Cohomological VC-density: upper bounds and applications

ASL Session on Combinatorics and Logic,
2025 ASL North American meeting,
New Mexico State University,
Las Cruces, NM
May 14, 2025

Saugata Basu
(Joint work with Deepam Patel)

available at arxiv.org/abs/2411.09670

Department of Mathematics
Purdue University, West Lafayette, IN, USA

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Classical VC-density

Definition of VC-density

Let Y be a set and $\mathcal{X} \subset 2^Y$ a set of subsets of Y. For $Y' \subset Y$, we set

$$S(Y'; \mathcal{X}) = \{Y' \cap \mathbf{X} \mid \mathbf{X} \in \mathcal{X}\}.$$

We use $Y' \subset_n Y$ to denote $Y' \subset Y$ and $\mathrm{card}(Y') = n$.

We denote

$$\nu_{\mathcal{X}}(n) := \max_{Y' \subset_n Y} \mathrm{card}(S(Y'; \mathcal{X})).$$

(Sauer-Shelah lemma) Either for all $n > 0$, $\nu_{\mathcal{X}}(n) = 2^n$, or there exists $c, d > 0$ such that $\nu_{\mathcal{X}}(n) < c \cdot n^d$, for all $n > 0$ (here c, d are independent of n).

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NIP structures and bound on the VC-density

In model theory one only considers definable families $\mathcal{X}$ of subsets of some definable set Y.

Suppose that X, Y are definable subsets in some model $\mathbf{M}$ of some theory $\mathbf{T}$, and $H \subset X \times Y$ a definable subset.

We denote

$$\mathcal{X} := \{H_x \mid x \in X\},\ \mathcal{Y} := \{H_y \mid y \in Y\}$$

where

$$H_x = \pi_Y(\pi_X^{-1}(x) \cap H), H_y = \pi_X(\pi_Y^{-1}(y) \cap H),$$

and $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are the projection maps.

We call $\mathrm{vcd}_{\mathcal{X}}$ the *VC-codensity* of $\mathcal{Y}$.

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Upper bounds on the VC-codensity

For a large class of NIP theories (ACF, ACVF, RCF, and o-minimal theories) with a good notion of dimension:

$$\mathrm{vcd}_{\mathcal{X}} \leq \dim X. \tag{1.1}$$

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0/1-patterns

In order to bound $\mathrm{vcd}(\mathcal{X})$, it is often useful to consider the dual family $\mathcal{Y}$ of subsets of X, and bound the number of realizable 0/1-patterns for any n members of this family.

Given any set S, and $\mathcal{F} \subset_n 2^S$, a 0/1-pattern on $\mathcal{F}$ is an element of $\{0,1\}^{\mathcal{F}}$. We say that a 0/1-pattern $\sigma \in \{0,1\}^{\mathcal{F}}$ is realizable if and only if $\exists s \in S, \chi_F(s) = \sigma(F)$ for all $F \in \mathcal{F}$, where we denote by χ_F the characteristic function of F. We will denote by $\mathcal{R}(\sigma) \subset S$ the realization of σ i.e. $\mathcal{R}(\sigma) = \{s \in X \mid \chi_F(s) = \sigma(F), F \in \mathcal{F}\}$.

Denote by $\hat{\nu}_{\mathcal{Y}}(n)$ the maximum number of realizable 0/1-patterns where the maximum is taken over all finite subsets $\mathcal{Y}' \subset_n \mathcal{Y}$.

Then, $\nu_{\mathcal{X}}(n) = \hat{\nu}_{\mathcal{Y}}(n)$. In particular, the VC-codensity of $\mathcal{Y}$ (or equivalently $\mathrm{vcd}_{\mathcal{X}} = \limsup_n \frac{\log \hat{\nu}_{\mathcal{Y}}(n)}{\log n}$.

