A SHARPER ESTIMATE ON THE BETTI NUMBERS OF SETS DEFINED BY QUADRATIC INEQUALITIES

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ABSTRACT. In this paper we consider the problem of bounding the Betti numbers, $b_i(S)$, of a semi-algebraic set $S \subset \mathbb{R}^k$ defined by polynomial inequalities $P_1 \geq 0, \ldots, P_s \geq 0$, where $P_i \in \mathbb{R}[X_1, \ldots, X_k]$ and $\deg(P_i) \leq 2$, for $1 \leq i \leq s$. We prove that for $0 \leq i \leq k-1$,

$$b_i(S) \leq \frac{1}{2}\left(\sum_{j=0}^{min\{s,k-i\}} \binom{s}{j}\binom{k+1}{j}2^j\right).$$

In particular, for $2 \leq s \leq \frac{k}{2}$, we have

$$b_i(S) \leq \frac{1}{2}3^s\binom{k+1}{s} \leq \frac{1}{2}\left(\frac{3e(k+1)}{s}\right)^s.$$

This improves the bound of $k^{O(s)}$ proved by Barvinok in [2]. This improvement is made possible by a new approach, whereby we first bound the Betti numbers of non-singular complete intersections of complex projective varieties defined by generic quadratic forms, and use this bound to obtain bounds in the real semi-algebraic case.

1. Introduction

The topological complexity of semi-algebraic sets, measured by their Betti numbers (ranks of their singular homology groups), has been the subject of many investigations. For any topological space X, we will denote by $b_i(X) = b_i(X, \mathbb{Z}_2)$ the i-th Betti number of X with $\mathbb{Z}_2$-coefficients, and we will denote by $b(X)$ the sum $\sum_{i \geq 0} b_i(X)$. Note that, since the homology groups of a semi-algebraic set $S \subset \mathbb{R}^k$ are finitely generated, it follows from the Universal Coefficients Theorem, that $b_i(S, \mathbb{Z}_2) \geq b_i(S, \mathbb{Z})$, where $b_i(S, \mathbb{Z})$ are the ordinary Betti numbers of S with integer coefficients (see [16], Corollary 3.A6 (b)). Hence, the bounds proved in this paper also apply to the ordinary Betti numbers. (The use of $\mathbb{Z}_2$ coefficients is necessitated by our use of Smith inequalities in the proof of the main theorem.)

The initial result on bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovskii [20], Thom [21] and Milnor [19]. They proved:

Theorem 1.1. [20, 21, 19] *Let*

$$\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$$

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with $\deg(P_i) \leq d$, $1 \leq i \leq s$ *and let* $S \subset \mathbb{R}^k$ *be the set defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0.$$

Then,

$$b(S) = O(sd)^k.$$

Notice that the above bound is exponential in k and this exponential dependence is unavoidable (see Example 1.2 below). See also [5, 13, 9] for more recent work extending the above bound to more general classes of semi-algebraic sets.

1.1. **Semi-algebraic Sets Defined by Quadratic Inequalities.** In this paper we consider a restricted class of semi-algebraic sets - namely, semi-algebraic sets defined by quadratic inequalities. Since sets defined by linear inequalities have no interesting topology, sets defined by quadratic inequalities can be considered to be the simplest class of semi-algebraic sets which can have non-trivial topology. Such sets are in fact quite general, as every semi-algebraic set can be defined by (quantified) formulas involving only quadratic polynomials (at the cost of increasing the number of variables and the size of the formula). Moreover, as in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large as can be seen in the following example.

Example 1.2. The set $S \subset \mathbb{R}^k$ defined by

$$X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$$

has $b_0(S) = 2^k$.

However, it turns out that for a semi-algebraic set $S \subset \mathbb{R}^k$ defined by s quadratic inequalities, it is possible to obtain upper bounds on the Betti numbers of S which are polynomial in k and exponential only in s. The first such result was proved by Barvinok who proved the following theorem.

Theorem 1.3. [2] *Let* $S \subset \mathbb{R}^k$ *be defined by* $P_1 \geq 0, \ldots, P_s \geq 0$, $\deg(P_i) \leq 2, 1 \leq i \leq s$. *Then,* $b(S) \leq k^{O(s)}$.

Theorem 1.3 is proved using a duality argument that interchanges the roles of k and s, and reduces the original problem to that of bounding the Betti numbers of a semi-algebraic set in $\mathbb{R}^s$ defined by $k^{O(1)}$ polynomials of degree at most k. One can then use Theorem 1.1 to obtain a bound of $k^{O(s)}$. The constant hidden in the exponent of the above bound is at least two. Also, the bound in Theorem 1.3 is polynomial in k but exponential in s. The exponential dependence on s is unavoidable as remarked in [2], but the implied constant (which is at least two) in the exponent of Barvinok's bound is not optimal.

