Numerical simulation of waves in non-isothermal poroelastic media

Department of Mathematics, Purdue University, Bridge to Research Seminar, March 20, 2023

Juan E. Santos

Hohai University, Nanjing, China, Universidad de Buenos Aires, Argentina, and Purdue University, Indiana, USA.

March 19, 2023

Waves in non-isothermal poroelastic media. I

- The analysis of the seismic response in fluid saturated poroelastic media uses the non-isothermal wave equation, which includes the temperature, stress and deformation fields.
- The model, which combines the Biot (J.App.Phys., 1957) and Lord-Shulman theories (J.Mech.Phys.Sol., 1967), predicts the propagation of four waves: two compressional P waves, one fast (P1) and one slow (diffusive) (P2), a slow (diffusive) thermal (T) wave, and a shear (S) wave (not coupled with the T-wave).
- The T wave is coupled with both P-waves (Sharma (J.Earth.Sys.Sci, 2008), Carcione et al.(J.Geophys.Res., 2019).
- The model assumes that the temperature in the porous solid and in the fluid is the same.

Waves in non-isothermal poroelastic media. II

- Santos et. al. (J.Math.Anal.App., 2021) demonstrated the existence and uniqueness of the solution of an initial boundary value problem (IBVP) for the Biot/Lord-Shulman formulation in linear thermo-poroelastic isotropic media.

- The solution of the IBVP is given in terms of displacements of the solid and fluid phases and temperature.

- The Finite Element Method (FEM) presented here provides a tool to study the physics of wave propagation in this type of medium, including mesoscopic loss effects (mode conversion of fast waves to T-waves at mesoscopic-scale heterogeneities).

Mathematical model. I

Consider a fluid-saturated poroelastic medium, and assume that the whole aggregate is isotropic.

θ: increment of temperature above a reference absolute temperature T_0 for the state of zero stress and strain.

$\mathbf{u}^s = (u_i^s)$, $\mathbf{u}^f = (u_i^f)$: average particle displacement vectors of the solid and relative fluid phases, respectively.

$\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$

$\varepsilon(\mathbf{u}^s) = (\varepsilon_{ij}(\mathbf{u}^s))$, $\boldsymbol{\sigma}(\mathbf{u}, \theta) = (\sigma_{ij}(\mathbf{u}, \theta))$: strain and stress tensors of the solid and bulk material, respectively

$p_f = p_f(\mathbf{u}, \theta)$: fluid pressure

Non-isothermal constitutive relations

$$\sigma_{ij}(\mathbf{u},\theta) = 2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_u \nabla\cdot\mathbf{u}^s + B\nabla\cdot\mathbf{u}^f - \beta\,\theta),$$
$$p_f(\mathbf{u},\theta) = -B\,\nabla\cdot\mathbf{u}^s - M\nabla\cdot\mathbf{u}^f + \beta_f\theta.$$

μ is the dry-material shear modulus,

$M = \left(\dfrac{\alpha-\phi}{K_s} + \dfrac{\phi}{K_f}\right)^{-1}$, ϕ: porosity $\alpha = 1 - K_m/K_s$

$B = \alpha\ M, \quad \lambda_u = \lambda + \alpha^2 M$

K_s, K_m and K_f: bulk moduli of the grains, solid frame and fluid, respectively.

β and β_f: positive thermoelasticity coefficients of the bulk material and fluid, respectively, $\beta > \beta_f$.

Biot's dynamical equations

$$\rho_b \ddot{\mathbf{u}}^s + \rho_f \ddot{\mathbf{u}}^f - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \theta) = \mathbf{f}^s, \tag{1}$$

$$\rho_f \ddot{\mathbf{u}}^s + g \ddot{\mathbf{u}}^f + \frac{\eta}{\kappa} \dot{\mathbf{u}}^f + \nabla p_f(\mathbf{u}, \theta) = \mathbf{f}^f. \tag{2}$$

ρ_b, ρ_f: mass density of the bulk material and the fluid

η: fluid viscosity κ:permeability

$g = \dfrac{T \rho_f}{\phi}$: mass coupling parameter, (T: tortuosity)

$\mathbf{f}^s, \mathbf{f}^f$: external sources in the frame and fluid, respectively

REMARK: Ignoring external sources and acceleration terms, eqn. (1) is the equilibrium equation of the bulk material and eqn.(2) is Darcy's law.

