Effective viscoelastic medium from the composite model

Juan E. Santos [a,c,*] Patricia M. Gauzellino [b] José M. Carcione [d]

[a] *IGPUBA, Universidad de Buenos Aires, Argentina*

[b] *Universidad Nacional de La Plata, Argentina*

[c] *Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, Indiana, 47907-2067, USA; santos@math.purdue.edu*

[d] *OGS, Trieste, Italy*

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* Corresponding author, e-mail: santos@math.purdue.edu

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Abstract

An effective viscoelastic medium is derived using harmonic FE numerical experiments in a fluid-saturated composite poroelastic solid.

Key words: , Poroelasticity, Finite element methods, Effective viscoelastic media

1 Introduction

To be written

2 The stress-strain relations

Let the Fourier transform in the time variable be defined as usual by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\; dt, \tag{1}$$

where ω denotes the angular frequency.

Let Ω be an elementary cube of poroelastic material composed of two porous solid phases, referred to by the subscripts or superscripts 1 and 3, saturated by a single-phase fluid phase indicated by the subscript or superscript 2. Thus, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Let V_i denote the volume of the phase Ω_i and V_b and V_{sm} the bulk volume of Ω and the solid matrix $\Omega_{sm} = \Omega_1 \cup \Omega_3$, so that

$$V_{sm} = V_1 + V_3, \qquad V_b = V_1 + V_2 + V_3.$$

Let $S_1 = \dfrac{V_1}{V_{sm}}$ and $S_3 = \dfrac{V_3}{V_{sm}}$, denote the two solid fractions of the composite matrix.

We also define the effective porosity as

$$\phi = \frac{V_2}{V_b}.$$

Let $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(3)}$ be the averaged solid and fluid displacements over the bulk material. Here $\mathbf{u}^{(2)}$ is defined such that on any face F of the cube Ω

$$\int_F \phi \mathbf{u}^{(2)} \cdot \boldsymbol{\nu} d\sigma$$

is the amount of fluid displaced through F, while

$$\int_F S_1 \mathbf{u}^{(1)} \cdot \boldsymbol{\nu} d\sigma \qquad \text{and} \qquad \int_F S_3 \mathbf{u}^{(3)} \cdot \boldsymbol{\nu} dF$$

represent the displacements in the two solid parts of F, respectively. Here $\boldsymbol{\nu} = (\nu_j)$ denotes the unit outward normal to F and dF the surface measure on F.

Let $\boldsymbol{\tau}^{(1)} = (\tau_{ij}^{(1)})$ and $\boldsymbol{\tau}^{(3)} = (\tau_{ij}^{(3)})$ denote the stress tensors in Ω_1 and Ω_3 averaged over the bulk material Ω, respectively, and let p_f denote the fluid pressure. These quantities describe small changes with respect to reference values corresponding to an initial equilibrium state. Let us also introduce the tensors

$$\tau_{ij}^{(1,T)} = \tau_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}, \quad \tau_{i,j}^{(3,T)} = \tau_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}, \tag{2}$$

associated with the total stresses in Ω_1 and Ω_3, respectively.

Let

$$\mathbf{w} = \phi\Big(\mathbf{u}^{(2)} - S_1 \mathbf{u}^{(1)} - S_3 \mathbf{u}^{(3)}\Big), \tag{3}$$

where $\zeta = -\nabla \cdot \mathbf{w}$ represents the change in fluid content and

$$\epsilon_{ij}(u^{(m)}) = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right), \quad m = 1, 3,$$

denotes the strain tensor in Ω_m with linear invariant $\theta_m = \epsilon_{ii}(u^{(m)})$.

The stress-strain relations are

$$\tau_{ij}^{(1,T)} = \Big[\lambda_1 \theta_1 - B_1 \zeta + D_3 \theta_3\Big]\delta_{ij} + 2\mu_1 \epsilon_{ij}^{(1)} + \mu_{1,3} \epsilon_{ij}^{(3)}, \tag{4}$$

$$\tau_{ij}^{(3,T)} = \Big[\lambda_3 \theta_3 - B_2 \zeta + D_3 \theta_1\Big]\delta_{ij} + 2\mu_3 \epsilon_{ij}^{(3)} + \mu_{1,3} \epsilon_{ij}^{(1)}, \tag{5}$$

$$p_f = -B_1 \theta_1 - B_2 \theta_3 + M\zeta. \tag{6}$$

3 The equations of motion

The equations of motion stated in the space-frequency domain are

$$
\begin{aligned}
&-\omega^2 p_{11}\mathbf{u}^{(1)} - \omega^2 p_{12}\mathbf{u}^{(2)} - \omega^2 p_{13}\mathbf{u}^{(3)} + i\omega f_{11}\mathbf{u}^{(1)} - i\omega f_{12}\mathbf{u}^{(2)} - i\omega f_{11}\mathbf{u}^{(3)} \\
&\qquad = \nabla\cdot\boldsymbol{\tau}^{(1,T)}, && (7)\\
&-\omega^2 p_{12}\mathbf{u}^{(1)} - \omega^2 p_{22} bu^{(2)} - \omega^2 p_{23}\mathbf{u}^{(3)} - i\omega f_{12}\mathbf{u}^{(1)} + i\omega f_{22}\mathbf{u}^{(2)} + i\omega f_{12}\mathbf{u}^{(3)} \\
&\qquad = -\nabla p_f, && (8)\\
&-\omega^2 p_{13}\mathbf{u}^{(1)} - \omega^2 p_{23}\mathbf{u}^{(2)} - \omega^2 p_{33}\mathbf{u}^{(3)} - i\omega f_{11}\mathbf{u}^{(1)} + i\omega f_{12}\mathbf{u}^{(2)} + i\omega f_{11}\mathbf{u}^{(3)} \\
&\qquad = \nabla\cdot\boldsymbol{\tau}^{(3,T)}, \qquad i = 1,2,3. && (9)
\end{aligned}
$$

4 Determination of the stiffneses p_{33}

In order to determine the coefficients p_{33} we proceed as follows. We will solve (7)-(9) in the 2D case on a reference square $\Omega = (0, L)^2$ with boundary Γ in the (x_1, x_3)-plane.

Set $\Gamma = \Gamma^L \cup \Gamma^B \cup \Gamma^R \cup \Gamma^T$, where

$$
\begin{aligned}
&\Gamma^L = \{(x_1, x_3) \in \Gamma : x_1 = 0\}, \quad \Gamma^R = \{(x_1, x_3) \in \Gamma : x_1 = L\},\\
&\Gamma^B = \{(x_1, x_3) \in \Gamma : x_3 = 0\}, \quad \Gamma^T = \{(x_1, x_3) \in \Gamma : x_3 = L\}.
\end{aligned}
$$

Denote by $\boldsymbol{\nu}$ the unit outer normal on Γ and let $\boldsymbol{\chi}$ be a unit tangent on Γ so that $\{\boldsymbol{\nu}, \boldsymbol{\chi}\}$ is an orthonormal system on Γ. It follows how to obtain the stiffness components.

To determine p_{33}, we solve (7)-(9) in Ω with the following boundary conditions

$$
\begin{aligned}
&\boldsymbol{\tau}^{(1,T)}(\mathbf{u})\boldsymbol{\nu}\cdot\boldsymbol{\nu} = -\Delta P_1, \quad (x_1, x_3) \in \Gamma^T, && (10)\\
&\boldsymbol{\tau}^{(3,T)}(\mathbf{u})\boldsymbol{\nu}\cdot\boldsymbol{\nu} = -\Delta P_3, \quad (x_1, x_3) \in \Gamma^T, && (11)\\
&\boldsymbol{\tau}^{(1,T)}(\mathbf{u})\boldsymbol{\nu}\cdot\boldsymbol{\chi} = 0, \quad (x_1, x_3) \in \Gamma, && (12)\\
&\boldsymbol{\tau}^{(3,T)}(\mathbf{u})\boldsymbol{\nu}\cdot\boldsymbol{\chi} = 0, \quad (x_1, x_3) \in \Gamma, && (13)\\
&\mathbf{u}^{(1)}\cdot\boldsymbol{\nu} = 0, \quad (x_1, x_3) \in \Gamma \setminus \Gamma^T, && (14)\\
&\mathbf{u}^{(3)}\cdot\boldsymbol{\nu} = 0, \quad (x_1, x_3) \in \Gamma\setminus\Gamma^T, && (15)\\
&\mathbf{w}\cdot\boldsymbol{\nu} = 0, \quad (x_1, x_3) \in \Gamma. && (16)
\end{aligned}
$$

In this experiment $\epsilon_{11}(\mathbf{u}^{(1)}) = \epsilon_{22}(\mathbf{u}^{(1)} = \epsilon_{11}(\mathbf{u}^{(3)}) = \epsilon_{22}(\mathbf{u}^{(3)} = \nabla\cdot\mathbf{w} = 0$ and this experiment determines p_{33} as follows.

Denoting by V the original volume of the sample, its (complex) oscillatory volume change, $\Delta V(\omega)$, we note that

$$\frac{\Delta V(\omega)}{V} = -\frac{\Delta P}{p_{33}(\omega)}, \tag{17}$$

valid in the quasistatic case.

After solving (7)-(9) with the boundary conditions (10)-(16), the vertical displacements $u_3^{(1)}(x, L, \omega)$ of the solid frame on Γ^T allow us to obtain an average vertical displacement $\widehat{u}_3^{(1)}(\omega)$ suffered by the boundary Γ^T.

The numerical tests show that $u_3^{(1)}(x, L, \omega)$ gives the proper volume change.

Then, for each frequency ω, the volume change produced by the compressibility test can be approximated by $\Delta V(\omega) \approx L\widehat{u}_3^{(1)}(\omega)$, which enable us to compute $p_{33}(\omega)$ by using the relation (17).

To determine p_{55}, let us consider the solution of (7)-(9)in Ω with the following boundary conditions

$$-\boldsymbol{\tau}^{(1,T)}(\mathbf{u})\boldsymbol{\nu} = \mathbf{g}_1, \quad (x_1, x_3) \in \Gamma^T \cup \Gamma^L \cup \Gamma^R, \tag{18}$$
$$-\boldsymbol{\tau}^{(3,T)}(\mathbf{u})\boldsymbol{\nu} = \mathbf{g}_3, \quad (x_1, x_3) \in \Gamma^T \cup \Gamma^L \cup \Gamma^R, \tag{19}$$
$$\mathbf{u}^{(1)} = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{20}$$
$$\mathbf{u}^{(3)} = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{21}$$
$$\mathbf{w} \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma \tag{22}$$

where

$$\mathbf{g}_1 = \begin{cases} (0, \Delta G_1), & (x_1, x_3) \in \Gamma^L, \\ (0, -\Delta G_1), & (x_1, x_3) \in \Gamma^R, \\ (-\Delta G_1, 0), & (x_1, x_3) \in \Gamma^T. \end{cases} \tag{23}$$

$$\mathbf{g}_3 = \begin{cases} (0, \Delta G_3), & (x_1, x_3) \in \Gamma^L, \\ (0, -\Delta G_3), & (x_1, x_3) \in \Gamma^R, \\ (-\Delta G_3, 0), & (x_1, x_3) \in \Gamma^T. \end{cases} \tag{24}$$

The change in shape of the rock sample allows to recover $c_{55}(\omega)$ by using the relation

$$tg(\theta(\omega)) = \frac{\Delta G_1}{p_{55}(\omega)}, \tag{25}$$

where $\theta(\omega)$ is the departure angle between the original positions of the lateral boundaries and those after applying the shear stresses.

The horizontal displacements $u_1^{(1)}(x_1, L, \omega)$ at the top boundary Γ^T allow us to obtain, for each frequency, an average horizontal displacement $\widehat{u}_1^{(1)}(\omega)$

suffered by the boundary Γ^T. This average value allows us to approximate the change in shape suffered by the sample, given by $tg(\theta(\omega)) \approx \widehat{u}_1^{(1)}(\omega)/L$, which from (25) let us estimate $p_{55}(\omega)$.

5 A variational formulation

In order to state a variational formulation we need to introduce some notation. For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle\cdot,\cdot\rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ and $|\cdot|_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$.

