The Hadwiger transversal theorem for pseudolines

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Abstract

We generalize the Hadwiger theorem on line transversals to collections of compact convex sets in the plane to the case where the sets are connected and the transversals form an arrangement of pseudolines. The proof uses the embeddability of pseudoline arrangements in topological affine planes.

In 1940 Santaló showed **[12]**, by an example, that Vincensini's proof **[13]** of an extension of Helly's theorem was incorrect. Vincensini claimed to have proven that for any finite collection S of at least three compact convex sets in the plane, any three of which were met by a line, there must exist a line meeting all the sets. This would have constituted an extension of the planar Helly theorem **[10]**, which showed that the same assertion holds if "line" is replaced by "point." The Santaló example was later extended by Hadwiger and Debrunner **[9]** to show that even if the convex sets are disjoint the conclusion still may not hold.

In 1957, however, Hadwiger showed that the conclusion of the theorem *is* valid if the hypothesis is strengthened by imposing a consistency condition on the order in which the triples of sets are met by transversals:

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Theorem (Hadwiger **[8]**). *If $B_1, \ldots, B_n$ is a family of disjoint compact convex sets in the plane with the property that for any $1 \le i < j < k \le n$ there is a line meeting each of B_i, B_j, B_k in that order, then there is a line meeting all the sets B_i.*

In 1988, the second and fourth authors found a generalization of Hadwiger's theorem to the case of hyperplane transversals **[4]**, and this in turn was subsequently extended in **[11]** and **[14]**, culminating in the following result:

Theorem (Anderson-Wenger **[1]**). *Let $\mathcal{A}$ be a finite collection of connected sets in $\mathbb{R}^d$. $\mathcal{A}$ has a hyperplane transversal if and only if for some k, $0 \le k < d$, there exists a rank $k+1$ acyclic oriented matroid structure on $\mathcal{A}$ such that every $k+2$ members of $\mathcal{A}$ are met by an oriented k-flat consistently with that oriented matroid structure.*

Our purpose in this paper is to extend the original Hadwiger theorem in a different direction—replacing "lines" by "pseudolines." A *pseudoline* in the affine plane is simply the homeomorphic image of a line. If that were all, the theorem would be true trivially: for any finite collection of sets there is a pseudoline meeting them in any prescribed order! (Of course this needs a suitable interpretation in the case where the sets are not mutually disjoint; see below.) But to reflect more accurately the properties of sets of lines in the plane, one insists that all the pseudolines under consideration form an *arrangement*, which means that they are finite in number, that any two meet exactly once, where they cross, and (for technical reasons) that they do not all pass through the same point.[1] (For examples of pseudoline arrangements that are not isomorphic, in a natural sense, to arrangements of straight lines, see, e.g., **[3]**.) Furthermore, given a pseudoline arrangement $\mathcal{A}$ we say that a pseudoline l *extends* $\mathcal{A}$ if $\mathcal{A} \cup \{l\}$ is also an arrangement of pseudolines. Thus the theorem we are going to prove is the following:

Theorem 1. *Suppose $B_1, \ldots, B_n$ is a family of connected compact sets in the plane such that for each $1 \le i < j < k \le n$ there is a pseudoline l_{ijk} meeting each of B_i, B_j, B_k at points p_i, p_j, p_k, not necessarily distinct, contained in B_i, B_j, B_k, respectively, with p_j lying between p_i and p_k on l_{ijk}. Suppose further that the pseudolines l_{ijk} constitute an* arrangement *$\mathcal{A}$. Then there exists a pseudoline l that extends the arrangement $\mathcal{A}$ and meets each set B_i.*

As in Wenger's generalization **[14]**, we do not assume the sets to be disjoint or even convex, merely connected. And in fact we will prove Theorem 1 by generalizing Wenger's proof, and by using the following result on topological planes:

[1] This is actually the definition of a "pseudoline arrangement" in the projective plane, while in the affine plane one allows pseudolines also to be "parallel"; in a finite arrangement, however, pseudolines can always be perturbed slightly to meet "at finite distance," and we will assume this whenever convenient.

Theorem (Goodman-Pollack-Wenger-Zamfirescu **[7]**). *Any arrangement of pseudolines in the projective plane can be extended to a topological projective plane.*

Here, a *topological projective plane* means $\mathbb{P}^2$, together with a distinguished collection $\mathcal{L}$ of pseudolines, one for each pair of points, varying continuously with the points, any two meeting (and crossing) exactly once. If we call a topological projective plane with one of its distinguished pseudolines removed a *topological affine plane* (TAP), the theorem above can trivially be modified to read: *Any arrangement of pseudolines in the affine plane can be extended to a TAP*. We will use it in this form.

For background on pseudoline arrangements as well as on geometric transversal theory, the interested reader may consult the following surveys: **[2, 3, 5, 6, 15, 16]**.

