On the Combinatorial and Topological Complexity of a Single Cell

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Abstract

The problem of bounding the *combinatorial complexity* of a single connected component (a single cell) of the complement of a set of n geometric objects in R^k of constant description complexity is an important problem in computational geometry which has attracted much attention over the past decade. It has been conjectured that the combinatorial complexity of a single cell is bounded by a function much closer to $O(n^{k-1})$ rather than $O(n^k)$ which is the bound for the combinatorial complexity of the whole arrangement. Till now, this was known to be true only for $k \leq 3$ and only for some special cases in higher dimensions.

A classic result in real algebraic geometry due to Oleinik-Petrovsky, Thom and Milnor, bounds the *topological complexity* (the sum of the Betti numbers) of basic semi-algebraic sets. However, till now no better bounds were known if we restricted attention to a single connected component of a basic semi-algebraic set.

In this paper, we show how these two problems are related. We prove a new bound on the sum of the Betti numbers of one connected component of a basic semi-algebraic set which is an improvement over the Oleinik-Petrovsky-Thom-Milnor bound. This also implies that the topological complexity of a single cell, measured by the sum of the Betti numbers, is bounded by $O(n^{k-1})$. The techniques used for proving this topological result combined with those developed by Halperin and Sharir for the single cell problem in three dimensions allow us to prove a bound of $O(n^{k-1+\epsilon})$ on the combinatorial complexity of a single cell.

Finally, we show that under a certain natural geometric assumption on the objects (namely that, whenever they intersect the intersection is robustly transversal on average) it is possible to prove a bound of $O(n^{k-1})$ on the combinatorial complexity of a single cell.

1 Introduction

Arrangements of a finite collection of geometric objects and their combinatorial and algorithmic properties have been fundamental objects of study in computational geometry. The most general setting considered in the computational geometry literature [1] is that of an arrangement, $A(\Sigma)$, of a set $\Sigma = \{S_1, \ldots, S_n\}$ of n *surface patches* in R^k.

Each surface patch S_i is a *closed* semi-algebraic set contained in a hypersurface $Z(Q_i)$ (the algebraic set defined by $Q_i = 0$) and defined by a first-order quantifier-free formula involving a family of polynomials, $\{P_{i,1}, \ldots, P_{i,r}\}$. We will assume that the degrees of all the polynomials, $Q_i, P_{i,j}$ are bounded by d. Also, the parameters r, d, k are treated as constants as is usual in the computational geometry literature.

A *cell* is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches. Thus, a k-dimensional cell is a connected component of the complement of $S_1, \ldots, S_n$. The *combinatorial complexity* of the arrangement is the total number of cells of all dimensions. The combinatorial complexity of an ℓ-dimensional cell C is the number of cells of dimension less than ℓ which are contained in the relative boundary of C.

Usually, some additional *geometric* assumptions are made about the surface patches. In some of our theorems we will impose one or both of the following two conditions T1 and T2.

T1 Each surface patch is part of a smooth hypersurface and whenever surface patches intersect they do so transversally. Thus, the intersection of any ℓ surface patches is either empty or a $(k-\ell)$-dimensional manifold (with boundary). This also

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implies that no $k+1$ of the surface patches have a non-empty intersection.

T2 For any k-dimensional cell C, the points of $\bar{C}$ (the closure of C) that are common to k of these hypersurfaces are isolated. They are called the vertices of C. If x is a vertex of C common to k patches $S_{i_1}, \ldots, S_{i_k}$, then the unit normals $N_1, \ldots, N_k$ (chosen with any orientations) to these hypersurfaces at p, span an (k-dimensional) angle of at least b, for some *real* constant $b > 0$.

Note that this latter condition is a natural strengthening of the requirement that the surfaces meet transversally at the vertices. If they are required to meet transversally then the solid angle will be positive. By requiring that the solid angle be at least b we are imposing the extra condition that the transversality be robust. Also, even though the condition T2 is stated in *worst case* terms, in our applications we only require the *weaker* assumption that the *average* angle at the vertices made by the normals is at least b where the average is taken over all the vertices as well as the orientations of the unit normals.