Let us introduce the following closed subspace of $[H^1(\Omega)]^2$:

$$\mathcal{W}_{33}(\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma^L \cup \Gamma^R \cup \Gamma^B\},$$

$$\mathcal{W}_{55}(\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = 0 \text{ on } \Gamma^B\}.$$

Also, let

$$H_0(\text{div};\Omega) = \{\mathbf{v} \in H(\text{div};\Omega) : \mathbf{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma\},$$

$$H^1(\text{div};\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^2 : \nabla\cdot\mathbf{v} \in H^1(\Omega)\},$$

and for $(I,J) = (3,3), (5,5)$ let

$$\mathcal{Z}_{IJ}(\Omega) = \mathcal{W}_{IJ}(\Omega) \times H_0(\text{div};\Omega) \times \mathcal{W}_{IJ}(\Omega)\ .$$

To obtain our variational formulation associated with p_{33}, set $\mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right) \in \mathcal{Z}_{33}(\Omega)$ Then multiply equations (7) by $\mathbf{v}^{(1)} \in \mathcal{W}_{33}(\Omega)$, equation (9) by $\mathbf{v}^{(3)} \in$

$\mathcal{W}_{33}(\Omega)$ and equation (8) by $v^{(2)} \in H_0(\mathrm{div};\Omega)$, integrate by parts using the boundary conditions (10)-(16) and add the resulting equations to get *the weak form*: find $\mathbf{u}^{(33)} = (\mathbf{u}^{(1,33)}, \mathbf{u}^{(2,33)}, \mathbf{u}^{(3,33)}) \in \mathcal{Z}_{33}(\Omega)$ such that

$$
\begin{aligned}
&\Lambda(\mathbf{u}^{(33)}, \mathbf{v}) = -\omega^2 \left(p_{11}\mathbf{u}^{(1,33)} + p_{12}\mathbf{u}^{(2,33)} + p_{13}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&+i\omega \left(f_{11}\mathbf{u}^{(1,33)} - f_{12}\mathbf{u}^{(2,33)} - f_{11}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&-\omega^2 \left(p_{12}\mathbf{u}^{(1,33)} + p_{22}\mathbf{u}^{(2,33)} + p_{23}\ddot{\mathbf{u}}^{(3,33)}, v^{(2)}\right) \\
&+i\omega \left(-f_{12}\mathbf{u}^{(1,33)} + f_{22}\mathbf{u}^{(2,33)} + f_{12}\mathbf{u}^{(3,33)}, v^{(2)}\right) \\
&-\omega^2 \left(p_{13}\mathbf{u}^{(1,33)} + p_{23}\mathbf{u}^{(2,33)} + p_{33}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+i\omega \left(-f_{11}\mathbf{u}^{(1,33)} + f_{12}\mathbf{u}^{(2,33)} + f_{11}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+\sum_{pq} \left(\tau_{pq}^{(1,T)}(\mathbf{u}^{(33)}),\, \varepsilon_{pq}(\mathbf{v}^{(1)})\right) - \left(p_f(\mathbf{u}^{(33)}), \nabla \cdot v^{(2)}\right) \\
&+\sum_{pq} \left(\tau_{pq}^{(3,T)}(\mathbf{u}^{(33)}),\, \varepsilon_{pq}(\mathbf{v}^{(3)})\right) \\
&= -\left(\Delta P_1, \mathbf{v}^{(1)} \cdot \boldsymbol{\nu}\right) - \left(\Delta P_3, \mathbf{v}^{(3)} \cdot \boldsymbol{\nu}\right), \quad \forall \quad \mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right) \in \mathcal{Z}_{33}(\Omega).
\end{aligned}
\tag{26}
$$

Similarly, to obtain our variational formulation associated with p_{55}, set $\mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right) \in \mathcal{Z}_{55}(\Omega)$. Then multiply equations (7) by $\mathbf{v}^{(1)} \in \mathcal{W}_{55}(\Omega)$, equation (9) by $\mathbf{v}^{(3)} \in \mathcal{W}_{55}(\Omega)$ and equation (8) by $v^{(2)} \in H_0(\mathrm{div};\Omega)$, integrate by parts using the boundary conditions (18)-(22) and add the resulting equations to get *the weak form*: find $\mathbf{u}^{(55)} = (\mathbf{u}^{(1,55)}, \mathbf{u}^{(2,55)}, \mathbf{u}^{(3,55)}) \in \mathcal{Z}_{55}(\Omega)$ such that

$$
\begin{aligned}
&\Lambda(\mathbf{u}^{(55)}, \mathbf{v}) = -\omega^2 \left(p_{11}\mathbf{u}^{(1,55)} + p_{12}\mathbf{w} + p_{13}\mathbf{u}^{(3,55)}, v^{(1)}\right) \\
&+i\omega \left(f_{11}\mathbf{u}^{(1,55)} - f_{12}\mathbf{u}^{(2,55)} - f_{11}\mathbf{u}^{(3,55)}, v^{(1)}\right) \\
&-\omega^2 \left(p_{12}\mathbf{u}^{(1,55)} + p_{22}\mathbf{u}^{(2,55)} + p_{23}\ddot{\mathbf{u}}^{(3,55)}, v^{(2)}\right) \\
&+i\omega \left(-f_{12}\mathbf{u}^{(1,55)} + f_{22}\mathbf{u}^{(2,55)} + f_{12}\mathbf{u}^{(3,55)}, v^{(2)}\right) \\
&-\omega^2 \left(p_{13}\mathbf{u}^{(1,55)} + p_{23}\mathbf{u}^{(2,55)} + p_{33}\mathbf{u}^{(3,55)}, v^{(3)}\right) \\
&+i\omega \left(-f_{11}\mathbf{u}^{(1,55)} + f_{12}\mathbf{u}^{(2,55)} + f_{11}\mathbf{u}^{(3,55)}, v^{(3)}\right) \\
&+\sum_{pq} \left(\tau_{pq}^{(1,T)}(\mathbf{u}^{(55)}),\, \varepsilon_{pq}(\mathbf{v}^{(1)})\right) - \left(p_f(\mathbf{u}^{(55)}), \nabla \cdot v^{(2)}\right) \\
&+\sum_{pq} \left(\tau_{pq}^{(3,T)}(\mathbf{u}^{(55)}),\, \varepsilon_{pq}(\mathbf{v}^{(3)})\right) \\
&= \left(\mathbf{g}_1, \mathbf{v}^{(1)}\right) + \left(\mathbf{g}_3, \mathbf{v}^{(3)}\right), \quad \forall \quad \mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right) \in \mathcal{Z}_{55}(\Omega).
\end{aligned}
\tag{27}
$$

6 Finite element formulation

Let $\mathcal{T}^h(\Omega)$ be a non-overlapping partition of Ω into rectangles Ω_j of diameter bounded by h such that $\overline{\Omega} = \cup_J^J \overline{\Omega}_j$. Denote by $\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k$ the common side of two adjacent rectangles Ω_j and Ω_k. Also, let $\Gamma_j = \partial\Omega_j \cap \Gamma$.

We employ the space of globally continuous piecewise bilinear polynomials, to approximate each component of the solid displacement $\mathbf{u}^s$, while the vector part of the Raviart-Thomas-Nedelec space of zero order is used to approximate the fluid displacement vector $\mathbf{u}^f$ [27]. More specifically, let

$$\mathcal{W}_{33}^h(\Omega) = \{\mathbf{v}^s : \mathbf{v}^s|_{\Omega_j} \in [P_{1,1}(\Omega_j)]^2,\ \mathbf{v}^s \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma^L \cup \Gamma^R \Gamma^B\} \cap [C^0(\overline{\Omega})]^2,$$

$$\mathcal{W}_{55}^h(\Omega) = \{\mathbf{v}^s : \mathbf{v}^s|_{\Omega_j} \in [P_{1,1}(\Omega_j)]^2,\ \mathbf{v}^s \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma^B\} \cap [C^0(\overline{\Omega})]^2$$

be the FE spaces to approximate the solid displacement, and let

$$\mathcal{V}^h(\Omega) = \{\mathbf{v}^f \in H(\mathrm{div};\Omega) : \mathbf{v}^f|_{\Omega_j} \in P_{1,0}(\Omega_j) \times P_{0,1}(\Omega_j),\ \mathbf{v}^f \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\}$$

be the space to approximate the fluid displacement vector. Here $P_{s,t}$ denotes the polynomials of degree not greater than s in x_1 and not greater than t in x_3.

Then, for $(I,J) = (3,3),(5,5)$ let

$$\mathcal{Z}_{IJ}^h(\Omega) = \mathcal{W}_{IJ}^h(\Omega) \times \mathcal{V}^h(\Omega) \times \mathcal{W}_{IJ}^h(\Omega).$$

Next, for $(I,J) = (3,3),(5,5)$ let

$$\Pi_{IJ}^h : [H^{3/2}(\Omega)]^2 \to \mathcal{W}_{IJ}^h(\Omega)$$

be the interpolant operators associated with the spaces $\mathcal{W}_{IJ}^h$. More specifically, the degrees of freedom associated with $\Pi_{IJ}^h \mathbf{v}$ are the vertexes of the rectangles Ω_j and if b is a common node of the adjacent rectangles Ω_j and Ω_k then $\left(\Pi_{IJ}^h \boldsymbol{\varphi}\right)_j (b) = \left(\Pi_{IJ}^h \boldsymbol{\varphi}\right)_k (b)$, where $\left(\Pi_{IJ}^h \boldsymbol{\varphi}\right)_j$ denotes the restriction of the interpolant $\Pi_{IJ}^h \boldsymbol{\varphi}$ of $\boldsymbol{\varphi}$ to Ω_j.

Also, let

$$Q^h : H_0^1(\mathrm{div};\Omega) \to \mathcal{V}^h(\Omega)$$

be the projection defined by

$$\Big\langle (Q^h\boldsymbol{\psi} - \boldsymbol{\psi})\cdot\boldsymbol{\nu}, 1\Big\rangle_B = 0, \quad B = \Gamma_{jk} \text{ or } B = \Gamma_j.$$

The approximating properties of Π^h_{IJ} and Q^h are [27]

$$\|\boldsymbol{\varphi} - \Pi^h_{IJ}\boldsymbol{\varphi}\|_0 + h\|\boldsymbol{\varphi} - \Pi^h_{IJ}\varphi\|_1 \le Ch^{3/2}\|\boldsymbol{\varphi}\|_{3/2}, \tag{28}$$

$$\|\boldsymbol{\psi} - Q^h\boldsymbol{\psi}\|_0 \le Ch\|\boldsymbol{\psi}\|_1, \tag{29}$$

$$\|\nabla\cdot(\boldsymbol{\psi} - Q^h\boldsymbol{\psi})\|_0 \le Ch\left(\|\boldsymbol{\psi}\|_1 + \|\nabla\cdot\boldsymbol{\psi}\|_1\right). \tag{30}$$

Now, we formulate the FE procedures to determine the stiffnesses p_{IJ}'s as follows:

- $p_{33}(\omega)$: find $\mathbf{u}^{(h,33)} \in \mathcal{Z}^h_{33}(\Omega)$ such that

$$\Lambda(\mathbf{u}^{(h,33)}, \mathbf{v}) = -\left(\Delta P_1, \mathbf{v}^{(1)}\cdot\boldsymbol{\nu}\right) - \left(\Delta P_3, \mathbf{v}^{(3)}\cdot\boldsymbol{\nu}\right), \ \forall\, \mathbf{v} \in \mathcal{Z}^h_{33}(\Omega). \tag{31}$$

- $p_{55}(\omega)$: find $\mathbf{u}^{(h,55)} \in \mathcal{Z}^h_{55}(\Omega)$ such that

$$\Lambda(\mathbf{u}^{(h,55)}, \mathbf{v}) = \left(\mathbf{g}_1, \mathbf{v}^{(1)}\right) + \left(\mathbf{g}_3, \mathbf{v}^{(3)}\right), \qquad \forall\, \mathbf{v} \in \mathcal{Z}^h_{55}(\Omega), \tag{32}$$

  where $\mathbf{g}_1$ and $\mathbf{g}_3$ are defined in (23)-(24).

To approximate each component $u_j^{(1)}, u_j^{(3)}$ of the solid displacement vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(3)}$ take a reference rectangle $\widehat{R} = [0,1]^2$ and consider bilinear polinomials $\mathcal{V}(\widehat{R})$ Then we define

$$\mathcal{V}(\widehat{R}) = \mathrm{Span}\Big\{\varphi^{BL}, \varphi^{BR}, \varphi^{TR}, \varphi^{TL}\Big\}.$$

For example, φ^{BL} is a bilinear polynomial taking the value one at the botttom left corner of $\widehat{R}$ and vanishing at the other three corners of $\widehat{R}$.

To approximate the fluid displacement vector $\mathbf{u}^{(2)}$ we choose the vector part of the Raviart-Thomas-Nedelec space [27,28] of zero order defined on $\widehat{R}$ as follows. The four degrees of freedom associated with each fluid displacement vector are the values of the normal components at the mid points $\xi^l, l = L, R, B, T$ of the faces of $\widehat{R}$. Thus, defining the local basis

$$\psi^L(x) = 1 - x,\ \psi^R(x) = x,\ \psi^B(z) = 1 - z,\ \psi^T(z) = z,$$

we have that

$$\mathcal{W}(\widehat{R}) = \mathrm{Span}\Big\{(\psi^L(x),0),(\psi^R(x),0),(0,\psi^B(z)),(0,\psi^T(z))\Big\}.$$

Now, our finite element approximations $\mathbf{U}^{(2)}$ to $u^{(2)}$, $\mathbf{U}^{(1)} = (U_1^{(1)}, U_3^{(1)})$ to $u^{(1)} = (u_1^{(1)}, u_3^{(1)})$ and $\mathbf{U}^{(3)} = (U_1^{(3)}, U_3^{(3)})$ to $u^{(3)} = (u_1^{(3)}, u_3^{(3)})$ in the reference element $\widehat{R}$ are represented as follows:

$$\begin{aligned}
\mathbf{U}^{(2)} &= U^L(\psi^L(x),0) + U^R(\psi^R(x),0) + U^B(0,\psi^B(z)) + U^T(0,\psi^T(z)),\\
\mathbf{U}_j^{(1)} &= U_m^{1,L}\varphi^{BL}(x,z) + U_j^{1,B}\varphi^{BR}(x,z) + U_j^{1,R}\varphi^{TR}(x,z) + U_j^{1,T}\varphi^{TL}(x,z),\\
\mathbf{U}_j^{(3)} &= U_m^{3,L}\varphi^{BL}(x,z) + U_j^{3,B}\varphi^{BR}(x,z) + U_j^{3,R}\varphi^{TR}(x,z) + U_j^{3,T}\varphi^{TL}(x,z),\ \ j = 1,3,
\end{aligned}$$

By properly scaling the given basis elements we construct the spaces $\mathcal{V}^h$ and $\mathcal{W}^h$ used to represent the approximating functions $\mathbf{U}^{(1)}$, $\mathbf{U}^{(3)}$ and $\mathbf{U}^{(2)}$ for the solid and fluid displacement vectors on each element Ω_j.

