Computing the First Few Betti Numbers of Semi-algebraic Sets in Single Exponential Time

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Abstract

For every fixed $\ell > 0$, we describe a singly exponential algorithm for computing the first ℓ Betti number of a given semi-algebraic set. More precisely, we describe an algorithm that given a semi-algebraic set $S \subset \mathrm{R}^{\mathrm{k}}$ a semi-algebraic set defined by a Boolean formula with atoms of the form $P > 0, P < 0, P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{k}}]$, computes $b_0(S), \ldots, b_\ell(S)$. The complexity of the algorithm is $(sd)^{k^{O(\ell)}}$, where where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. Previously, singly exponential time algorithms were known only for computing the Euler-Poincaré characteristic, the zero-th and the first Betti numbers.

Key words: Semi-algebraic sets, Bett numbers, single exponential complexity

1 Introduction

Let R be a real closed field and $S \subset \mathrm{R}^{\mathrm{k}}$ a semi-algebraic set defined by a Boolean formula with atoms of the form $P > 0, P < 0, P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{k}}]$ (we call such a set a $\mathcal{P}$-semi-algebraic set). It is well known (18; 19; 17; 22; 1) that the topological complexity of S (measured by the various Betti numbers of S) is bounded by $O(sd)^k$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. Note that these bounds are singly exponential in k. More precise bounds on the individual Betti numbers of S appear in (2). Even though the Betti numbers of S are bounded singly exponentially in k, there

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is no known algorithm for producing a singly exponential sized triangulation of S (which would immediately imply a singly exponential algorithm for computing the Betti numbers of S). In fact, designing a singly exponential time algorithm for computing the Betti numbers of semi-algebraic sets is one of the outstanding open problems in algorithmic semi-algebraic geometry. More recently, determining the exact complexity of computing the Betti numbers of semi-algebraic sets has attracted the attention of computational complexity theorists (8), who are interested in developing a theory of counting complexity classes for the Blum-Shub-Smale model of real Turing machines.

Doubly exponential algorithms (with complexity $(sd)^{2^{O(k)}}$) for computing all the Betti numbers are known, since it is possible to obtain a triangulation of S in doubly exponential time using cylindrical algebraic decomposition (10; 5). In the absence of singly exponential time algorithms for computing triangulations of semi-algebraic sets, algorithms with single exponential complexity are known only for the problems of testing emptiness (20; 3), computing the zero-th Betti number (i.e. the number of semi-algebraically connected components of S) (13; 9; 12; 4), as well as the Euler-Poincaré characteristic of S (1). Very recently a singly exponential time algorithm has been developed for the problem of computing the first Betti number of a given semi-algebraic set (6).

In this paper we describe, for each fixed number $\ell > 0$, a singly exponential algorithm for computing the first ℓ Betti numbers of a given semi-algebraic set $S \subset \mathrm{R}^{\mathrm{k}}$. We remark that using Alexander duality, we immediately get a singly exponential algorithm for computing the top ℓ Betti numbers too. However, the complexity of our algorithm becomes doubly exponential if we want to compute the middle Betti numbers of a semi-algebraic set using it.

There are two main ingredients in our algorithm for computing the first ℓ Betti numbers of a given closed semi-algebraic set. The first ingredient is a result proved in (6), which enables us to compute a singly exponential sized covering of the given semi-algebraic set consisting of closed, acyclic semi-algebraic sets, in single exponential time. (A closed bounded semi-algebraic set X is acyclic if its cohomology groups, $H^i(X, \mathbb{Q})$ is 0 for all $i > 0$ and $H^0(X, \mathbb{Q}) = \mathbb{Q}$.) The number and the degrees of the polynomials used to define the sets in this covering are also bounded singly exponentially.

The second ingredient, which is the main contribution of this paper, is an algorithm which uses the covering algorithm recursively and computes in singly exponential time a complex whose homology groups are isomorphic to the first ℓ homology groups of the input set. This complex is of singly exponential size.

The main result of the paper is the following.
Main Result: For any given ℓ, there is an algorithm that takes as input a description of a $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^{\mathrm{k}}$, and outputs $b_0(S), \ldots, b_\ell(S)$.

The complexity of the algorithm is $(sd)^{k^{O(\ell)}}$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. Note that the complexity is singly exponential in k for every fixed ℓ.

The paper is organized as follows. In Section 2, we recall some basic definitions from algebraic topology and fix notations. In Section 3 we describe the construction of the complexes which allows us to compute the the first ℓ Betti numbers of a given semi-algebraic set. In Section 4 we recall the inputs, outputs and complexities of a few algorithms described in detail in (6), which we use in our algorithm. Finally, in Section 5 we describe our algorithm for computing the first ℓ Betti numbers, prove its correctness as well as the complexity bounds.

2 Mathematical Preliminaries

In this section, we recall some basic facts about semi-algebraic sets as well as the definitions of complexes and double complexes of vector spaces, and fix some notations.

2.1 Semi-algebraic sets and their homology groups

Let R be a real closed field. If $\mathcal{P}$ is a finite subset of $\mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}]$, we write the set of zeros of $\mathcal{P}$ in R^k as

$$\mathrm{Z}(\mathcal{P}, \mathrm{R}^\mathrm{k}) = \{\mathrm{x} \in \mathrm{R}^\mathrm{k} \mid \bigwedge_{\mathrm{P} \in \mathcal{P}} \mathrm{P}(\mathrm{x}) = 0\}.$$

We denote by $B(0, r)$ the open ball with center 0 and radius r.

Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}]$, $Z = \mathrm{Z}(\mathcal{Q}, \mathrm{R}^\mathrm{k})$, and $Z_r = Z \cap B(0, r)$. A *sign condition* on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization of the sign condition* σ *over* Z, $\mathcal{R}(\sigma, Z)$, is the basic semi-algebraic set

$$\{x \in \mathrm{R}^\mathrm{k} \mid \bigwedge_{\mathrm{Q} \in \mathcal{Q}} \mathrm{Q}(\mathrm{x}) = 0 \wedge \bigwedge_{\mathrm{P} \in \mathcal{P}} \mathrm{sign}(\mathrm{P}(\mathrm{x})) = \sigma(\mathrm{P})\}.$$

The *realization of the sign condition* σ *over* Z_r, $\mathcal{R}(\sigma, Z_r)$, is the basic semi-algebraic set $\mathcal{R}(\sigma, Z) \cap B(0, r)$. For the rest of the paper, we fix an open ball $B(0, r)$ with center 0 and radius r big enough so that, for every sign condition σ, $\mathcal{R}(\sigma, Z)$ and $\mathcal{R}(\sigma, Z_r)$ are homeomorphic. This is always possible by the local conical structure at infinity of semi-algebraic sets ((7), page 225).

