tag{4.151}$$

Then applying the curl operator to equations (4.122) we obtain the equations governing the propagation of rotational waves:

$$\rho\ddot{\boldsymbol{\kappa}}^s + \rho_f \ddot{\boldsymbol{\kappa}}^f = \mu\Delta\boldsymbol{\kappa}^s, \qquad \rho_f\ddot{\boldsymbol{\kappa}}^s + g\ddot{\boldsymbol{\kappa}}^f + \eta\mathbb{K}^{-1}\dot{\boldsymbol{\kappa}}^f = 0. \tag{4.152}$$

Let us consider a plane rotational wave of angular frequency $\omega = 2\pi f$ and wave number $\ell = \ell_r + i\ell_i$ travelling in the x_1–direction:

$$\begin{aligned} \kappa^s &= C_1^{(\ell)} e^{i(\ell x_1 - \omega t)} = C_1^{(\ell)} e^{-\ell_i x_1} e^{i\ell_r (x_1 - \frac{\omega}{\ell_r} t)}, \\ \kappa^f &= C_2^{(\ell)} e^{i(\ell x_1 - \omega t)} = C_2^{(\ell)} e^{-\ell_i x_1} e^{i\ell_r (x_1 - \frac{\omega}{\ell_r} t)}. \end{aligned} \tag{4.153}$$

Substitution in (4.152) yields

$$\begin{aligned} &\text{i)} \quad -\omega^2 [C_1\rho + C_2\rho_f] = -\ell^2 \mu C_1, \\ &\text{ii)} \quad -\omega^2 \left[C_1\rho_f + C_2 g - \frac{\eta\mathbb{K}^{-1}}{i\omega} C_2 \right] = 0. \end{aligned} \tag{4.154}$$

From (4.154.ii), we have that

$$C_1\rho_f + C_2\left(g + i\frac{\eta\mathbb{K}^{-1}}{\omega}\right) = 0,$$

so that

$$C_2 = -\frac{\rho_f}{g + i\frac{\eta\mathbb{K}^{-1}}{\omega}} C_1. \tag{4.155}$$

Using (4.155) in (4.154.i),

$$C_1\rho + \rho_f\left(- \frac{\rho_f}{g + \frac{i\eta\mathbb{K}^{-1}}{\omega}} \right) C_1 = \left(\frac{\ell}{\omega}\right)^2 \mu C_1.$$

Thus,

$$\left(\rho - \frac{\rho_f^2}{g + i\frac{\eta\mathbb{K}^{-1}}{\omega}}\right) \Big/ \mu = \left(\frac{\ell}{\omega}\right)^2.$$

This can be written as

$$\left(\rho\left(g + i\frac{\eta\mathbb{K}^{-1}}{\omega}\right) - \rho_f^2\right) \Big/ \mu = \left(\frac{\ell}{\omega}\right)^2$$

or

$$\frac{\rho g - \rho_f^2}{\mu} + i\frac{\eta\mathbb{K}^{-1}}{\omega\mu} \equiv E_r + i\,E_i = \left(\frac{\ell}{\omega}\right)^2 = \frac{1}{\beta^2}. \tag{4.156}$$

In the nondissipative case, the shear phase wave velocity is given by

$$\beta = \sqrt{\frac{\mu}{\rho g - \rho_f^2}} = \frac{\omega}{|\ell_r^s|}.$$

Now consider the dissipative case. From (4.156), we get

$$E_r + i\,E_i = \frac{1}{\omega^2}(\ell_r + i\ell_i)^2 = \frac{1}{\omega^2}(\ell_r^2 - \ell_i^2 + 2i\ell_i\ell_r).$$

Thus,

$$\text{(4.157)} \qquad \begin{aligned} \text{i)} \quad & \ell_r^2 - \ell_i^2 = E_r\omega^2, \\ \text{ii)} \quad & 2\ell_r\ell_i = E_i\omega^2. \end{aligned}$$

To determine the phase velocity and attenuation of the shear waves, we need to determine ℓ_r and ℓ_i. From (4.157.ii), since

$$\text{(4.158)} \qquad \ell_i = \frac{E_i\omega^2}{2\ell_r},$$

substitution in (4.157.i) yields

$$\ell_r^2 - \frac{E_i^2\omega^4}{4\ell_r^2} = E_r\omega^2.$$

Thus, multiplying by $(\ell_r)^2$, we obtain

$$\ell_r^4 - E_r\omega^2\ell_r^2 - \frac{E_i^2\omega^4}{4} = 0.$$

From this equation,

$$\text{(4.159)} \qquad \ell_r^2 = \frac{E_r\omega^2 \pm \omega^2\sqrt{E_r^2 + E_i^2}}{2}.$$

Since $\ell_r^2 > 0$, we choose the positive sign in (4.159). Thus,

$$\text{(4.160)} \qquad \ell_r^2 = \frac{\omega^2}{2}\left(E_r + \sqrt{E_r^2 + E_i^2}\right).$$

Now, using (4.160) in (4.157),

$$\ell_i^2 = \ell_r^2 - E_r\omega^2 = \omega^2\left(\frac{\sqrt{E_r^2 + E_i^2} - E_r}{2}\right).$$

Since ℓ_i needs to be nonnegative to have physically meaningful solutions, we choose

$$\text{(4.161)} \qquad \ell_i^{(s)} = \frac{\omega}{\sqrt{2}}((E_r^2 + E_i^2)^{\frac{1}{2}} - E_r)^{\frac{1}{2}}.$$

Now using (4.158), we find

$$\text{(4.162)} \qquad \ell_r^{(s)} = \frac{E_i\omega^2}{2\ell_i^{(s)}}.$$

Finally, the phase velocity $\nu^{(s)}$ and attenuation factor $b^{(s)}$ are defined as before by

$$\nu^{(s)} = \frac{\omega}{|\ell_r^{(s)}|}, \quad b^{(s)} = (2\pi) \cdot 8.685889(\ell_i^{(s)}/|\ell_r^{(s)}|). \tag{4.163}$$

In the table below, we give values of the material constants for several types of sandstones [6], [13]:

Pore Fluid

	Viscosity	*Density*	*Bulk Modulus*
Water	1.0 cp (centipoise)	1 gr/cm^3	2.25 10^{10}dynes/cm^2
Methane (4500 PSI, 275°F)	.022 cp	.1398 gr/cm^3	.05543378 10^{10}dynes/cm^2
Gas (after [6])	.015 cp	.1 gr/cm^3	.022 10^{10}dynes/cm^2
Oil	.4 cp	.7 gr/cm^3	.57 10^{10}dynes/cm^2

Solid Grains

Density	Bulk Modulus
2.65 gr/cm^3	3.79 10^{11} dynes/cm^2

Solid Matrix

	Porosity	*Permeability*	*Dilational Velocity*	*Shear Velocity*
Berea Sandstone	.19	200 md (millidarcies)	3670 m/sec	2175 m/sec
Teapot Sandstone	.297	1900 md	3048 m/sec	1865 m/sec
Fox-Hill Sandstone	.074	32.5 md	4450 m/sec	2515 m/sec

Elastic Constants for Saturated Berea Sandstone (Units: 10^{10} dynes/cm^2)

Fluid	k_c	μ	B	M
Water	19.0866	10.1542	6.2493	10.5136
Oil	16.399	10.1542	1.7279	2.9069
Gas [6]	15.4128	10.1542	.06874	.1156
Methane	15.4747	10.1542	.1728	.2908

CHAPTER 5
EQUATIONS OF MOTION FOR FLUID-SATURATED POROUS MEDIA IN THE HIGH-FREQUENCY RANGE

The equations of motion (4.96) were derived under the assumption that the flow inside the poral space is of Poiseuille type. This assumption breaks down if the frequency exceeds a certain critical value f_t. This can be seen by the following argument.

Consider a plane boundary in the presence of an infinitely–extended viscous fluid oscillating harmonically in its own plane.

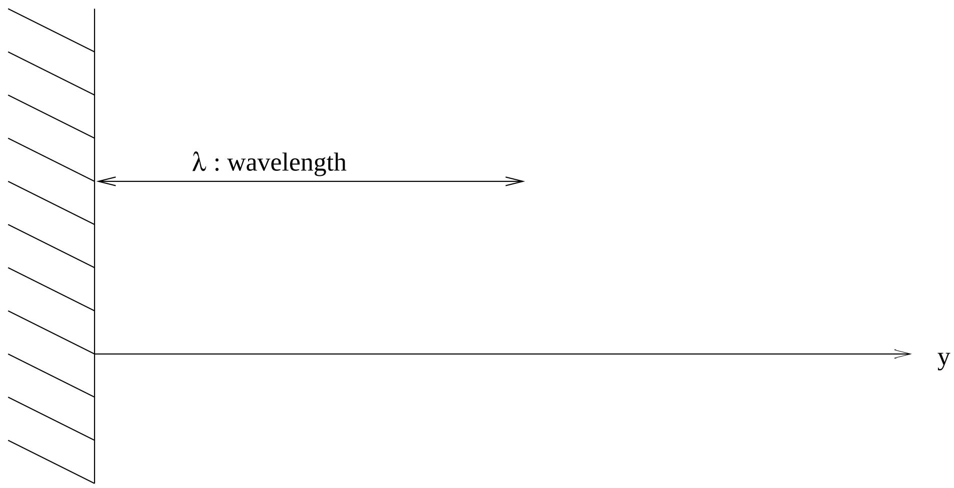


FIGURE 14

To describe the viscous fluid motion, we have the general Navier–Stokes equations [18]

$$\rho_f \frac{D\mathbf{v}}{Dt} = -\nabla p_f + \nabla\cdot\tau + \rho_f \mathbf{g} \tag{5.1}$$

where

$$\tau_{ij} = 2\eta d_{ij} + \left[\left(\kappa - \frac{2}{3}\eta\right)\nabla\cdot\mathbf{v}\right]\delta_{ij} \tag{5.2}$$

$=$ stress tensor in the fluid (Newtonian Fluid),

$\eta =$ shear coefficient of viscosity,

$\kappa =$ bulk coefficient of viscosity,

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) : \text{ rate of strain tensor.} \tag{5.3}$$

Assuming that the fluid is incompressible, $\nabla\cdot\mathbf{v} = 0$ and

$$\tau_{ij} = 2\eta d_{ij}. \tag{5.4}$$

In this case, for constant viscosity η,

$$\nabla\cdot\tau = \eta 2 \frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \eta\left[\frac{\partial^2 v_i}{\partial x_j} + \frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right] = \eta\Delta v_i.$$

Then, in the incompressible case, we get the Navier–Stokes equations

$$\rho_f \Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Local acceleration}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{\text{Convective acceleration}} \Big) = - \underbrace{\nabla p_f}_{\text{pressure force per unit volume}} + \underbrace{\eta \Delta \mathbf{v}}_{\text{viscous force per unit volume}} + \underbrace{\rho \mathbf{g}}_{\text{body force per unit volume}} . \tag{5.5}$$

In applying (5.5) to the present case, we ignore the convective acceleration, gravity effects, and pressure gradients and consider that the velocity is only $x_2 = y$–dependent. Thus, (5.5) reduces to

$$\rho_f \frac{\partial v}{\partial t} = \eta \frac{\partial^2 v}{\partial y^2}. \tag{5.6}$$

Now we search for solutions of (5.6) of the form

$$v = A e^{i(\omega t - \ell y)} = A e^{i(\omega t - (\ell_r + i\ell_i) y)} = e^{\ell_i y} e^{i(\omega t - \ell_r y)}. \tag{5.7}$$

Substitution of (5.7) into (5.6) yields the relation

$$\rho_f i\omega = -\eta \ell^2 = -\eta(\ell_r^2 - \ell_i^2 + 2i\ell_i\ell_r).$$

Set

$$\nu = \frac{\eta}{\rho_f} = \text{ kinematic viscosity.}$$

Thus,

$$\begin{aligned} &\text{i)} \quad \ell r^2 = \ell_i^2, \\ &\text{ii)} \quad 2\ell_i\ell_r = \frac{\rho_f}{\eta}\omega = \frac{\omega}{\nu}. \end{aligned} \tag{5.8}$$

Therefore, $|\ell_r| = |\ell_i|$, and taking modulus in (5.8.ii) we have that

$$|\ell_i|^2 = \frac{\omega}{2\nu}. \tag{5.9}$$

Since ℓ_i must be nonpositive to have a physically–meaningful solution, from (5.9) we have that

$$\ell_i = -\sqrt{\frac{\omega}{2\nu}}, \tag{5.10}$$

and, consequently, $|\ell_r| = \sqrt{\frac{\omega}{2\nu}}$. Thus, for motion in the positive y–direction, $\ell_r > 0$, and from (5.7) we see that the velocity of the fluid parallel with the plane at a distance y from the plane is

$$v(y,t) = e^{i(\omega t - \sqrt{\frac{\omega}{2\nu}} y)} e^{-\sqrt{\frac{\omega}{2\nu}} y}. \tag{5.11}$$