7 The effective viscoelastic solid

let $\rho_m, m = 1,2,3$ denote the mass density of each solid and fluid constituent in Ω. Also let

$$\phi_1 = \frac{V_1}{V_b} \quad \text{and} \quad \phi_3 = \frac{V_3}{V_b}$$

be the fractions of the two solid phases in the bulk material. The mass density of the effective viscoelastic material is given by the arithmetic average

$$\rho = \phi_1\rho_1 + \phi\rho_2 + \phi_3\rho_3$$

8 Numerical Examples

desde aqui hay que comenzar a definir, validacion y ejemplos, fractal ice content, fractal shale-sandstoen distributions etc.

Lo de abajo es para el codigo

9 The Algebraic Problem for $\mathbf{u} = \mathbf{u}^{(33)}$

Let

$$
\begin{aligned}
&\Lambda(\mathbf{u}^{(33)}, \mathbf{v}) = -\omega^2 \left(p_{11}\mathbf{u}^{(1,33)} + p_{12}\mathbf{u}^{(2,33)} + p_{13}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&+i\omega \left(f_{11}\mathbf{u}^{(1,33)} - f_{12}\mathbf{u}^{(2,33)} - f_{11}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&-\omega^2 \left(p_{12}\mathbf{u}^{(1,33)} + p_{22}\mathbf{u}^{(2,33)} + p_{23}\ddot{\mathbf{u}}^{(3,33)}, v^{(2)}\right) \\
&+i\omega \left(-f_{12}\mathbf{u}^{(1,33)} + f_{22}\mathbf{u}^{(2,33)} + f_{12}\mathbf{u}^{(3,33)}, v^{(2)}\right) \\
&-\omega^2 \left(p_{13}\mathbf{u}^{(1,33)} + p_{23}\mathbf{u}^{(2,33)} + p_{33}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+i\omega \left(-f_{11}\mathbf{u}^{(1,33)} + f_{12}\mathbf{u}^{(2,33)} + f_{11}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+\Lambda_1(\mathbf{u}^{(33)},\, \mathbf{v}^{(1)}) + \Lambda_2(\mathbf{u}^{(33)},\, \mathbf{v}^{(2)}) + \Lambda_3(\mathbf{u}^{(33)},\, \mathbf{v}^{(1)}) \\
&= -\left(\Delta P_1, \mathbf{v}^{(1)} \cdot \boldsymbol{\nu}\right) - \left(\Delta P_3, \mathbf{v}^{(3)} \cdot \boldsymbol{\nu}\right), \quad \mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right),
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
&\Lambda_1(\mathbf{u},\, \mathbf{v}^{(1)}) = \Big(2\mu\varepsilon_{11}(\mathbf{u}^{(1)}) + \lambda_1 \left(\varepsilon_{11}(\mathbf{u}^{s}) + \varepsilon_{33}(\mathbf{u}^{(1)}\right) + B_1 \nabla \cdot \mathbf{u}^{(2)} \qquad (34)\\
&+ \qquad D_3 \left(\varepsilon_{11}(\mathbf{u}^{(3)}) + \varepsilon_{33}(\mathbf{u}^{(3)})\right) + \mu_{13}\varepsilon_{11}(\mathbf{u}^{(3)}), \varepsilon_{11}(\mathbf{v}^{(1)})\Big)\\
&+ \Big(2\mu\varepsilon_{22}(\mathbf{u}^{(1)}) + \lambda_1 \left(\varepsilon_{11}(\mathbf{u}^{s}) + \varepsilon_{33}(\mathbf{u}^{(1)}\right) + B_1 \nabla \cdot \mathbf{u}^{(2)}\\
&+ \qquad D_3 \left(\varepsilon_{11}(\mathbf{u}^{(3)}) + \varepsilon_{33}(\mathbf{u}^{(3)})\right) + \mu_{13}\varepsilon_{22}(\mathbf{u}^{(3)}), \varepsilon_{22}(\mathbf{v}^{(1)})\Big)\\
&+2 \left(2\mu_1\varepsilon_{12}(\mathbf{u}^{(1)}) + \mu_{13}\varepsilon_{12}(\mathbf{u}^{(3)}), \varepsilon_{12}(\mathbf{v}^{(1)}\right)\\
&= \left((\lambda_1 + 2\mu_1)\,\frac{\partial u_1^{(1)}}{\partial x}, \frac{\partial v_1^{(1)}}{\partial x}\right) + \left(\lambda_1 \frac{\partial u_2^{(1)}}{\partial y}, \frac{\partial v_1^{(1)}}{\partial x}\right)\\
&+ \left((\lambda_1 + 2\mu_1)\,\frac{\partial u_2^{(1)}}{\partial y}, \frac{\partial v_2^{(1)}}{\partial y}\right) + \left(\lambda_1 \frac{\partial u_1^{(1)}}{\partial x}, \frac{\partial v_2^{(1)}}{\partial y}\right)\\
&+ \left(\mu_1 \frac{\partial u_1^{(1)}}{\partial y}, \frac{\partial v_1^{(1)}}{\partial y}\right) + \left(\mu_1 \frac{\partial u_2^{(1)}}{\partial x}, \frac{\partial v_1^{(1)}}{\partial y}\right) + \left(\mu_1 \frac{\partial u_1^{(1)}}{\partial y}, \frac{\partial v_2^{(1)}}{\partial x}\right) + \left(\mu_1 \frac{\partial u_2^{(1)}}{\partial x}, \frac{\partial v_2^{(1)}}{\partial x}\right)\\
&+\frac{1}{2}\left(\mu_{13} \frac{\partial u_1^{(3)}}{\partial y}, \frac{\partial v_1^{(1)}}{\partial y}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial u_2^{(3)}}{\partial x}, \frac{\partial v_1^{(1)}}{\partial y}\right)\\
&+\frac{1}{2}\left(\mu_{13} \frac{\partial u_1^{(3)}}{\partial y}, \frac{\partial v_2^{(1)}}{\partial x}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial u_2^{(3)}}{\partial x}, \frac{\partial v_2^{(1)}}{\partial x}\right)\\
&+ \left((D_3 + \mu_{13})\,\frac{\partial u_1^{(3)}}{\partial x}, \frac{\partial v_1^{(1)}}{\partial x}\right) + \left(D_3 \frac{\partial u_2^{(3)}}{\partial y}, \frac{\partial v_1^{(1)}}{\partial x}\right)\\
&+ \left((D_3 \frac{\partial u_1^{(3)}}{\partial x}, \frac{\partial v_2^{(1)}}{\partial y}\right) + \left((D_3 + \mu_{13})\,\frac{\partial u_2^{(3)}}{\partial y}, \frac{\partial v_2^{(1)}}{\partial y}\right)\\
&+ \left(B_1 \left(\frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial u_2^{(2)}}{\partial y}\right), \frac{\partial v_1^{(1)}}{\partial x}\right)\\
&+ \left(B_1 \left(\frac{\partial u_1^{2)}}{\partial x} + \frac{\partial u_2^{(2)}}{\partial y}\right), \frac{\partial v_2^{(1)}}{\partial y}\right)
\end{aligned}
$$

$$\Lambda_3(\mathbf{u},\,\mathbf{v}^{(3)}) = \left((\lambda_3+2\mu_3)\,\frac{\partial u_1^{(3)}}{\partial x}, \frac{\partial v_1^{(3)}}{\partial x}\right) + \left(\lambda_3 \frac{\partial u_2^{(3)}}{\partial y}, \frac{\partial v_1^{(3)}}{\partial x}\right) \tag{35}$$
$$+ \left((\lambda_3+2\mu_3)\,\frac{\partial u_2^{(3)}}{\partial y}, \frac{\partial v_2^{(3)}}{\partial y}\right) + \left(\lambda_3 \frac{\partial u_1^{(3)}}{\partial x}, \frac{\partial v_2^{(3)}}{\partial y}\right)$$
$$+ \left(\mu_3 \frac{\partial u_1^{(3)}}{\partial y}, \frac{\partial v_1^{(3)}}{\partial y}\right) + \left(\mu_3 \frac{\partial u_2^{(3)}}{\partial x}, \frac{\partial v_1^{(3)}}{\partial y}\right) + \left(\mu_3 \frac{\partial u_1^{(3)}}{\partial y}, \frac{\partial v_2^{(3)}}{\partial x}\right) + \left(\mu_3 \frac{\partial u_2^{(3)}}{\partial x}, \frac{\partial v_2^{(3)}}{\partial x}\right)$$
$$+\frac{1}{2}\left(\mu_{13} \frac{\partial u_1^{(1)}}{\partial y}, \frac{\partial v_1^{(3)}}{\partial y}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial u_2^{(1)}}{\partial x}, \frac{\partial v_1^{(3)}}{\partial y}\right)$$
$$+\frac{1}{2}\left(\mu_{13} \frac{\partial u_1^{(1)}}{\partial y}, \frac{\partial v_2^{(3)}}{\partial x}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial u_2^{(1)}}{\partial x}, \frac{\partial v_2^{(3)}}{\partial x}\right)$$
$$+ \left((D_3+\mu_{13})\,\frac{\partial u_1^{(1)}}{\partial x}, \frac{\partial v_1^{(3)}}{\partial x}\right) + \left(D_3 \frac{\partial u_2^{(1)}}{\partial y}, \frac{\partial v_1^{(3)}}{\partial x}\right)$$
$$+ \left((D_3 \frac{\partial u_1^{(1)}}{\partial x}, \frac{\partial v_2^{(3)}}{\partial y}\right) + \left((D_3+\mu_{13})\,\frac{\partial u_2^{(1)}}{\partial y}, \frac{\partial v_2^{(3)}}{\partial y}\right)$$
$$+ \left(B_2\left(\frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial u_2^{(2)}}{\partial y}\right), \frac{\partial v_1^{(3)}}{\partial x}\right)$$
$$+ \left(B_2\left(\frac{\partial u_1^{2)}}{\partial x} + \frac{\partial u_2^{(2)}}{\partial y}\right), \frac{\partial v_2^{(3)}}{\partial y}\right).$$

$$\Lambda_2(\mathbf{u}, \mathbf{v}^n) = \left(B_1\theta_1 + B_2\theta_2 - M\xi, \nabla\cdot v^{(2)}\right) \tag{36}$$
$$= \left(B_1\left(\frac{\partial u_1^{(1)}}{\partial x} + \frac{\partial u_2^{(1)}}{\partial y}\right), \nabla\cdot\mathbf{v}^{(2)}\right)$$
$$+ \left(B_2\left(\frac{\partial u_1^{(3)}}{\partial x} + \frac{\partial u_2^{(3)}}{\partial y}\right), \nabla\cdot\mathbf{v}^{(2)}\right)$$
$$+ \left(M\left(\frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial u_2^{(2)}}{\partial y}\right), \nabla\cdot\mathbf{v}^{(2)}\right).$$

Then the FE problems associated with p_{33} is : find $\mathbf{u}^{(33)} = (\mathbf{u}^{(1,33)}, \mathbf{u}^{(2,33)}, \mathbf{u}^{(3,33)}) \in \mathcal{Z}_{33}(\Omega)$ such that

$$
\begin{aligned}
&\Lambda(\mathbf{u}^{(33)}, \mathbf{v}) = -\omega^2 \left(p_{11}\mathbf{u}^{(1,33)} + p_{12}\mathbf{u}^{(2,33)} + p_{13}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&+i\omega \left(f_{11}\mathbf{u}^{(1,33)} - f_{12}\mathbf{u}^{(2,33)} - f_{11}\mathbf{u}^{(3,33)}, v^{(1)}\right) \\
&-\omega^2 \left(p_{12}\mathbf{u}^{(1,33)} + p_{22}\mathbf{u}^{(2,33)} + p_{23}\ddot{\mathbf{u}}^{(3,33)}, v^{(2)}\right) \\
&+i\omega \left(-f_{12}\mathbf{u}^{(1,33)} + f_{22}\mathbf{u}^{(2,33)} + f_{12}\mathbf{u}^{(3,33)}, v^{(2)}\right) \\
&-\omega^2 \left(p_{13}\mathbf{u}^{(1,33)} + p_{23}\mathbf{u}^{(2,33)} + p_{33}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+i\omega \left(-f_{11}\mathbf{u}^{(1,33)} + f_{12}\mathbf{u}^{(2,33)} + f_{11}\mathbf{u}^{(3,33)}, v^{(3)}\right) \\
&+\Lambda_1(\mathbf{u}^{(33)},\, \mathbf{v}^{(1)}) + \Lambda_2(\mathbf{u}^{(33)},\, \mathbf{v}^{(2)}) + \Lambda_3(\mathbf{u}^{(33)},\, \mathbf{v}^{(1)}) \\
&= -\left(\Delta P_1, \mathbf{v}^{(1)} \cdot \boldsymbol{\nu}\right) - \left(\Delta P_3, \mathbf{v}^{(3)} \cdot \boldsymbol{\nu}\right), \quad \mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right), \mathbf{v} \in \mathcal{Z}^h_{33}(\Omega).
\end{aligned} \tag{37}
$$

We write the local 20×20 linear system associated with (37) on a square $\Omega = (0, h)^2$ in the x, z-plane. I am thinking on the z-axis pointing upwards to imagine the node BL the bottom left corner $(0, 0)$, the node BR the bottom right corner $(1, 0)$, the node TR the top right corner $(1, 1)$, and the node TL the top left corner $(0, 1)$. (I am moving counterclockwise).