Using Barvinok's result, as well as inequalities derived from the Mayer-Vietoris sequence, a polynomial bound (polynomial both in k and s) was proved in [5] on the top few Betti numbers of a set defined by quadratic inequalities. More precisely the following theorem is proved there.

Theorem 1.4. [5] *Let* $\ell > 0$ *and* $\mathbb{R}$ *a real closed field. Let* $S \subset R^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0,$$

with $\deg(P_i) \leq 2$. *Then,*

$$b_{k-\ell}(S) \leq \binom{s}{\ell} k^{O(\ell)}.$$

Notice that for fixed ℓ, the bound in Theorem 1.4 is polynomial in both s as well as k.

Apart from their intrinsic mathematical interest (in distinguishing the semi-algebraic sets defined by quadratic inequalities from general semi-algebraic sets), the bounds in Theorems 1.3 and 1.4 have motivated recent work on designing polynomial time algorithms for computing topological invariants of semi-algebraic sets defined by quadratic inequalities (see [3, 14, 6, 7, 10]). Traditionally an important goal in algorithmic semi-algebraic geometry has been to design algorithms for computing topological invariants of semi-algebraic sets, whose worst-case complexity matches the best upper bounds known for the quantity being computed. It is thus of interest to tighten the bounds on the Betti numbers of semi-algebraic sets defined by quadratic inequalities, as has been done recently in the case of general semi-algebraic sets (see for example [13, 5, 9]).

1.2. **Brief Outline of Our Method.** In this paper we use a new method to bound the Betti numbers of semi-algebraic sets defined by quadratic inequalities. Our method is to first bound the Betti numbers of complex projective varieties which are non-singular complete intersections defined by quadratic forms in general position. It is a well known fact from complex geometry, (see for instance, [17], pp. 122) that the Betti numbers of a complex projective variety which is a non-singular complete intersection depend only on the sequence of degrees of the polynomials defining the variety. Moreover, there exist precise formulas for the Betti numbers of such varieties, using well-known techniques from algebraic geometry (see Theorem 2.6 below).

Our strategy for bounding the Betti numbers of semi-algebraic sets in $\mathbb{R}^k$ defined by s quadratic inequalities is as follows. Using certain *infinitesimal deformations* we first reduce the problem to bounding the Betti numbers of another closed and *bounded* semi-algebraic set defined by a new family quadratic polynomials. We then use inequalities obtained from the Mayer-Vietoris exact sequence to further reduce the problem of bounding the Betti numbers of this new semi-algebraic set, to the problem of bounding the Betti numbers of the *real projective varieties* defined by each ℓ-tuple, $\ell \leq s$, of the new polynomials. The new family of polynomials also has the property that the *complex projective variety* defined by each ℓ-tuple, $\ell \leq k$, of these polynomials is a *non-singular complete intersection.* As mentioned above we have precise information about the Betti numbers of these complex complete intersections. An application of *Smith inequalities* then allows us to obtain bounds on the Betti numbers of the real parts of these varieties and as a result on the Betti numbers of the original semi-algebraic set. Because of the direct nature of our proof we are able to remove the constant in the exponent in the bounds proved in [2, 5] and this constitutes the main contribution of this paper.

Remark 1.5. We remark here that the technique used in this paper was proposed as a possible alternative method by Barvinok in [2], who did not pursue this further in that paper. Also, Benedetti, Loeser, and Risler [11] used a similar technique for proving upper bounds on the number of connected components of real algebraic sets in $\mathbb{R}^k$ defined by polynomials of degrees bounded by d. However, these bounds (unlike the situation considered in this paper) are exponential in k. Finally, there exists another possible method for bounding the Betti numbers of semi-algebraic sets defined by quadratic inequalities, using a spectral sequence argument due to Agrachev [1]. However, this method also produces a non-optimal bound of the form $k^{O(s)}$ (similar to Barvinok's bound) where the constant in the exponent is at least two. We omit the details of this argument referring the reader to [6] for an indication of the proof (where the case of computing, and as a result, bounding the Euler-Poincaré characteristics of such sets is worked out in full details).

We prove the following theorem.

Theorem 1.6. *Let* $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$, $s \leq k$. *Let* $S \subset \mathbb{R}^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0$$

with $\deg(P_i) \leq 2$. *Then, for* $0 \leq i \leq k-1$,

$$b_i(S) \leq \frac{1}{2} \left(\sum_{j=0}^{min\{s,k-i\}} \binom{s}{j} \binom{k+1}{j} 2^j \right).$$

In particular, for $2 \leq s \leq \frac{k}{2}$, *we have*

$$b_i(S) \leq \frac{1}{2} 3^s \binom{k+1}{s} \leq \frac{1}{2} \left(\frac{3e(k+1)}{s} \right)^s.$$

As a consequence of the proof of Theorem 1.6 we get a new bound on the sum of the Betti numbers, which we state for the sake of completeness.