Generalized heat equation

$$\tau\ c\ \ddot{\theta} + c\ \dot{\theta} - \nabla \cdot (\gamma \nabla \theta) + \beta T_0 \nabla \cdot \dot{\mathbf{u}}^s$$
$$+ \beta T_0 \nabla \cdot \dot{\mathbf{u}}^f + \tau \beta T_0 \nabla \cdot \ddot{\mathbf{u}}^s + \tau \beta T_0 \nabla \cdot \ddot{\mathbf{u}}^f = -q.$$

γ: bulk coefficient of heat conduction (thermal conductivity)

c: bulk specific heat of the unit volume in the absence of deformation

τ: Maxwell relaxation time, q: heat source.

These equations assume that the temperature in the solid and fluid phases is the same.

β, β_f, γ and c are considered parameters, obtained from experiments or from a specific theoretical model.

Plane wave analysis. Compressional waves. I

Apply the divergence operator in Biot's dynamical equations and set $e^j = \nabla \cdot \mathbf{u}^j, j = s, f$. Let

$$e^j = C_j \, e^{\mathrm{i}(\omega t - kx)}, \quad j = s, f, \quad \theta = C_\theta e^{\mathrm{i}(\omega t - kx)}.$$

a plane compressional wave of angular frequency ω and wave number $k = k_r + \mathrm{i}\, k_i$

$V_c = \dfrac{\omega^2}{k^2}$: square of the complex velocity. From the dynamic equations we get a linear system of three homogeneous equations for the amplitudes C_s, C_f, C_θ.

For non-trivial solutions the determinant of the associated matrix must vanish, which yields a cubic polynomial in V_c^2.

The phase velocity v_p and attenuation factor Q^{-1} are computed using the relations

$$v_p = \left[\mathrm{Re}(V_c^{-1})\right]^{-1}, \quad Q^{-1} = -\omega \mathrm{Im}(V_c^{-1}) \qquad (3)$$

Plane wave analysis. Material properties. I

The plane wave analysis is performed with nonzero coupling coefficients β and β_f (coupled case) with the material properties given in Table 1.

Table 1. Material Properties

Grain bulk modulus, K_s	35 GPa
density, ρ_s	2650 kg/m^3
Frame bulk modulus, K_m	1.7 GPa
shear modulus, μ_m	1.885 GPa
porosity, ϕ	0.3
permeability, κ	1 darcy
Fluid bulk modulus, K_f	2.4 GPa
density, ρ_f	1000 kg/m^3
viscosity, η_f	0.001 Pa $\cdot$ s
thermoelasticity coefficient, β_f	50000 kg/(m s^2 K)
Bulk specific heat, c	820 kg/(m s^2 K)
thermoelasticity coefficient, β	90000 kg/(m s^2 K)
absolute temperature, T_0	300 K
thermal conductivity, γ	4.5 $\times 10^6$ kg/m^3
relaxation time, τ	1.5 $\times 10^{-2}$ s

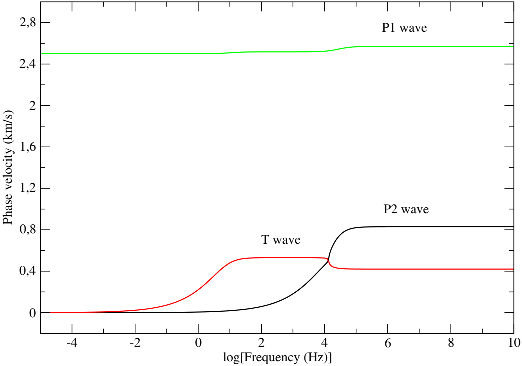

Figure 1. Phase velocity of the P1, P2 and T waves as a function of frequency for the coupled case.

Plane wave analysis. III

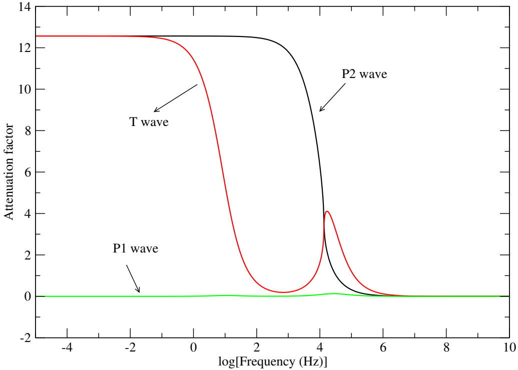

Figure 2. Attenuation factor of P1, P2 and T waves as a function of frequency for the coupled case. The two slow waves P2 and T are diffusive, strongly attenuated at low frequencies.