Since a plane wave $e^{i(\omega t-\frac{2\pi}{\lambda}y)}$ has wavelength λ; i.e., spatial period λ, from (5.11) we see that $v(y,t)$ has wavelength

$$\lambda = 2\pi\sqrt{\frac{2\nu}{\omega}}.$$

The quarter wavelength of the boundary layer is then

$$y_1 = \frac{\lambda}{4} = \pi\sqrt{\frac{\nu}{2\omega}}. \tag{5.12}$$

For a porous material, we may assume that Poiseuille flow breaks down when y_1 is on the order of diameter d of the pores; i.e., for

$$\pi\sqrt{\frac{\nu}{2\omega}} = y_1 < d,$$

or, equivalently,

$$\pi^2\frac{\nu}{2(2\pi f)} < d^2.$$

Then,

$$d^2 > \frac{\pi\nu}{4f},$$

so that

$$f > \frac{\pi\nu}{4d^2} = f_t. \tag{5.13}$$

For water at 15°C, $\rho_f = 1$ gr/cm^3,

$$\eta = 1cp = 10^{-2} \text{ poise } = 10^{-2}\frac{\text{dynes}}{\text{cm}^2}\text{sec}.$$

Then,

$$\frac{\eta}{\rho_f} = \frac{10^{-2}\text{cm}^2}{\text{sec}}.$$

Then, for $d = 10^{-2}$ cm,

$$f_t = \frac{\pi}{4}\frac{1}{10^{-4}\text{cm}^2}\left(\frac{10^{-2}\text{cm}^2}{\text{sec}}\right) = \frac{\pi}{4}\frac{10^2}{\text{sec}} \approx 100 \text{ Hz}.$$

For $d = 10^{-3}$ cm,

$$f_t = \frac{\pi}{4}10^4 \text{ Hz}.$$

To study the modifications to be introduced in the theory in the range of frequencies above f_t it is more convenient to use the form of Biot's equations given by (4.103) together with the constitutive equations (4.107); i.e.,

$$\begin{aligned} &\text{i)} \quad \rho_{11}\ddot{u}_i^s + \rho_{12}\ddot{\tilde{u}}_i^f - b(\dot{\tilde{u}}_i^f - \dot{u}_i^2) = \frac{\partial\sigma_{ij}}{\partial x_j}, \\ &\text{ii)} \quad \rho_{12}\ddot{u}_i^s + \rho_{22}\ddot{\tilde{u}}_i^f + b(\dot{\tilde{u}}_i^f - \dot{u}_i^s) = \frac{\partial\sigma}{\partial x_i}, \qquad i = 1,2,3, \end{aligned} \tag{5.14}$$

$$(5.15)\quad \begin{aligned} &\text{i)}\quad \sigma_{ij} = [Ae + Q\theta]\delta_{ij} + 2\mu\varepsilon_{ij}: && \text{force in solid per unit bulk volume,}\\ &\text{ii)}\quad \sigma = -\phi p_f = Qe + R\theta: && \text{force in the fluid per unit bulk volume.}\end{aligned}$$

In this part, we will follow the ideas in [4]. Ignore acceleration terms and consider motion only in the x_1–direction. Then, from (5.14) we have that

$$(5.16)\quad \begin{aligned} &\text{i)}\quad -b(\dot{\tilde{u}}_1^f - \dot{u}_1^s) = \frac{\partial \sigma_{11}}{\partial x_1},\\ &\text{ii)}\quad b(\dot{\tilde{u}}_1^f - \dot{u}_1^s) = -\phi\frac{\partial p_f}{\partial x_1}.\end{aligned}$$

If $p_f^1 < p_f^0$, then $\frac{\partial p_f}{\partial x_1} \approx \frac{p_f^1 - p_f^0}{\Delta x_1} < 0$, so that from (5.16.ii), $b(\dot{\tilde{u}}_1^f - \dot{u}_1^s) > 0$ and the relative flow rate $(\dot{\tilde{u}}_1^f - \dot{u}_1^s)$ is in the positive x_1–direction; i.e., flow to the right. Now from (5.16.i) we see that $G = b(\dot{\tilde{u}}_1^f - \dot{u}_1^s)$ is the friction force per unit bulk volume exerted by the fluid on the solid in the direction of the motion. To study the nature of this friction force G we analyze two limit cases.

First we consider the flow in a cylindrical duct of radius a as in Figure 15.

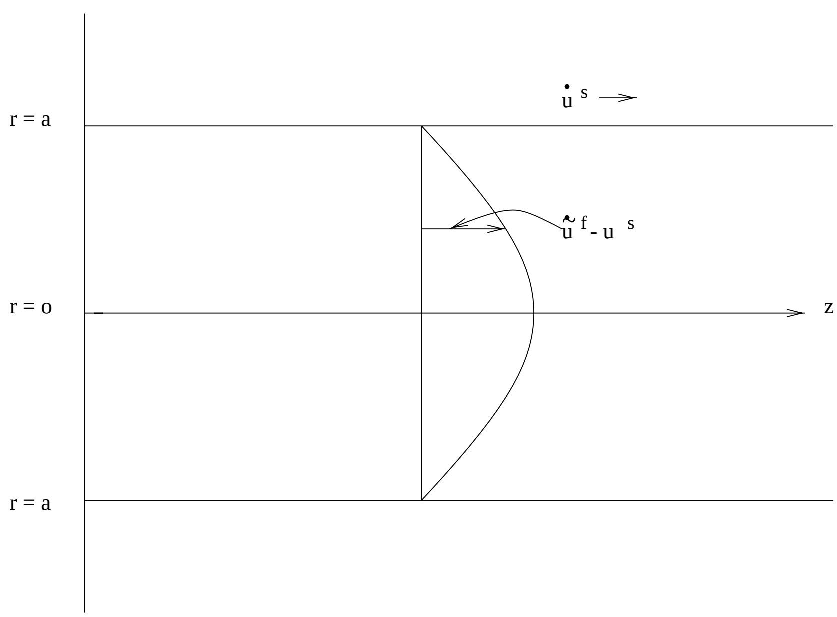


FIGURE 15

To describe this flow, we go back to the Navier–Stokes equations

$$\rho_f \frac{D\mathbf{v}}{Dt} = -\nabla p_f + \rho_f \mathbf{g} + \nabla \cdot \tau, \quad i = 1, 2, 3, \tag{5.17}$$

and write it in cylindrical coordinates. Recall that the strain tensor τ in cylindrical coordinates is given by [18, p. 146].

$$\begin{aligned}
&\tau_{rr} = 2\eta \frac{\partial v_r}{\partial r} + \left(\kappa - \frac{2}{3}\eta\right)\nabla \cdot \mathbf{v}, && \tau_{\theta\theta} = 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \left(\kappa - \frac{2}{3}\eta\right)\nabla \cdot \mathbf{v},\\
&\tau_{zz} = 2\eta\left(\frac{\partial v_z}{\partial z}\right) + \left(\kappa - \frac{2}{3}\eta\right)\nabla \cdot \mathbf{v}, && \tau_{r\theta} = \tau_{\theta r} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right),\\
&\tau_{\theta z} = \tau_{z\theta} = \eta\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right), && \tau_{zr} = \tau_{rz} = \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right).
\end{aligned} \tag{5.18}$$

Also recall that in cylindrical coordinates,

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}. \tag{5.19}$$

Now assume that the flow is only in the z–direction, that velocities are small, ignore gravity, pressure gradients, and velocity components normal to the boundaries of the cylinder ($\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0, \quad v_r = v_\theta = 0$). Also assume that the fluid is incompressible.

Then, the only nonzero component of the stress tensor τ is $\tau_{rz} = \eta\frac{\partial v_z}{\partial r}$ (force/unit area in the z–direction on surfaces $r = \text{const}$) and (5.17) reduces to the single equation

$$\begin{aligned}
\rho_f \dot{v}_z &= -\frac{\partial p_f}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = -\frac{\partial p_f}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta\frac{\partial v_z}{\partial r}\right)\\
&= -\frac{\partial p_f}{\partial z} + \eta\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v_z.
\end{aligned} \tag{5.20}$$

Since $\dot{\tilde{u}}_z^f = v_z =$ velocity of the fluid in the z–direction, we get the following equation for $\tilde{u}^f$:

$$\rho_f \ddot{\tilde{u}}_z^f = -\frac{\partial p_f}{\partial z} + \eta\Delta\dot{\tilde{u}}_z^f, \tag{5.21}$$

where

$$\Delta\dot{\tilde{u}}_z^f = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\dot{\tilde{u}}_z^f.$$

Set

$$v = \dot{\tilde{u}}_z^f - \dot{u}_z^s = \text{relative velocity of the fluid with respect to the well (in the } z\text{–direction).} \tag{5.22}$$

Using (5.22) in (5.21), we obtain

$$\rho_f(\dot{v} + \ddot{u}_z^s) = -\frac{\partial p_f}{\partial z} + \eta\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(v + \dot{u}_z^s) = -\frac{\partial p_f}{\partial z} + \eta\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v,$$

since u_z^s is independent of r. Therefore,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{\rho_f}{\eta}\dot{v} = \frac{1}{\eta}\left(\frac{\partial p_f}{\partial z} + \rho_f \ddot{u}_z^s\right) = -\frac{\rho_f}{\eta}\left(-\frac{1}{\rho_f}\frac{\partial p_f}{\partial z} - \ddot{u}_z^s\right).$$

Setting

$$\rho_f X \equiv -\frac{\partial p}{\partial z} - \rho_f \ddot{u}_z^s \quad \text{(equivalent external volume force independent of } r\text{)},$$

we obtain

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{\nu}\dot{v} = -\frac{X}{\mu}. \tag{5.23}$$

Now consider that v and X in (5.23) are oscillatory functions of the form

$$v = v(r)e^{i\omega t}, \quad X = X(z)e^{i\omega t}. \tag{5.24}$$

Then, from (5.23) we have

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{i\omega}{\nu}v = -\frac{X}{\nu}. \tag{5.25}$$

To solve (5.25), we define

$$h(r) = v(r) - \frac{X(z)}{i\omega}. \tag{5.26}$$

Then, h satisfies the homogeneous equation

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{i\omega}{\nu}h = 0$$

or

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \left(i\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right)^2 h = 0, \tag{5.27}$$

which is a Bessel's differential equation for h. The general solution of (5.27) is a linear combination of $J_0(i(\frac{i\omega}{\nu})^{\frac{1}{2}}r)$ and $Y_0(i(\frac{i\omega}{\nu})^{\frac{1}{2}}r)$; i.e., Bessel functions of zero order of the first and second kind. We choose the one that is finite at $r = 0$; i.e., J_0. Thus, using (5.26), the solution of (5.25) is given by

$$v(r, \omega) = C\, J_0\left(i\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} r\right) + \frac{X}{i\omega}. \tag{5.28}$$

The constant C can be determined by asking the continuity of the velocity at the duct well; i.e.,

$$v(r = a) = 0, \tag{5.29}$$

so that

$$C = -\frac{X}{i\omega}\frac{1}{J_0(i(\frac{i\omega}{\nu})^{\frac{1}{2}}a)}. \tag{5.30}$$

Using (5.30) in (5.28), we obtain

$$v(r,\omega) = \frac{X}{i\omega}\left[1 - \frac{J_0(i(\frac{i\omega}{\nu})^{\frac{1}{2}}r)}{J_0(i(\frac{i\omega}{\nu})^{\frac{1}{2}}a)}\right]. \tag{5.31}$$

For the general theory, we need to compute the average flow over the cross–section. Thus, we compute

$$V_{av}(\omega) = \frac{1}{\pi a^2}\int_0^a v(r,\omega)2\pi r\,dr. \tag{5.32}$$

Recall that

$$J_0(ii^{\frac{1}{2}}z) = \operatorname{ber} z + i\operatorname{bei} z \tag{5.33}$$

where $\operatorname{ber} z$ $\operatorname{bei} z$ are the Kelvin functions of the first kind and zero order. Next, use that [12],

$$\int_0^z z\operatorname{ber} z\,dz = z\operatorname{bei}' z, \qquad \int_0^z z\operatorname{bei} z\,dz = -z\operatorname{ber}' z. \tag{5.34}$$

Set

$$k_1 = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}, \quad k = ak_1.$$

Recall that $[\nu] = [\frac{\text{cm}^2}{\text{sec}}]$ so that $[a(\frac{\omega}{\nu})^{\frac{1}{2}}] = [\text{cm}(\frac{1}{\text{sec}}\frac{\text{sec}}{\text{cm}^2})^{\frac{1}{2}}]$; i.e., k is dimensionless. Then,

$$\begin{aligned}
V_{av}(\omega) &= \frac{X}{i\omega}\frac{1}{\pi a^2}\int_0^a\left[1 - \frac{J_0(ii^{\frac{1}{2}}k_1 r)}{J_0(ii^{\frac{1}{2}}ak_1)}\right]2\pi r\,dr\\
&= \frac{X}{i\omega}\frac{1}{\pi a^2}\left[\pi a^2 - \frac{2\pi}{J_0(ii^{\frac{1}{2}}ak_1)}\int_0^a[\operatorname{ber}(k_1r) + i\operatorname{bei}(k_1r)]r\,dr\right]\\
&= \frac{X}{i\omega}\left[1 - \frac{2}{a^2J_0(ii^{\frac{1}{2}}ak_1)}\int_0^a[\operatorname{ber}(k_1r) + i\operatorname{bei}(k_1r)]k_1r\,d(k_1r)\frac{1}{k_1^2}\right]\\
&= \frac{X}{i\omega}\left[1 - \frac{2}{a^2J_0(ii^{\frac{1}{2}}ak_1)k_1^2}(k_1a\operatorname{bei}'(k_1a) - i\widehat{k}_1a\operatorname{ber}'(k_1a))\right]\\
&= \frac{X}{i\omega}\left[1 - \frac{2}{aJ_0(ii^{\frac{1}{2}}ak_1)k_1}(-i)(\operatorname{ber}' k + i\operatorname{bei}' k)\right]\\
&= \frac{X}{i\omega}\left[1 - \frac{2}{ik}\frac{(\operatorname{ber}' k + i\operatorname{bei}' k)}{(\operatorname{ber} k + i\operatorname{bei} k)}\right].
\end{aligned}$$