Corollary 1.7. *Let* $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$, $s \leq k$. *Let* $S \subset \mathbb{R}^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0$$

with $\deg(P_i) \leq 2$. *Then,*

$$b(S) \leq \frac{1}{2} k \left(\sum_{j=0}^{s} \binom{s}{j} \binom{k+1}{j} 2^j \right).$$

The rest of the paper is organized as follows. In Section 2, we recall some well known results from complex algebraic geometry on the Betti numbers of non-singular complex projective varieties which are complete intersections, and also some classical results from algebraic topology which we need for the proof of our main result. In Section 3, we prove Theorem 1.6. Finally, in Section 4 we state some open problems.

2. Mathematical Preliminaries

In this section we recall a few basic facts about Betti numbers and complex projective varieties which are non-singular complete intersections, as well as fix some notations.

Throughout the paper, $\mathbb{P}^k_{\mathbb{R}}$ (respectively, $\mathbb{P}^k_{\mathbb{C}}$) denotes the real (respectively, complex) projective space of dimension k, $\mathbf{S}^k_r$ (resp., $\mathbf{B}^{k+1}_r$) denotes the sphere (resp., closed ball) centered at the origin and of radius r in $\mathbb{R}^{k+1}$. For any polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$, we denote by $P^h \in \mathbb{R}[X_1, \ldots, X_{k+1}]$ the homogenization of P with respect to X_{k+1}.

For any family of polynomials $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathbb{R}[X_1, \ldots, X_k]$, and $Z \subset \mathbb{R}^k$, we denote by $\mathrm{Zer}(\mathcal{P}, Z)$ the set of common zeros of $\mathcal{P}$ in Z. Moreover, for any family of homogeneous polynomials $\mathcal{Q} = \{Q_1, \ldots, Q_\ell\} \subset \mathbb{R}[X_1, \ldots, X_{k+1}]$, we denote by $\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathbb{R}})$ (resp., $\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathbb{C}})$) the set of common zeros of $\mathcal{Q}$ in $\mathbb{P}^k_{\mathbb{R}}$ (resp., $\mathbb{P}^k_{\mathbb{C}}$).

2.1. **Some Results from Algebraic Topology.** Let

$$\mathcal{Q} = \{Q_1, \ldots, Q_\ell\} \subset \mathbb{R}[X_1, \ldots, X_{k+1}]$$

be a set of homogeneous polynomials. Denote by $\mathcal{Q}_J$ the set $\{Q_j | j \in J\}$ for $J \subset \{1, \ldots, \ell\}$. We have the following inequality which is a consequence of the Mayer-Vietoris exact sequence.

Proposition 2.1. *Let $Z \subset \mathbb{R}^{k+1}$. For $0 \leq i \leq k-1$,*

$$b_i(\bigcup_{j=1}^{\ell} \mathrm{Zer}(Q_j, Z)) \leq \sum_{j=1}^{i+1} \sum_{J \subset \{1,\ldots,\ell\}, |J|=j} b_{i-j+1}(\mathrm{Zer}(\mathcal{Q}_J, Z)).$$

Proof. See [5], Lemma 2. □

We also use the well-known Alexander duality theorem which relates the Betti numbers of a compact subset of a sphere to those of its complement.

Theorem 2.2 (Alexander Duality)**.** *Let $r > 0$. For any closed subset $A \subset S^k_r$,*

$$H_i(S^k_r \setminus A) \approx \tilde{H}^{k-i-1}(A),$$

where $\tilde{H}^i(A)$, $0 \leq i \leq k-1$, denotes the reduced cohomology group of A.

Proof. See [18], Theorem 6.6. □

Finally, we state a version of the Smith inequality which plays a crucial role in the proof of the main theorem. Recall that for any compact topological space equipped with an involution, inequalities derived from the *Smith exact sequences* allows one to bound the *sum* of the Betti numbers (with $\mathbb{Z}_2$ coefficients) of the fixed point set of the involution by the sum of the Betti numbers (again with $\mathbb{Z}_2$ coefficients) of the space itself (see for instance, [22], pp. 131). In particular, we have for a complex projective variety defined by real forms, with the involution taken to be complex conjugation, the following theorem.

Theorem 2.3 (Smith inequality)**.** *Let $\mathcal{Q} \subset \mathbb{R}[X_1, \ldots, X_{k+1}]$ be a family of homogeneous polynomials. Then,*

$$b(\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathbb{R}})) \leq b(\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathbb{C}})).$$

2.2. **Complete Intersection Varieties.**

Definition 2.4. A projective variety X of codimension n is a *non-singular complete intersection* if it is the intersection of n non-singular hypersurfaces that meet transversally at each point of the intersection.