Inequality (1.1) (and its dual version which gives bounds on realizable 0/1-patterns) play a fundamental role in many applications in combinatorics, discrete geometry as well as in theoretical computer science.

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Homological underpinnings

Often bounds on the number of realizable 0/1-patterns and consequently on VC-codensity are best proved using (algebraic) topological methods.

For example, for o-minimal structures (following the notation of previous page) – assuming H is closed and bounded (for convenience):

$$\begin{aligned}
\hat{\nu}_{\mathcal{Y}}(n) &\leq \max_{\mathcal{Y}' \subset_n \mathcal{Y}} \operatorname{card}(\{\sigma \in \{0,1\}^{\mathcal{Y}'} \mid \mathcal{R}(\sigma) \neq \emptyset\}) \\
&\leq \max_{\mathcal{Y}' \subset_n \mathcal{Y}} \sum_{\sigma \in \{0,1\}^{\mathcal{Y}'}} b_0(\mathcal{R}(\sigma)) \\
&\leq C_H \cdot n^{\dim X}.
\end{aligned}$$

It follows immediately that the VC-codensity of $\mathcal{Y}$ is bounded by $\dim X$.

In fact more is true – for any $p, 0 \leq p \leq \dim X$:

$$\max_{\mathcal{Y}' \subset_n \mathcal{Y}} \sum_{\sigma \in \{0,1\}^{\mathcal{Y}'}} b_p(\mathcal{R}(\sigma)) \leq C_H \cdot n^{\dim X - p}.$$

Is there a VC-density implication for this result ? I don't know ... but clearly the higher degree homology groups have a role.

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From points to curves (or higher dimensional definable sets)

With an eye towards topology we want to assign meaning to the "non-shattering function" $\nu_{\mathcal{X}}$ in the case of finite sets of definable sets (from some fixed definable family) instead of points. Will be more precise later ... but here is a picture to keep in mind.

In discrete geometry going from "arrangements" of finite sets of points to higher dimensional sets or varieties is a very natural thing to do with many precedents.

Generalizing Helly's theorem for convex sets (which can be thought of as as a theorem about point transversals) to higher dimensional transversals (Hadwiger (1957), Goodman and Pollack (1988)) is a generalization of this type.

Another example is Guth's generalization (2015) of the Guth-Katz polynomial partitioning theorem (2014) (which is about partitioning finite sets of points) to partitioning of varieties (of possibly positive dimension).

Many other examples ...

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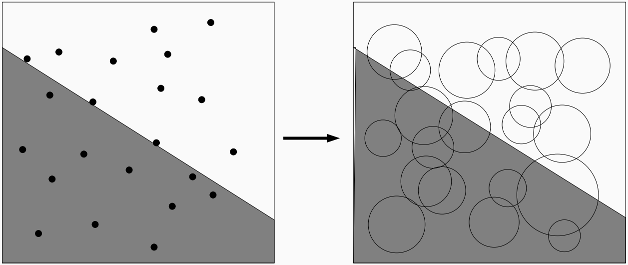

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Cohomological VC-density

Preliminary

How to interpret $\mathrm{card}(S(Y';\mathcal{X}))$ in the setting where elements of Y' are allowed to be definable subsets of Y ?

Suppose now that Y is a topological space and $\mathcal{X}$ is a set of subspaces of Y.

Given any finite subset $Y' \subset Y$, the different intersections $Y' \cap \mathbf{X}, \mathbf{X} \in \mathcal{X}$ are each characterized by the image of the linear map

$$\mathrm{H}_0(Y' \cap \mathbf{X}, \mathbb{Q}) \to \mathrm{H}_0(Y', \mathbb{Q}).$$