Set

$$T(k) = \frac{\text{ber}' k + i\,\text{bei}' k}{\text{ber}\, k + i\,\text{bei}\, k}. \tag{5.35}$$

Then,

$$V_{av}(\omega) = \frac{1}{i\omega}\left(1 - \frac{2}{ik}T(\kappa)\right)X, \quad k = a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}. \tag{5.36}$$

Next, since the unit outward normal at the surface $r = a$ is pointing in the negative r–direction, by our sign convection the stress τ_{rz} at the wall is given by

$$\begin{aligned} \tau_{rz}|_{r=a} &= -\eta\frac{\partial v_z}{\partial r}|_{r=a} = -\eta\frac{\partial \dot{\tilde{u}}^f}{\partial r}|_{r=a} \\ \text{(since } u^s \text{ is independent of } r) &= -\eta\frac{\partial}{\partial r}(\dot{\tilde{u}}^f - u^s)|_{r=a} = -\eta\frac{\partial v}{\partial r}|_{r=a}. \end{aligned}$$

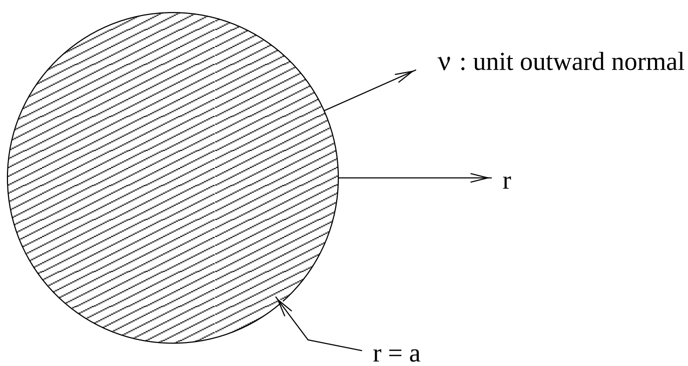


FIGURE 16

Then, using (5.31),

$$\tau = \tau_{rz}|_{z=a} = -\eta\frac{d}{dr}\left[\frac{X}{i\omega}\left(1 - \frac{J_0(i^{\frac{3}{2}}(\frac{\omega}{\nu})^{\frac{1}{2}}r)}{J_0(i^{\frac{3}{2}}(\frac{\omega}{\nu})^{\frac{1}{2}}a)}\right]\right|_{r=a} = -\eta\frac{X}{i\omega}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\frac{[i^{\frac{3}{2}}J_1(i^{\frac{3}{2}}k)]}{J_0(i^{\frac{3}{2}}k)}, \tag{5.37}$$

where we have used that $J_0'(z) = -J_1(z)$. Then,

$$\frac{d}{dr}J_0\left(i^{\frac{3}{2}}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}r\right) = -i^{\frac{3}{2}}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}J_1\left(i^{\frac{3}{2}}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}r\right)\Big).$$

But,

$$i^{\frac{3}{2}}J_1(i^{\frac{3}{2}}\kappa) = -i^{\frac{3}{2}}J_0'(i^{\frac{3}{2}}k) = -i^{\frac{3}{2}}i^{-\frac{3}{2}}(ber'k + i\,\text{bei}'\,k).$$

Then,

$$\tau = \eta\frac{X}{i\omega}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}T(k). \tag{5.38}$$

Next, let us consider a unit length cylinder as in Figure 17.

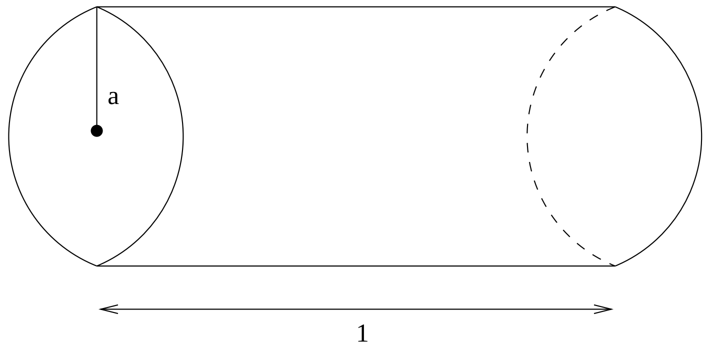


FIGURE 17

The wetted surface area is equal to $2\pi a.1$ and, consequently,

$$2\pi a \tau = \text{ total friction force at the wall per unit length.} \tag{5.39}$$

Thus,

$$\frac{2\pi a \tau}{\pi a^2} = \tau \frac{2}{a} = \text{ total friction force at the wall per unit volume.} \tag{5.40}$$

Now from (5.36) and (5.37)

$$\frac{2\pi a \tau}{V_{av}} = \frac{\eta 2\pi a (\frac{\omega}{\nu})^{\frac{1}{2}} T(k)}{1 - \frac{2}{ik} T(k)} = 2\pi\eta \frac{kT(k)}{1 - \frac{2}{ik} T(k)} = 8\pi\eta F(k), \tag{5.41}$$

where

$$F(k) = \frac{1}{4} \frac{kT(k)}{1 - \frac{2}{ik} T(k)}. \tag{5.42}$$

Now, from (5.40) and (5.41) we see that

$$\begin{aligned} \tau \frac{2}{a} &= \frac{2\pi a \tau}{\pi a^2} \frac{V_{av}}{V_{av}} = \frac{8\eta}{a^2} F(k) V_{av} \\ &= \text{ total friction force at the wall per unit volume.} \end{aligned} \tag{5.43}$$

Next, consider a unit cube of bulk material of porosity ϕ consisting of parallel cylinders as in Figure 18.

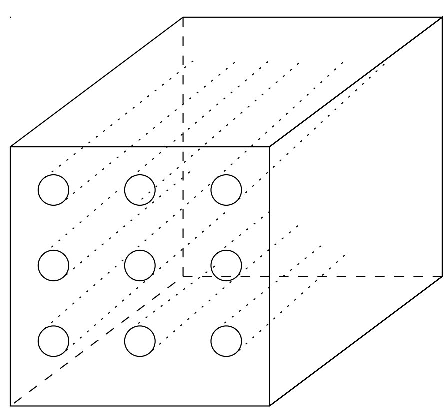


FIGURE 18

In this case, $\tau\frac{2}{a}\phi$ represents the total friction force per unit volume of bulk material and from (5.43) we have

$$\tau\frac{2}{a}\phi = \frac{8\eta}{a^2}\phi F(k)V_{av} = \left(\frac{8\eta}{a^2\phi}\right)\phi^2 F(k)V_{av}. \tag{5.44}$$

The coefficient $\frac{8\eta}{a^2\phi}$ can be associated with the Poiseuille flow in the unit cube above with the following argument. Consider steady state flow first; i.e., constant velocity, in a cylinder as in Figure 19.

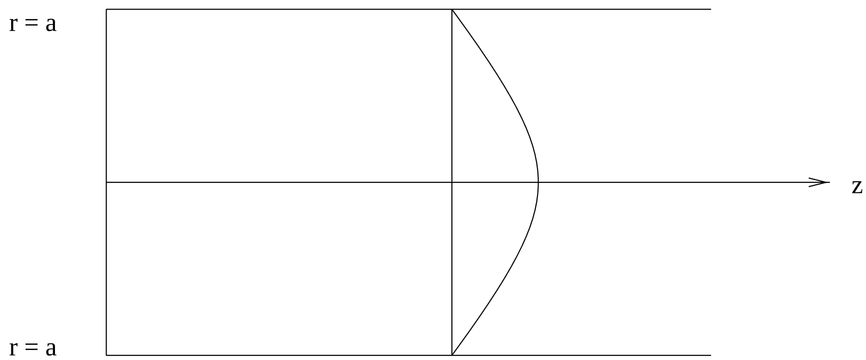


FIGURE 19

According to (5.20), since $\dot{v}_z = 0$, we need to solve

$$\eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p_f}{\partial z} \tag{5.45}$$

or

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{1}{\eta}\frac{\partial p_f}{\partial z}r. \tag{5.46}$$

Integration in r yields

$$r\frac{\partial v_z}{\partial r} = -\frac{1}{\eta}\frac{\partial p_f}{\partial z}\frac{r^2}{2} + C_1.$$

Thus,

$$\frac{\partial v_z}{\partial r} = -\frac{1}{\eta}\frac{\partial p_f}{\partial z}\frac{r}{2} + \frac{C_1}{r}.$$

Integration in r again give us

$$v_z(r) = -\frac{1}{\eta}\frac{\partial p_f}{\partial z}\frac{r^2}{4} + C_1\ln r + C_2. \tag{5.47}$$

Since at $r = 0$ v_z must be finite, we conclude that $C_1 = 0$.

Next, imposing the nonslipping boundary condition; i.e., the cylinder wall is not moving,

$$v_z(r = a) = 0, \tag{5.48}$$

we see that

$$0 = -\frac{1}{\eta}\frac{\partial p_f}{\partial z}\frac{a^2}{4} + C_2.$$

Thus,

$$v_z(r) = \frac{a^2}{4\eta}\frac{\partial p_f}{\partial z}\left(1 - \frac{r^2}{a^2}\right) \tag{5.49}$$

indicating that the velocity profile is parabolic.

A useful quantity is the volumetric flow rate given by

$$\begin{aligned} Q = \overline{V}_z &= \int_0^{2\pi}\int_0^a v_z(r) r \, dr \, d\theta = \frac{a^2}{4\eta}\frac{\partial p_f 2\pi}{\partial z}\int_0^a \left(r - \frac{r^3}{a^2}\right) dr \\ &= \frac{\pi a^2}{2\eta}\frac{\partial p_f}{\partial z}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a = -\frac{\pi a^4}{8\eta}\frac{\partial p_f}{\partial z}. \end{aligned}$$

The relation

$$Q = -\frac{\pi a^4}{8\eta}\frac{\partial p_f}{\partial z} \tag{5.50}$$

is known as the Hagen–Poiseuille law, experimentally shown in the early 19th century.

Now assume that in the unit cube of bulk material we have n parallel cylinders of radius a per unit cross section. Then,

$$q = -\frac{n\pi a^4}{8\eta}\frac{\partial p_f}{\partial z}. \tag{5.51}$$

Let us compute the porosity ϕ in the unit cube. Each cylinder has pore volume

$$PV_c = \pi a^2 \cdot \underbrace{1}_{\substack{\text{length of} \\ \text{the cylinder}}} .$$

Thus, the total pore volume for n–cylinders PV is given by

$$PV = n\pi a^2.$$

Next, by definition,

$$\phi = \frac{PV}{V_b} = \frac{n\pi a^2}{1} = n\pi a^2. \tag{5.52}$$

Thus,

$$n = \frac{\phi}{\pi a^2}. \tag{5.53}$$

Using (5.53) in (5.51) we see that

$$Q = -\frac{\phi}{\pi a^2}\frac{\pi a^4}{8\eta}\frac{\partial p_f}{\partial z} = -\frac{a^2\phi}{8\eta}\frac{\partial p_f}{\partial z}. \tag{5.54}$$

This shows that the coefficient $\left(\frac{8\eta}{a^2\phi}\right)$ in (5.44) is associated with the Hagen–Poiseuille flow. Defining

$$\mathbb{K} = \frac{a^2\phi}{8}, \tag{5.55}$$

we still can write

$$Q = -\frac{\mathbb{K}}{\eta}\frac{\partial p_f}{\partial z} \tag{5.56}$$

which is the form of Darcy's law for this type of flow.

Finally, using (5.55) we can write (5.44) in the form

$$\tau\frac{2}{a}\phi = \frac{\eta}{\mathbb{K}}\phi^2 F(\kappa)V_{av} = \text{total friction force at the wall per unit volume of bulk material.} \tag{5.57}$$

Next, we observe that identifying V_{av} with $\dot{\tilde{u}}_1^f - \dot{u}_1^s$, and $G = b(\dot{\tilde{u}}_1^f - \dot{u}_1^s)$ with $\tau\frac{2}{a}\phi$, we see that

$$b = \frac{\eta}{\mathbb{K}}\phi^2 F(k); \tag{5.58}$$

i.e., the coefficient b representing the ratio of the total friction force at the wall per unit volume to the average relative flow has a frequency correction factor $F(k)$ multiplying the value $\frac{\eta}{\mathbb{K}}\phi^2$ associated with the low–frequency equations for laminar flow.