A closed and bounded semi-algebraic set $S \subset \mathrm{R}^{\mathrm{k}}$ is semi-algebraically triangulable (see (5)), and we denote by $H_i(S)$ the i-th simplicial homology group of S with rational coefficients. The groups $H_i(S)$ are invariant under semi-algebraic homeomorphisms and coincide with the corresponding singular homology groups when $\mathrm{R} = \mathbb{R}$. We denote by $b_i(S)$ the i-th Betti number of S (that is, the dimension of $H_i(S)$ as a vector space), and $b(S)$ the sum $\sum_i b_i(S)$. For a closed but not necessarily bounded semi-algebraic set $S \subset \mathrm{R}^{\mathrm{k}}$, we will denote by $H_i(S)$ the i-th simplicial homology group of $S \cap \overline{B(0,r)}$, where r is sufficiently large. The sets $S \cap \overline{B(0,r)}$ are semi-algebraically homeomorphic for all sufficiently large $r > 0$, by the local conical structure at infinity of semi-algebraic sets, and hence this definition makes sense.

The definition of homology groups of arbitrary semi-algebraic sets in R^{k} requires some care and several possibilities exist. In this paper, we define the homology groups of realizations of sign conditions as follows.

Let R denote a real closed field and R' a real closed field containing R. Given a semi-algebraic set S in R^k, the *extension* of S to R', denoted $\mathrm{Ext}(S, \mathrm{R}')$, is the semi-algebraic subset of R'^k defined by the same quantifier free formula that defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle (5).

Now, let $S \subset \mathrm{R}^{\mathrm{k}}$ be a $\mathcal{P}$-semialgebraic set, where $\mathcal{P} = \{P_1, \ldots, P_s\}$ is a finite subset of $\mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{k}}]$. Let $\phi(X)$ be a quantifier-free formula defining S. Let $P_i = \sum_\alpha a_{i,\alpha} X^\alpha$ where the $a_{i,\alpha} \in \mathrm{R}$. Let $A = (\ldots, A_{i,\alpha}, \ldots)$ denote the vector of variables corresponding to the coefficients of the polynomials in the family $\mathcal{P}$, and let $a = (\ldots, a_{i,\alpha}, \ldots) \in \mathrm{R}^{\mathrm{N}}$ denote the vector of the actual coefficients of the polynomials in $\mathcal{P}$. Let $\psi(A, X)$ denote the formula obtained from $\phi(X)$ by replacing each coefficient of each polynomial in $\mathcal{P}$ by the corresponding variable, so that $\phi(X) = \psi(a, X)$. It follows from Hardt's triviality theorem for semi-algebraic mappings (14), that there exists, $a' \in \mathbb{R}_{\mathrm{alg}}^N$ such that denoting by $S' \subset \mathbb{R}_{\mathrm{alg}}^k$ the semi-algebraic set defined by $\psi(a', X)$, the semi-algebraic set $\mathrm{Ext}(S', \mathrm{R})$, has the same homeomorphism type as S. We define the homology groups of S to be the singular homology groups of $\mathrm{Ext}(S', \mathbb{R})$. It follows from the Tarski-Seidenberg transfer principle, and the corresponding property of singular homology groups, that the homology groups defined this way are invariant under semi-algebraic homotopies. It is also clear that this definition is compatible with the simplicial homology for closed, bounded semi-algebraic sets, and the singular homology groups when the ground field is $\mathbb{R}$. Finally it is also clear that, the Betti numbers are not changed after extension: $b_i(S) = b_i(\mathrm{Ext}(S, \mathrm{R}'))$.

2.2 Complex of Vector Spaces

A sequence $\{C^p\}$, $p \in \mathbb{Z}$, of $\mathbb{Q}$-vector spaces together with a sequence $\{\delta^p\}$ of homomorphisms $\delta^p : C^p \to C^{p+1}$ for which $\delta^{p-1} \circ \delta^p = 0$ for all p is called a complex.

Ths homology groups, $H^p(C^\bullet)$ are defined by,

$$H^p(C^\bullet) = Z^p(C)/B^p(C),$$

where $B^p(C^\bullet) = \mathrm{Im}(\delta^{p-1})$, and $Z^p(C^\bullet) = \mathrm{Ker}(\delta^p)$.

The homology groups, $H^*(C^\bullet)$, are all $\mathbb{Q}$-vector spaces (finite dimensional if the vector spaces C^p's are themselves finite dimensional). We will henceforth omit reference to the field of coefficients $\mathbb{Q}$ which is fixed throughout the rest of the paper.

Given two complexes, $C^\bullet = (C^p, \delta^p)$ and $D^\bullet = (D^p, \delta^p)$, a homomorphism of complexes, $\phi : C^\bullet \to D^\bullet$, is a sequence of homomorphisms $\phi^p : C^p \to D^p$ for which $\delta^p \circ \phi^p = \phi^{p+1} \circ \delta^p$ for all p.

In other words, the following diagram is commutative.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C^p & \xrightarrow{\delta^p} & C^{p+1} & \longrightarrow & \cdots \\
 & & \Big\downarrow \phi^p & & \Big\downarrow \phi^{p-1} & & \\
\cdots & \longrightarrow & D^p & \xrightarrow{\delta^p} & D^{p+1} & \longrightarrow & \cdots
\end{array}$$

A homomorphism of complexes, $\phi : C^\bullet \to D^\bullet$, induces homorphisms, $\phi^* : H^*(C^\bullet) \to H^*(D^\bullet)$. The homomorphism ϕ is called a *quasi-isomorphism* if the homomorphism ϕ^* is an isomorphism.

2.3 Double Complexes

In this section, we recall the basic notions of a double complex of vector spaces and associated spectral sequences. A *double complex* is a bi-graded vector space,

$$C^{\bullet,\bullet} = \bigoplus_{p,q \in \mathbb{Z}} C^{p,q},$$

with co-boundary operators $d : C^{p,q} \to C^{p,q+1}$ and $\delta : C^{p,q} \to C^{p+1,q}$ and such that $d\delta + \delta d = 0$. We say that $C^{\bullet,\bullet}$ is a first quadrant double complex, if it satisfies the condition that $C^{p,q} = 0$ if either $p < 0$ or $q < 0$. Double complexes lying in other quadrants are defined in an analogous manner.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{0,2} \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,2} \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,2} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
C^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
C^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

The complex defined by

$$\mathrm{Tot}^n(C^{\bullet,\bullet}) = \bigoplus_{p+q=n} C^{p,q},$$

with differential

$$\mathrm{D}^n = d \pm \delta : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^{n+1}(C^{\bullet,\bullet}),$$

is denoted by $\mathrm{Tot}^\bullet(C^{\bullet,\bullet})$ and called the *associated total complex of* $C^{\bullet,\bullet}$.

$$
\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
{} \arrow[r, "\delta"] & C^{p-1,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
{} \arrow[r, "\delta"] & C^{p-1,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
{} \arrow[r, "\delta"] & C^{p-1,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
& \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}
$$

2.4 Spectral Sequences

A *spectral sequence* is a sequence of bigraded complexes $(E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})$ such that the complex E_{r+1} is obtained from E_r by taking its homology with respect to d_r (that is $E_{r+1} = H_{d_r}(E_r)$).