Then, for any $Y' \subset_n Y$, and $\mathbf{X} \in \mathcal{X}$, $Y' \cap \mathbf{X}$ is determined by the kernel of the homomorphism $\mathrm{H}^0(Y', \mathbb{Q}) \to \mathrm{H}^0(Y' \cap \mathbf{X}, \mathbb{Q})$. With notation as above, let

$$\mathbb{S}^0(Y';\mathcal{X}) = \{\ker(\mathrm{H}^0(Y', \mathbb{Q}) \to \mathrm{H}^0(Y' \cap \mathbf{X}, \mathbb{Q})) \subset \mathrm{H}^0(Y', \mathbb{Q}) \mid \mathbf{X} \in \mathcal{X}\}.$$

Observe:

$$\mathrm{card}(S(Y';\mathcal{X})) = \mathrm{card}(\mathbb{S}^0(Y';\mathcal{X})).$$

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How to interpret $\mathrm{card}(S(Y';\mathcal{X}))$ in the setting where elements of Y' are allowed to be definable subsets of Y ?

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Higher degree VC-density

The last observation that is at the heart of our generalization of VC-density to higher (cohomological) degrees.

Let $\mathcal{Z} \subset 2^Y$ be another set of subspaces of Y, $\mathcal{Z}_0 \subset_n \mathcal{Z}$, and let $\bigcup \mathcal{Z}_0$ denote $\bigcup_{\mathbf{Z} \in \mathcal{Z}_0} \mathbf{Z}$.

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In all cases mentioned before (using appropriate topology and cohomology theory – euclidean (for RCF, ACF_0, o-minimal expansions of $\mathbb{R}$, étale for ACF_p etc.)

$$\mathrm{card}(\mathbb{S}^p(\mathcal{Z}_0; \mathcal{X})) < \infty,$$

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The finiteness of $\mathbb{S}^p(\mathcal{Z}_0; \mathcal{X})$ motivates the following definition – that of the degree-p VC-density of the pair $(\mathcal{X}, \mathcal{Z})$:

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Upper bound on degree p VC-density

Theorem 1

For every pair of definable families $(\mathcal{X}, \mathcal{Z})$ of proper definable subsets of some definable set Y, in an algebraically closed field or in an o-minimal structure and $p \geq 0$,

$$\mathrm{vcd}^{p}_{\mathcal{X},\mathcal{Z}} \leq (p+1) \cdot \dim X.$$

Inequality (1.1) (in the theories mentioned above) is then recovered as a special case with $p = 0$ and $\mathcal{Z} = \{\{y\} \mid y \in Y\}$.

Note that our notion of $\mathrm{vcd}^{0}_{\mathcal{X},\mathcal{Z}}$ is in fact more general than the classical VC-density $\mathrm{vcd}_{\mathcal{X}}$, since we allow $\mathcal{Z}$ to be a more general definable family, rather than just Y itself.

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Higher order cohomological VC-density

Higher order dependence

Shelah (2014) generalized the notion of the independence-property of formulas to higher orders.

More precisely, a formula $\phi(X; Y^{(0)}, \ldots, Y^{(q-1)})$ is said to be *q-independent* if for every $n > 0$, there exists $Z^{(i)} \subset_n \in \mathbf{M}^{|Y^{(i)}|}, 0 \leq i \leq q-1$, such that for every subset $S \subset Z^{(0)} \times \cdots \times Z^{(q-1)}$, there exists $x_S \in \mathbf{M}^{|X|}$, such that for every $(y_0, \ldots, y_{q-1}) \in Z^{(0)} \times \cdots \times Z^{(q-1)}$ $\mathbf{M} \models \phi(x_S, y_0, \ldots, y_{q-1})$ if and only $(y_0, \ldots, y_{q-1}) \in S$.

A theory $\mathbf{T}$ has the property NIP_q if every formula is not q-independent.

It is obvious that the property NIP_q implies NIP_{q+1}, and the property $\mathrm{NIP}_1 = \mathrm{NIP}$.