We now introduce some notions that will be used in the proof of the theorem.

Since $\mathbb{P}^2$ can be modeled by a closed circular disk Δ with antipodal points on the boundary $\partial\Delta$ identified, we will model our TAP by using int Δ, the interior of Δ, and call two pseudolines "parallel" if they meet on $\partial\Delta$. (From now on, whenever we speak of "pseudolines" in the TAP, we will mean members of the distinguished family of pseudolines constituting its "lines.") An *arrangement* of pseudolines is thus a finite set of Jordan arcs, each joining a pair of antipodal points of $\partial\Delta$, any two meeting (and crossing) exactly once, or possibly at their endpoints (the "parallel" case).

We will also speak of *directed* pseudolines, which corresponds to specifying one of the antipodal points where the pseudoline meets $\partial\Delta$. Thus it will make sense to say: let p be a point on $\partial\Delta$ and let l_p be a pseudoline in the direction p. Further, when we direct a pseudoline, we specify a positive and negative open "pseudohalfspace" bounded by that line, determined with respect to a fixed orientation of Δ. We denote these halfspaces by $H_+(l_p)$ and $H_-(l_p)$, respectively; see Figure 1.

Now let A and B be two connected compact sets in our TAP and let $p \in \partial\Delta$. If there is a pseudoline in the direction of p that contains points $a \in A$ and $b \in B$, with either $a = b$ or a preceding b on the pseudoline, we say that p is an (AB)-*transversal direction*. If there is a pseudoline l_p that strictly separates A and B such that $A \subset H_+(l_p)$ and $B \subset H_-(l_p)$, we say that p is a (AB)-*separating direction*.

Notice that a given direction can be both an (AB)-transversal direction and a (BA)-transversal direction; even the same pseudoline, in fact, can meet A before B and B before A in this sense.

Notice also that given a pair A, B, each direction p is either a transversal direction or a separating direction for A, B; this follows by a simple continuity argument, sweeping a pseudoline in direction p across the TAP.

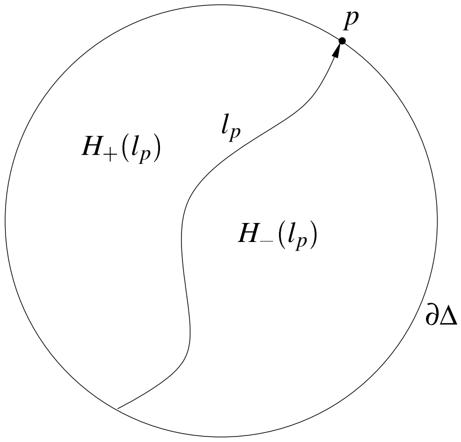

Figure 1

Finally, notice that if there is an (AB)-separating direction p, then no direction q can be both an (AB)-transversal direction and a (BA)-transversal direction. This follows from the fact that if two pseudolines have the same direction q, they must cross a given pseudoline l in direction p the same way: both from $H_+(l)$ to $H_-(l)$, or both from $H_-(l)$ to $H_+(l)$.

It then follows from the definition of a TAP and the compactness of our sets that the set T_{AB} of (AB)-transversal directions is a closed arc of $\partial\Delta$: If A and B have a point in common then clearly $T_{AB} = \partial\Delta$. If not, consider any two distinct directions $p_1, p_2 \in T_{AB}$. For $i = 1,2$ choose points $a_i \in A$, $b_i \in B$ along a pseudoline l_i in direction p_i, with a_i preceding b_i, as well as parametrized arcs $a(t) \subset A$ and $b(t) \subset B$ from a_1 to a_2 (resp. b_1 to b_2). By continuity, the set of directions $\overrightarrow{a(t)b(t)}$ must contain one of the two arcs on $\partial\Delta$ joining p_1 and p_2. It follows that the set T_{AB} is itself an arc (possibly all of $\partial\Delta$), and this must be closed by the compactness of the sets A and B.

We have thus proved the following:

Lemma 2. *Let A and B be connected compact sets in the plane. Then*

$$\partial\Delta = T_{AB} \cup S_{AB} \cup T_{BA} \cup S_{BA},$$

where $T_{AB} = -T_{BA}$ is the closed arc corresponding to the (AB)-transversal directions, and $S_{AB} = -S_{BA}$ is the open arc corresponding to the (AB)-separating directions. (Note that S_{AB} can be empty.)