There are two spectral sequences, ${}'E_*^{p,q}, {}''E_*^{p,q}$, (corresponding to taking row-wise or column-wise filtrations respectively) associated with a first quadrant double complex $C^{\bullet,\bullet}$, which will be important for us. Both of these converge to $H^*(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}))$. This means that the homomorphisms, d_r are eventually zero, and hence the spectral sequences stabilize, and

$$\bigoplus_{p+q=i} {}'E_\infty^{p,q} \cong \bigoplus_{p+q=i} {}''E_\infty^{p,q} \cong H^i(\mathrm{Tot}^\bullet(C^{\bullet,\bullet})),$$

for each $i \geq 0$.

The first terms of these are ${}'E_1 = H_d(C^{\bullet,\bullet}), {}'E_2 = H_d H_\delta(C^{\bullet,\bullet})$, and ${}''E_1 = H_\delta(C^{\bullet,\bullet}), {}''E_2 = H_d H_\delta(C^{\bullet,\bullet})$.

Given two (first quadrant) double complexes, $C^{\bullet,\bullet}$ and $\bar{C}^{\bullet,\bullet}$, a homomorphism

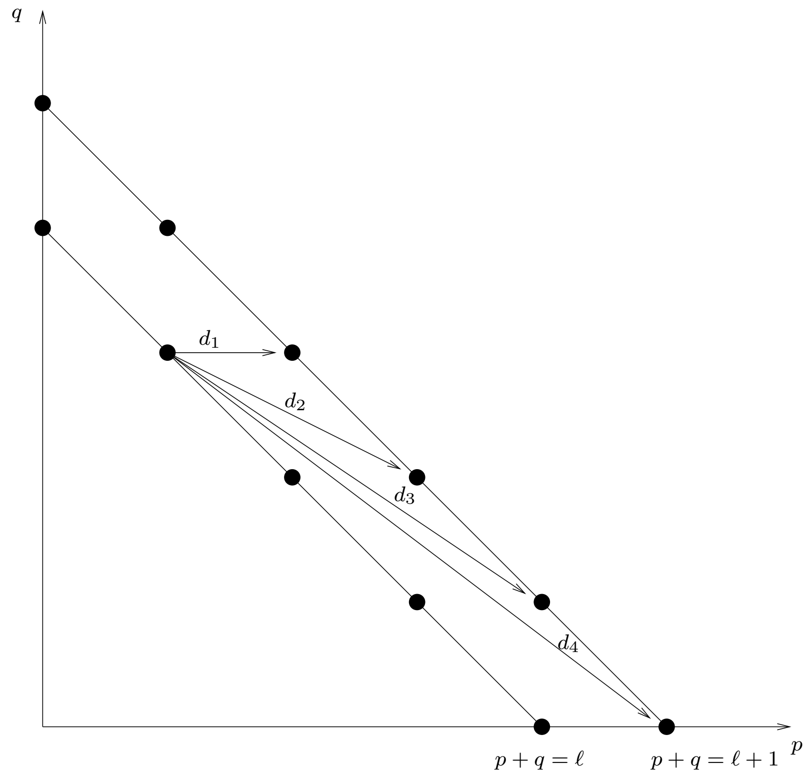

Fig. 1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$

of double complexes,

$$\phi : C^{\bullet,\bullet} \longrightarrow \bar{C}^{\bullet,\bullet},$$

is a collection of homomorphisms, $\phi^{p,q} : C^{p,q} \longrightarrow \bar{C}^{p,q}$, such that the following diagrams commute.

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{\delta} & C^{p+1,q} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p+1,q} \\ \bar{C}^{p,q} & \xrightarrow{\delta} & \bar{C}^{p+1,q} \end{array}$$

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d} & C^{p,q+1} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p,q+1} \\ \bar{C}^{p,q} & \xrightarrow{d} & \bar{C}^{p,q+1} \end{array}$$

A homomorphism of double complexes,

$$\phi : C^{\bullet,\bullet} \longrightarrow \bar{C}^{\bullet,\bullet},$$

induces homomorphisms, $'\phi_s : {}'E_s \longrightarrow {}'\bar{E}_s$ (respectively, $''\phi_s : {}''E_s \longrightarrow {}''\bar{E}_s$) between the associated spectral sequences (corresponding either to the row-wise or column-wise filtrations). For the precise definition of homomorphisms of spectral sequences, see (16). We will need the following useful fact (see (16), page 66, Theorem 3.4 for a proof).

Proposition 2.1 *If* $'\phi_s$ *(respectively,* $''\phi_s$*) is an isomorphism for some* $s \geq 1$*, then* $'E_r^{p,q}$ *and* $'\bar{E}_r^{p,q}$ *(repectively,* $''E_r^{p,q}$ *and* $''\bar{E}_r^{p,q}$ *) are isomorphic for all* $r \geq s$*. In particular, the induced homomorphism,*

$$\phi : \mathrm{Tot}^\bullet(C^{\bullet,\bullet}) \to \mathrm{Tot}^\bullet(\bar{C}^{\bullet,\bullet})$$

is a quasi-isomorphism.

2.5 The Mayer-Vietoris Double Complex

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Note that the intersections of any number of the sub-complexes, A_i, is again a sub-complex of A. We will denote by $A_{i_0,\ldots,i_p}$ the sub-complex $A_{i_0} \cap \cdots \cap A_{i_p}$.

Let $C^i(A)$ denote the $\mathbb{Q}$-vector space of i co-chains of A, and $C^\bullet(A)$, the complex

$$\cdots \to C^{q-1}(A) \xrightarrow{d} C^q(A) \xrightarrow{d} C^{q+1}(A) \to \cdots$$

where $d : C^q(A) \to C^{q+1}(A)$ are the usual co-boundary homomorphisms. More precisely, given $\omega \in C^q(A)$, and a $q+1$ simplex $[a_0, \ldots, a_{q+1}] \in A$,

$$d\omega([a_0, \ldots, a_{q+1}]) = \sum_{0 \leq i \leq q+1} (-1)^i \omega([a_0, \ldots, \hat{a}_i, \ldots, a_{q+1}]) \tag{1}$$

(here and everywhere else in the paper ˆ denotes omission). Now extend $d\omega$ to a linear form on all of $C_{q+1}(A)$ by linearity, to obtain an element of $C^{q+1}(A)$.

The connecting homomorphisms are "generalized" restrictions and will be defined below.