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Let $Y^{(0)}, \ldots, Y^{(q-1)}$ be sets and for each $i, 0 \leq i \leq q-1$, $\mathcal{X}^{(i)} \subset 2^{Y^{(i)}}$, and let $\bar{\mathcal{X}} = (\mathcal{X}^{(0)}, \ldots, \mathcal{X}^{(q-1)})$. For any tuples of subsets $\bar{Y}' = (Y^{(0)'}, \ldots, Y^{(q-1)'})$ where $Y^{(i)'} \subset Y^{(i)}$, we set

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where the intersection is defined component-wise.

Denote

$$\nu_{\bar{\mathcal{X}},q}(n) = \max_{\bar{Y}' \subset_n \bar{Y}} \mathrm{card}(\bar{S}(\bar{Y}'; \bar{\mathcal{X}})),$$

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Fix $q \geq 1$. Suppose now that $Y^{(0)}, \ldots, Y^{(q-1)}$ are topological spaces, $\mathcal{X}$ a set of subspaces of $Y^{(0)} \times \cdots \times Y^{(q-1)}$, and for each $i, 0 \leq i \leq q-1$, $\mathcal{Z}^{(i)} \subset 2^{Y^{(i)}}$ be a set of subspaces of $Y^{(i)}$. We denote $\bar{\mathcal{Z}} = (\mathcal{Z}^{(0)}, \ldots, \mathcal{Z}^{(q-1)})$. Let for each $i, 0 \leq i \leq q-1$ $\mathcal{Z}_0^{(i)} \subset_n \mathcal{Z}^{(i)}$, and let $\bigcup \bar{\mathcal{Z}}_0$ denote $\prod_{0 \leq i \leq q-1} \bigcup_{\mathbf{Z}^{(i)} \in \mathcal{Z}_0^{(i)}} \mathbf{Z}^{(i)}$.

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$$\nu^{p,q}_{\mathcal{X},\bar{\mathcal{Z}}}(n) = \sup_{\mathcal{Z}_0^{(i)} \subset_n \mathcal{Z}^{(i)}, 0 \leq i \leq q-1} \mathrm{card}(\mathbb{S}^p(\bar{\mathcal{Z}}_0; \mathcal{X})).$$

Higher degree, higher order VC-density

Our higher degree notion of VC-density generalizes to the setting of higher order VC-density as well.

Fix $q \geq 1$. Suppose now that $Y^{(0)}, \ldots, Y^{(q-1)}$ are topological spaces, $\mathcal{X}$ a set of subspaces of $Y^{(0)} \times \cdots \times Y^{(q-1)}$, and for each $i, 0 \leq i \leq q-1$, $\mathcal{Z}^{(i)} \subset 2^{Y^{(i)}}$ be a set of subspaces of $Y^{(i)}$. We denote $\bar{\mathcal{Z}} = (\mathcal{Z}^{(0)}, \ldots, \mathcal{Z}^{(q-1)})$. Let for each $i, 0 \leq i \leq q-1$ $\mathcal{Z}_0^{(i)} \subset_n \mathcal{Z}^{(i)}$, and let $\bigcup \bar{\mathcal{Z}}_0$ denote $\prod_{0 \leq i \leq q-1} \bigcup_{\mathbf{Z}^{(i)} \in \mathcal{Z}_0^{(i)}} \mathbf{Z}^{(i)}$.

For each $p \geq 0$, define

$$\mathbb{S}^p(\bar{\mathcal{Z}}_0; \mathcal{X}) = \{\ker(\mathrm{H}^p(\bigcup \bar{\mathcal{Z}}_0, \mathbb{Q}) \to \mathrm{H}^p(\bigcup \bar{\mathcal{Z}}_0 \cap \mathbf{X}, \mathbb{Q})) \mid \mathbf{X} \in \mathcal{X}\}.$$

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Bound on higher order, higher degree VC-density

We have following generalization of Theorem 1.