To complete the proof of Theorem 1, we extend the arrangement $\mathcal{A}$ to a topological affine plane. We want to show first that there is a direction $p \in \partial\Delta$ that is a transversal direction for every pair B_i, B_j. For each pair B_i, B_j, let S_{ij} be the open arc of (B_iB_j)-separating directions. Now define the following antipodal sets:

$$\mathcal{S}_+ = \bigcup_{i<j} S_{ij} \quad , \quad \mathcal{S}_- = \bigcup_{i<j} S_{ji} \quad .$$

If there is no point $p \in \partial\Delta$ that is a transversal direction for every pair B_i, B_j then we must have $\partial\Delta = \mathcal{S}_+ \cup \mathcal{S}_-$. But since $\mathcal{S}_+$ and $\mathcal{S}_-$ are open sets that cover $\partial\Delta$ there must be a point $p \in \mathcal{S}_+ \cap \mathcal{S}_-$. But then we would have pseudolines l_1 and l_2, both directed toward p, and sets B_i, B_j, B_k, B_l with $i < j$ and $k < l$, such that $B_i \subset h_+(l_1)$, $B_j \subset h_-(l_1)$, $B_k \subset h_-(l_2)$, and $B_l \subset h_+(l_2)$. It is then easy to check that there would always be some triple that violates the transversal assumption; see Figure 2 for a typical case.

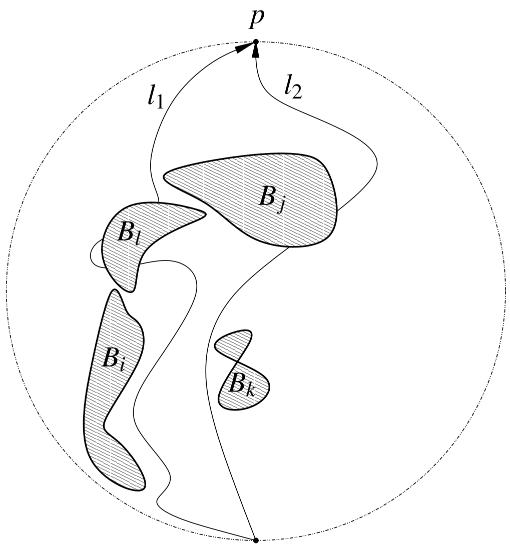

Figure 2: If $i < k$, $\nexists\, l_{ikl}$; if $k < i$, $\nexists\, l_{kij}$

This means that there is a direction $q \in \partial\Delta$ that is a transversal direction for every pair B_i, B_j. It follows that q is not a separating direction for any pair B_i, B_j, so that a pseudoline in direction q sweeping through the TAP must pass simultaneously through all the sets B_i at some point. This completes the proof.

Remarks:

1. It is not hard to see that Theorem 1 is equivalent to the following.

Theorem 3. *Suppose $\mathcal{L}$ is an arrangement of pseudolines in the affine plane. For each triple $i < j < k$ in $[1,n]$, select three (not necessarily distinct) points belonging to the same pseudoline of $\mathcal{L}$, and label them i, j, k, with the point labeled j between* *the other two (or possibly equal to one or both). Then there is a pseudoline l extending the arrangement $\mathcal{L}$ such that for each $i \in [1,n]$ there are points labeled i in both (closed) halfspaces bounded by l.*

2. As in the original Hadwiger theorem, one cannot strengthen the conclusion of Theorem 1 to include the assertion that the common transversal meets the sets in the order $1,2,\dots,n$ (see **[14]** for an example). But it is easily seen that, as in **[14]**, that stronger assertion follows if we are willing to assume that every *six* of the sets are met in a consistent order; the argument is the same, *mutatis mutandis*.

Theorem 4. *Suppose $B_1,\dots,B_n$ is a family of at least six connected compact sets in the plane such that for each $1 \le f < g < h < i < j < k \le n$ there is a pseudoline l_{fghijk} meeting each of B_f,B_g,B_h,B_i,B_j,B_k at points p_f,p_g,p_h,p_i,p_j,p_k, not necessarily distinct, contained in B_f,B_g,B_h,B_i,B_j,B_k, respectively, and occurring in that order on l_{fghijk}. Suppose further that the pseudolines l_{fghijk} constitute an* arrangement $\mathcal{A}$. *Then there exists a pseudoline l that extends the arrangement $\mathcal{A}$ and meets each all the sets $B_1,\dots,B_n$ in that order.*

The example in **[14]** showing that the number 6 in the corresponding result for straight lines and convex sets is tight does not seem correct. Here is an example, however, showing that the result would fail for a collection $B_1,\dots,B_6$ of convex sets if we assumed only that every five were met in a consistent order; here every five sets have a transversal meeting them in numerical order, but all six do not:

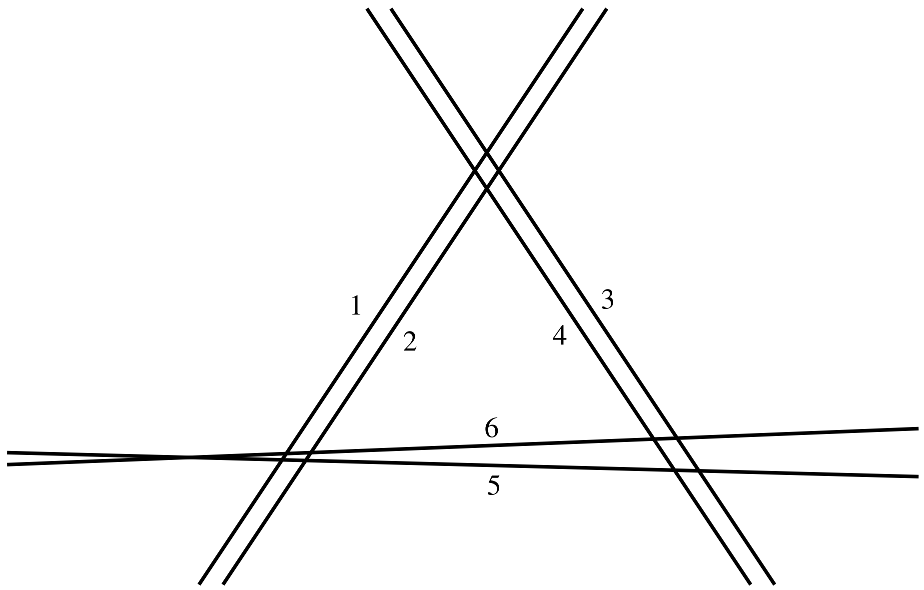

Figure 3

3. In the process of proving Theorem 1, we have actually proven the following (stronger) theorem about TAPs:

Theorem 5. *If $B_1, \ldots, B_n$ is a family of connected compact sets in a topological affine plane $\mathcal{P}$ with the property that for any $1 \leq i < j < k \leq n$ there is a pseudoline of $\mathcal{P}$ meeting each of B_i, B_j, B_k in that order, then there is a pseudolineline of $\mathcal{P}$ meeting all the sets B_i.*

This raises the question: What other transversal theorems extend to TAPs?

4. Finally, what about higher dimensions? The notion of 'topological plane' extends only trivially to dimension ≥ 3, since, as is well-known, Desargues's theorem holds automatically in higher dimensions and any d-dimensional "topological projective space" is consequently isomorphic to the usual projective space $\mathbb{P}^d$. Nevertheless, one may ask: Does Theorem 1 extend in some way, in dimension > 2, to a result about (finite) arrangements of pseudohyperplane transversals?

References

[1] L. Anderson and R. Wenger. Oriented matroids and hyperplane transversals. *Adv. in Math.* 119 (1996), 117–125.

[2] J. Eckhoff. Helly, Radon, and Carathéodory type theorems. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, Volume A, North-Holland, Amsterdam, 1993, pages 389–448.

[3] J. E. Goodman. Pseudoline arrangements. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, Boca Raton, 2004, pages 97–128.

[4] J. E. Goodman and R. Pollack. Hadwiger's transversal theorem in higher dimensions. *J. Amer. Math. Soc.* 1 (1988), 301–309.

[5] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, Volume 10 of *Algor. Combin.*, Springer-Verlag, Berlin, 1993, pages 163–198.

[6] B. Grünbaum. *Arrangements and Spreads*. Amer. Math. Soc., Providence, 1972.

[7] J. E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. *Amer. Math. Monthly* 101 (1994), 866–878.

[8] H. Hadwiger. Über Eibereiche mit gemeinsamer Treffgeraden. *Portugal. Math.* 6 (1957), 23–29.

[9] H. Hadwiger, H. Debrunner, and V. Klee. *Combinatorial Geometry in the Plane*. Holt, Rinehart & Winston, New York, 1964.

[10] E. Helly. Über Mengen konvexen Körper mit gemeinschaftlichen Punkten. *Jber. Deutsch. Math. Verein 32* (1923), 175–176.

[11] R. Pollack and R. Wenger. Necessary and sufficient conditions for hyperplane transversals. *Combinatorica 10* (1990), 307–311.

[12] L. Santaló. Un teorema sobre conjuntos de paralelepipedos de aristas paralelas. *Publ. Inst. Mat. Univ. Nac. Litoral 2* (1940), 49–60.

[13] P. Vincensini. Figures convexes et variétés linéaires de l'espace euclidien à n dimensions. *Bull. Sci. Math. 59* (1935), 163–174.

[14] R. Wenger. A generalization of Hadwiger's theorem to intersecting sets. *Discrete Comput. Geom. 5* (1990), 383–388.

[15] R. Wenger. Progress in geometric transversal theory. In B. Chazelle, J. E. Goodman, and R. Pollack, editors, *Advances in Discrete and Computational Geometry*, Volume 223 of *Contemp. Math.*, Amer. Math. Soc., Providence, 1999, pages 375–393.

[16] R. Wenger. Helly-type theorems and geometric transversals. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, Boca Raton, 2004, pages 73–96.