Let us define the 4 local basis for each component of the solid dispalcement vectors $u^{(1)}, v^{(3)}$:

$$
\varphi^{BL}(x, z) = (1 - \frac{x}{h_{xj}})(1 - \frac{z}{h_{zk}}), \tag{38}
$$

$$
\varphi^{BR}(x, z) = (\frac{x}{h_{xj}})(1 - \frac{z}{h_{zk}}), \tag{39}
$$

$$
\varphi^{TL}(x, z) = (1 - \frac{x}{h_{xj}})(\frac{z}{h_{zk}}), \tag{40}
$$

$$
\varphi^{TR}(x, z) = (\frac{x}{h_{xj}})(\frac{z}{h_{zk}}), \tag{41}
$$

and the 4 local basis for the fluid u^f:

$$
\psi^{L}(x, z) = 1 - \frac{x}{h_{xj}} \tag{42}
$$

$$
\psi^{R}(x, z) = \frac{x}{h_{xj}} \tag{43}
$$

$$
\psi^{T}(x, z) = \frac{z}{h_{zk}}, \tag{44}
$$

$$
\psi^{B}(x, z) = 1 - \frac{z}{h_{zk}}, \tag{45}
$$

Let us use the notation $\mathbf{u}^{(1)} = (u_1^{(1)}, u_3^{1)})$, $\mathbf{u}^{2)} = (u_1^{(2)}, u_3^{(2)})$, $\mathbf{u}^{(3)} = (u_1^{(3)}, u_3^{3)})$, and set

$$u_1^{(1)}(x,z,\omega) = u_1^{(1,BL)}(\omega)\varphi^{BL}(x,z) + u_1^{(1,BR)}(\omega)\varphi^{BR}(x,z) + u_1^{(1,TR)}(\omega)\varphi^{TR}(x,z) + u_1^{(1,TL)}(\omega)\varphi^{TL}(x,z),$$

$$u_3^{(1)}(x,z,\omega) = u_3^{(1,BL)}(\omega)\varphi^{BL}(x,z) + u_3^{(1,BR)}(\omega)\varphi^{BR}(x,z) + u_3^{(1,TR)}(\omega)\varphi^{TR}(x,z) + u_3^{(1,TL)}(\omega)\varphi^{TL}(x,z)$$

$$u_1^{(3)}(x,z,\omega) = u_1^{(3,BL)}(\omega)\varphi^{BL}(x,z) + u_1^{(3,BR)}(\omega)\varphi^{BR}(x,z) + u_1^{(3,TR)}(\omega)\varphi^{TR}(x,z) + u_1^{(3,TL)}(\omega)\varphi^{TL}(x,z),$$

$$u_3^{(3)}(x,z,\omega) = u_3^{(3,BL)}(\omega)\varphi^{BL}(x,z) + u_3^{(3,BR)}(\omega)\varphi^{BR}(x,z) + u_3^{(3,TR)}(\omega)\varphi^{TR}(x,z) + u_3^{(3,TL)}(\omega)\varphi^{TL}(x,z)$$

$$u_1^{(2)} = u^{(2,L)}(\psi^L, 0) + u^{(2,R)}(\psi^R, 0)$$

$$u_3^{(2)} = u^{(2,B)}(0, \psi^B) + u^{(2,T)}(0, \psi^T).$$

The $u_1^{(1,BL)}(\omega), \cdots, u_3^{(3,TL)}(\omega), u^{(2,L)}, u^{(2,R)}, u^{(2,B)}, u^{(2,T)}$ are the coefficients in the 20×20 linear system to be defined next.

To get the equation for the first unknown $u_1^{(1,BL)}$, choose

$v^{(1)} = (v_1^{(1)}, v_3^{(1)}) = (\varphi^{BL}(x,z), 0)$ and $v^{(2)} = (0,0)$, $v^{(3)} = (0,0)$ in (37) and note that

$$\begin{aligned}
&\varepsilon_{11}((\varphi^{BL}(x,y),0)) = \frac{\partial \varphi^{BL}(x,y)}{\partial x},\\
&\varepsilon_{13}((\varphi^{BL}(x,y),0)) = \frac{1}{2}\frac{\partial \varphi^{BL}(x,y)}{\partial y}\\
&\varepsilon_{33}((\varphi^{BL}(x,y),0)) = 0,\\
&\nabla \cdot ((\varphi^{BL}(x,y),0)) = \frac{\partial \varphi^{BL}(x,y)}{\partial x}.
\end{aligned}$$

Then we get the equation

$$
\begin{aligned}
&-\omega^2\left(p_{11}\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{1,BR}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{12}\left[u^{(2,L)}\psi^{L}+u^{(2,R)}\psi^{R}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{13}\left[u_1^{(3,BL)}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&+i\omega\left(f_{11}\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-i\omega\left(f_{12}\left[u^{(2,L)}\psi^{L}+u^{(2,R)}\psi^{R}\right],\varphi^{BL}\right)\\
&-i\omega\left(f_{11}\left[u_1^{(3,BL)}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&\left((\lambda_1+2\mu_1)\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\lambda_1\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\mu_1\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\mu_1\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\left((D_3+\mu_{13})\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\left(D_3\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(B_1\frac{\partial\left[u^{(2,L)}\psi^{L}+[u^{(2,R)}\psi^{R}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(B_1\frac{\partial\left[[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right]}{\partial z},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad=-\left\langle\Delta P_1,(\varphi^{BL}(x,z),0)\cdot\nu\right\rangle_{\Gamma^T}=0
\end{aligned}
\tag{46}
$$

Remark: in this equation the right-hand side vanishes since the normal component of $(\varphi^{BL}(x,z),0)$ on the top boundary vanishes.

Let us number the unknowns as follows:

$$1 \to u_1^{(1,BL)}, 2 \to u_1^{(1,BR)}, 3 \to u_1^{(1,TR)}, 4 \to u_1^{(1,TL)},$$

$$5 \to u_3^{(1,BL)}, 6 \to u_3^{(1,BR)}, 7 \to u_3^{(1,TR)}, 8 \to u_3^{(1,TL)},$$

$$9 \to u^{2,L}, \quad 10 \to u^{2,R}, \quad 11 \to u^{2,B}, \quad 12 \to u^{T}.$$

$$13 \to u_1^{(3,BL)}, 14 \to u_1^{(3,BR)}, 15 \to u_1^{(3,TR)}, 16 \to u_1^{(3,TL)},$$

$$17 \to u_3^{(3,BL)}, 18 \to u_3^{(3,BR)}, 19 \to u_3^{(3,TR)}, 20 \to u_3^{(3,TL)},$$

Collecting in (46) the coefficients multiplying the unknowns $u_1^{1,BL}(\omega), \cdots, u_3^{1,TL}$ etc we get:

$$
\begin{aligned}
&\left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(1,BL)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(1,BR)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(1,TR)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(1,TL)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(1,BL)} \qquad (47)\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(1,BR)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(1,TR)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(1,TL)}\\
&\left[-\omega^2\left(p_{12}\psi^{L},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{L},\varphi^{BL}\right)+\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u^{(2,L)}\\
&\left[-\omega^2\left(p_{12}\psi^{R},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{R},\varphi^{BL}\right)+\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u^{(2,R)}\\
&+\left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)u^{(2,B)}\\
&+\left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)u^{(2,T)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(3,BL)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(3,BR)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(3,TR)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(3,TL)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(3,BL)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(3,BR)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(3,TR)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(3,TL)}\\
&=-\left\langle\Delta P_1,(\varphi^{BL}(x,z),0)\cdot\nu\right\rangle_{\Gamma^T}=0 \qquad (48)
\end{aligned}
$$

Thus we get an equation of the form

$$
\begin{aligned}
&a_{1,1}u_1^{(1,BL)} + a_{1,2}u_1^{(1,BR)} + a_{1,3}u_1^{(1,TR)} + a_{1,4}u_1^{(1,TL)} + a_{1,5}u_3^{(1,BL)} + a_{1,6}u_3^{(1,BR)} + a_{1,7}u_3^{(1,TR)} + a_{1,8}u_3^{(1,TL)} \\
&+a_{1,9}u^{(2,L)} + a_{1,10}u^{(2,R)} + a_{1,11}u^{(2,B)} + a_{1,12}u^{(2,T)} + \\
&a_{1,13}u_1^{(3,BL)} + a_{1,14}u_1^{(3,BR)} + a_{1,15}u_1^{(3,TR)} + a_{1,16}u_1^{(3,TL)} + a_{1,17}u_3^{(3,BL)} + a_{1,18}u_3^{(3,BR)} + a_{1,19}u_3^{(3,TR)} \qquad (49) \\
&\quad = -\left\langle \Delta P_1, (\varphi^{BL}(x,z), 0) \cdot \nu \right\rangle_{\Gamma^T} = 0 + a_{1,20}u_3^{(3,TL)}.
\end{aligned}
$$

The coefficient $a_{1j}, j = 1, \cdots, 16$ in (47) are

$$
\begin{aligned}
a_{1,1} &= \left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,2} &= \left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,3} &= \left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,4} &= \left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,5} &= \left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,6} &= \left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,7} &= \left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,8} &= \left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{1,9} &= -\omega^2\left(p_{12}\psi^{L},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{L},\varphi^{BL}\right)+\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{1,10} &= -\omega^2\left(f_{12}\psi^{R},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{R},\varphi^{BL}\right)+\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{1,11} &= \left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{1,12} &= \left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{1,13} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,14} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,15} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,16} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,17} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,18} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,19} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{1,20} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]
\end{aligned}
\tag{50}
$$

Next, taking the test functions $v^{(1)} = (\varphi^{BR}, 0), v^{(1)} = (\varphi^{TR}, 0), v^{(1)} = (\varphi^{TL}, 0), v^{(2)} = (0,0), v^{(3)} = (0,0)$ in (37) and noting that

$$\begin{aligned}
\varepsilon_{11}((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{BR}(x,y)}{\partial x},\\
\varepsilon_{13}((\varphi^{BR}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{BR}(x,y)}{\partial y}\\
\varepsilon_{33}((\varphi^{BR}(x,y),0)) &= 0,\\
\nabla \cdot ((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{BR}(x,y)}{\partial x}.
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{11}((\varphi^{TR}(x,y),0)) &= \frac{\partial \varphi^{TR}(x,y)}{\partial x},\\
\varepsilon_{13}((\varphi^{TR}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{TR}(x,y)}{\partial y}\\
\varepsilon_{33}((\varphi^{TR}(x,y),0)) &= 0,\\
\nabla \cdot ((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{TR}(x,y)}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{11}((\varphi^{TL}(x,y),0)) &= \frac{\partial \varphi^{TL}(x,y)}{\partial x},\\
\varepsilon_{13}((\varphi^{TL}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{TL}(x,y)}{\partial y}\\
\varepsilon_{33}((\varphi^{TL}(x,y),0)) &= 0,\\
\nabla \cdot ((\varphi^{TL}(x,y),0)) &= \frac{\partial \varphi^{TL}(x,y)}{\partial x}
\end{aligned}$$

we get equations identical to (47) but changing φ^{BL} by $\varphi^{BR}, \varphi^{TR}$ and φ^{TL}, respectively, in all the right-hand parts of the inner products.