Fix an j tuple of natural numbers $\bar{d} = (d_1, \ldots, d_j)$. Let $X_{\mathbb{C}} = \mathrm{Zer}(\{Q_1, \ldots, Q_j\}, \mathbb{P}^k_{\mathbb{C}})$, such that the degree of Q_i is d_i, denote a complex projective variety of codimension j which is a non-singular complete intersection.

Let $b(j, k, \bar{d})$ denote the sum of the Betti numbers with $\mathbb{Z}_2$ coefficients of $X_{\mathbb{C}}$. This is well defined since the Betti numbers only depend only on the degree sequence and not on the specific $X_{\mathbb{C}}$.

The function $b(j, k, \bar{d})$ satisfies the following (see [11]):

$$b(j, k, \bar{d}) = \begin{cases} c(j, k, \bar{d}) & \text{if } k - j \text{ is even,} \\ 2(k - j + 1) - c(j, k, \bar{d}) & \text{if } k - j \text{ is odd,} \end{cases}$$

where

$$c(j,k,\bar{d}) = \begin{cases} k+1 & \text{if } j=0, \\ d_1 \dots d_j & \text{if } j=k, \\ d_k c(j-1,k-1,(d_1,\dots,d_{k-1})) - (d_k-1)c(j,k-1,\bar{d}) & \text{if } 0<j<k. \end{cases}$$

In the special case when each $d_i = 2$, we denote by $b(j,k) = b(j,k,(2,\dots,2))$. We then have the following recurrence for $b(j,k)$.

$$b(j,k) = \begin{cases} q(j,k) & \text{if } k-j \text{ is even,} \\ 2(k-j+1) - q(j,k) & \text{if } k-j \text{ is odd,} \end{cases}$$

where

$$q(j,k) = \begin{cases} k+1 & \text{if } j=0, \\ 2^j & \text{if } j=k, \\ 2q(j-1,k-1) - q(j,k-1) & \text{if } 0<j<k. \end{cases}$$

Next, we show some properties of $q(j,k)$.

Lemma 2.5.

(1) $q(1,k) = k + 1/2(1-(-1)^k)$ *and* $q(2,k) = (-1)^k k + k$.
(2) *For* $2 \le j \le k$, $|q(j,k)| \le 2^{j-1}\binom{k}{j-1}$.
(3) *For* $2 \le j \le k$ *and* $k-j$ *odd,* $2(k-j+1) - q(j,k) \le 2^{j-1}\binom{k}{j-1}$.

Proof. The first part is shown by two easy computations and note that $2(k-2+1) - q(2,k) = 2k-2$ if $k-2$ is odd. Hence, we can assume that the statements are true for $k-1$ and that $3 \le j < k$. Note that for the special case $j = k-1$, we have that $2^{k-1} \le 2^{k-2}\binom{k-1}{k-2}$ since $k>2$.Then,

$$\begin{aligned} |q(j,k)| &= |2q(j-1,k-1) - q(j,k-1)| \\ &\le 2|q(j-1,k-1)| + |q(j,k-1)| \\ &\le 2 \cdot 2^{j-2}\binom{k-1}{j-2} + 2^{j-1}\binom{k-1}{j-1} \\ &= 2^{j-1}\binom{k}{j-1}. \end{aligned}$$

and, for $k-j$ odd,

$$\begin{aligned} 2(k-j+1) - q(j,k) &= 2(k-j+1) - 2q(j-1,k-1) + q(j,k-1) \\ &\le |2((k-1)-(j-1)+1) - q(j-1,k-1)| \\ &\quad +|q(j-1,k-1)| + |q(j,k-1)| \\ &\le 2^{j-2}\binom{k-1}{j-2} + 2^{j-2}\binom{k-1}{j-2} + 2^{j-1}\binom{k-1}{j-1} \\ &\le 2^{j-1}\left(\binom{k-1}{j-2} + \binom{k-1}{j-1}\right) = 2^{j-1}\binom{k}{j-1}. \end{aligned}$$

□

Hence, we get the following bound for $b(j,k)$.

Theorem 2.6.

(1) $b(1,k) = \begin{cases} q(0,k-1) & \textit{if } k \textit{ is even,} \\ q(0,k) & \textit{if } k \textit{ is odd,} \end{cases}$

(2) $b(j,k) \le 2^{j-1}\binom{k}{j-1}$, *for* $2 \le j \le k$.