One important problem that has remained open till now has been to bound the combinatorial complexity of a single k-dimensional cell in an arrangement of n surface patches. Various special cases have been considered by researchers [14, 2, 15, 10]. In the special case, when each surface patch is a hyperplane, the combinatorial complexity of a single cell (which is a convex polytope with at most n facets) is bounded by $n^{\lfloor k/2 \rfloor}$ by the upper bound theorem [11]. In the more general situation the prevailing conjecture is that the combinatorial complexity of a single cell is bounded by $O(n^{k-1}\beta(n))$ for some extremely slow growing function $\beta(n)$. This is known for $k=2$ (see [8]). The best known result for $k=3$ is due to Halperin and Sharir [10] who proved a bound of $O(n^{2+\epsilon})$ for every $\epsilon > 0$. In higher dimensions bounds close to $O(n^{k-1})$ are known only for certain special cases. For instance, a $O(n^{k-1}\log n)$ bound was proved by Aronov and Sharir [2] (see also [15]) for the combinatorial complexity of a single cell in an arrangement of n $(k-1)$-simplices in R^k. These bounds do not extend to general surface patches. The general problem was open in higher dimensions. Note also, that using the lower bound construction given by Wiernik and Sharir [17] one can prove a lower bound of $\Omega(n^{k-1}\beta(n))$, and hence a $O(n^{k-1})$ bound is not possible. For a survey of the numerous algorithmic applications of these bounds in computational geometry and robotics see [1, 9].

Another problem that has received considerable attention from researchers interested in real algebraic geometry, is to bound the topological complexity of semi-algebraic sets. The first and classical result in this area is due to Oleinik and Petrovsky [13], Thom [16] and Milnor [12], who independently proved that the sum of the Betti numbers, $\beta_i(S)$, of a basic closed semi-algebraic set $S \in R^k$ defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $deg(P_i) \leq d, 1 \leq i \leq s$, is bounded by $O(sd)^k$. This bound was extended to arbitrary closed semi-algebraic sets in [6], where it is shown that the sum of the Betti numbers of a closed semi-algebraic set defined in terms of s polynomials of degree d, which is contained in a real variety of dimension k' defined by a polynomial of degree at most d, is bounded by $s^{k'}O(d)^k$.

These bounds are essentially tight as one can easily define a basic semi-algebraic set with s polynomials and degrees bounded by d which has $(sd)^k$ connected components. However, until now no attempt has been made to study the topological complexity of a single connected component of a basic semi-algebraic set. In analogy to the single cell combinatorial complexity results in computational geometry, one might conjecture that the sum of the Betti numbers of a single connected component of a basic semi-algebraic set is bounded by $s^{k-1}O(d)^k$. One cannot hope to do much better as it is easy to construct a basic semi-algebraic set defined in terms of s polynomials of degree d such that it has one connected component the sum of whose Betti numbers is $\Omega(sd)^{k-1}$.

2 New Results

First, we bound the topological complexity of a single connected component of a basic semi-algebraic set.

Theorem 1 *Let $C \subset R^k$ be a connected component of a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with the degrees of the polynomials P_i bounded by d. Then the sum of the Betti numbers of C is bounded by $\binom{s}{k-1}O(d)^k$.*

Actually, our technique proves the following stronger version of the above theorem.

Theorem 2 *Let $C_1, \ldots, C_m \subset R^k$ be m different connected components of a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with the degrees of the polynomials P_i bounded by d. Then $\sum_{i,j}\beta_i(C_j)$ is bounded by $m + \binom{s}{k-1}O(d)^k$.*

For single cells we have the following results.

Let C be a k-dimensional cell in an arrangement of n surface patches $S_1, \ldots, S_n$. A corollary of theorem 1 is the following:

Corollary 1 *The sum of the Betti numbers of C is bounded by $O(n^{k-1})$.*

Note that no general position assumption is required for the above corollary.

We are also interested in bounding the combinatorial complexity of C. In order to do this, it is enough to bound the number of vertices (the 0-dimensional cells) of C (see [1]).

Theorem 3 *Let C be a k-dimensional cell in an arrangement of n surface patches $S_1, \ldots, S_n$ in R^k satisfying conditions T1 and T2. Then, the number of vertices and hence the combinatorial complexity of C is bounded by $O(b^{-1}n^{k-1})$.*

As before, we can prove a bound of $O(b^{-1}n^{k-1}+m)$ on the sum of the complexities of m different cells. This immediately gives us a bound of $O(b^{-1}n^{k-1})$ on the combinatorial complexity of the *zone* of an algebraic variety of degree d in an arrangement of n surface patches in R^k.