Thus, we will get equations of the form

$$
\begin{aligned}
&a_{2,1}u_1^{(1,BL)} + a_{2,2}u_1^{(1,BR)} + a_{2,3}u_1^{(1,TR)} + a_{2,4}u_1^{(1,TL)} + a_{2,5}u_3^{(1,BL)} + a_{2,6}u_3^{(1,BR)} + a_{2,7}u_3^{(1,TR)} + a_{2,8}u_3^{(1,TL)} \\
&+a_{2,9}u^{(2,L)} + a_{2,10}u^{(2,R)} + a_{2,11}u^{(2,B)} + a_{2,12}u^{(2,T)} + \\
&a_{2,13}u_1^{(3,BL)} + a_{2,14}u_1^{(3,BR)} + a_{2,15}u_1^{(3,TR)} + a_{2,16}u_1^{(3,TL)} + a_{2,17}u_3^{(3,BL)} + a_{2,18}u_3^{(3,BR)} \\
&+a_{2,19}u_3^{(3,TR)} + a_{2,20}u_3^{(3,TL)} = 0.
\end{aligned}
$$

$$
\begin{aligned}
&a_{3,1}u_1^{(1,BL)} + a_{3,2}u_1^{(1,BR)} + a_{3,3}u_1^{(1,TR)} + a_{3,4}u_1^{(1,TL)} + a_{3,5}u_3^{(1,BL)} + a_{3,6}u_3^{(1,BR)} + a_{3,7}u_3^{(1,TR)} + a_{3,8}u_3^{(1,TL)} \\
&+a_{3,9}u^{(2,L)} + a_{3,10}u^{(2,R)} + a_{3,11}u^{(2,B)} + a_{3,12}u^{(2,T)} + \\
&a_{3,13}u_1^{(3,BL)} + a_{3,14}u_1^{(3,BR)} + a_{3,15}u_1^{(3,TR)} + a_{3,16}u_1^{(3,TL)} + a_{3,17}u_3^{(3,BL)} + a_{3,18}u_3^{(3,BR)} \\
&+a_{3,19}u_3^{(3,TR)} + a_{3,20}u_3^{(3,TL)} = 0.
\end{aligned}
$$

$$
\begin{aligned}
&a_{4,1}u_1^{(1,BL)} + a_{4,2}u_1^{(1,BR)} + a_{4,3}u_1^{(1,TR)} + a_{4,4}u_1^{(1,TL)} + a_{4,5}u_3^{(1,BL)} + a_{4,6}u_3^{(1,BR)} + a_{4,7}u_3^{(1,TR)} + a_{4,8}u_3^{(1,TL)} \\
&+a_{4,9}u^{(2,L)} + a_{4,10}u^{(2,R)} + a_{4,11}u^{(2,B)} + a_{4,12}u^{(2,T)} + \\
&a_{4,13}u_1^{(3,BL)} + a_{4,14}u_1^{(3,BR)} + a_{4,15}u_1^{(3,TR)} + a_{4,16}u_1^{(3,TL)} + a_{4,17}u_3^{(3,BL)} + a_{4,18}u_3^{(3,BR)} \\
&+a_{4,19}u_3^{(3,TR)} + a_{4,20}u_3^{(3,TL)} = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
a_{2,1} &= \left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,2} &= \left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,3} &= \left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,4} &= \left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,5} &= \left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,6} &= \left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,7} &= \left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,8} &= \left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right]\\
a_{2,9} &= -\omega^2\left(p_{12}\psi^{L},\varphi^{BR}\right)-i\omega\left(f_{12}\psi^{L},\varphi^{BR}\right)+\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{2,10} &= -\omega^2\left(p_{12}\psi^{R},\varphi^{BR}\right)-i\omega\left(f_{12}\psi^{R},\varphi^{BR}\right)+\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{2,11} &= \left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{2,12} &= \left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{2,13} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,14} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,15} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,16} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,17} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,18} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,19} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]\\
a_{2,20} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
a_{3,1} &= \left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,2} &= \left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,3} &= \left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,4} &= \left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,5} &= \left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{13,6} &= \left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,7} &= \left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,8} &= \left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{3,9} &= -\omega^2\left(p_{12}\psi^{L},\varphi^{TR}\right)-i\omega\left(f_{12}\psi^{L},\varphi^{TR}\right)+\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{3,10} &= -\omega^2\left(f_{12}\psi^{R},\varphi^{TR}\right)-i\omega\left(f_{12}\psi^{R},\varphi^{TR}\right)+\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{3,11} &= \left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{3,12} &= \left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{3,13} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,14} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,15} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,16} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,17} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,18} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,19} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{3,20} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
a_{4,1} &= \left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{TL}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{TL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,2} &= \left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{TL}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{TL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,3} &= \left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{TL}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{TL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,4} &= \left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{TL}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{TL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,5} &= \left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,6} &= \left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,7} &= \left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,8} &= \left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{4,9} &= -\omega^2\left(p_{12}\psi^{L},\varphi^{TL}\right)-i\omega\left(f_{12}\psi^{L},\varphi^{TL}\right)+\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{4,10} &= -\omega^2\left(f_{12}\psi^{R},\varphi^{TL}\right)-i\omega\left(f_{12}\psi^{R},\varphi^{TL}\right)+\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{4,11} &= \left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{4,12} &= \left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{4,13} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{TL}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{TL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,14} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{TL}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{TL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,15} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{TL}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{TL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,16} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{TL}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{TL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,17} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,18} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,19} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{4,20} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]
\end{aligned}
\tag{53}
$$

To get the fifth equation for the unknown $x_5 = u_3^{1,BL}$ take the test function $v^{(1)} = (0, \varphi^{BL}(x,z)) = (v_1^{(1)}, v_3^{(1)}), v^{(2)} = V^{(3)} = (0,0)$ in (37) and note that

$$\begin{aligned}
\varepsilon_{33}((0,\varphi^{BL}(x,y))) &= \frac{\partial \varphi^{BL}(x,z)}{\partial y}, \\
\varepsilon_{13}((0,\varphi^{BL}(x,z))) &= \frac{1}{2}\frac{\partial \varphi^{BL}(x,z)}{\partial x} \\
\varepsilon_{1}((0,\varphi^{BL}(x,y))) &= 0, \\
\nabla \cdot (0,\varphi^{BL}(x,y)) &= \frac{\partial \varphi^{BL}(x,z)}{\partial y}
\end{aligned}$$

Then we get

$$-\omega^2\left(p_{11}\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{1,BR}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)$$

$$-\omega^2\left(p_{12}\left[u^{(2,B)}\psi^{L}+u^{(2,T)}\psi^{R}\right],\varphi^{BL}\right)$$

$$-\omega^2\left(p_{13}\left[u_3^{(3,BL)}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)$$

$$+i\omega\left(f_{11}\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)$$

$$-i\omega\left(f_{12}\left[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right],\varphi^{BL}\right)$$

$$-i\omega\left(f_{11}\left[u_3^{(3,BL)}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)$$

$$\begin{aligned}
&\left((\lambda_1+2\mu_1)\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\lambda_1\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\mu_1\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\mu_1\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\left((D_3+\mu_{13})\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\left(D_3\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(B_1\frac{\partial\left[u^{(2,L)}\psi^{L}+[u^{(2,R)}\psi^{R}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(B_1\frac{\partial\left[[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad=-\left\langle\Delta P,(0,\varphi^{BL}(x,y))\cdot\nu\right\rangle_{\Gamma^T}=0
\end{aligned}\tag{54}$$

since φ^{BL} vanishes on Γ^T.

Collecting, we get

$$
\begin{aligned}
&\left[\left(\lambda_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(1,BL)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(1,BR)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(1,TR)}\\
&+\left[\left(\lambda_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_1^{(1,TL)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{BL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(1,BL)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{BR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(1,BR)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{TR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(1,TR)}\\
&+\left[-\omega^2\left(p_{11}\varphi^{TL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\left((\lambda_1+2\mu_1)\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u_3^{(1,TL)}\\
&+\left[\left(B_1\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u^{(2,L)}\\
&+\left[\left(B_1\frac{\psi^{R}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]u^{(2,R)}\\
&+\left[-\omega^2\left(p_{12}\psi^{B},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{B},\varphi^{BL}\right)+\left(B_1\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u^{(2,B)}\\
&+\left[-\omega^2\left(p_{12}\psi^{T},\varphi^{BL}\right)-i\omega\left(f_{12}\psi^{T},\varphi^{BL}\right)+\left(B_1\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u^{(2,T)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(3,BL)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(3,BR)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_1^{(3,TR)}\\
&+\left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(3,TL)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(3,BL)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(3,BR)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(3,TR)}\\
&+\left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]u_3^{(3,TL)}\\
&\equiv a_{5,1}u_1^{(1,BL)}+a_{5,2}u_1^{(1,BR)}+a_{5,3}u_1^{(1,TR)}+a_{5,4}u_1^{(1,TL)}+a_{5,5}u_3^{(1,BL)}+a_{5,6}u_3^{(1,BR)}+a_{5,7}u_3^{(1,TR)}+a_{5,8}u_3^{(1,TL)}\\
&\quad+a_{5,9}u^{(2,L)}+a_{5,10}u^{(2,R)}+a_{5,11}u^{(2,B)}+a_{5,12}u^{(2,T)}+
\end{aligned}
$$

where

$$
\begin{aligned}
a_{5,1} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,2} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,3} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,4} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,5} &= \left[-\omega^2\left(p_{11}\varphi^{BL}, \varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{BL}, \varphi^{BL}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,6} &= \left[-\omega^2\left(p_{11}\varphi^{BR}, \varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{BR}, \varphi^{BL}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,7} &= \left[-\omega^2\left(p_{11}\varphi^{TR}, \varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{TR}, \varphi^{BL}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,8} &= \left[-\omega^2\left(p_{11}\varphi^{TL}, \varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{TL}, \varphi^{BL}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,9} &= \left[\left(B_1 \frac{\partial \psi^{L}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,10} &= \left[\left(B_1 \frac{\psi^{R}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right)\right] \\
a_{5,11} &= \left[-\omega^2\left(p_{12}\psi^{B}, \varphi^{BL}\right) - i\omega\left(f_{12}\psi^{B}, \varphi^{BL}\right) + \left(B_1 \frac{\partial \psi^{B}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,12} &= \left[-\omega^2\left(p_{12}\psi^{T}, \varphi^{BL}\right) - i\omega\left(f_{12}\psi^{T}, \varphi^{BL}\right) + \left(B_1 \frac{\partial \psi^{T}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,13} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,14} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,15} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,16} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,17} &= \left[-\omega^2\left(p_{13}\varphi^{BL}, \varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{BL}, \varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,18} &= \left[-\omega^2\left(p_{13}\varphi^{BR}, \varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{BR}, \varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,19} &= \left[-\omega^2\left(p_{13}\varphi^{TR}, \varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{TR}, \varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right] \\
a_{5,20} &= \left[-\omega^2\left(p_{13}\varphi^{TL}, \varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{TL}, \varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BL}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BL}}{\partial y}\right)\right]
\end{aligned} \tag{55}
$$

Next, taking the test functions $v^{(1)} = (0, \varphi^{BR}(x, y)), v^{(1)} = (0, \varphi^{TR}(x, y)), v^{(1)} = (0, \varphi^{TL}(x, y)), v^{(2)} = v^{(3)} = (0, 0)$ in (37) we get the sixth, seventh and eigth equations with coefficients $a_{6,j}, a_{7,j}, a_{8,j}, j = 1, \cdots, 20$ defined as those in (55), changing the test function φ^{BL} in all the right parts of the innner products by $\varphi^{BR}, \varphi^{TR}$ or φ^{TL}, respectively.

Thus the coefficients $a_{6,j}, a_{7,j}, a_{8,j}, j = 1, \cdots, 20$ are:

$$
\begin{aligned}
a_{6,1} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,2} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,3} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,4} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,5} &= \left[-\omega^2\left(p_{11}\varphi^{BL}, \varphi^{BR}\right) + i\omega\left(f_{11}\varphi^{BL}, \varphi^{BR}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,6} &= \left[-\omega^2\left(p_{11}\varphi^{BR}, \varphi^{BR}\right) + i\omega\left(f_{11}\varphi^{BR}, \varphi^{BR}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,7} &= \left[-\omega^2\left(p_{11}\varphi^{TR}, \varphi^{BR}\right) + i\omega\left(f_{11}\varphi^{TR}, \varphi^{BR}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,8} &= \left[-\omega^2\left(p_{11}\varphi^{TL}, \varphi^{BR}\right) + i\omega\left(f_{11}\varphi^{TL}, \varphi^{BR}\right) + \left((\lambda_1 + 2\mu_1)\frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,9} &= \left[\left(B_1 \frac{\partial \psi^{L}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,10} &= \left[\left(B_1 \frac{\psi^{R}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right)\right] \\
a_{6,11} &= \left[-\omega^2\left(p_{12}\psi^{B}, \varphi^{BR}\right) - i\omega\left(f_{12}\psi^{B}, \varphi^{BR}\right) + \left(B_1 \frac{\partial \psi^{B}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,12} &= \left[-\omega^2\left(p_{12}\psi^{T}, \varphi^{BR}\right) - i\omega\left(f_{12}\psi^{T}, \varphi^{BR}\right) + \left(B_1 \frac{\partial \psi^{T}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,13} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,14} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,15} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,16} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,17} &= \left[-\omega^2\left(p_{13}\varphi^{BL}, \varphi^{BR}\right) - i\omega\left(f_{11}\varphi^{BL}, \varphi^{BR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,18} &= \left[-\omega^2\left(p_{13}\varphi^{BR}, \varphi^{BR}\right) - i\omega\left(f_{11}\varphi^{BR}, \varphi^{BR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,19} &= \left[-\omega^2\left(p_{13}\varphi^{TR}, \varphi^{BR}\right) - i\omega\left(f_{11}\varphi^{TR}, \varphi^{BR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right] \\
a_{6,20} &= \left[-\omega^2\left(p_{13}\varphi^{TL}, \varphi^{BR}\right) - i\omega\left(f_{11}\varphi^{TL}, \varphi^{BR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{BR}}{\partial x}\right) + \left((D_3 + \mu_{13})\frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{BR}}{\partial y}\right)\right]
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
a_{7,1} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,2} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,3} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,4} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,5} &= \left[-\omega^2\left(p_{11}\varphi^{BL}, \varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{BL}, \varphi^{TR}\right) + \left((\lambda_1+2\mu_1) \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,6} &= \left[-\omega^2\left(p_{11}\varphi^{BR}, \varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{BR}, \varphi^{TR}\right) + \left((\lambda_1+2\mu_1) \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,7} &= \left[-\omega^2\left(p_{11}\varphi^{TR}, \varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{TR}, \varphi^{TR}\right) + \left((\lambda_1+2\mu_1) \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,8} &= \left[-\omega^2\left(p_{11}\varphi^{TL}, \varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{TL}, \varphi^{TR}\right) + \left((\lambda_1+2\mu_1) \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,9} &= \left[\left(B_1 \frac{\partial \psi^{L}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,10} &= \left[\left(B_1 \frac{\psi^{R}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right)\right] \\
a_{7,11} &= \left[-\omega^2\left(p_{12}\psi^{B}, \varphi^{TR}\right) - i\omega\left(f_{12}\psi^{B}, \varphi^{TR}\right) + \left(B_1 \frac{\partial \psi^{B}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,12} &= \left[-\omega^2\left(p_{12}\psi^{T}, \varphi^{TR}\right) - i\omega\left(f_{12}\psi^{T}, \varphi^{TR}\right) + \left(B_1 \frac{\partial \psi^{T}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,13} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,14} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,15} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,16} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,17} &= \left[-\omega^2\left(p_{13}\varphi^{BL}, \varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{BL}, \varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left((D_3+\mu_{13}) \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,18} &= \left[-\omega^2\left(p_{13}\varphi^{BR}, \varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{BR}, \varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left((D_3+\mu_{13}) \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,19} &= \left[-\omega^2\left(p_{13}\varphi^{TR}, \varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{TR}, \varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left((D_3+\mu_{13}) \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right] \\
a_{7,20} &= \left[-\omega^2\left(p_{13}\varphi^{TL}, \varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{TL}, \varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TR}}{\partial x}\right) + \left((D_3+\mu_{13}) \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TR}}{\partial y}\right)\right]
\end{aligned}
\tag{57}
$$