Next we will consider the motion of a fluid in a two–dimensional duct; i.e., the space limited by two parallel boundaries when these boundaries are subject to an oscillatory motion and when an oscillatory pressure gradient acts at the same time on the fluid (see Figure 20). Again we start with the Navier–Stokes equations

$$\rho_f\frac{Dv}{Dt} = -\nabla p_f + \rho\mathbf{g} + \nabla\cdot\tau, \tag{5.59}$$

but now in cartesian coordinates. According to [18, p. 145], in cartesian coordinates the strain tensor τ in the fluid is given by

$$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \left[\left(\kappa - \frac{2}{3}\eta\right)\nabla\cdot v\right]\delta_{ij}. \tag{5.60}$$

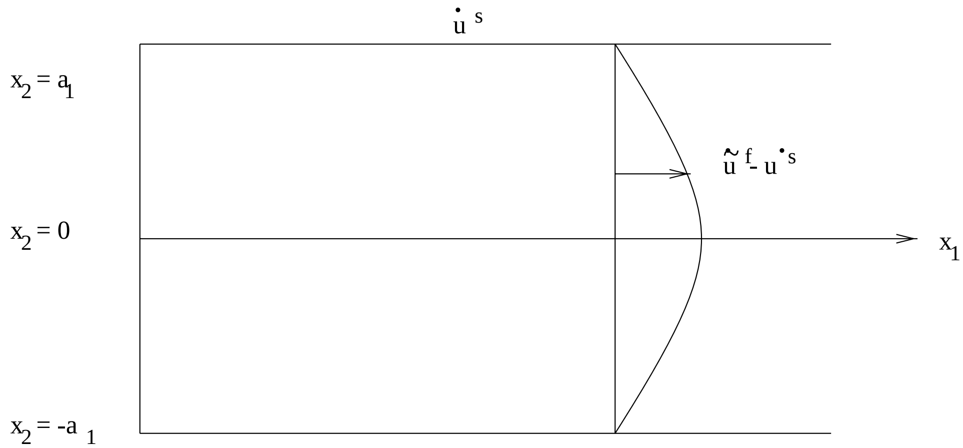


FIGURE 20

Now assume that the flow is only in the x_1–direction, that velocities are small, ignore gravity and pressure gradients and velocity components normal to the boundaries, so that

$$\frac{\partial p_f}{\partial x_2} = \frac{\partial p_f}{\partial x_3} = 0, \quad v_2 = v_3 = 0.$$

Also, assume that the fluid is incompressible. In this case, the only nonzero stress component is

(5.61) $$\tau_{12} = \eta \frac{\partial v_1}{\partial x_2} (\text{force per unit area in the } x_1\text{–direction on planes } x_2 = \text{const}),$$

and (5.59) reduces to

(5.62) $$\rho_f \dot{v}_1 = -\frac{\partial p_f}{\partial x_1} + \eta \frac{\partial^2 V_1}{\partial x_2^2}.$$

Since $\dot{\tilde{u}}_1^f = v_1$, we see that the equation of motion for the fluid is

$$\rho_f \ddot{\tilde{u}}_1^f = -\frac{\partial p_f}{\partial x_1} + \eta \frac{\partial^2 \dot{\tilde{u}}_1^f}{\partial x_2^2}.$$

Set

(5.63) $$U_1 = \dot{\tilde{u}}_1^f - \dot{u}_1^s = \text{ relative fluid velocity with respect to the wall.}$$

Then,

(5.64) $$\rho_f \dot{U}_1 = -\frac{\partial p_f}{\partial x_1} + \eta \frac{\partial^2}{\partial x_2^2}(u_1 + \dot{u}_1^s) - \rho_f \ddot{u}_1^s = -\frac{\partial p_f}{\partial x_1} + \eta \frac{\partial^2 U_1}{\partial x_2^2} - \rho \ddot{u}_1^s,$$

since u_1^s is independent of x_2. Set

(5.65) $$\rho_f X = -\frac{\partial p_f}{\partial x_1} - \rho_f \ddot{u}_1^s.$$

Then (5.64) reduces to

$$\dot{U}_1 = X + \nu \frac{\partial^2 U_1}{\partial x_2^2}. \tag{5.66}$$

Now assume that

$$U_1 = U_1(x_2)e^{i\omega t}, \quad X = X(x_1)e^{i\omega t}.$$

Then (5.66) yields

$$\nu \frac{\partial^2 U_1}{\partial X_2^2} - i\omega U_1 = -X. \tag{5.67}$$

Set

$$h = U_1 - \frac{X}{i\omega}. \tag{5.68}$$

Then h satisfies the homogeneous equation

$$\frac{\partial^2 h}{\partial x_2^2} - \frac{i\omega}{\nu} h = 0. \tag{5.69}$$

The general solution of (5.69) is

$$h(x_2) = C_1 e^{(\frac{i\omega}{\nu})^{\frac{1}{2}} x_2} + C_2 e^{-(\frac{i\omega}{\nu})^{\frac{1}{2}} x_2}.$$

Using that h must be symmetric in x_2; i.e., $h(x_2) = h(-x_2)$, we see that

$$C_1 = C_2 = C$$

and, consequently,

$$h(x_2) = C \cosh \left[\left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} x_2 \right]$$

so that from (5.68),

$$U_1(x_2) = \frac{X}{i\omega} + C \cosh \left[\left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} x_2 \right]. \tag{5.70}$$

Using the nonslip boundary condition

$$U_1(x_2 = \pm a_1) = 0$$

we can determine the constant C in (5.70):

$$C = -\frac{X}{i\omega} \frac{1}{\cos[(\frac{i\omega}{\nu})^{\frac{1}{2}} a_1]}. \tag{5.71}$$

Using (5.71) in (5.70) we see that

$$U_1(x_2) = \frac{X}{i\omega}\left[1 - \frac{\cosh((\frac{i\omega}{\nu})^{\frac{1}{2}} x_2)}{\cosh((\frac{i\omega}{\nu})^{\frac{1}{2}} a_1)}\right]. \tag{5.72}$$

Next we compute the average velocity of the fluid through the cross–section and the friction force at the well. First,

$$\begin{aligned} U_{1_{av}} &= \frac{1}{2a_1}\int_{-a_1}^{a_1} U_1(x_2)dx_2 = \frac{1}{2a_1}\left[\int_{-a_1}^{a_1}\left(1 - \frac{\cosh((\frac{i\omega}{\nu})^{\frac{1}{2}} x_2)}{\cosh((\frac{i\omega}{\nu})^{\frac{1}{2}} a_1)}\right)dx_2\right]\frac{X}{i\omega} \\ &= \frac{1}{2a_1}\frac{X}{i\omega}\left[2a_1 - \frac{1}{\cosh((\frac{i\omega}{\nu})^{\frac{1}{2}} a_1)}\sinh\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} x_2\Big|_{-a_1}^{\sigma_1} \cdot \left(\frac{\nu}{i\omega}\right)^{\frac{1}{2}}\right] \\ &= \frac{X}{i\omega}\left[1 - \frac{1}{a_1}\left(\frac{\nu}{i\omega}\right)^{\frac{1}{2}}\tanh\left(\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} a_1\right)\right]. \end{aligned}$$

Thus,

$$U_{1_{av}} = \frac{X}{i\omega}\left[1 - \frac{1}{a_1}\left(\frac{\nu}{i\omega}\right)^{\frac{1}{2}}\tanh\left(\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} a_1\right)\right]. \tag{5.73}$$

Next, the friction stress τ_{12} at the wall $x_2 = -a_1$ is

$$\tau = \tau_{12}\Big|_{x_2=-a_1} = \eta\frac{\partial v_1}{\partial x_2}\Big|_{x_2=-a_1}.$$

Note that the unit outward normal to $x_2 = -a_1$ is on the positive x_2–direction inside the conduct (see Figure 21).

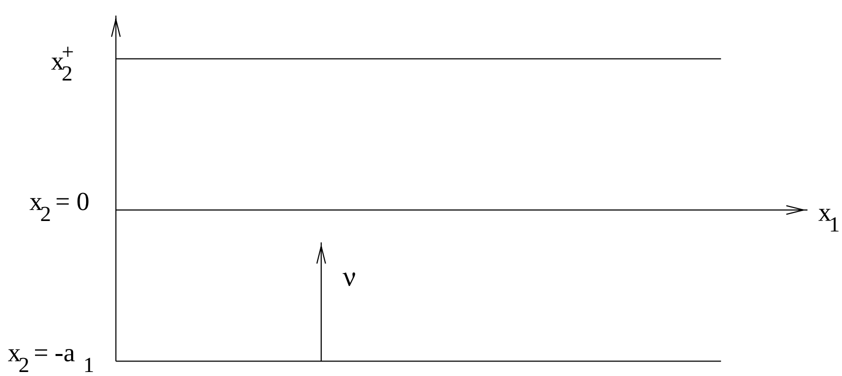


FIGURE 21

Since $v_1 = \widetilde{\dot{u}}^f$ and $\dot{u}_1^s$ is independent of x_2, and consequently, $\frac{\partial v_1}{\partial x_2} = \frac{\partial U_1}{\partial x_2}$, we have that

$$\tau = \eta\frac{\partial U_1}{\partial x_2}\Big|_{x_2=-a_1} = \eta\frac{X}{i\omega}\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\tanh\left[\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} a_1\right]. \tag{5.74}$$

Set

$$k_1 = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} a_1 \qquad \text{(dimensionless variable).} \tag{5.75}$$

Next, as in (5.41), we compute the ratio of the total friction force 2τ exerted by the fluid on a unit length well and the averaged velocity of the fluid relative to the wall. Since

$$2: \text{ total wetted surface,}$$

then,

$$2\tau: \text{ total friction force/unit length,}$$

so that

$$\frac{2\tau}{U_{1_{av}}} = \frac{2\eta i^{\frac{1}{2}}(\frac{\omega}{\nu})^{\frac{1}{2}} \tanh[i^{\frac{1}{2}}k_1]\frac{a_1}{a_1}}{1 - \frac{1}{a_1}(\frac{\nu}{i\omega})^{\frac{1}{2}} \tanh[i^{\frac{1}{2}}k_1]} = \frac{2\eta}{a_1} \cdot \frac{i^{\frac{1}{2}}k_1 \tanh(i^{\frac{1}{2}}k_1)}{1 - \frac{1}{i^{\frac{1}{2}}k_1}\tanh(i^{\frac{1}{2}}k_1)}.$$

Set

$$F_1(k_1) = \frac{1}{3}\frac{i^{\frac{1}{2}}k_1 \tanh(i^{\frac{1}{2}}k_1)}{1 - \frac{1}{i^{\frac{1}{2}}k_1}\tanh(i^{\frac{1}{2}}k_1)}. \tag{5.76}$$

Then,

$$\frac{2\tau}{U_{1_{av}}} = \frac{6\eta}{a_1}F_1(k_1). \tag{5.77}$$

With the same argument then in (5.58) we see that in this case

$$b = \frac{6\eta}{a_1}F_1(k_1) = \frac{\eta}{\frac{a_1}{6}\phi^2}\phi^2 F_1(k_1) = \frac{\eta}{\mathbb{K}}\phi^2 F_1(k_1),$$

where

$$\mathbb{K} = \frac{a_1\phi^2}{6}.$$

Next, a plot of the real and imaginary parts of $F(k)$ and $F_1(k_1)$ show that [4]

$$\operatorname{Re}(F_1(k_1)) \simeq \operatorname{Re}\left(F\left(\frac{4}{3}k\right)\right), \qquad \operatorname{Im}(F_1(k_1)) \simeq \operatorname{Im}\left(F\left(\frac{4}{3}k\right)\right). \tag{5.78}$$

Thus, when the pores have the shape of narrow slits, the associated frequency dependent function may be taken the same as for circular pores with a radius $a = \frac{4}{3}a_1$.

These cases correspond to extreme shapes in the cross–section of the pores; i.e., when they are close to circles (cylindrical ducts) or very flat ellipses (plane slits). Thus, in the two extreme cases, the effect of frequency on the correcting factor $F(k)$ is the same except for a change in scale of the dimensionless variable k. It is natural to assume that if it is true for the extreme cases, it will also be true for intermediate cases.

Biot claims [4] that there exist a universal function $F(k)$ that can be adopted to represent the frequency effect with a nondimensional parameter

$$k = a_p \left(\frac{\omega}{\nu} \right)^{\frac{1}{2}} \tag{5.79}$$

where a_p is the pore–size parameter depending on size and pore geometry.

To estimate a_p we may proceed as follows ([10], [9]).

Let us define the hydraulic radius m by

$$m = \frac{\text{volume filled with fluid}}{\text{wetted surface}}. \tag{5.80}$$

For a unit cylinder as in Figure 17, the wetted surface is equal to $2\pi a$ and the volume filled with fluid is equal to πa^2, so that we have the relation

$$m = \frac{\pi a^2}{2\pi a} = \frac{a}{2}; \tag{5.81}$$

i.e.,

$$a = 2m. \tag{5.82}$$

Since the frequency correction factor function $F(k)$ is not heavily dependent of the shape of the pores, Hovem *et al,* [10] suggests to replace a in the function $F(k)$ by the pore–size parameter

$$a_p = 2m. \tag{5.83}$$

Now for media composed of regular grains, the hydraulic radius m can be related to the rock permeability by the Kozeny–Carman equation (see [17], [1])

$$\mathbb{K} = \phi m^2 / A_0. \tag{5.84}$$

The coefficient A_0 is known as the Kozeny–Carman constant, which for glass beads, Ottawa sand, Panama City sand, and other sandstones has been found to be approximately equal to 5 [10].