The *generalized Mayer-Vietoris sequence* is the following exact sequence of vector spaces. (Here and everywhere else in the paper $\oplus$ denotes the direct sum of vector spaces).

$$0 \longrightarrow C^\bullet(A) \xrightarrow{r} \oplus_{i_0} C^\bullet(A_{i_0}) \xrightarrow{\delta^1} \oplus_{i_0<i_1} C^\bullet(A_{i_0,i_1}) \xrightarrow{\delta^2} \cdots$$

$$\xrightarrow{\delta^p} \oplus_{i_0<\cdots<i_{p+1}} C^\bullet(A_{i_0,\ldots,i_{p+1}}) \xrightarrow{\delta^{p+1}} \cdots$$

where r is induced by restriction and the connecting homomorphisms δ are described below.

Given an $\omega \in \oplus_{i_0<\cdots<i_p} C^q(A_{i_0,\ldots,i_p})$ we define $\delta(\omega)$ as follows:

First note that $\delta\omega \in \oplus_{i_0<\cdots<i_{p+1}} C^q(A_{i_0,\ldots,i_{p+1}})$, and it suffices to define $(\delta\omega)_{i_0,\ldots,i_{p+1}}$ for each $(p+2)$-tuple $0 \le i_0 < \cdots < i_{p+1} \le n$. Note that, $(\delta\omega)_{i_0,\ldots,i_{p+1}}$ is a linear form on the vector space, $C_q(A_{i_0,\ldots,i_{p+1}})$, and hence is determined by its values on the q-simplices in the complex $A_{i_0,\ldots,i_{p+1}}$. Furthermore, each q-simplex, $s \in A_{i_0,\ldots,i_{p+1}}$ is automatically a simplex of the complexes $A_{i_0,\ldots,\hat{i_i},\ldots i_{p+1}}$, $0 \le i \le p+1$.

We define,

$$(\delta\omega)_{i_0,\ldots,i_{p+1}}(s) = \sum_{0\le j\le p+1} (-1)^i \omega_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}(s)$$

(here and everywhere else in the paper $\hat{}$ denotes omission).

The fact that the generalized Mayer-Vietoris sequence is exact is classical (see (2) for example).

We now define the Mayer-Vietoris double complex of A, which we will denote by $\mathcal{N}^{\bullet,\bullet}(A)$. $\mathcal{N}^{\bullet,\bullet}(A)$ is the double complex defined by,

$$\mathcal{N}^{p,q}(A) = \bigoplus_{\alpha_0<\cdots<\alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

The horizontal differentials are as defined above. The vertical differentials are those induced by the ones in the different complexes, $C^\bullet(A_{\alpha_0,\ldots,\alpha_p})$.

$\mathcal{N}^{\bullet,\bullet}(A)$ is depicted in the following figure.

$$\begin{tikzcd}
 & & \\
\oplus_{\alpha_0} C^2(A_{\alpha_0}) \arrow[r] \arrow[u] & \oplus_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) \arrow[r] \arrow[u] & \ldots \\
\oplus_{\alpha_0} C^1(A_{\alpha_0}) \arrow[r] \arrow[u] & \oplus_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) \arrow[r] \arrow[u] & \ldots \\
\oplus_{\alpha_0} C^0(A_{\alpha_0}) \arrow[r] \arrow[u] & \oplus_{\alpha_0<\alpha_1} C^0(A_{\alpha_0,\alpha_1}) \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

The following proposition is classical (see (2) for a proof) and follows from the exactness of the generalized Mayer-Vietoris sequence.

Proposition 2.2 *The spectral sequences, $'E_r, ''E_r$, associated to $\mathcal{N}^{\bullet,\bullet}(A)$ converge to $H^*(A,\mathbb{Q})$ and thus,*

$$H^*(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))) \cong H^*(A,\mathbb{Q}).$$

Moreover, the homomorphism $\psi : C^\bullet(A) \to \mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))$ induced by the homomorphism r (in the generalized Mayer-Vietoris sequence) is a quasi-isomorphism.

3 Double complexes associated to certain coverings

Let $\mathcal{P}$ be a finite subset of $\mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}]$. A $\mathcal{P}$-closed formula is a formula constructed as follows:

For each $P \in \mathcal{P}$,

$$P = 0, P \geq 0, P \leq 0,$$

are $\mathcal{P}$-closed formulas.

If Φ_1 and Φ_2 are $\mathcal{P}$-closed formulas, $\Phi_1 \wedge \Phi_2$ and $\Phi_1 \vee \Phi_2$ are $\mathcal{P}$-closed formulas.

Clearly, $\mathcal{R}(\Phi) = \{x \subset \mathrm{R}^\mathrm{k} \mid \Phi(\mathrm{x})\}$, the realization of a $\mathcal{P}$-closed formula Φ, is a closed semi-algebraic set and we call such a set a $\mathcal{P}$-closed semi-algebraic set.

In this section, we consider a fixed family of polynomials, $\mathcal{P} \subset \mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}]$, as well as a fixed $\mathcal{P}$-closed and bounded semi-algebraic set, $S \subset \mathrm{R}^\mathrm{k}$. We also fix a number, $\ell, 0 \leq \ell \leq k$.

We identify certain closed and bounded semi-algebraic subsets of S (which we call the admissible subsets of S). We associate to each admissible subset $X \subset S$, its level denoted $\mathrm{level}(X)$, with $\mathrm{level}(S) = 0$. For each such admissible subset, $X \subset S$, we define a double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, such that

$$H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X))) \cong H^i(X, \mathbb{Q}),\ 0 \leq i \leq \ell - \mathrm{level}(X).$$

The main idea behind the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ is as follows. Given any covering of X by closed semi-algebraic set there is associated to it a double complex (the Mayer-Vietoris double complex) arising from the generalized Mayer-Vietoris exact sequence (see (2)). If the sets occuring in the covering of X are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row. The homology groups of the associated total complex of the Mayer-Vietoris double complex are isomorphic to those of X and thus in order to compute $b_0(X), \ldots, b_{\ell-\mathrm{level}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex. However, computing (even the truncated) Mayer-Vietoris double complex directly within a singly exponential time complexity is not possible by any known method, since we are unable to compute triangulations of semi-algebraic sets in singly exponential time. However, making use of the covering construction recursively, we are able to compute another double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex and hence has isomorphic homology groups (see Proposition 3.2 below). The construction of $\mathcal{M}^{\bullet,\bullet}(X)$ is possible in singly exponential time since the coverings can be computed in singly exponential time.

3.1 Admissible sets and Coverings

Given any closed and bounded semi-algebraic set $X \subset \mathrm{R}^{\mathrm{k}}$, we will denote by $\mathcal{C}'(X)$, a fixed covering of X by a finite family of closed, bounded and acyclic semi-algebraic sets (we will use the construction in (6) to compute such a covering).