Proof. Follows from Lemma 2.5. $\square$

Proposition 2.7. *Let* $\varepsilon > 0$ *and let* $P_\varepsilon = (\frac{2}{\varepsilon})^2 - \sum_{i=1}^{k+1} X_i^2$*. Then, there exist a family* $\mathcal{H}_\varepsilon = \{H_{\varepsilon,1},\ldots,H_{\varepsilon,s}\} \subset \mathbb{R}[X_1,\ldots,X_{k+2}]$*,* $s \le k$*, of positive definite quadratic forms such that* $\mathrm{Zer}(\mathcal{H}_{\varepsilon,J} \cup \{P_\varepsilon^h\}, \mathbb{P}_{\mathbb{C}}^{k+1})$ *is a non-singular complete intersection for every* $J \subset \{1,\ldots,s\}$*.*

Proof. Note that for any family $\mathcal{H} = \{H_1,\ldots,H_s\} \subset \mathbb{R}[X_1,\ldots,X_{k+2}]$ of quadratic forms such that their coefficients are algebraically independent over $\mathbb{Q}$, $\mathrm{Zer}(\mathcal{H}_J, \mathbb{P}_{\mathbb{C}}^{k+1})$, $J \subset \{1,\ldots,s\}$, is a non-singular complete intersection by Bertini's Theorem (see [15], Theorem 17.16). Moreover, recall that the set of positive definite quadratic forms is open in the set of quadratic forms (over $\mathbb{R}$). Thus, we can choose for every $\varepsilon > 0$ a family $\mathcal{H}_\varepsilon = \{H_{\varepsilon,1},\ldots,H_{\varepsilon,s}\} \subset \mathbb{R}[X_1,\ldots,X_{k+2}]$, $s \le k$, of positive definite quadratic forms such that their coefficients are algebraically independent over $\mathbb{Q}(\varepsilon)$, and $\mathrm{Zer}(\mathcal{H}_{\varepsilon,J} \cup \{P_\varepsilon^h\}, \mathbb{P}_{\mathbb{C}}^{k+1})$ will be a non-singular complete intersection for every $J \subset \{1,\ldots,s\}$. $\square$

The following proposition allows to replace a set of real quadratic forms by another family obtained by infinitesimal perturbations of the original family and whose zero sets are non-singular complete intersections in complex projective space.

Proposition 2.8. *Let* $\varepsilon > 0$ *and let*

$$P_\varepsilon = (\frac{2}{\varepsilon})^2 - \sum_{i=1}^{k+1} X_i^2.$$

Let

$$\mathcal{Q} = \{Q_1,\ldots,Q_s\} \subset \mathbb{R}[X_1,\ldots,X_{k+2}],$$

$s \le k$*, be a set of quadratic forms and let*

$$\mathcal{H}_\varepsilon = \{H_{\varepsilon,1},\ldots,H_{\varepsilon,s}\} \subset \mathbb{R}[X_1,\ldots,X_{k+2}]$$

be a family of positive definite quadratic forms such that $\mathrm{Zer}(\mathcal{H}_{\varepsilon,J} \cup \{P_\varepsilon^h\}, \mathbb{P}_{\mathbb{C}}^{k+1})$ *is a non-singular complete intersection for every* $J \subset \{1,\ldots,s\}$*.*

For $t \in \mathbb{C}$*, let*

$$\tilde{\mathcal{Q}}_{\varepsilon,t} = \{\tilde{Q}_{\varepsilon,t,1},\ldots,\tilde{Q}_{\varepsilon,t,s}\} \text{ with}$$
$$\tilde{Q}_{\varepsilon,t,i} = (1-t)Q_i + tH_{\varepsilon,i}.$$

Then, for all sufficiently small $\delta > 0$*, and any* $J \subset \{1,\ldots,s\}$*,*

$$\mathrm{Zer}(\tilde{\mathcal{Q}}_{\varepsilon,\delta,J} \cup \{P_\varepsilon^h\}, \mathbb{P}_{\mathbb{C}}^{k+1})$$

is a non-singular complete intersection.

Proof. Let $J \subset \{1,\ldots,s\}$, and let $T_J \subset \mathbb{C}$ be defined by,

$$T_J = \{t \in \mathbb{C} \mid \mathrm{Zer}(\tilde{\mathcal{Q}}_{\varepsilon,t,J}, \mathbb{P}_{\mathbb{C}}^{k+1}) \text{ is a non-singular complete intersection }\}.$$

Clearly, T_J contains 1. Moreover, since being a non-singular complete intersection is stable condition, T_J must contain an open neighborhood of 1 in $\mathbb{C}$ and so must $T = \cap_{J \subset \{1,\ldots,s\}} T_J$. Finally, the set T is constructible, since it can be defined by a first order formula. Since a constructible subset of $\mathbb{C}$ is either finite or the complement of a finite set (see for instance, [9], Corollary 1.25), T must contain an interval $(0,t_0), t_0 > 0$. $\square$