Thus, using (5.84) in (5.83) we have

$$a_p = 2(\mathbb{K} A_0 / \phi)^{\frac{1}{2}}, \tag{5.85}$$

which is the desired form of the pore–size parameter a_p.

Asymptotic properties of $F(k)$:

$$\begin{aligned} &F(k) = \frac{1}{4} \frac{kT(k)}{1 - \frac{2}{ik} T(k)} = F_r(k) + iF_i(k), \qquad T(k) = \frac{\operatorname{ber} k + i \operatorname{bei} k}{\operatorname{ber}' k + i \operatorname{bei}' k}, \\ &F(k) \xrightarrow[k \to \infty]{} \frac{1}{4} k \frac{1}{\sqrt{2}} (1 + i), \end{aligned} \tag{5.86}$$

so that it behaves like $\omega^{\frac{1}{2}}$ for large values of ω. Also,

$$F(k) \xrightarrow[k\to 0]{} 1 + i\frac{k^2}{24} \tag{5.87}$$

so that at low frequencies the low–frequency coefficient is recovered.

Now we write the high–frequency form of Biot's equation (5.14) in the space–frequency domain:

$$\begin{aligned} &\text{i)} \quad -\omega^2\rho_{11}\widehat{u}_i^s - \rho_{12}\omega^2\widehat{\widetilde{u}}_i^f - i\omega bF(k)(\widehat{\widetilde{u}}_i^f - \widehat{u}_i^s) = \frac{\partial\widehat{\sigma}_{ij}}{\partial x_j}, \\ &\text{ii)} \quad -\omega^2\rho_{12}\widehat{u}_i^s - \omega^2\rho_{22}\widehat{\widetilde{u}}_i^f + i\omega bF(k)(\widehat{\widetilde{u}}_i^f - \widehat{u}_i^s) = -\phi\frac{\partial p_f}{\partial x_i}, \end{aligned} \tag{5.88}$$

$$\widehat{\sigma}_{ij} = [A\hat{e} + Q\widehat{\theta}]\delta_{ij} + 2\mu\varepsilon_{ij}(\widehat{u}^s), \qquad \widehat{\sigma} = -\phi\widehat{p}_f = Q\hat{e} + R\widehat{\theta}. \tag{5.89}$$

Let us now write (5.88) using the variables $\widehat{u}^s$ and $\widehat{u}^f = \phi(\widehat{\widetilde{u}}^f - \widehat{u}^s)$. Recall that

$$\rho_{11} + \rho_{12} = (1-\phi)\rho_s, \qquad \rho_{12} + \rho_{22} = \phi\rho_f,$$

and that

$$\widehat{\widetilde{u}}^f = \frac{1}{\phi}\widehat{u}^f + \widehat{u}^s.$$

Thus, adding (5.88.i) and (5.88.ii) we obtain

$$\begin{aligned} \frac{\partial\widehat{\sigma}_{ij}}{\partial x_j} - \phi\frac{\partial\widehat{p}_f}{\partial x_i} = \frac{\partial\widehat{\tau}_{ij}}{\partial x_j} &= -\omega^2(1-\phi)\rho_s\widehat{u}_i^s - \omega^2\phi\rho_f\widehat{\widetilde{u}}_i^f \\ &= -\omega^2(1-\phi)\rho_s\widehat{u}_i^s - \omega^2\phi\rho_f\left(\frac{1}{\phi}\widehat{u}_i^f + \widehat{u}_i^s\right) \\ &= -\omega^2([1-\phi]\rho_s + \phi\rho_f)\widehat{u}^s - \omega^2\rho_f\widehat{u}^f \\ &= -\omega^2\rho\widehat{u}^s - \omega^2\rho_f\widehat{u}^f. \end{aligned}$$

Hence,

$$-\omega^2\rho\widehat{u}^s - \omega^2\rho_f\widehat{u}^f = \frac{\partial\widehat{\tau}_{ij}}{\partial x_j}. \tag{5.90}$$

Next, from (5.88.ii) we see that

$$-\omega^2\frac{\rho_{12}}{\phi}\widehat{u}_i^s - \omega^2\frac{\rho_{22}}{\phi}\left(\frac{1}{\phi}\widehat{u}_i^f + \widehat{u}_i^s\right) + i\omega b\frac{1}{\phi}[F_r(k) + iF_i(k)]\frac{1}{\phi}\widehat{u}_i^f = \frac{\partial\widehat{p}_f}{\partial x_i}.$$

Then, since $\frac{b}{\phi^2} = \frac{\eta}{\mathbb{K}}$,

$$-\omega^2\left(\frac{\rho_{12}+\rho_{22}}{\phi}\right)\widehat{u}_i^s - \omega^2\left(\frac{\rho_{22}}{\phi^2} + \frac{F_i(k)}{\omega}\frac{\eta}{\mathbb{K}}\right)\widehat{u}_i^f + i\omega\frac{\eta}{\mathbb{K}}F_r(k)\widehat{u}_i^f = \frac{\partial\widehat{p}_f}{\partial x_i}. \tag{5.91}$$

According to (4.103),

$$\rho_{22} = \phi^2 g.$$

Next, following [6], [2], we write

$$g = S\rho_f/\phi, \tag{5.92}$$

where S is called the structure factor. According to [2],

$$S = \frac{1}{2}\left(1 + \frac{1}{\phi}\right). \tag{5.93}$$

Thus, we finally write the equation in the form

$$\begin{aligned} &\text{i)} \quad -\omega^2\rho\widehat{u}_i^s - \omega^2\rho_f\widehat{u}_i^f = \frac{\partial\widehat{\tau}_{ij}}{\partial x_j}, \\ &\text{ii)} \quad -\omega^2\rho_f\widehat{u}_i^s - \omega^2 g(\omega)\widehat{u}_i^f + i\omega b(\omega)\widehat{u}_i^f = -\frac{\partial p_f}{\partial x_i}, \quad 1 \le i \le 3, \end{aligned} \tag{5.94}$$

where

$$\begin{aligned} &\text{i)} \quad g(\omega) = \frac{S\rho_f}{\phi} + \frac{F_i(k)}{\omega}\frac{\eta}{\mathbb{K}}, \\ &\text{ii)} \quad b(\omega) = \frac{\eta}{k}F_r(k). \end{aligned} \tag{5.95}$$

CHAPTER 6
DERIVATION OF ABSORBING BOUNDARY CONDITIONS FOR ELASTIC SOLIDS AND FLUID-SATURATED POROUS SOLIDS

First we consider an elastic body Ω and follow the ideas in [11] and [16].

Consider a small disturbance originated in a restricted portion of an elastic solid medium Ω. We may assume that the disturbed portion is bounded at any instant by a surface S. If the medium is isotropic and if the disturbance involves dilatation, we may expect that the surface S moves normally to itself with velocity $(\frac{\lambda+2\mu}{\rho})^{1/2}$. If the disturbance involves rotation without dilation, we may expect the velocity of the surface to be $(\frac{\mu}{\rho})^{1/2}$.

Then, let us assume that the surface moves normally to itself with velocity $\underline{\underline{c}}$ and let us seek the conditions that must be satisfied at the moving surface.

Let $u^c = (u_i^c)_{1\le i\le 3}$ be the displacement vector. On one side Ω_1 of the surface S at time t, the medium is disturbed and $u^c \neq 0$. On the other side, $(\Omega_2)u^c \equiv 0$. We take the velocity c to be directed from the first side Ω_1 into the second side Ω_2 so that the disturbance spreads into parts of the medium Ω_2 that were previously undisturbed.

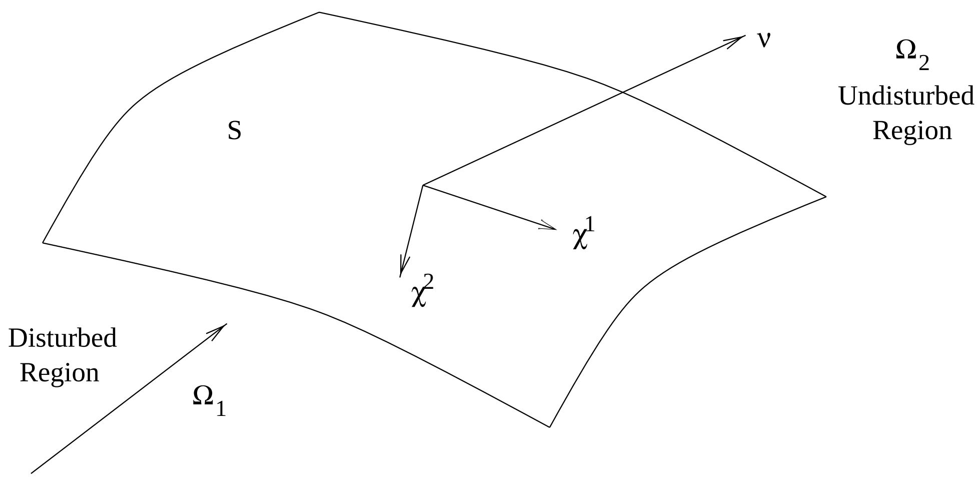


FIGURE 22

Since $\mathbf{u}^c$ is continuous across S, $\mathbf{u}^c$ must vanish on S; i.e.,

$$\mathbf{u}^c \equiv 0 \text{ on } S. \tag{6.1}$$

Consequently,

$$\frac{\partial u_i^c}{\partial x} = \nabla u_i^c \cdot \boldsymbol{\chi} = 0, \tag{6.2}$$

for any $\boldsymbol{\chi}$ in the plane defined by

$$\mathbf{z} \cdot \boldsymbol{\nu} = 0, \tag{6.3}$$

where $\boldsymbol{\nu}$ is the normal to S at the point $\mathbf{0}$.

Then it must necessarily be

$$\nabla u_i^c = \gamma \boldsymbol{\nu}. \tag{6.4}$$

Then,

$$\nabla u_i^c \cdot \boldsymbol{\nu} = \gamma \boldsymbol{\nu} \cdot \boldsymbol{\nu} = \gamma;$$

i.e.,

$$\gamma = \frac{\partial u_i^c}{\partial \boldsymbol{\nu}}. \tag{6.5}$$

Now from (6.4) and (6.5),

$$\left(\frac{\partial u_i^c}{\partial x_1}, \frac{\partial u_i^c}{\partial x_2}, \frac{\partial u_i^c}{\partial x_3} \right) = \frac{\partial u_i^c}{\partial \boldsymbol{\nu}} \cdot (\nu_1, \nu_2, \nu_3).$$

Thus,

$$\frac{\partial u_i^c}{\partial \boldsymbol{\nu}} = \frac{\frac{\partial u_i^c}{\partial x_1}}{\nu_1} = \frac{\frac{\partial u_i^c}{\partial x_2}}{\nu_2} = \frac{\frac{\partial u_i^c}{\partial x_3}}{\nu_3}. \tag{6.6}$$

Next, we observe that the equation

$$u_i^c(x_1, x_2, x_3, t) = 0 \quad \text{on } S \tag{6.7}$$

must be satisfied to the first order in δt when for (x_1, x_2, x_3, t) we substitute

$$(x_1 + c\delta t \nu_1, x_2 + c\delta t \nu_2, x_3 + c\delta t \nu_3, t + dt).$$

Thus,

$$\begin{aligned} &u_i^c(x_1 + c\delta t\nu_1, x_2 + c\delta t\nu_2, x_3 + c\delta t\nu_3, t + \delta t) \\ &\quad = u_i^c(x_1, x_2, x_3, t) + \frac{\partial u_i^c}{\partial x_j}\nu_j c\delta t + \frac{\partial u_i^c}{\partial t}\delta t = 0. \end{aligned}$$

Hence,

$$\frac{\partial u_i^c}{\partial t} + c\nabla u_i^c \cdot \boldsymbol{\nu} = \frac{\partial u_i^c}{\partial t} + c\frac{\partial u_i^c}{\partial \nu} = 0 \quad \text{on } S. \tag{6.8}$$

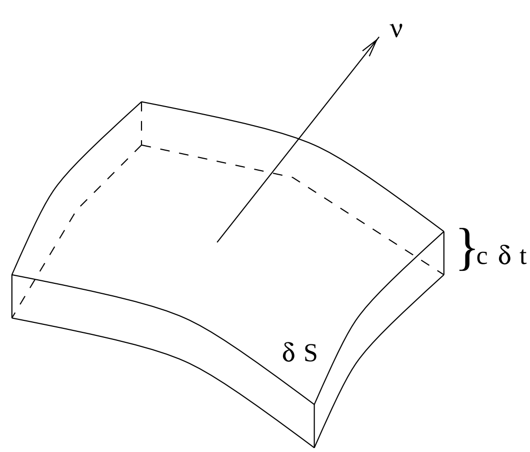


FIGURE 23

Now from (6.6) and (6.8) we obtain the relations

$$\frac{\frac{\partial u_i^c}{\partial x_j}}{\nu_j} = -\frac{1}{c}\frac{\partial u_i^c}{\partial t}, \text{ on } S, \quad 1 \le i \le 3, \quad (j \text{ not summed}). \tag{6.9}$$

In (6.9), the derivatives need to be computed from the side Ω_1 where there is a disturbance at time t. The dynamical conditions which hold at the surface S are found by considering the change in momentum of a thin slice of the medium in a neighborhood of S. We mark out a small area δS of S and consider the prismatic element bounded by S, by the normals to S at the edge of δS, and by a surface parallel to S at a distance $c\delta t$ from it (see Figure 23). The volume V_s of the prismatic element is $V_s = c\delta t\,\delta S$. According to the conservation of linear momentum, using (1.32) for $A = \rho$,

$$\frac{d}{dt}\int_{V_s} \rho \dot{u}_i^c dv_s = \int_{\delta S} F_{s,i} ds.$$

Thus, using (6.7),

$$\begin{aligned} &\int_t^{t+\delta t}\left(\frac{d}{dt}\int_{V_s} \rho \dot{u}_i^c dv_s\right) dt \\ &\simeq \rho\Big[\dot{u}_i^c(x_1,x_2,x_3,t+\delta t) - \dot{u}_i^c(x_1,x_2,x_3,t)\Big] V_s \\ &= \rho \dot{u}_i^c(x_1,x_2,x_3,t+\delta t) c\delta t \delta S = \int_t^{t+\delta t}\int_{\delta S} F_{s,i} ds. \end{aligned} \tag{6.10}$$

This shows that the change of momentum is equal to the time integral of the tractions across δS. The traction F_s acts across the surface S normal to ν upon the matter on that side of the surface towards which $\boldsymbol{\nu}$ is drawn.