(58)

$$
\begin{aligned}
a_{8,1} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,2} &= \left[\left(\lambda_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,3} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,4} &= \left[\left(\lambda_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,5} &= \left[-\omega^2\left(p_{11}\varphi^{BL}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BL}, \varphi^{TL}\right) + \left((\lambda_1 + 2\mu_1) \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,6} &= \left[-\omega^2\left(p_{11}\varphi^{BR}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BR}, \varphi^{TL}\right) + \left((\lambda_1 + 2\mu_1) \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,7} &= \left[-\omega^2\left(p_{11}\varphi^{TR}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TR}, \varphi^{TL}\right) + \left((\lambda_1 + 2\mu_1) \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,8} &= \left[-\omega^2\left(p_{11}\varphi^{TL}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TL}, \varphi^{TL}\right) + \left((\lambda_1 + 2\mu_1) \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right) + \left(\mu_1 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,9} &= \left[\left(B_1 \frac{\partial \psi^{L}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,10} &= \left[\left(B_1 \frac{\psi^{R}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right)\right] \\
a_{8,11} &= \left[-\omega^2\left(p_{12}\psi^{B}, \varphi^{TL}\right) - i\omega\left(f_{12}\psi^{B}, \varphi^{TL}\right) + \left(B_1 \frac{\partial \psi^{B}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,12} &= \left[-\omega^2\left(p_{12}\psi^{T}, \varphi^{TL}\right) - i\omega\left(f_{12}\psi^{T}, \varphi^{TL}\right) + \left(B_1 \frac{\partial \psi^{T}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,13} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,14} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,15} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,16} &= \left[\frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left(D_3 \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,17} &= \left[-\omega^2\left(p_{13}\varphi^{BL}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BL}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left((D_3 + \mu_{13}) \frac{\partial \varphi^{BL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,18} &= \left[-\omega^2\left(p_{13}\varphi^{BR}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BR}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{BR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left((D_3 + \mu_{13}) \frac{\partial \varphi^{BR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,19} &= \left[-\omega^2\left(p_{13}\varphi^{TR}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TR}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TR}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left((D_3 + \mu_{13}) \frac{\partial \varphi^{TR}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right] \\
a_{8,20} &= \left[-\omega^2\left(p_{13}\varphi^{TL}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TL}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13} \frac{\partial \varphi^{TL}}{\partial x}, \frac{\partial \varphi^{TL}}{\partial x}\right) + \left((D_3 + \mu_{13}) \frac{\partial \varphi^{TL}}{\partial y}, \frac{\partial \varphi^{TL}}{\partial y}\right)\right]
\end{aligned}
\tag{59}
$$

Next take the test function $v^{(1)} = (0,0), v^{2)} = (\psi^L(x,y),0), v^{(3)} = (0,0)$ in (37) and note that

$$\varepsilon_{33}(v^{(1)}) = \varepsilon_{13}(v^{(1)}) = \varepsilon_{11}(v^{(1)}) = 0, \qquad \nabla\cdot(\psi^L,0) = \frac{\partial\psi^L(x,y)}{\partial x}$$

to get

$$\begin{aligned}
&-\omega^2\left(p_{12}\left[u_1^{(1,BL)}\varphi^{BL} + u_1^{1,BR}\varphi^{BR} + u_1^{(1,TR)}\varphi^{TR} + u_1^{(1,TL)}\varphi^{TL}\right],\psi^L\right) \\
&-\omega^2\left(p_{22}\left[u^{(2,L)}\psi^L + u^{(2,R)}\psi^R\right],\psi^L\right) \\
&-\omega^2\left(p_{23}\left[u_1^{(3,BL)} + u_1^{(3,BR)}\varphi^{BR} + u_1^{(3,TR)}\varphi^{TR} + u_1^{(3,TL)}\varphi^{TL}\right],\psi^L\right)) \\
&-i\omega\left(f_{12}\left[u_1^{(1,BL)}\varphi^{BL} + u_1^{(1,BR)}\varphi^{BR} + u_1^{(1,TR)}\varphi^{TR} + u_1^{(1,TL)}\varphi^{TL}\right],\psi^L\right)) \\
&+i\omega\left(f_{22}\left[u^{(2,L)}\psi^L + u^{(2,R)}\psi^R\right],\psi^L\right) \\
&+i\omega\left(f_{12}\left[u_1^{(3,BL)} + u_1^{(3,BR)}\varphi^{BR} + u_1^{(3,TR)}\varphi^{TR} + u_1^{(3,TL)}\varphi^{TL}\right],\psi^L\right) \\
&\left(B_1\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL} + u_1^{(1,BR)}\varphi^{BR} + u_1^{(1,TR)}\varphi^{TR} + u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad+\left(B_1\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL} + u_3^{(1,BR)}\varphi^{BR} + u_3^{(1,TR)}\varphi^{TR} + u_3^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad+\left(B_2\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL} + u_1^{(3,BR)}\varphi^{BR} + u_1^{(3,TR)}\varphi^{TR} + u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad+\left(B_2\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL} + u_3^{(3,BR)}\varphi^{BR} + u_3^{(3,TR)}\varphi^{TR} + u_3^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad+\left(M\frac{\partial\left[u^{(2,L)}\psi^L + [u^{(2,R)}\psi^R\right]}{\partial x},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad+\left(M\frac{\partial\left[[u^{(2,B)}\psi^B + u^{(2,T)}\psi^T\right]}{\partial y},\frac{\partial\psi^L}{\partial x}\right) \\
&\quad= -\left\langle(0,0)\cdot\nu\right\rangle_{\Gamma^T} = 0.
\end{aligned} \tag{60}$$

Collecting, the coefficients $a_{9j}, j = 1,\cdots,20$ for the 9th equation in (60) are :

$$
\begin{aligned}
a_{91} &= \left[-\omega^2\left(p_{12}\varphi^{BL},\psi^L\right) - i\omega\left(f_{12}\varphi^{BL},\psi^L\right) + \left(B_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{92} &= \left[-\omega^2\left(p_{12}\varphi^{BR},\psi^L\right) - i\omega\left(f_{12}\varphi^{BR},\psi^L\right) + \left(B_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{93} &= \left[-\omega^2\left(p_{12}\varphi^{TR},\psi^L\right) - i\omega\left(f_{12}\varphi^{TR},\psi^L\right) + \left(B_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{94} &= \left[-\omega^2\left(p_{12}\varphi^{TL},\psi^L\right) - i\omega\left(f_{12}\varphi^{TL},\psi^L\right) + \left(B_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{95} &= \left[\left(B_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{96} &= \left[\left(B_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{97} &= \left[\left(B_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{98} &= \left[\left(B_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,9} &= \left[-\omega^2\left(p_{22}\psi^L,\psi^L\right) + i\omega(f_{22}\psi^L,\psi^L) + \left(M\frac{\partial\psi^L}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,10} &= \left[(-\omega^2\left(p_{22}\psi^R,\psi^L\right) + i\omega(f_{22}\psi^R,\psi^L) + \left((M\frac{\partial\psi^R}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,11} &= \left(M\frac{\partial\psi^B}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\\
a_{9,12} &= \left(M\frac{\partial\psi^T}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\\
a_{9,13} &= \left[(-\omega^2\left(p_{23}\varphi^{BL},\psi^L\right) + i\omega(f_{12}\varphi^{BL},\psi^L) + \left((B_2\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,14} &= \left[(-\omega^2\left(p_{23}\varphi^{BR},\psi^L\right) + i\omega(f_{12}\varphi^{BR},\psi^L) + \left((B_2\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,15} &= \left[(-\omega^2\left(p_{23}\varphi^{TR},\psi^L\right) + i\omega(f_{12}\varphi^{TR},\psi^L) + \left((B_2\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,16} &= \left[(-\omega^2\left(p_{23}\varphi^{TL},\psi^L\right) + i\omega(f_{12}\varphi^{TL},\psi^L) + \left((B_2\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^L}{\partial x}\right)\right]\\
a_{9,17} &= \left((B_2\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\\
a_{9,18} &= \left((B_2\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\\
a_{9,19} &= \left((B_2\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)\\
a_{9,20} &= \left((B_2\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^L}{\partial x}\right)
\end{aligned}
\tag{61}
$$

Next take the test functions $v^{(1)} = (0,0), v^{(2)} = (\psi^R(x,y), 0), v^{(3)} = (0,0)$ in (37) and note that

$$\varepsilon_{33}(v^{(1)}) = \varepsilon_{13}(v^{(1)}) = \varepsilon_{11}(v^{(1)}) = 0, \qquad \nabla \cdot (\psi^R, 0) = \frac{\partial \psi^R(x,y)}{\partial x}.$$

Then the 10th equations is obtained changing ψ^L by ψ^R in all right hand side inner products in (61)

Thus the coefficient $a_{10j}, j = 1, \cdots, 20$ for the 9th equation are

$$
\begin{aligned}
a_{10,1} &= \left[-\omega^2\left(p_{12}\varphi^{BL},\psi^R\right) - i\omega\left(f_{12}\varphi^{BL},\psi^R\right) + \left(B_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,2} &= \left[-\omega^2\left(p_{12}\varphi^{BR},\psi^R\right) - i\omega\left(f_{12}\varphi^{BR},\psi^R\right)\left(B_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,3} &= \left[-\omega^2\left(p_{12}\varphi^{TR},\psi^R\right) - i\omega\left(f_{12}\varphi^{TR},\psi^R\right)\left(B_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,4} &= \left[-\omega^2\left(p_{12}\varphi^{TL},\psi^R\right) - i\omega\left(f_{12}\varphi^{TL},\psi^R\right)\left(B_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,5} &= \left[\left(B_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,6} &= \left[\left(B_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,7} &= \left[\left(B_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,8} &= \left[\left(B_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,9} &= \left[-\omega^2\left(p_{22}\psi^L,\psi^R\right) + i\omega(f_{22}\psi^L,\psi^R) + \left(M\frac{\partial\psi^L}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,10} &= \left[(-\omega^2\left(p_{22}\psi^R,\psi^R\right) + i\omega(f_{22}\psi^R,\psi^R) + \left((M\frac{\partial\psi^R}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,11} &= \left(M\frac{\partial\psi^B}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\\
a_{10,12} &= \left(M\frac{\partial\psi^T}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\\
a_{10,13} &= \left[(-\omega^2\left(p_{23}\varphi^{BL},\psi^R\right) + i\omega(f_{12}\varphi^{BL},\psi^R) + \left((B_2\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,14} &= \left[(-\omega^2\left(p_{23}\varphi^{BR},\psi^R\right) + i\omega(f_{12}\varphi^{BR},\psi^R) + \left((B_2\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,15} &= \left[(-\omega^2\left(p_{23}\varphi^{TR},\psi^R\right) + i\omega(f_{12}\varphi^{TR},\psi^R) + \left((B_2\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,16} &= \left[(-\omega^2\left(p_{23}\varphi^{TL},\psi^R\right) + i\omega(f_{12}\varphi^{TL},\psi^R) + \left((B_2\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^R}{\partial x}\right)\right]\\
a_{10,17} &= \left((B_2\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\\
a_{10,18} &= \left((B_2\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\\
a_{10,19} &= \left((B_2\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)\\
a_{10,20} &= \left((B_2\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^R}{\partial x}\right)
\end{aligned}
\tag{62}
$$