Next, using the topological tools developed in this paper we are able to extend Halperin and Sharir's bound on the combinatorial complexity of a single cell in R^3 to all higher dimensions.

Theorem 4 *Let C be a k-dimensional cell in an arrangement of n surface patches $S_1, \ldots, S_n$ in R^k, satisfying condition T1. Then the combinatorial complexity of C is bounded by $O(n^{k-1+\epsilon})$ for every $\epsilon > 0$.*

In corollary 2 and theorems 3 and 4 , the constants depend on the parameters r, d, k used in the definition of the surface patches and also on ϵ. The dependence on the degree d is of the order of $2^{O(d)^k}$.

3 Mathematical Preliminaries

In the following we will perturb polynomials by various *infinitesimals* so that our geometric objects live over the field of Puiseux series in these infinitesimals. We write $R\langle\zeta\rangle$ for the real closed field of Puiseux series in ζ with coefficients in R [3]. The sign of such a Puiseux series agrees with the sign of the coefficient of the lowest degree term in ζ. This order makes ζ positive and smaller than any positive element of R. The map eval_ζ maps an element of $R\langle\zeta\rangle$ bounded over R (one that has no negative powers of ζ) to its constant term. An element of $R\langle\zeta\rangle$ is infinitesimal over R if it is mapped by eval_ζ to 0. In particular ζ is infinitesimal over R. If S is a semi-algebraic subset of R^k we denote by $S_{R\langle\zeta\rangle}$ the subset of $R\langle\zeta\rangle^k$ defined by the same equalities and inequalities that define S. We refer the reader to [4] for more details on the use of infinitesimals. The main topological results that we will require are the two basic lemmas of stratified Morse theory which we now proceed to state. (Note that stratified Morse theory is available only over the reals and not over general real closed fields. We will continue to use the infinitesimal notation. However, a positive infinitesimal will denote a sufficiently small positive real number.)

For the rest of the paper we will denote by π the projection map onto the first co-ordinate. For any set $S \subset R^k$ we let S_x (resp. $S_{\leq x}$) denote $S \cap \pi^{-1}(x)$ (resp. $S \cap \pi^{-1}((-\infty, x]))$. A Whitney stratification of a space X is a decomposition of X into sub-manifolds called strata, which satisfy certain frontier conditions, (see [7] page 37). In particular, given a compact set bounded by a smooth algebraic hypersurface, the boundary and the interior form a Whitney stratification.

Now, let X be a compact Whitney stratified subset of R^k, and f a restriction to X of a smooth function. A critical point of f is defined to be a critical point of the restriction of f to any stratum, and a critical value of f is the value of f at a critical point. A function is called a Morse function if it has only non-degenerate critical points when restricted to each strata, and all its critical values are distinct.(There is an additional non-degeneracy condition which states that the differential of f at a critical point p of a strata S should not annihilate any limit of tangent spaces to a stratum other than S. However, in our simple situation this will always be true.)

The first fundamental result of stratified Morse theory is the following.

Theorem 5 *(Goresky-MacPherson, [7]) As c varies in the open interval between two adjacent critical values, the topological type of $X \cap f^{-1}((-\infty, c])$ remains constant.*

Stratified Morse theory actually gives a recipe for describing the topological change in $X \cap f^{-1}((-\infty, c])$ as c crosses a critical value. This is given in terms of Morse data, which consists of a pair of topological spaces $(A, B), A \supset B$, with the property that as c crosses the critical value $v = f(p)$, the change in $X \cap f^{-1}((-\infty, c])$ can be described by gluing in A along B.

In stratified Morse theory the Morse data is presented as a product of two pairs, called the tangential Morse data and the normal Morse data. The notion of product of pairs is the standard one in topology, namely

$$(A, B) \times (A', B') = (A \times A', A \times B' \cup B \times A').$$

The tangential Morse data at a critical point p is then given by $(B^\lambda \times B^{k-\lambda}, (\partial B^\lambda) \times B^{k-\lambda})$ where B^k is the closed k-dimensional disk, ∂ is the boundary map, and λ is the index of the Hessian matrix of f (restricted to the stratum containing p) at p.

Let $p = (p_1, \ldots, p_k)$, be a critical point in some ℓ-dimensional stratum S of a stratified subset Z of R^k.