Now recall the convention about notation: If δS is a small area of the plane normal to $\boldsymbol{\nu}$ at the point $\mathbf{0}$, that portion of the body which is on the side of the plane towards which $\boldsymbol{\nu}$ is drawn acts upon the portion on the other side with a force at the point $\mathbf{0}$ specificed by

$$\overset{\nu}{T}_i = \sigma_{ij}\nu_j.$$

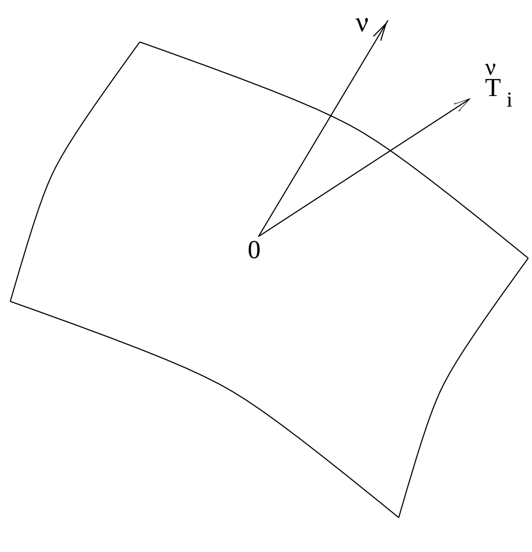


FIGURE 24

Thus, the traction F_s is given by

$$F_{s,i} = -\sigma_{ij}\nu_j. \tag{6.11}$$

Using (6.11), (6.10) can be written as follows:

$$\rho\dot{u}_i^c(x_1,x_2,x_3,t+\delta t)c\delta t\,\delta S = \int_t^{t+\delta t}\int_{\delta S}(-\sigma_{ij}\nu_j) \approx \delta t(-\sigma_{ij}\nu_j)\delta S. \tag{6.12}$$

Dividing by $\delta t\,\delta S$ and taking limit when δt, $\delta S \to 0$, we obtain

$$\rho c\dot{u}_i^c = -\sigma_{ij}\nu_j = -\frac{\partial W}{\partial \varepsilon_{ij}}\nu_j, \qquad 1 \le i \le j, \tag{6.13}$$

or, in vector notation,

$$\rho c\dot{\mathbf{u}}^c = -\sigma\boldsymbol{\nu} = -\mathcal{F}_s, \text{ on } S. \tag{6.14}$$

Let $\boldsymbol{\chi}^1$ and $\boldsymbol{\chi}^2$ be two tangent vectors at the point $\mathbf{0} \in S$. Then from (6.14) we get the three equations

$$\rho c\dot{\mathbf{u}}^c \cdot \boldsymbol{\nu} = -\sigma\boldsymbol{\nu}\boldsymbol{\nu}, \qquad \rho c\dot{\mathbf{u}}^c \cdot \boldsymbol{\chi}^1 = -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^1, \qquad \rho c\dot{\mathbf{u}}^c \cdot \boldsymbol{\chi}^2 = -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^2. \tag{6.15}$$

Set

$$\begin{aligned} \mathbf{v}^c &= (v_1^c, v_2^c, v_3^c)^t, \\ v_1^c &= \frac{1}{c}\dot{\mathbf{u}}^c \cdot \boldsymbol{\nu} = \frac{1}{c}\dot{u}_i^c\nu_i, \\ v_2^c &= \frac{1}{c}\dot{\mathbf{u}}^c \cdot \boldsymbol{\chi}^1 = \frac{1}{c}\dot{u}_i^c x_i^1, \\ v_3^c &= \frac{1}{c}\dot{\mathbf{u}}^c \cdot \boldsymbol{\chi}^2 = \frac{1}{c}\dot{u}_i^c x_i^2. \end{aligned} \tag{6.16}$$

In the new variables, equations (6.15) become

$$c^2\rho v_1^c = -\sigma\boldsymbol{\nu}\boldsymbol{\nu}, \qquad c^2\rho v_2^c = -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^1, \qquad c^2\rho v_3^c = -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^2. \tag{6.17}$$

Next we write the right–hand side of (6.17) in terms of the variables v_1^c, v_2^c, and v_3^c. For that purpose, we first note that using (6.9) we can write $\varepsilon_{ij}(u^c)$ on the surface S in the form

$$\varepsilon_{ij}(\mathbf{u}^c) = \frac{1}{2}\left(\frac{\partial u_i^c}{\partial x_j} + \frac{\partial u_j^c}{\partial x_i}\right) = -\frac{1}{2}\left(\nu_j\frac{1}{c}\dot{u}_i^c + \nu_i\frac{1}{c}\dot{u}_j^c\right). \tag{6.18}$$

Then, using the constitutive equations (1.41),

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{ii}(u^c)^c + 2\mu\varepsilon_{ij}(u^c) \\ &= \lambda\delta_{ij}\left(-\frac{1}{c}\nu_i\dot{u}_i^c\right) - -\mu\left(\nu_j\frac{1}{c}\dot{u}_i^c + \nu_i\frac{1}{c}\dot{u}_j^c\right) \\ &= -\lambda v_1^c\delta_{ij} - \mu\left(\nu_j\frac{1}{c}\dot{u}_i^c + \nu_i\frac{1}{c}\dot{u}_j^c\right). \end{aligned} \tag{6.19}$$

Consequently,

$$\begin{aligned}\sigma\boldsymbol{\nu\nu} &= \sigma_{ij}\nu_i\nu_j \\ &= -\lambda v_1^c\delta_{ij}\nu_i\nu_j - \mu\frac{1}{c}\dot{u}_i^c\nu_j\nu_j\nu_i - \mu\frac{1}{c}\dot{u}_j^c\nu_i\nu_i\nu_j \\ &= -\lambda v^c + 1 - 2\mu v_1^c = -(\lambda+2\mu)v_1^c,\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}\sigma\boldsymbol{\nu\chi}^1 &= \sigma_{ij}\nu_i\chi_j^1 \\ &= -\lambda v_1\delta_{ij}\nu_i\chi_j^1 - \mu\frac{1}{c}\dot{u}_i^c\nu_j\nu_i\chi_j^1 - \mu\frac{1}{c}\dot{u}_j^c\nu_i\nu_i\chi_j^1 = -\mu v_2^c.\end{aligned} \tag{6.21}$$

Similarly,

$$\sigma\boldsymbol{\nu\chi}^2 = -\mu v_3^c. \tag{6.22}$$

Next note that

$$\begin{aligned}\varepsilon_{ij}\nu_i\nu_j &= -\frac{1}{2}\left(\frac{1}{c}\dot{u}_i^c\nu_j + \frac{1}{c}\dot{u}_j^c\nu_i\right)\nu_i\nu_j \\ &= -\frac{1}{2}\left(\frac{1}{c}\dot{u}_i^c\nu_i\nu_j\nu_u + \frac{1}{c}\dot{u}_j^c\nu_j\nu_i\nu_i\right) \\ &= -\frac{1}{2}(v_1^c + v_1^c) = -v_1^c,\end{aligned} \tag{6.23}$$

$$\begin{aligned}\varepsilon_{ij}\nu_i\chi_j^1 &= -\frac{1}{2}\left(\frac{1}{c}\dot{u}_i^c\nu_j + \frac{1}{c}\dot{u}_j^c\nu_i\right)\nu_i\chi_j^1 \\ &= -\frac{1}{2}\left(\frac{1}{c}\dot{u}_i^c\nu_i\underbrace{\nu_j\chi_j^1}_{=0} + \frac{1}{c}\dot{u}_j^c\chi_j^1\nu_i\nu_i\right) = -\frac{1}{2}v_2^c.\end{aligned} \tag{6.24}$$

Similarly,

$$\varepsilon_{ij}\nu_i\chi_j^2 = -\frac{1}{2}v_3^c. \tag{6.25}$$

Let us compute the strain energy density $W(\varepsilon_{ij})$ on the surface S in terms of the variables $(v_i^c)_{1\le i\le 3}$. To simplify the calculations, let us assume that we have changed coordinates so that $\boldsymbol{\nu} = (1,0,0)$, $\boldsymbol{\chi}^1 = (0,1,0)$, and $\boldsymbol{\chi}^2 = (0,0,1)$. Then, from (6.16),

$$v_1^c = \frac{1}{c}\dot{u}_i^c\nu_i = \frac{1}{c}\dot{u}_1^c, \qquad v_2^c = \frac{1}{c}\dot{u}_i^c\chi_i^1 = \frac{1}{c}\dot{u}_2^c, \qquad v_3^c = \frac{1}{c}\dot{u}_i^c\chi_i^2 = \frac{1}{c}\dot{u}_3^c.$$

Also, from (618) and (6.19),

$$
\begin{aligned}
\varepsilon_{11} &= -\frac{1}{2}\left(\nu_1 \frac{1}{c}\dot{u}_1^c + \nu_1 \frac{1}{c}\dot{u}_1^c\right) = -v_1^c, \\
\varepsilon_{12} &= -\frac{1}{2}\left(\nu_2 \frac{1}{c}\dot{u}_1^c + \nu_1 \frac{1}{c}\dot{u}_2^c\right) = -\frac{1}{2}v_2^c, \\
\varepsilon_{13} &= -\frac{1}{2}\left(\nu_3 \frac{1}{c}\dot{u}_1^c + \nu_1 \frac{1}{c}\dot{u}_3^c\right) = -\frac{1}{2}v_3^c, \\
\varepsilon_{23} &= -\frac{1}{2}\left(\nu_3 \frac{1}{c}\dot{u}_2^c + \nu_2 \frac{1}{c}\dot{u}_3^c\right) = 0, \\
\varepsilon_{22} &= \varepsilon_{33} = 0, \\
\sigma_{11} &= \lambda\varepsilon_{11} + 2\mu\varepsilon_{11} = -(\lambda + 2\mu)v_1^c, \\
\sigma_{12} &= 2\mu\varepsilon_{12} = -2\mu\frac{1}{2}v_2^c = -\mu v_2^c, \\
\sigma_{13} &= 2\mu\varepsilon_{13} = -\mu v_3^c, \\
\sigma_{23} &= \sigma_{22} = \sigma_{33} = 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
2W|_S = 2\Pi(\mathbf{v}^c) &= \sigma_{11}\varepsilon_{11} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13} \\
&= (\lambda + 2\mu)(v_1^c)^2 + 2\mu(v_2^c)^2 + 2\mu(v_3^c)^2.
\end{aligned}
$$

Thus,

$$
\Pi(\mathbf{v}^c) = \frac{1}{2}(\mathbf{v}^c)^t E \mathbf{v}^c, \tag{6.26}
$$

where

$$
E = \begin{bmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{6.27}
$$

Since

$$
\frac{\partial \pi}{\partial v_1^c} = (\lambda + 2\mu)v_1^c, \qquad \frac{\partial \pi}{\partial v_2^c} = \mu v_2^c, \qquad \frac{\partial \pi}{\partial v_3^c} = \mu v_3^c,
$$

from (6.21)–(6.23), we see that (6.15) can also be written in the form

$$
\rho c^2 v_1^c = -\frac{\partial \Pi}{\partial v_1^c}, \qquad \rho c^2 v_2^c = -\frac{\partial \Pi}{\partial v_2^c}, \qquad \rho c^2 v_3^c = -\frac{\partial \Pi}{\partial v_3^c} \tag{6.28}
$$

or in vector form

$$
c^2 \rho \mathbf{v}^c = \frac{\partial \Pi}{\partial \mathbf{v}^c} = -\mathcal{F}_s = E\mathbf{v}^c. \tag{6.29}
$$

Next write (6.29) in the form

$$c^2\rho^{1/2}\mathbf{v}^c = \rho^{-1/2}E\rho^{-1/2}\rho^{1/2}\mathbf{v}^c.$$

Set

$$\overline{\mathbf{v}}^c = \rho^{1/2}\mathbf{v}^c, \qquad S = \rho^{-1/2}E\rho^{-1/2}.$$