We have that, $V \subset X$ for each $V \in \mathcal{C}'(X)$ and $X = \cup_{V \in \mathcal{C}'(X)} V$. We will index the sets in $\mathcal{C}'(X)$ as $V_1, \ldots, V_{n_X}$ where $n_X = \#\mathcal{C}'(X)$, and for $1 \leq \alpha_0 < \cdots < \alpha_p \leq n_X$, we will denote $V_{\alpha_0,\ldots,\alpha_p} = \bigcap_{0 \leq i \leq p} V_{\alpha_i}$. For $I \subset J \subset \{1, \ldots, n_X\}$ we will call V_I an *ancestor* of V_J and X an ancestor of all the V_I's. We will henceforth transitively close the ancestor relation, so that ancestor of an ancestor is also an ancestor. Moreover, if $\{U_i\}_{i \in I}, \{V_j\}_{j \in J}$ are two families of admissible sets such that, $U = \cap_{i \in I} U_i$ and $V = \cap_{j \in J} V_j$ are both admissible, and such that for every $j \in J$ there exists $i \in I$ such that U_i is an ancestor of V_j, then U is an ancestor of V.

We now associate to certain closed semi-algebraic subsets X of S (which we call the admissible subsets of S), a covering, $\mathcal{C}(X)$, of X by closed, bounded, acyclic semi-algebraic sets, obtained by enlarging the covering $\mathcal{C}'(X)$. We also associate an integer to each admissible X, which we will call the level of X (denoted $\mathrm{level}(X)$). We emphasize that the admissible subsets are to be considered as indexed sets and we will consider two equal sets with distinct indices to be distinct.

The set S itself is admissible of level 0 and $\mathcal{C}(S) = \mathcal{C}'(S)$. All intersections of the sets in $\mathcal{C}(S)$ taken upto $\ell + 2$ at a time are admissible and have level 1.

The admissible subsets of S are the smallest family of subsets of S containing the above sets and satisfying the following. For any admissible subset $X \subset S$ at level i, we define $\mathcal{C}(X)$ as follows. Let $\{Y_1, \ldots, Y_N\}$ be the set of admissible sets which are ancestors of X. Then,

$$\mathcal{C}(X) = \bigcup_{U_i \in \mathcal{C}(Y_i), 1 \leq i \leq N} \mathcal{C}'(U_1 \cap \cdots \cap U_N \cap X).$$

All intersections of the sets in $\mathcal{C}(X)$ taken at most $\ell - i + 2$ at a time are admissible, have level $i + 1$, and have X as an ancestor. For $I \subset J \subset \{1, \ldots, n_X\}$, V_I is an ancestor of V_J and X is an ancestor of all the V_I's. Moreover, for

$V \in \mathcal{C}'(U_1 \cap \cdots \cap U_N \cap X)$, each U_i is an ancestor of V. This clearly implies that each $V \in \mathcal{C}(X)$ has a unique ancestor in each $\mathcal{C}(Y_i)$ (namely, U_i).

Now, suppose that we have a procedure for computing $\mathcal{C}'(X)$, for any given $\mathcal{P}'$-closed and bounded semi-algebraic set, X, where $\#\mathcal{P}' = m$ and $\deg(P) \leq D$, for $P \in \mathcal{P}'$. Moreover, suppose that the number and the degrees of the polynomials appearing in the output of this procedure is bounded by $(mD)^{k^{O(1)}}$. Using the above procedure for computing $\mathcal{C}'(X)$, and the definition of admissible sets we have the following quantitative bounds on admissible sets which is crucial in proving the complexity bound of our algorithm.

Proposition 3.1 *Let $S \subset \mathrm{R}^{\mathrm{k}}$ be a $\mathcal{P}$-closed semi-algebraic set, where $\mathcal{P} \subset \mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{k}}]$ is a family of s polynomials of degree at most d. Then the number of admissible sets, the number of polynomials used to define them, the degrees of these polynomials, are all bounded by $(sd)^{k^{O(\ell)}}$.*

Proof: The claim is clear for admissible sets at level 0. The proposition follows easily by induction on the level and the quantitative bounds on $\mathcal{C}'(X)$ stated above. □

3.2 Double Complex Associated to a Covering

Given the different coverings described above, we now associate to each admissible set $X \subset S$ a double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, satisfying the following:

(1)
$$H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q}), \text{ for } 0 \leq i \leq \ell - \mathrm{level}(X). \qquad (2)$$

(2) For every admissible set Y, such that X is an ancestor of Y, and $\mathrm{level}(X) = \mathrm{level}(Y)$, a restriction homomorphism: $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$, which induces the restriction homomorphisms between the cohomology groups:
$$r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$
for $0 \leq i \leq \ell - \mathrm{level}(X)$ via the isomorphisms in (2).

We now describe the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ and prove that it has the properties stated above. The double complex $\mathcal{M}^{\bullet,\bullet}(X)$ is constructed inductively using induction on $\mathrm{level}(X)$:

The base case is when $\mathrm{level}(X) = \ell$. In this case the double complex, $\mathcal{M}^{\bullet,\bullet}(X)$ is defined by:

$$\begin{aligned}
\mathcal{M}^{0,0}(X) &= \bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}), \\
\mathcal{M}^{1,0}(X) &= \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in \mathcal{C}(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0,\alpha_1}), \\
\mathcal{M}^{p,q}(X) &= 0, \text{ if } q > 0 \text{ or } p > 1.
\end{aligned}$$

Here $C^0(Y)$ is the $\mathbb{Q}$-vector space of $\mathbb{Q}$ valued locally constant functions on Y.

This is shown diagramatically below.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u] & \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in \mathcal{C}(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0,\alpha_1}) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The only non-trivial homomorphism in the above complex,

$$\delta : \bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}) \longrightarrow \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in \mathcal{C}(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0,\alpha_1})$$

is defined by $\delta(x)_{\alpha_0,\alpha_1} = (x_{\alpha_1} - x_{\alpha_0})|_{U_{\alpha_0,\alpha_1}}$ for each $x \in \bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0})$.

For every admissible set Y, such that X is an ancestor of Y, and $\mathrm{level}(X) = \mathrm{level}(Y) = \ell$, we define $r_{X,Y} : \mathcal{M}^{0,0}(X) \to \mathcal{M}^{0,0}(Y)$, as follows.

Recall that, $\mathcal{M}^{0,0}(X) = \bigoplus_{U \in \mathcal{C}(X)} C^0(U)$, and $\mathcal{M}^{0,0}(Y) = \bigoplus_{V \in \mathcal{C}(Y)} C^0(V)$. Also, by definition of $\mathcal{C}(Y)$, we have that for each $V \in \mathcal{C}(Y)$ there is a unique $U \in \mathcal{C}(X)$ (which we will denote by $a(V)$) such that U is an ancestor of V.

