Bi-Lipschitz classification of surface germs

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(joint work with Lev Birbrair, Alexandre Fernandes)

Let X be a germ at the origin of $\mathbf{R^n}$ of a two-dimensional surface definable in a polynomially bounded o-minimal structure over $\mathbf{R}$. The outer metric on X, such that $\operatorname{dist}(x, y) = |y - x|$, is induced from $\mathbf{R^n}$. Two such surface germs, X and Y, are bi-Lipschitz equivalent if there exists a homeomorphism $h : (X, 0) \to (Y, 0)$ such that the outer metric on X is equivalent to the metric on X induced from Y. The goal is to find a discrete (no moduli in definable families) invariant of bi-Lipschitz equivalence class of definable surface germs. We start with a simpler problem of bi-Lipschitz classification of germs at the origin of definable functions $f : \mathbf{R^2} \to \mathbf{R}$ up to contact equivalence. Two such function germs f and g are bi-Lipschitz contact equivalent if there exists a bi-Lipschitz homeomorphism $h : (\mathbf{R^3}, \mathbf{0}) \to (\mathbf{R^3}, \mathbf{0})$ commuting with the projection $\mathbf{R^3} \to \mathbf{R^2}$ and mapping the graph of f to the graph of g. The invariant consists of a partition of the neighborhood of the origin of $\mathbf{R^2}$ into Hölder triangles T_j (defined up to bi-Lipschitz equivalence and uniquely determined by the cyclically ordered sequence of the exponents β_j of T_j), and for each j the sign $s_j \in \{+, -, 0\}$ of $f|_{T_j}$, the interval Q_j of the exponents q of f on the arcs $\gamma \subset T_j$, and the affine width function $\mu_j(q) = a_j q + c_j$ on Q_j that measures how much an arc $\gamma \subset T_j$ may be deformed so that the exponent q of $f|_\gamma$ does not change. Such a partition is called a “pizza” with “slices” T_j and “toppings” β_j, s_j, Q_j, $\mu_j(q)$. These data (with some additional constraints due to continuity of f constitute a complete discrete invariant of the bi-Lipschitz contact equivalence class of function-germs in $\mathbf{R^2}$. For a germ X of a definable two-dimensional surface, a similar (but more complicated combinatorially) canonical (up to bi-Lipschitz equivalence) partition into normally embedded Holder triangles T_j can be defined, so that any two triangles of the partition are either “transversal” (not tangent to each other) or “coherent” (bi-Lipschitz equivalent to a slice of a pizza and a graph of a definable function over that slice). The discrete invariant of the bi-Lipschitz equivalence class of X consists in the combinatorial structure of the partition, the tangency orders q_{jk} between the boundary arcs of any two triangles T_j and T_k of the partition, and the toppings β_{jk}, Q_{jk}, $\mu_{jk}(q)$ associated with each pair of coherent triangles T_j and T_k (work in progress).