Thus, (6.29) can be written in the form

$$c^2\overline{\mathbf{v}}^c = S\overline{\mathbf{v}}^c. \tag{6.30}$$

Also, in terms of $\overline{\mathbf{v}}^c$, the strain energy density on S can be written in the form

$$\begin{aligned}\overline{\Pi}(\overline{\mathbf{v}}^c) = \Pi(\mathbf{v}^c) &= \frac{1}{2}(\mathbf{v}^c)^t E\mathbf{v}^c \\ &= \frac{1}{2}[\rho^{1/2}(\mathbf{v}^c)^t]\rho^{-1/2}E\rho^{-1/2}(\rho^{1/2}\mathbf{v}^c) = \frac{1}{2}\overline{\mathbf{v}}^c S\overline{\mathbf{v}}^c.\end{aligned} \tag{6.31}$$

Let $(c_i)_{1\le i\le 3}$ be the three positive wave speeds satisfying (6.30); i.e., solutions of

$$\det(S - c^2 I) = 0. \tag{6.32}$$

They are equal to $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $c_2 = c_3 = \sqrt{\frac{\mu}{\rho}}$, corresponding to the compressional and shear modes of propagation, respectively. Let $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ be the set of orthonormal vectors associated with c_1, c_2, and c_3, respectively, and let N be the matrix containing as rows the eigenvectors $\mathbf{N}_i$ and let $\wedge$ be the diagonal matrix containing the eigenvalues c_i^2, $1 \le i \le 3$ of S, so that

$$S = N^t \wedge N. \tag{6.33}$$

Next, let $\mathbf{z} = (\dot{\mathbf{u}}\cdot\boldsymbol{\nu}, \dot{\mathbf{u}}\cdot\boldsymbol{\chi}^1, \dot{\mathbf{u}}\cdot\boldsymbol{\chi}^2)^t$ be a velocity vector on the surface S due to the simultaneous normal arrival of waves of velocities $(c_i)_{1\le i\le 3}$. Since the $\mathbf{N}_i$'s are orthonormal, we can write

$$\overline{\mathbf{z}} = \rho^{1/2}\mathbf{z} = \sum_{i=1}^{3}[\mathbf{N}_i, \rho^{1/2}\mathbf{z}]_\rho \mathbf{N}_i. \tag{6.34}$$

Set

$$\overline{\mathbf{z}}^{c_i} = \rho^{1/2}\mathbf{z}^{c_i} = \frac{1}{c_i}[\mathbf{N}_i, \rho^{1/2}\mathbf{z}]_e \mathbf{N}_i, \tag{6.35}$$

where $[\cdot,\cdot]_e$ denotes the euclidean inner product. Then $\overline{z}^{c_i}$ satisfies the equation (c.f., 6.30)

$$S\overline{\mathbf{z}}^{c_i} = c_i^2\overline{\mathbf{z}}^{c_i}, \tag{6.36}$$

and the strain energy associated with $\overline{z}^{c_i}$ satisfies the relation (c.f. (6.31))

$$\overline{\Pi}(\overline{\mathbf{z}}^{c_i}) = \frac{1}{2}(\overline{\mathbf{z}}^{c_i})^t S\overline{\mathbf{z}}^{c_i}. \tag{6.37}$$

Now using (6.29) and (6.31) we see that the force $\mathcal{F}_i$ on S associated with $\overline{z}^{c_i}$ satisfies the relation

$$\rho^{1/2}\frac{\partial\overline{\Pi}}{\partial\overline{\mathbf{z}}^{c_i}} = \rho^{1/2}S\overline{\mathbf{z}}^{c_i} = \rho^{1/2}\rho^{-1/2}E\rho^{-1/2}\rho^{1/2}\overline{\mathbf{z}}^{c_i} = E\mathbf{z}^{c_i} = -\mathcal{F}^{(i)}. \tag{6.38}$$

Now the total force $\mathcal{F}_s$ on the surface S is equal to the sum of the forces $\mathcal{F}_i$; i.e.,

$$\mathcal{F}_s = \sum_{i=1}^{3}\mathcal{F}^{(i)} = -\sum_{i=1}^{3}\rho^{1/2}S\overline{\mathbf{z}}^{c_i}. \tag{6.39}$$

On the other hand, we can also write

$$\rho^{-1/2}\mathcal{F}_s = \sum_{i=1}^{3}[\mathbf{N}_i, \rho^{-1/2}\mathcal{F}_s]_e\mathbf{N}_i. \tag{6.40}$$

Consequently, since $S\overline{\mathbf{z}}^{c_i} = c_i^2\overline{\mathbf{z}}^{c_i}$ is a vector in the direction of $\mathbf{N}_i$, from (6.39) and (6.40) we have that

$$S\overline{\mathbf{z}}^{c_i} = -[\mathbf{N}_i, \rho^{-1/2}\mathcal{F}_s]_e\mathbf{N}_i, \quad 1 \leq i \leq 3. \tag{6.41}$$

Now from (6.35),

$$c_i^2\overline{\mathbf{z}}^{c_i} = c_i[\mathbf{N}_i, \rho^{1/2}\mathbf{z}]_e\mathbf{N}_i. \tag{6.42}$$

Also, from (6.36) and (6.41),

$$c_i^2\overline{z}^{c_i} = S\overline{z}^{c_i} = -[\mathbf{N}_i, \rho^{-1/2}\mathcal{F}_s]_e\mathbf{N}_i, \quad 1 \leq i \leq 3. \tag{6.43}$$

Thus, from (6.42)–(6.43) we have that

$$c_i[\mathbf{N}_i, \rho^{1/2}\mathbf{z}]_e = -[\mathbf{N}_i, \rho^{-1/2}\mathcal{F}_s]_e, \quad 1 \leq i \leq 3. \tag{6.44}$$

In matrix form, the equation above becomes

$$-N\rho^{-1/2}\mathcal{F}_s = \wedge^{1/2}N\rho^{1/2}\mathbf{z}. \tag{6.45}$$

Multiplying (6.45) by $\rho^{1/2}N^t$, we obtain

$$\begin{aligned} -\mathcal{F}_s &= \rho^{1/2}N^t \wedge^{1/2} N\rho^{1/2}\mathbf{z}, \\ \text{(c.f. (6.33))} &= \rho^{1/2}S^{1/2}\rho^{1/2}\mathbf{z} = \rho S^{1/2}\mathbf{z} = \mathcal{B}\mathbf{z}. \end{aligned} \tag{6.46}$$

The matrix $\mathcal{B}$ in the right hand side of (6.46) is positive definite and finally we write the first order absorbing boundary condition on S in the form

$$-\mathcal{F}_s = -(\sigma\boldsymbol{\nu}\boldsymbol{\nu}, -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^1, -\sigma\boldsymbol{\nu}\boldsymbol{\chi}^2) = \mathcal{B}(\dot{\mathbf{u}}\cdot\boldsymbol{\nu}, \dot{\mathbf{u}}\cdot\boldsymbol{\chi}^1, \dot{\mathbf{u}}\cdot\boldsymbol{\chi}^2). \tag{6.47}$$

Observation. Recall that in the two–dimensional case the work done at the interface $x_1 = 1$ averaged over a period is given by (see Figure 25)

$$\mathcal{F}_\ell = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} (\sigma_{11}\dot u_1 + \sigma_{12}\dot u_2)dt.$$

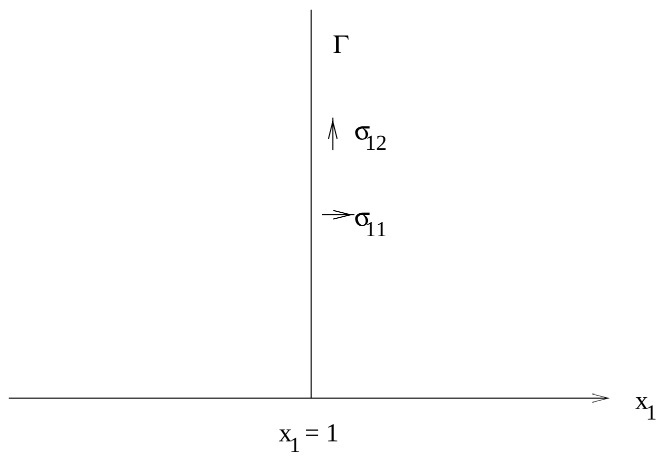


FIGURE 25

$\mathcal{F}_\ell$ represents the energy flux across the interface and since σ_{ij} and u_i are continuous, we have conservation of energy. Since

$$\varepsilon_{11} = \frac{1}{c}\dot u_1 = -v_1^c, \quad \varepsilon_{12} = -\frac{1}{2}\frac{1}{c}\dot u_2 = -\frac{1}{2}v_2^c$$

we may rewrite $\mathcal{F}_\ell$ in the form

$$\mathcal{F}_\ell = -\frac{1}{c}\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} (\sigma_{11}\varepsilon_{11} + 2\sigma_{12}\varepsilon_{12})dt = -\frac{1}{c}\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \Pi(v^c)dt.$$

For a wavefront arriving to , we may decompose the displacement u_i in the form

$$u_i = u_{i,p} + u_{i,s}$$

associated with the compressional and shear modes of propagation.

Then we can define the partial fluxes as:

$$F_{k,k} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} (\sigma_{11,k}\dot u_{1,k} + \sigma_{12,k}\dot u_{2,k})dt, \qquad k = p, s,$$

$$\mathcal{F}_{j,k} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} (\sigma_{11,j}\dot u_{1,k} + \sigma_{2,j}\dot u_{2,k})dt, \qquad j, k = p, s, \qquad j \neq k,$$

where $\sigma_{ij,p}$ denotes the stress associated with the compressional mode, likewise for $\sigma_{ij,s}$. It can be seen that $\mathcal{F}_{j,k} = 0$ for $j \neq k$ ([6]). This shows that the energy flux on , can be written as the sum of the energy fluxes associated with each type of wave, so that

$$\Pi(\mathbf{z}) = \sum_{i=1}^{3}\Pi(\mathbf{z}^{c_i}), \qquad \overline{\Pi}(\overline{\mathbf{z}}) = \sum_{i=1}^{3}\overline{\Pi}(\overline{\mathbf{z}}^{c_i}).$$

Thus, the force on , can be written as

$$-\mathcal{F}_s = \sum_{i=1}^{3}\frac{\partial\overline{\Pi}}{\partial\overline{\mathbf{z}}^{c_i}} = -\sum_{i=1}^{3}\mathcal{F}^{(i)}.$$

The Saturated Porous Solid Case.

Next, let us consider the case of an isotropic fluid–saturated porous solid where a small disturbance has originated on one side Ω_1 of the surface S, which is the boundary between the disturbed region Ω_1 and the undisturbed region Ω_2 if the wave front is arriving normally to S with velocity c. Following the ideas leading to (6.14), we see that the conservation of momentum on S can be written as

$$c\mathcal{A}\begin{pmatrix}\dot{\mathbf{u}}^{s,c}\\ \dot{\mathbf{u}}^{f,c}\end{pmatrix} = (-\tau_{ij}\nu_j, p_f\nu_i) = \left(-\frac{\partial W}{\partial\varepsilon_{ij}}\nu_j, \frac{\partial W}{\partial\xi}\nu_i\right), \qquad 1 \le i \le 3, \tag{6.48}$$

where

$$\mathcal{A} = \left(\begin{array}{c|c}\rho I & \rho_f I\\ \hline \rho_f I & gI\end{array}\right),$$

and I denotes the identity matrix in $R^{3\times 3}$. Then (6.48) can also be written in the form

$$\begin{aligned} &\text{i)} \quad c(\rho\dot{\mathbf{u}}^{s,c} + \rho_f\dot{\mathbf{u}}^{f,c}) = -\tau_{ij}\nu_j = -\frac{\partial W}{\partial\varepsilon_{ij}}\nu_j,\\ &\text{ii)} \quad c(\rho_f\dot{\mathbf{u}}^{s,c} + g\dot{\mathbf{u}}^{f,c}) = p_f\nu_i = \frac{\partial W}{\partial\xi}\nu_i, \quad \text{on } S.\end{aligned} \tag{6.49}$$

As before, let $\boldsymbol{\chi}^1$ and $\boldsymbol{\chi}^2$ be two tangent vectors at the point $O \in S$. Taking the inner product with $\boldsymbol{\chi}^1$ and $\boldsymbol{\chi}^2$ in (6.49.ii) we see that

$$c(\rho_f\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^k + g\dot{\mathbf{u}}^{f,c}\cdot\boldsymbol{\chi}^k) = 0, \quad k = 1,2,$$

so that

$$\dot{\mathbf{u}}^{f,c}\cdot\boldsymbol{\chi}^k = -g^{-1}\rho_f\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^k, \quad k = 1,2. \tag{6.50}$$

Hence, taking the inner product with $\boldsymbol{\nu}$ and $\boldsymbol{\chi}^1$, $\boldsymbol{\chi}^2$ in (6.49.i) and using (6.50) we obtain the equation

$$\begin{aligned} &c[\rho\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\nu} + \rho_f\dot{\mathbf{u}}^{f,c}\cdot\boldsymbol{\nu}] = -\tau\boldsymbol{\nu}\boldsymbol{\nu},\\ &c[\rho\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^k - g^{-1}\rho_f\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^k] = -\tau\boldsymbol{\nu}\mathbf{x}^k, \quad k = 1,2.\end{aligned} \tag{6.51}$$