Next take the test function $v^{(10} = (0,0), v^{(2)} = (0, \psi^B(x,z)), v^{(3)} = (0,0)$ in (37) and note that

$$\varepsilon_{33}(v^{(1)}) = \varepsilon_{13}(v^{(1)}) = \varepsilon_{11}(v^{(1)}) = 0, \qquad \nabla \cdot (0, \psi^B) = \frac{\partial \psi^B(x,y)}{\partial y}$$

to get

$$\begin{aligned}
&-\omega^2 \left(p_{12}\left[u_1^{(3,BL)}\varphi^{BL} + u_3^{1,BR}\varphi^{BR} + u_3^{(1,TR)}\varphi^{TR} + u_3^{(1,TL)}\varphi^{TL}\right], \psi^B\right) \\
&-\omega^2 \left(p_{22}\left[u^{(2,B)}\psi^B + u^{(2,B)}\psi^B\right], \psi^B\right) \\
&-\omega^2 \left(p_{23}\left[u_3^{(3,BL)} + u_3^{(3,BR)}\varphi^{BR} + u_3^{(3,TR)}\varphi^{TR} + u_3^{(3,TL)}\varphi^{TL}\right], \psi^B\right)) \\
&-i\omega \left(f_{12}\left[u_3^{(1,BL)}\varphi^{BL} + u_3^{(1,BR)}\varphi^{BR} + u_3^{(1,TR)}\varphi^{TR} + u_3^{(1,TL)}\varphi^{TL}\right], \psi^B\right)) \\
&+i\omega \left(f_{22}\left[u^{(2,B)}\psi^B + u^{(2,B)}\psi^B\right], \psi^B\right) \\
&+i\omega \left(f_{12}\left[u_3^{(3,BL)} + u_3^{(3,BR)}\varphi^{BR} + u_3^{(3,TR)}\varphi^{TR} + u_3^{(3,TL)}\varphi^{TL}\right], \psi^B\right) \\
&\left(B_1 \frac{\partial\left[u_1^{(1,BL)}\varphi^{BL} + u_1^{(1,BR)}\varphi^{BR} + u_1^{(1,TR)}\varphi^{TR} + u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad + \left(B_1 \frac{\partial\left[u_3^{(1,BL)}\varphi^{BL} + u_3^{(1,BR)}\varphi^{BR} + u_3^{(1,TR)}\varphi^{TR} + u_3^{(1,TL)}\varphi^{TL}\right]}{\partial x}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad + \left(B_2 \frac{\partial\left[u_1^{(3,BL)}\varphi^{BL} + u_1^{(3,BR)}\varphi^{BR} + u_1^{(3,TR)}\varphi^{TR} + u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad + \left(B_2 \frac{\partial\left[u_3^{(3,BL)}\varphi^{BL} + u_3^{(3,BR)}\varphi^{BR} + u_3^{(3,TR)}\varphi^{TR} + u_3^{(3,TL)}\varphi^{TL}\right]}{\partial x}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad + \left(M \frac{\partial\left[u^{(2,L)}\psi^L + [u^{(2,R)}\psi^R\right]}{\partial x}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad + \left(M \frac{\partial\left[[u^{(2,B)}\psi^B + u^{(2,T)}\psi^T\right]}{\partial y}, \frac{\partial \psi^B}{\partial y}\right) \\
&\quad = -\left\langle (0,0)\cdot \nu \right\rangle_{\Gamma^T} = 0.
\end{aligned} \tag{63}$$

Collecting, the coefficients $a_{11,j}, j = 1, \cdots, 20$ of the the 11th equation are

$$
\begin{aligned}
a_{11,1} &= \left[\left(B_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,2} &= \left[\left(B_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,3} &= \left[\left(B_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,4} &= \left[\left(B_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,5} &= \left[-\omega^2\left(p_{12}\varphi^{BL},\psi^B\right)-i\omega\left(f_{12}\varphi^{BL},\psi^B\right)+\left(B_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,6} &= \left[-\omega^2\left(p_{12}\varphi^{BR},\psi^B\right)-i\omega\left(f_{12}\varphi^{BR},\psi^B\right)\left(B_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,7} &= \left[-\omega^2\left(p_{12}\varphi^{TR},\psi^B\right)-i\omega\left(f_{12}\varphi^{TR},\psi^B\right)\left(B_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,8} &= \left[-\omega^2\left(p_{12}\varphi^{TL},\psi^B\right)-i\omega\left(f_{12}\varphi^{TL},\psi^B\right)\left(B_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,9} &= \left(M\frac{\partial\psi^L}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,10} &= \left(M\frac{\partial\psi^R}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,11} &= \left[-\omega^2\left(p_{22}\psi^B,\psi^B\right)+i\omega(f_{22}\psi^B,\psi^B)+\left(M\frac{\partial\psi^B}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,12} &= \left[(-\omega^2\left(p_{22}\psi^T,\psi^B\right)+i\omega(f_{22}\psi^T,\psi^B)+\left((M\frac{\partial\psi^T}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,13} &= \left((B_2\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,14} &= \left(B_2\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,15} &= \left(B_2\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,16} &= \left(B_2\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^B}{\partial y}\right) \\
a_{11,17} &= \left[-\omega^2\left(p_{23}\varphi^{BL},\psi^B\right)+i\omega\left(f_{12}\varphi^{BL},\psi^B\right)+\left(B_2\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,18} &= \left[-\omega^2\left(p_{23}\varphi^{BR},\psi^B\right)+i\omega\left(f_{12}\varphi^{BR},\psi^B\right)+\left(B_2\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,19} &= \left[-\omega^2\left(p_{23}\varphi^{TR},\psi^B\right)+i\omega\left(f_{12}\varphi^{TR},\psi^B\right)+\left(B_2\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right] \\
a_{11,20} &= \left[-\omega^2\left(p_{23}\varphi^{TL},\psi^B\right)+i\omega\left(f_{12}\varphi^{TL},\psi^B\right)+\left(B_2\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^B}{\partial y}\right)\right]
\end{aligned} \tag{64}
$$

Next, take the test function $v^{(1)} = (0,0), v^{(2)} = (0, \psi^T(x,y)), v^{(3)} = (0,0)$ in (37) and note that

$$\varepsilon_{33}(v^{(1)}) = \varepsilon_{13}(v^{(1)}) = \varepsilon_{11}(v^{(1)}) = 0, \qquad \nabla \cdot (0, \psi^T) = \frac{\partial \psi^T(x,y)}{\partial y}.$$

The 12th equation is obtained changing ψ^B by ψ^T in all right hand side inner products in (63). Then the 12th equation has coefficients $a_{12,j}, j = 1, \cdots, 20$:

$$
\begin{aligned}
a_{12,1} &= \left[\left(B_1\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,2} &= \left[\left(B_1\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,3} &= \left[\left(B_1\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,4} &= \left[\left(B_1\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,5} &= \left[-\omega^2\left(p_{12}\varphi^{BL},\psi^T\right)-i\omega\left(f_{12}\varphi^{BL},\psi^T\right)+\left(B_1\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,6} &= \left[-\omega^2\left(p_{12}\varphi^{BR},\psi^T\right)-i\omega\left(f_{12}\varphi^{BR},\psi^T\right)\left(B_1\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,7} &= \left[-\omega^2\left(p_{12}\varphi^{TR},\psi^T\right)-i\omega\left(f_{12}\varphi^{TR},\psi^T\right)\left(B_1\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,8} &= \left[-\omega^2\left(p_{12}\varphi^{TL},\psi^T\right)-i\omega\left(f_{12}\varphi^{TL},\psi^T\right)\left(B_1\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,9} &= \left(M\frac{\partial\psi^L}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,10} &= \left(M\frac{\partial\psi^R}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,11} &= \left[-\omega^2\left(p_{22}\psi^B,\psi^T\right)+i\omega(f_{22}\psi^B,\psi^T)+\left(M\frac{\partial\psi^B}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,12} &= \left[(-\omega^2\left(p_{22}\psi^T,\psi^T\right)+i\omega(f_{22}\psi^T,\psi^T)+\left((M\frac{\partial\psi^T}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,13} &= \left((B_2\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,14} &= \left(B_2\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,15} &= \left(B_2\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,16} &= \left(B_2\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\psi^T}{\partial y}\right) \\
a_{12,17} &= \left[-\omega^2\left(p_{23}\varphi^{BL},\psi^T\right)+i\omega\left(f_{12}\varphi^{BL},\psi^T\right)+\left(B_2\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,18} &= \left[-\omega^2\left(p_{23}\varphi^{BR},\psi^T\right)+i\omega\left(f_{12}\varphi^{BR},\psi^T\right)+\left(B_2\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,19} &= \left[-\omega^2\left(p_{23}\varphi^{TR},\psi^T\right)+i\omega\left(f_{12}\varphi^{TR},\psi^T\right)+\left(B_2\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right] \\
a_{12,20} &= \left[-\omega^2\left(p_{23}\varphi^{TL},\psi^T\right)+i\omega\left(f_{12}\varphi^{TL},\psi^T\right)+\left(B_2\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\psi^T}{\partial y}\right)\right]
\end{aligned} \tag{65}
$$

Next take the test function $v^{(1)} = (0,0), v^{(2)} = (0,0), v^{3)} = (\varphi^{BL}(x,y), 0)$ in (37) and note that

$$\varepsilon_{33}(v^{(1)}) = \varepsilon_{13}(v^{(1)}) = \varepsilon_{11}(v^{(1)}) = 0, \qquad \nabla \cdot v^{(2)} = 0$$

to get

$$
\begin{aligned}
&-\omega^2\left(p_{13}\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{1,BR}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{23}\left[u^{(2,L)}\psi^{L}+u^{(2,R)}\psi^{R}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{33}\left[u_1^{(3,BL)}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-i\omega\left(f_{11}\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&+i\omega\left(f_{12}\left[u^{(2,L)}\psi^{L}+u^{(2,R)}\psi^{R}\right],\varphi^{BL}\right)\\
&+i\omega\left(f_{11}\left[u_1^{(3,BL)}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&\left((\lambda_3+2\mu_3)\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\lambda_3\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\mu_3\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\mu_3\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\left((D_3+\mu_{13})\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\left(D_3\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(B_2\frac{\partial\left[u^{(2,L)}\psi^{L}+[u^{(2,R)}\psi^{R}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(B_2\frac{\partial\left[[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right]}{\partial z},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad=-\left\langle\Delta P_3,(\varphi^{BL}(x,y),0)\cdot\nu\right\rangle_{\Gamma^T}=0,
\end{aligned}
\tag{66}
$$

since $\varphi^{BL}(x, y)$ vanishes on Γ^T.

Collecting, the coefficient $a_{13,j}, j = 1, \cdots, 20$ of the 13th equation are

$$
\begin{aligned}
a_{13,1} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,2} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,3} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,4} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BL}\right) - i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,5} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,6} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,7} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,8} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,9} &= \left[(-\omega^2\left(p_{23}\psi^{L},\varphi^{BL}\right) + i\omega(f_{12}\psi^{L},\varphi^{BL}) + \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,10} &= \left[(-\omega^2\left(p_{23}\psi^{R},\varphi^{BL}\right) + i\omega(f_{12}\psi^{R},\varphi^{BL}) + \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]\\
a_{13,11} &= \left((B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{13,12} &= \left((B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
a_{13,13} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,14} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,15} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,16} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BL}\right) + i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,17} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,18} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,19} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]\\
a_{13,20} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right]
\end{aligned}
\tag{67}
$$

Next, take the test functions $v^{(1)} = (0,0), v^{(2)} = (0,0), v^{(3)} = (\varphi^{BR}, 0), v^{(3)} = (\varphi^{TR}, 0), v^{(3)} = (\varphi^{TL}, 0), v^{(2)} = (0,0), v^{(1)} = (0,0)$ in (37) to get the 14th, 15th and 16th equations.