Plane wave analysis. IV

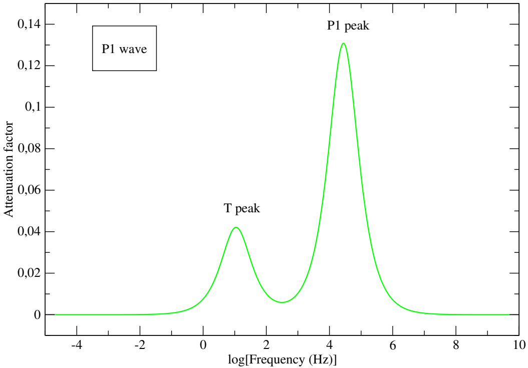

Figure 3. Attenuation factor of P1 wave as a function of frequency for the coupled case. The P1 wave has two Zener-like relaxation peaks, associated with the Biot and thermal loss mechanisms.

The Initial Bounday Value Problem (IBVP)

The solution of an IBVP for the thermo-poroelasticity equations was obtained using the Finite Element Method using C^0-piecewise linear polynomials to represent the solid and fluid phases and the temperature.

The equations are solved for the 1D case over an interval $\Omega = (0, L)$, L $=$ 166 m discretized using a uniform mesh.

Absorbing boundary conditions are used at the boundaries $\{0, L\}$.

Mesh size h: 0.175 m, time step dt: 7.95 $\times 10^{-3}$ ms.
The material properties are those in Table 1.

Numerical experiments. I

In the first experiments, the medium is uniform, initially at rest with a point dilatational source $\mathbf{f} = (\mathbf{f}^s, \mathbf{f}^s, q)$ located at x_s =1 m on the x-axis with time history

$$g(t) = \cos[2\pi f_0(t - 1.5/f_0)]\exp[-2f_0^2(t - 1.5/f_0)^2].$$

$f_0 = 200$ Hz: the dominant frequency. Displacements and temperature are recorded at 60 m from the source.
The experiments analyzed the coupled and uncoupled Cases considering the coupling coefficients β and β_f nonzero or null.

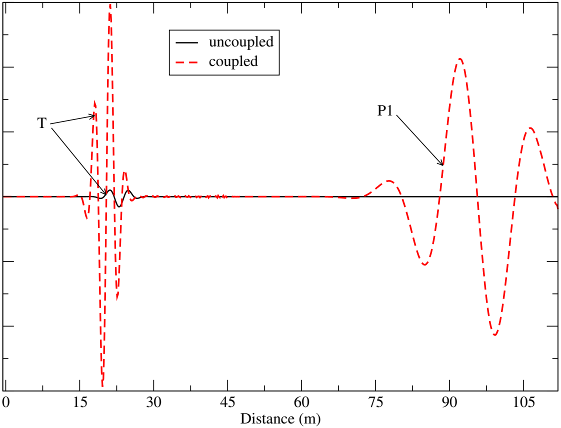

Figure 4. Snapshot of the temperature field at 48 ms. Uncoupled and coupled cases, non-zero viscosity. A P1 arrival is seen due to the coupling effect.

Numerical experiments. Uniform material as in Table 1, non-zero viscosity. III

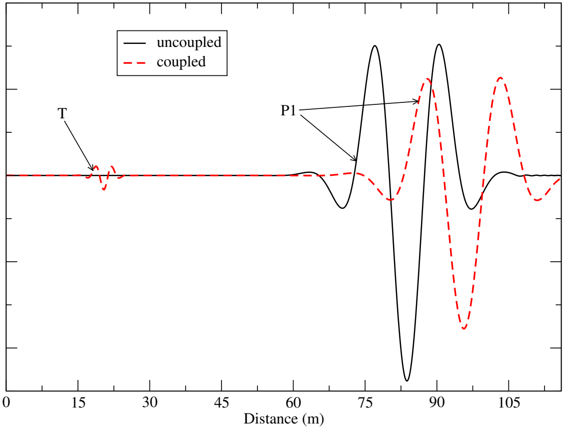

Figure 5. Snapshot of the particle displacement of the frame at 48 ms for the uncoupled and coupled cases with non-zero viscosity. P1 waves travel faster in the coupled case.

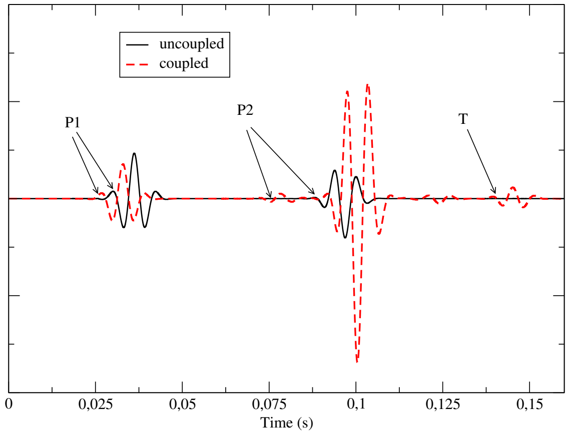

Figure 6. Time history of the particle displacement of the frame, null viscosity, uncoupled and coupled cases. P1 and P2 waves arrive earlier in the coupled case as compared with the uncoupled one.