For $x \in \mathcal{M}^{0,0}(X)$ and $V \in \mathcal{C}(Y)$ we define,

$$r_{X,Y}(x)_V = x_{a(V)}|_V.$$

We define $r_{X,Y} : \mathcal{M}^{1,0}(X) \to \mathcal{M}^{1,0}(Y)$, in a similar manner. More precisely, for $x \in \mathcal{M}^{0,0}(X)$ and $V, V' \in \mathcal{C}(Y)$, we define

$$r_{X,Y}(x)_{V,V'} = x_{a(V),a(V')}|_{V \cap V'}.$$

(The inductive step) In general the $\mathcal{M}^{p,q}(X)$ are defined as follows using induction on $\mathrm{level}(X)$ and with $n = \ell - \mathrm{level}(X) + 1$.

$$\begin{aligned}
&\mathcal{M}^{0,0}(X) = \textstyle\bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} \ C^0(U_{\alpha_0}), && \\
&\mathcal{M}^{0,q}(X) = 0, && 0 < q, \\
&\mathcal{M}^{p,q}(X) = \textstyle\bigoplus_{\alpha_0 < \cdots < \alpha_p,\, U_{\alpha_i} \in \mathcal{C}(X)} \ \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})), && 0 < p,\ 0 < p+q \leq n, \\
&\mathcal{M}^{p,q}(X) = 0, && \text{else.}
\end{aligned}$$

The double complex $\mathcal{M}^{\bullet,\bullet}(X)$ is shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 & \cdots & 0 \\
0 \arrow[r] \arrow[u] & \oplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{n-1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) \arrow[r, "\delta"] \arrow[u] & 0 \arrow[u] & \cdots & 0 \arrow[u] \\
0 \arrow[r] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{n-2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) \arrow[r, "\delta"] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{n-2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) \arrow[u, "d"] & \cdots & 0 \arrow[u, "d"] \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 \arrow[r] & \oplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) \arrow[r, "\delta"] & \oplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) & \cdots & 0 \\
0 \arrow[r] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) \arrow[r, "\delta"] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) \arrow[u, "d"] & \cdots & 0 \arrow[u, "d"] \\
\oplus_{U_{\alpha_0} \in \mathcal{C}_X} C^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) \arrow[r, "\delta"] \arrow[u, "d"] & \oplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) \arrow[u, "d"] & \cdots & \oplus_{\alpha_0<\cdots<\alpha_n} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_n})) \arrow[u, "d"]
\end{tikzcd}$$

The vertical homomorphisms, d, in $\mathcal{M}^{\bullet,\bullet}(X)$ are those induced by the differentials in the various

$$\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})), U_{\alpha_i} \in \mathcal{C}(X).$$

The horizontal ones are defined by generalized restriction as follows (using the fact that the restriction homomorphisms, $r_{U,V}$, are defined for all admissible sets U, V with $\mathrm{level}(U) = \mathrm{level}(V) > \mathrm{level}(X)$, by induction). We define

$$\delta : \bigoplus_{\alpha_0 < \cdots < \alpha_p} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \longrightarrow \bigoplus_{\alpha_0 < \cdots < \alpha_{p+1}} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_{p+1}}))$$

by

$$\delta(x)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum_{0\leq i\leq p+1} (-1)^i r_{U_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{p+1}},U_{\alpha_0,\ldots,\alpha_{p+1}}}(x_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{p+1}})$$

for $x \in \bigoplus_{\alpha_0<\cdots<\alpha_p} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))$, noting that for each $i, 0 \leq i \leq p+1$, $U_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{p+1}}$ is an ancestor of $U_{\alpha_0,\ldots,\alpha_{p+1}}$, and $U_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{p+1}}$, $U_{\alpha_0,\ldots,\alpha_{p+1}}$ have the same levels.

Now let, $Y \subset X$ be admissible sets with X an ancestor of Y and $\mathrm{level}(X) = \mathrm{level}(Y)$. We define the restriction homomorphism,

$$r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$$

as follows.

As before, for $x \in \mathcal{M}^{0,0}(X)$ and $V \in \mathcal{C}(Y)$ we define,

$$r^{0,0}_{X,Y}(x)_V = x_{a(V)}|_V.$$

For $0 < p, 0 < p+q \leq \ell - \mathrm{level}(X)+1$, we define $r^{p,q}_{X,Y} : \mathcal{M}^{p,q}(X) \to \mathcal{M}^{p,q}(Y)$, component-wise as follows.

Let $x \in \mathcal{M}^{p,q}(X) = \bigoplus_{\alpha_0<\cdots<\alpha_p,\, U_{\alpha_i}\in\mathcal{C}(X)} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))$. We define,

$$r^{p,q}_{X,Y}(x)_{\beta_0,\ldots,\beta_p} = \oplus_{i+j=q} r^{i,j}_{a(V_{\beta_0,\ldots,\beta_p}),V_{\beta_0,\ldots,\beta_p}} x_{a(V_{\beta_0}),\ldots,a(V_{\beta_p})},$$

where $a(V_{\beta_0,\ldots,\beta_p}) = \cap_{0\leq i\leq p} a(V_{\beta_i})$. Note that,

$$\mathrm{level}(a(V_{\beta_0,\ldots,\beta_p})) = \mathrm{level}(V_{\beta_0,\ldots,\beta_p}) = \mathrm{level}(X)+1.$$

It is easy to verify by induction on $\mathrm{level}(X)$ that, $\mathcal{M}^{\bullet,\bullet}(X)$ defined as above, is indeed a double complex, that is the homomorphisms d and δ satisfy the equations,

$$d^2 = \delta^2 = 0, d\circ\delta + \delta\circ d = 0.$$

We now prove properties 1 and 2 of the various $\mathcal{M}^{\bullet,\bullet}(X)$.

Proposition 3.2 *For each admissible subset $X \subset S$ the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ satisfies the following properties:*

(1) $H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)),\mathbb{Q}) \cong H^i(X,\mathbb{Q})$ for $0 \leq i \leq \ell - \mathrm{level}(X)$.

(2) *For every admissible set Y, such that X is an ancestor of Y, and* $\mathrm{level}(X) = \mathrm{level}(Y)$*, the homomorphism,* $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$*, induces the restriction homomorphisms between the cohomology groups:*

$$r : H^i(X,\mathbb{Q}) \to H^i(Y,\mathbb{Q})$$

for $0 \leq i \leq \ell - \mathrm{level}(X)$ *via the isomorphisms in (1).*

The main idea behind the proof of Proposition 3.2 is as follows. For any admissible subset X of S, we consider a suitably fine semi-algebraic triangulation, Δ_X, of X. The triangulation Δ_X is fine enough such that, for each admissible subset U of X, Δ_X restricts to a semi-algebraic triangulation, Δ_U, of U.