The coefficients of the 14th, 15th and 16th equations are obtained by changing φ^{BL} by $\varphi^{BR}, \varphi^{TR}$ and φ^{TL}, respectively, in all the right-hand parts of the inner products in (67) and noting that

$$
\begin{aligned}
\varepsilon_{11}((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{BR}(x,y)}{\partial x}, \\
\varepsilon_{13}((\varphi^{BR}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{BR}(x,y)}{\partial y} \\
\varepsilon_{33}((\varphi^{BR}(x,y),0)) &= 0, \\
\nabla \cdot ((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{BR}(x,y)}{\partial x}.
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{11}((\varphi^{TR}(x,y),0)) &= \frac{\partial \varphi^{TR}(x,y)}{\partial x}, \\
\varepsilon_{13}((\varphi^{TR}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{TR}(x,y)}{\partial y} \\
\varepsilon_{33}((\varphi^{TR}(x,y),0)) &= 0, \\
\nabla \cdot ((\varphi^{BR}(x,y),0)) &= \frac{\partial \varphi^{TR}(x,y)}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{11}((\varphi^{TL}(x,y),0)) &= \frac{\partial \varphi^{TL}(x,y)}{\partial x}, \\
\varepsilon_{13}((\varphi^{TL}(x,y),0)) &= \frac{1}{2}\frac{\partial \varphi^{TL}(x,y)}{\partial y} \\
\varepsilon_{33}((\varphi^{TL}(x,y),0)) &= 0, \\
\nabla \cdot ((\varphi^{TL}(x,y),0)) &= \frac{\partial \varphi^{TL}(x,y)}{\partial x}.
\end{aligned}
$$

Thus, the coefficient $a_{14,j}, a_{15,j}, a_{16,j} j = 1, \cdots, 20$, are

$$
\begin{aligned}
a_{14,1} &= \bigg[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,2} &= \bigg[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,3} &= \bigg[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,4} &= \bigg[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,5} &= \bigg[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,6} &= \bigg[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,7} &= \bigg[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,8} &= \bigg[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,9} &= \bigg[(-\omega^2\left(p_{23}\psi^{L},\varphi^{BR}\right)+i\omega(f_{12}\psi^{L},\varphi^{BR})+\left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,10} &= \bigg[(-\omega^2\left(p_{23}\psi^{R},\varphi^{BR}\right)+i\omega(f_{12}\psi^{R},\varphi^{BR})+\left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\bigg]\\
a_{14,11} &= \left((B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{14,12} &= \left((B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\\
a_{14,13} &= \bigg[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,14} &= \bigg[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,15} &= \bigg[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,16} &= \bigg[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,17} &= \bigg[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,18} &= \bigg[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,19} &= \bigg[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]\\
a_{14,20} &= \bigg[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\bigg]
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
a_{15,1} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,2} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,3} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,4} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{TR}\right) - i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,5} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,6} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,7} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,8} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,9} &= \left[(-\omega^2\left(p_{23}\psi^{L},\varphi^{TR}\right) + i\omega(f_{12}\psi^{L},\varphi^{TR}) + \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,10} &= \left[(-\omega^2\left(p_{23}\psi^{R},\varphi^{TR}\right) + i\omega(f_{12}\psi^{R},\varphi^{TR}) + \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]\\
a_{15,11} &= \left((B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{15,12} &= \left((B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\\
a_{15,13} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,14} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,15} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,16} &= \left[-\omega^2\left(p_{33}\varphi^{TL},\varphi^{TR}\right) + i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,17} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,18} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,19} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]\\
a_{15,20} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right]
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
a_{16,1} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BL},\varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,2} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BR},\varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,3} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TR},\varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,4} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TL},\varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,5} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,6} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,7} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,8} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right) + \left(D_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,9} &= \left[(-\omega^2\left(p_{23}\psi^{L},\varphi^{TL}\right) + i\omega(f_{12}\psi^{L},\varphi^{TL}) + \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,10} &= \left[(-\omega^2\left(p_{23}\psi^{R},\varphi^{TL}\right) + i\omega(f_{12}\psi^{R},\varphi^{TL}) + \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{16,11} &= \left((B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{16,12} &= \left((B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right)\\
a_{16,13} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BL},\varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,14} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BR},\varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,15} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TR},\varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,16} &= \left[-\omega^2\left(p_{33}\varphi^{TL},\varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TL},\varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,17} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,18} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,19} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{16,20} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TL}}{\partial x}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TL}}{\partial y}\right)\right]
\end{aligned}
\tag{70}
$$

Next, take the test function $v^{(1)} = (0,0), v^{(2)} = (0,0), v^{(3)} = (0, \varphi^{BL})$ in (37) to get

$$
\begin{aligned}
&-\omega^2\left(p_{13}\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{1,BR}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{23}\left[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right],\varphi^{BL}\right)\\
&-\omega^2\left(p_{33}\left[u_3^{(3,BL)}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&-i\omega\left(f_{11}\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&+i\omega\left(f_{12}\left[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right],\varphi^{BL}\right)\\
&+i\omega\left(f_{11}\left[u_3^{(3,BL)}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right],\varphi^{BL}\right)\\
&\left((\lambda_3+2\mu_3)\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\lambda_3\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(\mu_3\frac{\partial\left[u_1^{(3,BL)}\varphi^{BL}+u_1^{(3,BR)}\varphi^{BR}+u_1^{(3,TR)}\varphi^{TR}+u_1^{(3,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&\quad+\left(\mu_3\frac{\partial\left[u_3^{(3,BL)}\varphi^{BL}+u_3^{(3,BR)}\varphi^{BR}+u_3^{(3,TR)}\varphi^{TR}+u_3^{(3,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\frac{1}{2}\left(\mu_{13}\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\\
&+\left((D_3+\mu_{13})\frac{\partial\left[u_3^{(1,BL)}\varphi^{BL}+u_3^{(1,BR)}\varphi^{BR}+u_3^{(1,TR)}\varphi^{TR}+u_3^{(1,TL)}\varphi^{TL}\right]}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&+\left(D_3\frac{\partial\left[u_1^{(1,BL)}\varphi^{BL}+u_1^{(1,BR)}\varphi^{BR}+u_1^{(1,TR)}\varphi^{TR}+u_1^{(1,TL)}\varphi^{TL}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(B_2\frac{\partial\left[u^{(2,L)}\psi^{L}+[u^{(2,R)}\psi^{R}\right]}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad+\left(B_2\frac{\partial\left[[u^{(2,B)}\psi^{B}+u^{(2,T)}\psi^{T}\right]}{\partial z},\frac{\partial\varphi^{BL}}{\partial y}\right)\\
&\quad=-\left\langle\Delta P_3,(\varphi^{BL}(x,y),0)\cdot\nu\right\rangle_{\Gamma^T}=0,
\end{aligned}
\tag{71}
$$

since $\varphi^{BL}(x, y)$ vanishes on Γ^T.

Collecting, we get the coefficients $a_{17,j}, j = 1, \cdots, 20$ of the get the 17 equation:

$$\begin{aligned}
a_{17,1} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,2} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,3} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,4} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,5} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,6} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,7} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,8} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BL}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,9} &= \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) \\
a_{17,10} &= \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right) \\
a_{17,11} &= \left[-\omega^2\left(p_{23}\psi^{B},\varphi^{BL}\right)+i\omega\left(f_{12}\psi^{B},\varphi^{BL}\right)+\left(B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,12} &= \left[-\omega^2\left(p_{23}\psi^{T},\varphi^{BL}\right)+i\omega\left(f_{12}\psi^{T},\varphi^{BL}\right)+\left(B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)\right] \\
a_{17,13} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,14} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,15} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,16} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,17} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BL}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,18} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BL}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,19} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BL}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right] \\
a_{17,20} &= \left[-\omega^2\left(p_{33}\varphi^{TL},\varphi^{BL}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BL}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BL}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BL}}{\partial x}\right)\right]
\end{aligned} \tag{72}$$

Next, take the test functions $v^{(1)} = (0,0), v^{(2)} = (0,0), v^{(3)} = (0,\varphi^{BR},0), v^{(3)} = (0,\varphi^{TR},0), v^{(3)} = (0,\varphi^{TL},0)$ in (37) to get the 18th, 9th and 20th equations.

The coefficients of the 18th, 19th and 20th equations are obtained by changing φ^{BL} by φ^{BR} , φ^{TR} and φ^{TL}, respectively, in all right hand side inner products in (72) Thus the coefficients $a_{18,j}, a_{19,j}, a_{20,j}, j = 1, \cdots, 20$ are

$$
\begin{aligned}
a_{18,1} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,2} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,3} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,4} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,5} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,6} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,7} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,8} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{BR}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,9} &= \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right) \\
a_{18,10} &= \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right) \\
a_{18,11} &= \left[-\omega^2\left(p_{23}\psi^{B},\varphi^{BR}\right)+i\omega\left(f_{12}\psi^{B},\varphi^{BR}\right)+\left(B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,12} &= \left[-\omega^2\left(p_{23}\psi^{T},\varphi^{BR}\right)+i\omega\left(f_{12}\psi^{T},\varphi^{BR}\right)+\left(B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)\right] \\
a_{18,13} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,14} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,15} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,16} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,17} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,18} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,19} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right] \\
a_{18,20} &= \left[-\omega^2\left(p_{33}\varphi^{TL},\varphi^{BR}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{BR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{BR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{BR}}{\partial x}\right)\right]
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
a_{19,1} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,2} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{18,3} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,4} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left(D_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,5} &= \left[-\omega^2\left(p_{13}\varphi^{BL},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,6} &= \left[-\omega^2\left(p_{13}\varphi^{BR},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,7} &= \left[-\omega^2\left(p_{13}\varphi^{TR},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,8} &= \left[-\omega^2\left(p_{13}\varphi^{TL},\varphi^{TR}\right)-i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right)+\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)+\left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,9} &= \left((B_2\frac{\partial\psi^{L}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) \\
a_{19,10} &= \left((B_2\frac{\partial\psi^{R}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right) \\
a_{19,11} &= \left[-\omega^2\left(p_{23}\psi^{B},\varphi^{TR}\right)+i\omega\left(f_{12}\psi^{B},\varphi^{TR}\right)+\left(B_2\frac{\partial\psi^{B}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,12} &= \left[-\omega^2\left(p_{23}\psi^{T},\varphi^{TR}\right)+i\omega\left(f_{12}\psi^{T},\varphi^{TR}\right)+\left(B_2\frac{\partial\psi^{T}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)\right] \\
a_{19,13} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,14} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,15} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,16} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,17} &= \left[-\omega^2\left(p_{33}\varphi^{BL},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{BL},\varphi^{TR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,18} &= \left[-\omega^2\left(p_{33}\varphi^{BR},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{BR},\varphi^{TR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{BR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,19} &= \left[-\omega^2\left(p_{33}\varphi^{TR},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{TR},\varphi^{TR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TR}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right] \\
a_{19,20} &= \left[-\omega^2\left(p_{33}\varphi^{TL},\varphi^{TR}\right)+i\omega\left(f_{11}\varphi^{TL},\varphi^{TR}\right)+\left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial y},\frac{\partial\varphi^{TR}}{\partial y}\right)+\left(\mu_3\frac{\partial\varphi^{TL}}{\partial x},\frac{\partial\varphi^{TR}}{\partial x}\right)\right]
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
a_{20,1} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left(D_3\frac{\partial\varphi^{BL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right] \qquad (75)\\
a_{20,2} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left(D_3\frac{\partial\varphi^{BR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,3} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left(D_3\frac{\partial\varphi^{TR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,4} &= \left[\frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left(D_3\frac{\partial\varphi^{TL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,5} &= \left[-\omega^2\left(p_{13}\varphi^{BL}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BL}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,6} &= \left[-\omega^2\left(p_{13}\varphi^{BR}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{BR}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{BR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{BR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,7} &= \left[-\omega^2\left(p_{13}\varphi^{TR}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TR}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,8} &= \left[-\omega^2\left(p_{13}\varphi^{TL}, \varphi^{TL}\right) - i\omega\left(f_{11}\varphi^{TL}, \varphi^{TL}\right) + \frac{1}{2}\left(\mu_{13}\frac{\partial\varphi^{TL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right) + \left((D_3+\mu_{13})\frac{\partial\varphi^{TL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,9} &= \left((B_2\frac{\partial\psi^{L}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\\
a_{20,10} &= \left((B_2\frac{\partial\psi^{R}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right)\\
a_{20,11} &= \left[-\omega^2\left(p_{23}\psi^{B}, \varphi^{TL}\right) + i\omega\left(f_{12}\psi^{B}, \varphi^{TL}\right) + \left(B_2\frac{\partial\psi^{B}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,12} &= \left[-\omega^2\left(p_{23}\psi^{T}, \varphi^{TL}\right) + i\omega\left(f_{12}\psi^{T}, \varphi^{TL}\right) + \left(B_2\frac{\partial\psi^{T}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right)\right]\\
a_{20,13} &= \left[\left(\lambda_3\frac{\partial\varphi^{BL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,14} &= \left[\left(\lambda_3\frac{\partial\varphi^{BR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,15} &= \left[\left(\lambda_3\frac{\partial\varphi^{TR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,16} &= \left[\left(\lambda_3\frac{\partial\varphi^{TL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,17} &= \left[-\omega^2\left(p_{33}\varphi^{BL}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BL}, \varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{BL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,18} &= \left[-\omega^2\left(p_{33}\varphi^{BR}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{BR}, \varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{BR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{BR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,19} &= \left[-\omega^2\left(p_{33}\varphi^{TR}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TR}, \varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TR}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{TR}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]\\
a_{20,20} &= \left[-\omega^2\left(p_{33}\varphi^{TL}, \varphi^{TL}\right) + i\omega\left(f_{11}\varphi^{TL}, \varphi^{TL}\right) + \left((\lambda_3+2\mu_3)\frac{\partial\varphi^{TL}}{\partial y}, \frac{\partial\varphi^{TL}}{\partial y}\right) + \left(\mu_3\frac{\partial\varphi^{TL}}{\partial x}, \frac{\partial\varphi^{TL}}{\partial x}\right)\right]
\end{aligned}
$$

All the inner products

$$(\varphi^{\alpha}, \varphi^{\beta})$$

$$\left(\frac{\partial \varphi^{\alpha}}{\partial x}, \frac{\partial \varphi^{\beta}}{\partial x}\right)$$

$$\left(\frac{\partial \varphi^{\alpha}}{\partial z}, \frac{\partial \varphi^{\beta}}{\partial x}\right)$$

and

$$\left(\frac{\partial \varphi^{\alpha}}{\partial z}, \frac{\partial \varphi^{\beta}}{\partial z}\right)$$

in (46) for $\alpha, \beta = BL, BR, TR, TL$ have already been computed for the case of the classic biot case

$$(\varphi^{BL}, \varphi^{BL}) = \frac{h^2}{9}.$$

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