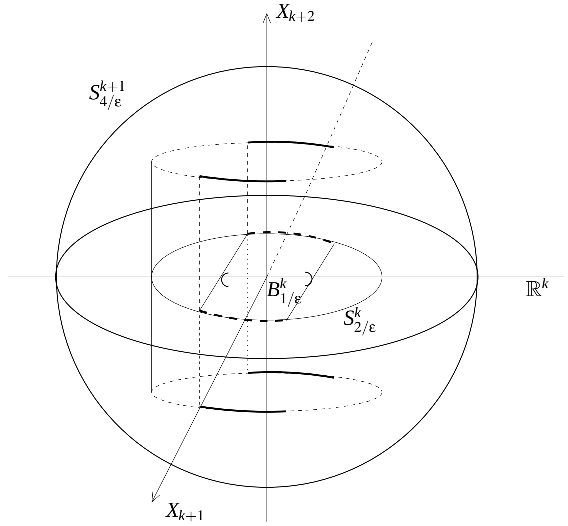

FIGURE 1. lifting the ball $\mathbf{B}^k_{1/\varepsilon}$ onto the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$

3. Proof of Theorem 1.6

In order to prove Theorem 1.6 we need what follows next:

Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$, $s \leq k$, with $\deg(P_i) \leq 2$, $1 \leq i \leq s$. Let $S \subset \mathbb{R}^k$ be the basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, and let

$$S_\varepsilon = S \cap \mathbf{B}^k_{1/\varepsilon}.$$

Proposition 3.1. *For all sufficiently small $\varepsilon > 0$, the homology groups of S and S_ε are isomorphic. Moreover, S_ε is bounded.*

Proof. The proof follows from Hardt's triviality theorem (see [8], Theorem 5.45.) and is similar to the proof of Lemma 1 in [4]. □

Before we continue, consider Figure 1 which will be helpful for the following. The cylinder $\mathbf{B}^k_{1/\varepsilon} \times \mathbb{R}$ above the ball $\mathbf{B}^k_{1/\varepsilon} \subset \mathbb{R}^k$ intersects the sphere $\mathbf{S}^k_{2/\varepsilon}$ in two disjoint copies (each homeomorphic to $\mathbf{B}^k_{1/\varepsilon}$). Each cylinder above those copies intersects the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ in two disjoint copies. Thus, there are four disjoint copies of $\mathbf{B}^k_{1/\varepsilon}$ on the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ (each homeomorphic to $\mathbf{B}^k_{1/\varepsilon}$). Notice that each such copy does not intersect the equator of the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ in $\mathbb{R}^{k+2}$ (i.e. the set $\mathbf{S}^{k+1}_{4/\varepsilon} \cap \mathrm{Zer}(X_{k+2}, \mathbb{R}^{k+2})$).

Let S^h_ε be the basic semi-algebraic set defined by $P^h_1 \geq 0, \ldots, P^h_s \geq 0$ contained in C_ε, where

$$C_\varepsilon = (\mathbf{B}^k_{1/\varepsilon} \times \mathbb{R}) \cap \mathbf{S}^k_{2/\varepsilon}.$$

Lemma 3.2. *For sufficiently small $\varepsilon > 0$ and $0 \leq i \leq k$, we have*

$$b_i(S_\varepsilon) = \frac{1}{2} b_i(S^h_\varepsilon).$$

Proof. Note that S_ε is bounded by Proposition 3.1 and S^h_ε is the projection from the origin of the set $S_\varepsilon \times \{1\} \subset \mathbb{R}^k \times \{1\}$ onto the unit sphere in $\mathbb{R}^{k+1}$. Since S_ε is bounded, the projection does not intersect the equator and consists of two disjoint copies (each homeomorphic to the set S_ε) in the upper and lower hemispheres. □

We now fix a sufficiently small $\varepsilon > 0$ and a family of polynomials that will be useful in what follows. Let

$$P = (\frac{2}{\varepsilon})^2 - \sum_{i=1}^{k+1} X_i^2.$$

By Proposition 2.7 we can choose a family $\mathcal{H} = \{H_1, \ldots, H_s\} \subset \mathbb{R}[X_1, \ldots, X_{k+2}]$ of positive definite quadratic forms such that $\mathrm{Zer}(\mathcal{H}_J \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{C}})$ is a non-singular complete intersection for every $J \subset \{1, \ldots, s\}$.

Let $\delta > 0$ and let $\tilde{P}_i = (1-\delta)P^h_i + \delta\tilde{H}_i$, $1 \le i \le s$, where $\tilde{H}_i = H_i(X_1, \ldots, X_{k+1}, 1)$. Note that $\tilde{H}_i$, $1 \le i \le s$, is positive definite since H_i is positive definite. Let $T_{\varepsilon,\delta} \subset \mathbb{R}^{k+1}$ (resp., $\bar{T}_{\varepsilon,\delta} \subset \mathbb{R}^{k+1}$) be the basic semi-algebraic set defined by $\tilde{P}_1 > 0, \ldots, \tilde{P}_s > 0$ (resp., $\tilde{P}_1 \ge 0, \ldots, \tilde{P}_s \ge 0$) contained in C_ε.