Theorem 2

For every $q \geq 1$, and every tuple of definable families $(\mathcal{X}, \mathcal{Z}^{(0)}, \ldots, \mathcal{Z}^{(q-1)})$ of proper definable subsets of some definable set Y, in any algebraically closed field or o-minimal structure, and $p \geq 0$,

$$\mathrm{vcd}^{p;q}_{\mathcal{X},\bar{\mathcal{Z}}} \leq (p+q) \cdot \dim X,$$

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Remarks

Notice that

$$\mathrm{vcd}^{p,1}_{\mathcal{X},\bar{\mathcal{Z}}} = \mathrm{vcd}^{p}_{\mathcal{X},\mathcal{Z}^{(0)}}.$$

Also it is immediate by taking $Y = Y^{(0)} \times \cdots \times Y^{(q-1)}$, and $\mathcal{Z} = \mathcal{Z}^{(0)} \times \cdots \times \mathcal{Z}^{(q-1)}$, and applying Theorem 1, that

$$\mathrm{vcd}^{p,q}_{\mathcal{X},\bar{\mathcal{Z}}} \leq q(p+1) \cdot \dim X.$$

Thus, if $p = 0$ or $q = 1$, Theorem 2 follows immediately from Theorem 1. In every other case, the bound in Theorem 2 is stronger than the one obtained by applying Theorem 1.

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Remarks (cont)

Note that the generalized VC-density $\mathrm{vcd}^{p;q}_{\mathcal{X},\bar{\mathcal{Z}}}(n)$ measures the 'complexity' of the definable family $\mathcal{X}$ against collections of n^q subsets of $Y^{(0)} \times \cdots \times Y^{(q-1)}$ of the special (product) form $\mathcal{Z}_0^{(0)} \times \cdots \times \mathcal{Z}_0^{(q-1)}$, where each $\mathcal{Z}_0^{(i)} \subset_n \mathcal{Z}^{(i)}$ (rather than against arbitrary subsets of size n^q of $\mathcal{Z}^{(0)} \times \cdots \times \mathcal{Z}^{(q-1)}$).

Since $p + q = q(p+1)$ whenever $p = 0$ or $q = 1$, the difference between these two classes of 'test' families of finite subsets, (i.e. finite sets of cardinality n^q of the special form $\mathcal{Z}_0^{(0)} \times \cdots \times \mathcal{Z}_0^{(q-1)}$, as opposed to general subsets of $\mathcal{Z}^{(0)} \times \cdots \times \mathcal{Z}^{(q-1)}$ having cardinality n^q) is reflected in our bound (Theorem 2) only for $p > 0$ (for $q > 1$).

It follows that for $p > 0$ and $q > 1$, $\mathrm{vcd}^{p;q}_{\mathcal{X},\bar{\mathcal{Z}}}$ are sensitive to the product structure of finite sets in a way that $\mathrm{vcd}^{q}_{\mathcal{X}}, q > 1$, are not.

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A few words about the proofs

- Uses theorems giving constructibility of certain sheaves (push forwards of constant sheaves) in the various categories considered.
- Cohomological descent arguments and the Mayer-Vietoris spectral sequence.
- Uses knowledge of inequality (1.1) (bound on the number of 0/1 patterns) in the various categories considered.

Applications and ongoing/future work

Applications

Higher degree analogs of well-known applications of VC-density.

Interpret $X \in_p Y$ as $\mathrm{Im}(\mathrm{H}_p(X) \to \mathrm{H}_p(Y)) \neq 0$ (or alternatively $\mathrm{H}_p(X) \to \mathrm{H}_p(Y)$ is injective).

Then, with certain restrictions on the families over real closed fields we can prove ...

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Some ongoing/future work

Topological distality and upper bounds on higher degree distal density.

Topological version of stability and bounds on higher degree versions of Morley rank that we define.

Topologically formulated (Szemeredi-Trotter style) incidence questions. For example, where ‘incidence’ of a curve on a surface might mean that the curve is not contractible inside the surface.

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Thank You!