Also, taking the inner product with $\boldsymbol{\nu}$ in (6.49.ii) we obtain

$$c[\rho_f\dot{\mathbf{u}}^{s,c}\boldsymbol{\nu} + g\dot{\mathbf{u}}^{f,c}\cdot\boldsymbol{\nu}] = p_f. \tag{6.52}$$

Set

$$\begin{aligned} &v_1^c = \frac{1}{c}\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\nu}, \qquad && v_2^c = \frac{1}{3}\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^1,\\ &v_2^c = \frac{1}{c}\dot{\mathbf{u}}^{s,c}\cdot\boldsymbol{\chi}^2, && v_4^c = \frac{1}{c}\dot{\mathbf{u}}^{f,c}\cdot\boldsymbol{\nu},\\ &\mathbf{v} = (v_1, v_2, v_3, v_4)^t, &&\\ &q = \rho - g^{-1}(\rho_f)^2. &&\end{aligned}$$

Then in the new variables, equations (6.51) and (6.52) become

$$\begin{aligned} &c^2[\rho v_1 + \rho_f v_4] = -\tau\boldsymbol{\nu}\boldsymbol{\nu}, && c^2 q v_2 = -\tau\boldsymbol{\nu}\boldsymbol{\chi}^1, \\ &c^2 q v_3 = -\tau\boldsymbol{\nu}\boldsymbol{\chi}^2, && c^2[\rho_f v_1 + g v_4] = p_f, \text{ on } S. \end{aligned} \tag{6.53}$$

Next we use the constitutive equations (c.f. (4.24)),

$$\tau_{ij} = (\lambda_c e - B\xi)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}^s), \qquad p_f = -Be + M\xi$$

with

$$e = \nabla\cdot\mathbf{u}^s, \qquad \xi = -\nabla\cdot\mathbf{u}^f,$$

and (6.18) to write the right hand side of (6.53) in terms of the variables $(v_i^c)_{1\le i\le 4}$. First note that

$$\begin{aligned} \xi &= -\nabla\cdot\mathbf{u}^{f,c} = -\varepsilon_{ii}(\mathbf{u}^{f,c}) \\ &= \frac{1}{2}\left(\nu_i\frac{1}{c}\mathbf{u}_i^{f,c} + \nu_i\frac{1}{c}u_i^{f,c}\right) = \mathbf{u}^{f,c}\cdot\boldsymbol{\nu} = v_4. \end{aligned}$$

Thus,

$$\begin{aligned} \tau_{ij} &= (\lambda_c\varepsilon_{ii}(\mathbf{u}^{s,c}) - Bv_4)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}^{s,c}) \\ &= -(\lambda_c v_1^c + Bv_4^c)\delta_{ij} - \mu\left(\nu_j\frac{1}{c}\dot{u}_i^{s,c} + \nu_i\frac{1}{c}\dot{u}_j^{s,c}\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \tau\boldsymbol{\nu}\boldsymbol{\nu} &= \tau_{ij}\nu_i\nu_j = -(\lambda_c v_1^c + Bv_4^c)\delta_{ij}\nu_i\nu_j - \mu\frac{1}{c}\dot{u}_i^{s,c}\nu_j\nu_j\nu_i - \mu\frac{1}{c}\dot{u}_j^{s,c}\nu_i\nu_i\nu_j \\ &= -(\lambda_c + 2\mu)v_1^c - Bv_4^c, \\ \tau\boldsymbol{\nu}\boldsymbol{\chi}^1 &= \tau_{ij}\nu_i\chi_j^1 = -(\lambda_c v_1^c + Bv_4^c)\delta_{ij}\nu_i\chi_j^1 - \mu\frac{1}{c}\dot{u}_i^{s,c}\nu_j\nu_i\chi_j^1 - \mu\frac{1}{c}\dot{u}_j^{2,c}\nu_i\nu_i x_j^1 \\ &= -\mu v_2^c, \\ \tau\boldsymbol{\nu}\boldsymbol{\chi}^2 &= -\mu v_3^c, \\ p_f &= Bv_1^c + Mv_4^c. \end{aligned}$$

Set

$$\widetilde{\mathcal{A}}_p = \begin{pmatrix} \rho & 0 & 0 & \rho_f \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ \rho_f & 0 & 0 & g \end{pmatrix}, \quad \widetilde{E}_p = \begin{pmatrix} \lambda_c + 2\mu & 0 & 0 & B \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ B & 0 & 0 & M \end{pmatrix}.$$

Then, in matrix form, equation (6.53) becomes

$$c^2\widetilde{\mathcal{A}}_p\mathbf{v}^c = \widetilde{E}_p\mathbf{v}^c. \tag{6.54}$$

Next, a calculation similar to that given for the elastic solid case shows that the strain energy density $W(\varepsilon_{ij},\xi)$ on the surface S can be written in terms of the variables $(v_i^c)_{1\le i\le 4}$ in the form

$$\Pi(\mathbf{v}^c)=\frac{1}{2}(\mathbf{v}^c)^t\widetilde{E}_p\mathbf{v}^c.$$

Thus, (6.54) can also be stated in the equivalent form

$$c^2\widetilde{\mathcal{A}}_p\mathbf{v}^c=\frac{\partial\Pi(\mathbf{v}^c)}{\partial\mathbf{v}^c}=\widetilde{E}_p\mathbf{v}^c=-\mathcal{F},\qquad \text{on } S, \tag{6.55}$$

where $\mathcal{F}=(\tau\boldsymbol{\nu}\boldsymbol{\nu},\tau\boldsymbol{\nu}\boldsymbol{\chi}^1,\tau\boldsymbol{\nu}\boldsymbol{\chi}^2,-p_f)^t$.

Set

$$\overline{\mathbf{v}}^c=\widetilde{\mathcal{A}}_p^{1/2}\mathbf{v}^c, \tag{6.56}$$

$$S=\widetilde{\mathcal{A}}_p^{-1/2}\widetilde{E}_p\widetilde{\mathcal{A}}_p^{-1/2}. \tag{6.57}$$

Then (6.55) becomes

$$S\overline{\mathbf{v}}^c=c^2\overline{\mathbf{v}}^c. \tag{6.58}$$

Also, in terms of $\overline{\mathbf{v}}^c$ the strain energy density on S can be written in the form

$$\begin{aligned}\Pi(\overline{\mathbf{v}}^c)&=\Pi(\mathbf{v}^c)=\frac{1}{2}(\mathbf{v}^c)^t\widetilde{E}_p\mathbf{v}^c\\&=\frac{1}{2}(\mathbf{v}^c)^t(\widetilde{\mathcal{A}}_p)^{1/2}(\widetilde{\mathcal{A}}_p)^{-1/2}\widetilde{E}_p(\widetilde{\mathcal{A}}_p)^{-1/2}(\widetilde{\mathcal{A}}_p)^{1/2}\mathbf{v}^c=\frac{1}{2}(\overline{\mathbf{v}}^c)^tS\overline{\mathbf{v}}^c.\end{aligned} \tag{6.59}$$

Let $(c_i)_{1\le i\le 4}$ be the four positive wave speeds satisfying (6.58); i.e., solutions of the equation

$$\det(S-c^2I)=0.$$

Two of these roots are

$$c_2=c_3=\left(\frac{\mu}{\rho-g^{-1}\rho_f}\right)^{1/2},$$

and they correspond to the shear modes of propagation. The other two roots are associated with the compressional models of propagation; i.e., the type I and type II compressional waves.

Next, let $\mathbf{N}_i$, $1\le i\le 4$, be the set of orthonormal eigenvectors corresponding to $(c_i)^2$, $1\le i\le 4$, and let $\mathbf{N}$ be the matrix containing the eigenvectors $\mathbf{N}_i$ of S as rows and $\wedge$ the diagonal matrix containing the eigenvalues $(c_i)^2$, $1\le i\le 4$, of S so that $S=N^t\wedge N$.

Next, let

$$\mathbf{z}=(\dot{\mathbf{u}}^s\cdot\boldsymbol{\nu},\dot{\mathbf{u}}^s\cdot\boldsymbol{\chi}^1,\dot{\mathbf{u}}^s\cdot\boldsymbol{\chi}^2,\dot{\mathbf{u}}^f\cdot\boldsymbol{\nu})^t$$

be a general velocity on the surface S due to the simultaneous arrival of waves of speeds (c_i), $1\le i\le 4$. Let

$$\overline{\mathbf{z}}=\widetilde{\mathcal{A}}^{1/2}\mathbf{z}. \tag{6.60}$$

Then we can write $\overline{\mathbf{z}}$ in the form

$$\overline{\mathbf{z}} = \sum_{i=1}^{4}[N_i, \overline{\mathbf{z}}]_e \mathbf{N}_i = \sum_{i=1}^{4}[N_i, \widetilde{\mathcal{A}}^{1/}\mathbf{z}]_e \mathbf{N}_i. \tag{6.61}$$

Set

$$\overline{\mathbf{z}}^{c_i} = \widetilde{\mathcal{A}}^{1/2}\mathbf{z}^{c_i} \equiv \frac{1}{c_i}[N_i, \widetilde{\mathcal{A}}^{1/2}\mathbf{z}]_e N_i, \quad 1 \le i \le 4. \tag{6.62}$$

Since $\overline{\mathbf{z}}^{c_i}$ is a multiple of N_i, we see that

$$S\overline{\mathbf{z}}^{c_i} = c_i^2 \overline{\mathbf{z}}^{c_i}, \tag{6.63}$$

and

$$\Pi(\overline{\mathbf{z}}^{c_i}) = \frac{1}{2}(\overline{\mathbf{z}}^{c_i})S\overline{\mathbf{z}}^{c_i}. \tag{6.64}$$

Also, using (6.55) we see that the force $\mathcal{F}_i$ on S associated with $\overline{\mathbf{z}}^{c_i}$ satisfies the equation

$$\begin{aligned} \widetilde{\mathcal{A}}_p^{1/2}\frac{\partial \overline{\Pi}}{\partial \overline{\mathbf{z}}^{c_i}} &= \widetilde{\mathcal{A}}_p^{1/2} S\overline{\mathbf{z}}^{c_i} = \widetilde{\mathcal{A}}_p^{1/2}\widetilde{\mathcal{A}}_p^{-1/2}\widetilde{E}_p\widetilde{\mathcal{A}}_p^{-1/2}\widetilde{\mathcal{A}}^{1/2}\mathbf{z}^{c_i} \\ &= \widetilde{E}_p \mathbf{z}^{c_i} = -\mathcal{F}^{(i)}. \end{aligned} \tag{6.65}$$

It is known that the interaction among the different types of waves arriving at an interface in a saturated porous medium is small compared with the total energy involved ([6], [15]). Neglecting such interactions, we can write the total strain energy density on S as the sum of the partial energies; i.e.,

$$\overline{\Pi}(\overline{\mathbf{z}}) = \sum_{i=1}^{4}\overline{\Pi}(\overline{\mathbf{z}}^{c_i}), \tag{6.66}$$

and the total force $\mathcal{F}$ on S as the sum of forces associated with each $\overline{\mathbf{z}}^{c_i}$ so that, according to (6.65),

$$\mathcal{F} = \sum_{i=1}^{4}\mathcal{F}^{(i)} = -\sum_{i=1}^{4}\widetilde{\mathcal{A}}_p^{1/2}S\overline{\mathbf{z}}^{c_i}. \tag{6.67}$$

On the other hand,

$$\widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F} = \sum_{i=1}^{4}[\mathbf{N}_i, \widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F}]_e \mathbf{N}_i. \tag{6.68}$$

Consequently,

$$S\overline{\mathbf{z}}^{c_i} = -[\mathbf{N}_i, \widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F}]_e \mathbf{N}_i, \quad 1 \le i \le 4. \tag{6.69}$$

Now using (6.62), (6.63), and (6.69), we see that

$$c_i^2 \overline{\mathbf{z}}^{c_i} = S\overline{\mathbf{z}}^{c_i} = c_i^2 \frac{1}{c_i}[\mathbf{N}_i, \widetilde{\mathcal{A}}_p^{1/2}\mathbf{z}]_e \mathbf{N}_i = -[N_i, \widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F}]_e \mathbf{N}_i, \qquad 1 \le i \le 4.$$

Thus,

$$c_i[\mathbf{N}_i, \widetilde{\mathcal{A}}_p^{1/2}\mathbf{z}] = -[\mathbf{N}_i, \widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F}]_e, \qquad 1 \le i \le 4. \tag{6.70}$$

In matrix form, the equation above becomes

$$-N\widetilde{\mathcal{A}}_p^{-1/2}\mathcal{F} = \wedge^{1/2} N \widetilde{\mathcal{A}}_p^{1/2}\mathbf{z}.$$

Hence, multiplying by $\mathcal{A}^{1/2}N^t = (N\mathcal{A}^{1/2})^t$, we obtain

$$-\mathcal{F} = [N\widetilde{\mathcal{A}}_p^{1/2}]^t \wedge^{1/2} [N\widetilde{\mathcal{A}}_p^{1/2}]\mathbf{z} \equiv \mathcal{B}_p \mathbf{z} \text{ on } S,$$

which are the first–order absorbing boundary conditions for S.

Note that $N\widetilde{\mathcal{A}}^{1/2}$ is nonsingular and, consequently, $\mathcal{B}_p$ is positive definite.

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