Also, let

$$\tilde{\mathcal{P}} = \{\tilde{P}_1, \ldots, \tilde{P}_s\}.$$

Lemma 3.3. *For all sufficiently small $0 < \delta < \varepsilon$ we have,*

1. *the homology groups of S^h_ε and $\bar{T}_{\varepsilon,\delta}$ are isomorphic,*
2. *the homology groups of $T_{\varepsilon,\delta}$ and $\bar{T}_{\varepsilon,\delta}$ are isomorphic.*

Proof. For the first part note that S_ε and $\bar{T}_{\varepsilon,\delta}$ have the same homotopy type using Lemma 16.17 in [8].

The second part is clear since by choosing any slightly smaller $0 < \delta' < \delta$, we have a retraction from $T_{\varepsilon,\delta}$ to $\bar{T}_{\varepsilon,\delta'}$. □

Now, let $T^h_{\varepsilon,\delta} \subset \mathbb{R}^{k+2}$ be the semi-algebraic set defined by $\tilde{P}^h_1 > 0, \ldots, \tilde{P}^h_s > 0$ contained in $\tilde{C}_\varepsilon$, where

$$\tilde{C}_\varepsilon = (\mathbf{B}^k_{1/\varepsilon} \times \mathbb{R}^2) \cap (\mathbf{S}^k_{2/\varepsilon} \times \mathbb{R}) \cap \mathbf{S}^{k+1}_{4/\varepsilon}.$$

Also, let

$$\tilde{\mathcal{P}}^h = \{\tilde{P}^h_1, \ldots, \tilde{P}^h_s\}.$$

Lemma 3.4. *For all sufficiently small $0 < \delta < \varepsilon$ and $0 \le i \le k$,*

1. $b_i(T_{\varepsilon,\delta}) = \frac{1}{2} b_i(T^h_{\varepsilon,\delta})$,
2. *for all $J \subset \{1, \ldots, s\}$,*
$$b_i\left(\mathrm{Zer}(\tilde{\mathcal{P}}^h_J, \tilde{C}_\varepsilon)\right) = 2 \cdot b_i\left(\mathrm{Zer}(\tilde{\mathcal{P}}^h_J \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{R}})\right),$$
3. *for all $J \subset \{1, \ldots, s\}$, $\mathrm{Zer}(\tilde{\mathcal{P}}^h_J \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{C}})$ is a non-singular complete intersection.*

Proof. First, observe that $T^h_{\varepsilon,\delta} \subset \tilde{C}_\varepsilon$ is the projection from the origin of $T_{\varepsilon,\delta} \times \{1\} \subset C_\varepsilon \times \{1\}$ onto the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ in $\mathbb{R}^{k+2}$. Note that $T^h_{\varepsilon,\delta}$ does not intersect the equator of the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ in $\mathbb{R}^{k+2}$ (i.e. the set $\tilde{C}_\varepsilon \cap \mathrm{Zer}(X_{k+2}, \mathbb{R}^{k+2})$), and consists of two disjoint copies (each homeomorphic to the set $T_{\varepsilon,\delta}$) in the upper and lower hemisphere.

For the second part, note that the set $\mathrm{Zer}(\tilde{\mathcal{P}}_J^h, \tilde{C}_\varepsilon)$ does not intersect the equator of the sphere $\mathbf{S}^{k+1}_{4/\varepsilon}$ in $\mathbb{R}^{k+2}$. Moreover, the two-fold covering $\pi : \mathbf{S}^{k+1}_{4/\varepsilon} \to \mathbb{P}^{k+1}_{\mathbb{R}}$ (obtained by identifying antipodal points) restricts to a homeomorphism on the upper and lower hemisphere.

The third part follows from Proposition 2.8. □

Proposition 3.5. *For all sufficiently small* $0 < \delta < \varepsilon$ *and for* $0 \le i \le k-1$, *we have*

$$b_i(T_{\varepsilon,\delta}) \le \sum_{j=0}^{min\{s,k-i\}} \binom{s}{j}\binom{k+1}{j} 2^j.$$

Proof. By Lemma 3.4 (1) it suffices to prove the statement for the set $T^h_{\varepsilon,\delta}$. Note that $\mathrm{Zer}(\tilde{\mathcal{P}}_J^h \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{C}})$ is a complete intersection for all $J \subset \{1,\ldots,s\}$ by Lemma 3.4 (3).