Numerical experiments. Uniform material as in Table 1, non-zero viscosity. V

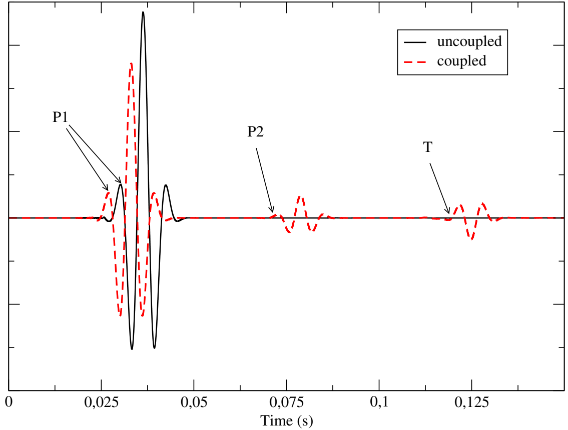

Figure 7. Time history of the particle displacement of the frame, non-zero viscosity, uncoupled and coupled cases. The P1 wave arrives earlier in the coupled case as compared with the uncoupled one. A P2 arrival is also observed in the coupled case, not present in the uncoupled one (black curve) due to its diffusive behavior.

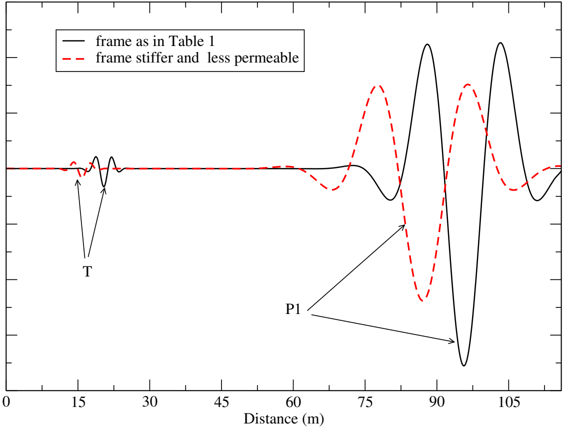

Figure 8. Frame snapshots for the uniform material in Table 1 and a stiffer, less permeable uniform material, with $K_m = 5.1$ GPa, $\mu = 5.565$ GPa, $\kappa = 0.5$ Darcy, the other properties as in Table 1. Coupled case, non-zero viscosity.

The last experiment compares time histories of the uniform medium in Table 1 with those of an inhomogeneous medium consisting of two intervals $I_1 = (0, l)$ and $I_2 = (l, L)$ with $l = 38$ m, $L = 116$ m, with different material properties in I_1 and I_2 .

In I_1 the material properties are those in Table 1, while in I_2 the material is stiffer and less permeable with $K_m = 5.1$ GPa, $\mu = 5.565$ GPa, $\kappa = 0.5$ Darcy and the other properties as in Table 1.

Time histories recorded inside the stiffer medium at 84 m from the source after crossing the interface l are compared with those corresponding to the uniform medium. The example analyzes the Coupled case for non-zero viscosity.

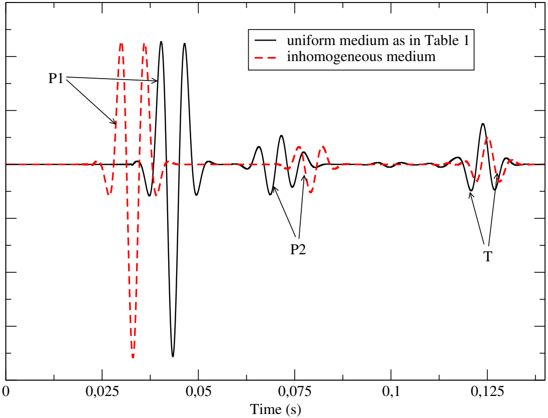

Figure 9. Time history of the particle displacement of the frame at the receiver inside the stiffer medium.
P1 waves arrives earlier and P2 and T waves later in the interface case as compared with the uniform material.

CONCLUSIONS

- **We presented a FEM to solve an IBVP in a non-isothermal poroelastic medium.**
- **In this type of media four waves can propagate, two compressisonal waves (P1, fast, and P2, diffusive, slow), a Thermal (diffusive) T wave and a Shear (S) wave.**
- **We analyzed the behavior of all waves, in particular the effect of coupling between the Thermal wave with P1 and P2 waves.**
- **The inclusion of thermal effects provides a tool to study attenuation and dispersion effects including mesoscopic loss effects related to Thermal waves.**
- **Thanks for your attention !!!!!.**