We denote by $\mathcal{N}_t(\Delta_X)$ the following truncated complex (denoting by $n_X = \ell - \mathrm{level}(X) + 1$),

$$\begin{aligned} \mathcal{N}_t^{p,q}(\Delta_X) &= \mathcal{N}^{p,q}(\Delta_X) \ 0 \leq p+q \leq n_X, \\ \mathcal{N}_t^{p,q}(\Delta_X) &= 0, \qquad\qquad \text{otherwise}, \end{aligned}$$

where $\mathcal{N}^{\bullet,\bullet}(\Delta_X)$ is the Mayer-Vietoris double complex defined in Section 2.5.

Since by Proposition 2.2 the spectral sequences associated to the double complex $\mathcal{N}^{\bullet,\bullet}(\Delta_X)$ converges to $H^*(X,\mathbb{Q})$, we have that

$$H^i(\mathrm{Tot}^\bullet(\mathcal{N}_t^{\bullet,\bullet}(\Delta_X)),\mathbb{Q}) \cong H^i(X,\mathbb{Q}),\ 0 \leq i \leq \ell - \mathrm{level}(X).$$

We then prove by induction on $\mathrm{level}(X)$ that for each admissible X there exists a double complex $D^{\bullet,\bullet}(X)$ and homomorphisms,

$$\begin{aligned} \phi_X &: \mathcal{M}^{\bullet,\bullet}(X) \to D^{\bullet,\bullet}(X) \\ \psi_X &: C^\bullet(\Delta_X) \to \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(X)) \end{aligned}$$

such that the induced homomorphism,

$$\phi_X : \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)) \to \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(X)),$$

as well as ψ_X are quasi-isomorphisms.

$$\begin{tikzcd} & \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(X)) & \\ \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)) \arrow[ur, "\phi_X"] & & C^\bullet(\Delta_X) \arrow[ul, "\psi_X"'] \end{tikzcd}$$

These quasi-isomorphisms will together imply that,

$$H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X))) \cong H^i(\mathrm{Tot}^\bullet(D^{\bullet,\bullet}(X))) \cong H^i(\mathrm{Tot}^\bullet(\mathcal{N}_t^{\bullet,\bullet}(\Delta_X))) \cong H^i(X,\mathbb{Q}),$$

for $0 \leq i \leq \ell - \mathrm{level}(X)$.

Proof of Proposition 3.2: The proof of the proposition is by induction on $\mathrm{level}(X)$. When $\mathrm{level}(X) = \ell$, we let $D^{\bullet,\bullet}(X) = \mathcal{N}_t^{\bullet,\bullet}(\Delta_X)$, and define the homomorphisms ϕ_X, ψ_X in the obvious manner.

Otherwise, by induction hypothesis for each $U_{\alpha_0}, \ldots, U_{\alpha_p}, U_{\alpha_{p+1}} \in \mathcal{C}(X), 0 \leq p \leq \ell - \mathrm{level}(X) + 2$, with $\alpha_0 < \ldots < \alpha_{p+1}$, there exists a double complex $D^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})$ and quasi-isomorphisms

$$\begin{aligned}\phi_{U_{\alpha_0,\ldots,\alpha_p}} &: \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \to \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))\\ \psi_{U_{\alpha_0,\ldots,\alpha_p}} &: C^\bullet(\Delta_X) \to \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})).\end{aligned}$$

We define $D^{\bullet,\bullet}(X)$ by,

$$D^{p,q}(X) = \bigoplus_{\alpha_0 < \cdots < \alpha_p,\, U_{\alpha_i} \in \mathcal{C}(X)} \mathrm{Tot}^q(D^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})),\ 0 \leq p+q \leq n_X.$$

The homomorphism ϕ_X is the one induced by the different $\phi_{U_{\alpha_0,\ldots,\alpha_p}}$ defined already by induction. In order to define the homomorphism ψ_X, we first define a homomorphism, $\psi'_X : \mathcal{N}_t^{\bullet,\bullet}(X) \to D^{\bullet,\bullet}(X)$ induced by the different $\psi_{U_{\alpha_0,\ldots,\alpha_p}}$, and compose the induced homomorphism, $\psi'_X : \mathrm{Tot}^\bullet(\mathcal{N}_t^{\bullet,\bullet}) \to \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(X))$, with the naturally defined quasi-isomorphism, $\psi''_X : C^\bullet(\Delta_X) \to \mathrm{Tot}^\bullet(\mathcal{N}_t^{\bullet,\bullet}(X))$ (see Proposition 2.2). $\square$

4 General Position and Coverings by Contractible Sets

In this section, we recall some results proved in (6) on constructing singly exponential sized covering of a given closed semi-algebraic set, by closed, acyclic semi-algebraic set. We recall the input, output and the complexity of the algorithms, referring the reader to (6) for all details including the proofs of correctness.

4.1 General Position

Let $Q \in \mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}]$ such that $\mathrm{Z}(Q, \mathrm{R}^\mathrm{k}) = \{\mathrm{x} \in \mathrm{R}^\mathrm{k} \mid \mathrm{Q}(\mathrm{x}) = 0\}$ is bounded. We say that a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ is in strong ℓ-general position with respect to Q if any $\ell + 1$ polynomials belonging to $\mathcal{P}$

have no zeros in common with Q in $\mathrm{R^k}$, and any ℓ polynomials belonging to $\mathcal{P}$ have at most a finite number of zeros in common with Q in $\mathrm{R^k}$.

4.2 Infinitesimals

In our algorithms we will use infinitesimal perturbations. In order to do so, we will extend the ground field R to, $\mathrm{R}\langle\varepsilon\rangle$, the real closed field of algebraic Puiseux series in ε with coefficients in R (5). The sign of a Puiseux series in $\mathrm{R}\langle\varepsilon\rangle$ agrees with the sign of the coefficient of the lowest degree term in ε. This induces a unique order on $\mathrm{R}\langle\varepsilon\rangle$ which makes ε infinitesimal: ε is positive and smaller than any positive element of R. When $a \in \mathrm{R}\langle\varepsilon\rangle$ is bounded by an element of R, $\lim_\varepsilon(a)$ is the constant term of a, obtained by substituting 0 for ε in a. We will also denote the field $\mathrm{R}\langle\varepsilon_1\rangle\cdots\langle\varepsilon_\mathrm{s}\rangle$ by $\mathrm{R}\langle\bar{\varepsilon}\rangle$, where $\varepsilon_1 \gg \varepsilon_2 \cdots \gg \varepsilon_s > 0$ are all infinitesimals.

4.3 Replacement by closed sets without changing homology

The following algorithm allows us to replace a given semi-algebraic set by a new one which is closed and defined by polynomials in general position without changing the homology groups. This construction is essentially due to Gabrielov and Vorobjov (11), with some modifications described in (6).