For $0 \le i \le k-1$,

$$\begin{aligned} b_i(T^h_{\varepsilon,\delta}) &\le b_i\left(\mathbf{S}^k_{4/\varepsilon} \setminus \bigcup_{i=1}^s \mathrm{Zer}(\tilde{P}_i^h, \tilde{C}_\varepsilon)\right) \\ &\le 1 + b_{k-1-i}\left(\bigcup_{i=1}^s \mathrm{Zer}(\tilde{P}_i^h, \tilde{C}_\varepsilon)\right), \end{aligned}$$

where the first inequality is a consequence of the fact that, $T^h_{\varepsilon,\delta}$ is an open subset of $\mathbf{S}^k_{4/\varepsilon} \setminus \bigcup_{i=1}^s \mathrm{Zer}(\tilde{P}_i^h, \tilde{C}_\varepsilon)$ and disconnected from its complement in $\mathbf{S}^k_{4/\varepsilon} \setminus \bigcup_{i=1}^s \mathrm{Zer}(\tilde{P}_i^h, \tilde{C}_\varepsilon)$, and the last inequality follows from Theorem 2.2 (Alexander Duality). It follows from Proposition 2.1, Lemma 3.4 (2) and Theorem 2.3 (Smith inequality), that

$$\begin{aligned} b_i(T^h_{\varepsilon,\delta}) &\le 1 + \sum_{j=1}^{k-i}\sum_{|J|=j} b_{k-i-j}\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J^h, \tilde{C}_\varepsilon)\right) \\ &= 1 + 2\cdot\sum_{j=1}^{k-i}\sum_{|J|=j} b_{k-i-j}\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J^h \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{R}})\right) \\ &\le 1 + 2\cdot\sum_{j=1}^{k-i}\sum_{|J|=j} b_{k-i-j}\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J^h \cup \{P^h\}, \mathbb{P}^{k+1}_{\mathbb{C}})\right). \end{aligned}$$

Note that for $j \le s$ the number of possible j-ary intersections is equal to $\binom{s}{j}$ and using Theorem 2.6, we conclude

$$\begin{aligned} b_i(T^h_{\varepsilon,\delta}) &\le 1 + 2\cdot\sum_{j=1}^{min\{s,k-i\}} \binom{s}{j} b(j+1,k+1) \\ &\le 2\cdot\sum_{j=0}^{min\{s,k-i\}} \binom{s}{j}\binom{k+1}{j} 2^j. \end{aligned}$$

The claim follows since $b_i(T_{\varepsilon,\delta}) = \frac{1}{2}\cdot b_i(T^h_{\varepsilon,\delta})$. □

We are now in a position to prove our main result.

Proof of Theorem 1.6. For all sufficiently small $0 < \delta < \varepsilon$ we have by Lemma 3.3 that the homology groups of S^h_ε and $T_{\varepsilon,\delta}$ are isomorphic. Moreover, for $0 \leq i \leq k-1$, $b_i(S) = \frac{1}{2} b_i(S^h_\varepsilon)$ by Proposition 3.1 and Lemma 3.2. Hence, the first part follows from Proposition 3.5.

The second part follows from an easy computation. □

Finally, we prove Corollary 1.7.

Proof of Corollary 1.7. Follows by applying the bound of Theorem 1.6 to each $b_i(S)$, $0 \leq i \leq k-1$. □

4. Conclusion and Open Problems

In this paper we have improved the upper bound proved by Barvinok on the Betti numbers of semi-algebraic sets in $\mathbb{R}^k$ defined by $s \leq \frac{k}{2}$ quadratic inequalities. The new bound is of the form $(O(k/s))^s$ improving the previous bound of $k^{O(s)}$ due to Barvinok. Using the fact that a complex non-singular complete intersection in $\mathbb{C}^k$ defined by s quadratic equations can be viewed as a real semi-algebraic set in $\mathbb{R}^{2k}$ defined $2s$ quadratic equations, it follows that the best bound on the sum of the Betti numbers of semi-algebraic sets defined by s quadratic inequalities in $\mathbb{R}^k$ cannot be better than $k^{O(s)}$. We conjecture that the exponent, s, in our bound is in fact optimal and an interesting open problem is to construct an example which meets our bound.

Another interesting problem in this context is to obtain a tighter bound on the number of connected components (that is on $b_0(S)$) for $S \subset \mathbb{R}^k$ defined by $s \leq k$ quadratic inequalities. It can be easily seen from the example of $S \subset \mathbb{R}^k$ defined by,

$$X_1(X_1 - 1) \geq 0, \ldots, X_s(X_s - 1) \geq 0,$$

that $b_0(S)$ can be as large as 2^s. However, we know of no examples where $b_0(S)$ is as large as $k^{\Omega(s)}$. Note that the Betti numbers of a non-singular complex complete intersection is concentrated in the "middle" dimension. Consequently, the Smith inequality gives bounds only on the sum of the Betti numbers of the corresponding real varieties. Because of this the method of the proof used in this paper has the drawback that it gives no way of proving better bounds on the individual (say the lowest or the highest) Betti numbers.

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