Algorithm 4.1 (Homology Preserving Modification to Closed)
Input : *a polynomial* $Q \in \mathrm{D}[X_1,\ldots,X_k]$ *such that* $\mathrm{Z}(Q,\mathrm{R^k}) \subset \mathrm{B}(0,1/\mathrm{c})$,
a finite set of s *polynomials*

$$\mathcal{P} = \{P_1,\ldots,P_s\} \subset \mathrm{D}[X_1,\ldots,X_k],$$

a semi-algebraic set X *defined by*

$$X = \cup_{\sigma\in\Sigma}\mathcal{R}(\sigma)$$

with $\Sigma \subset \mathrm{Sign}(Q,\mathcal{P})$.
Output : *A description of a* $\mathcal{P}'$*-closed and cbounded semi-algebraic subset,*

$$X' \subset \mathrm{Z}(Q,\mathrm{R}\langle\varepsilon,\varepsilon_1,\ldots,\varepsilon_{2\mathrm{s}}\rangle^\mathrm{k}),$$

with $\mathcal{P}' = \bigcup_{1\le i\le s, 1\le j\le 2s}\{P_i \pm \varepsilon_j\}$, *such that,*
$H_*(X') \cong H_*(X)$, *and*
the family of polynomials $\mathcal{P}'$ *is in* k'*-strong general position with respect to* $\mathrm{Z}(Q,\mathrm{R}\langle\varepsilon,\varepsilon_1,\ldots,\varepsilon_{2\mathrm{s}}\rangle^\mathrm{k})$, *where* k' *is the real dimension of* $\mathrm{Z}(Q,\mathrm{R}\langle\varepsilon,\varepsilon_1,\ldots,\varepsilon_{2\mathrm{s}}\rangle^\mathrm{k})$.
Procedure :

Step 1 *Let ε be an infinitesimal. Define $\tilde{T}$ as the intersection of* $\mathrm{Ext}(T, \langle\varepsilon\rangle)$ *with the ball of center* 0 *and radius* $1/\varepsilon$*. Define $\mathcal{P}$ as $\mathcal{Q} \cup \{\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) - 4, X_{k+1}\}$ Replace $\tilde{T}$ by the $\mathcal{P}$- semi-algebraic set S defined as the intersection of the cylinder $\tilde{T} \times \mathrm{R}\langle\varepsilon\rangle$ with the upper hemisphere defined by $\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) = 4, X_{k+1} \geq 0$.*

Step 2 *Using the Gabrielov-Vorobjov construction described in (6), replace S by a $\mathcal{P}'$-closed set, S'. Note that $\mathcal{P}'$ is in general position with respect to the sphere of center* 0 *and radius* $2/\varepsilon$*.*

Complexity: Let d be the maximum degree among the polynomials in $\mathcal{P}$. The total complexity is bounded by $s^{k+1}d^{O(k)}$ (see (6)). □

4.4 Algorithm for Computing Coverings by Contractible Sets

The following algorithm described in detail in (6) is used to a covering of a given closed and bounded semi-algebraic sets defined by polynomials in general position by closed, bounded and contractible semi-algebraic sets.

Algorithm 4.2 (Covering by Contractible Sets)

Input : *a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$ such that $\mathrm{Z}(Q, \mathrm{R}^{\mathrm{k}}) \subset \mathrm{B}(0, 1/\mathrm{c})$,*
a finite set of s polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong ℓ-general position on $\mathrm{Z}(Q, \mathrm{R}^{\mathrm{k}})$.

Output : *a finite family of polynomials $\mathcal{C} = \{Q_1, \ldots, Q_N\} \subset \mathrm{D}[X_1, \ldots, X_k]$,*
the finite family $\overline{\mathcal{C}} \subset \mathrm{D}[\bar{\varepsilon}][X_1, \ldots, X_k]$ (where $\bar{\varepsilon}$ denotes the infinitesimals $\varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_{2N} > 0$) defined by

$$\overline{\mathcal{C}} = \{Q \pm \varepsilon_i \mid Q \in \mathcal{C}, 1 \leq i \leq 2N\}.$$

a set of $\overline{\mathcal{C}}$-closed formulas $\{\phi_1, \ldots, \phi_M\}$ such that
each $\mathcal{R}(\phi_i, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\mathrm{k}})$ is acyclic,
their union $\cup_{1 \leq i \leq M} \mathcal{R}(\phi_i, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\mathrm{k}}) = \mathrm{Z}(\mathrm{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\mathrm{k}})$, and
each basic $\mathcal{P}$-closed subset of $\mathrm{Z}(Q, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\mathrm{k}})$ is a union of some subset of the $\mathcal{R}(\phi_i, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\mathrm{k}})$'s.

Complexity: The total complexity is bounded by $s^{(k+1)^2}d^{O(k^5)}$ (see (6)). □

5 Algorithm for computing the first ℓ Betti numbers of a semi-algebraic set

We are finally in a position to describe the main algorithm of this paper.

Algorithm 5.1 (First ℓ Betti Numbers of a $\mathcal{P}$ Semi-algebraic Set)

Input : *a polynomial* $Q \in \mathrm{D}[X_1, \ldots, X_k]$ *such that* $\mathrm{Z}(Q, \mathrm{R}^\mathrm{k}) \subset \mathrm{B}(0, 1/\mathrm{c})$,
a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$,
a formula defining a $\mathcal{P}$ *semi-algebraic set,* S*, contained in* $\mathrm{Z}(Q, \mathrm{R}^\mathrm{k})$.

Output : $b_0(S), \ldots, b_\ell(S)$.

Procedure :

Step 1 *Using Algorithm 4.1 (Homology Preserving Modification to Closed), replace* S *by a* $\mathcal{P}'$*-closed set,* S'*. Note that* $\mathcal{P}'$ *is in* k'*-general position with respect to* $\mathrm{Z}(Q, \mathrm{R}^\mathrm{k})$.

Step 2 *Compute using Algorithm 4.2 (Covering by Contractible Sets) repeatedly descriptions of the admissible subsets of* S*, their levels, and for each admissible subset* X*, the covering* $\mathcal{C}(X)$*. We also keep track of the ancestor relationships amongst the different admissible set.*

Step 3 *Starting from the admissible subsets at level* ℓ*, compute for each admissible* X*, the double complex* $\mathcal{M}^{\bullet,\bullet}(X)$.

Step 4 *For each* $i, 0 \leq i \leq \ell$*, output,*

$$b_i(S) = \mathrm{rank} H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X))).$$

Proof of correctness : The correctness of the algorithm is a consequence of the correctness of Algorithms 4.1 (Homology Preserving Modification to Closed), Algorithm 4.2 (Covering by Contractible Sets), and Proposition 3.2. □

Complexity analysis: Each step is clearly singly exponential from the complexity analysis of Algorithms 4.1 (Homology Preserving Modification to Closed), 4.2 (Covering by Contractible Sets), and Proposition 3.1 □

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