Let N' be any $(k-\ell)$-dimensional hyperplane passing through the point p which is transverse to S which intersects the stratum S locally at the single point p.

Let $B_p(\delta)$ be a closed ball centered at p of radius δ, for some sufficiently small δ. Then, the *normal slice*, $N(p)$ at the point p is defined to be,

$$N(p) = N' \cap Z \cap B_p(\delta).$$

Choose $\delta \gg \epsilon' > 0$.

Let $\ell^- = N(p) \cap B_p(\delta) \cap \pi^{-1}(p_1 - \epsilon')$. The normal Morse data has the homotopy type of the pair $(\mathrm{cone}(\ell^-), \ell^-)$.

The following theorem measures the change in topology as we cross a critical value.

Theorem 6 *(Goreski-MacPherson, [7], page 69) Let Z be a Whitney stratified space, $f: Z \to R$ a proper Morse function, and $[a,b] \subset R$ an interval which contains no critical values except for an isolated critical value $v \in (a,b)$ which corresponds to a critical point p which lies in some stratum S of Z. Let λ be the Morse index of the critical point p. Then, the space $Z_{\leq b}$ has the homotopy type of a space which is obtained from $Z_{\leq a}$ by attaching the pair $(B^\lambda, \partial B^\lambda) \times (\mathrm{cone}(\ell^-), \ell^-)$.*

In our situation (because of transversality assumptions) the normal Morse data is either empty or is homotopic to $(\mathrm{cone}(B^{k-\ell-1}), B^{k-\ell-1})$.

In the first case, the Morse data is just the tangential Morse data. In the second case, there is no change in the homotopy type as the pair that is being added is contractible.

The following example should be helpful.

Example: Consider the (non-smooth) solid torus in Figure 1 formed by rotating the shaded region enclosed by two curves, around the dotted line at the center. The resulting set has one 3-dimensional stratum, two 2-dimensional strata, and two 1-dimensional strata. There are four critical points p_1, p_2, p_3, p_4, with respect to the projection map onto X_1 co-ordinate, all on the 1-dimensional strata. The corresponding critical values are c_1, c_2, c_3, c_4. The critical points p_1, p_2 are of index 0, and the critical points p_3, p_4 are of index 1. The corresponding lower halflinks ℓ^- are shown in Figure 1 for each critical point. Notice, that the

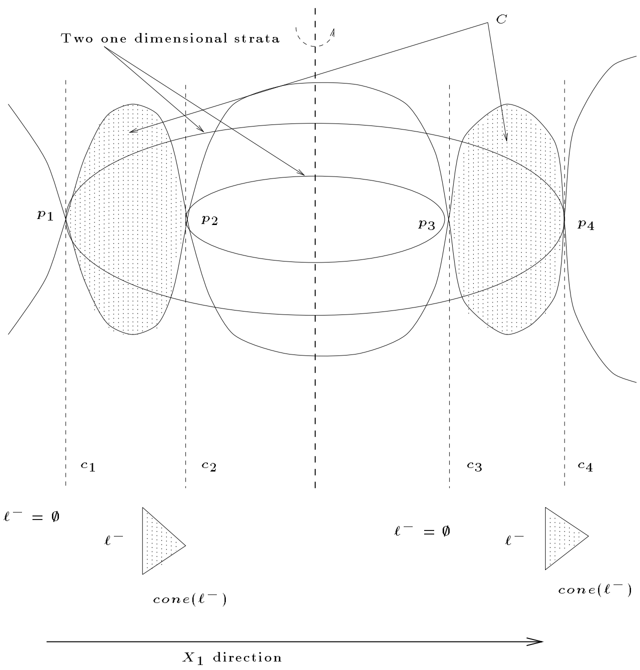


Figure 1: Example of a non-smooth torus in R^3.

normal data $(\mathrm{cone}(\ell^-), \ell^-)$ is contractible for the critical points p_2 and p_4. Thus no change in the homotopy type occurs as the sweep plane crosses c_2 and c_4. At c_1 and c_3 a zero and a one-dimensional ball gets added, respectively. This shows that the solid torus as the same homotopy type as S^1, as one would expect.

We next bound the number of critical points of the projection map π restricted to a stratum defined by a single polynomial equation.

Lemma 1 *Let $Q \in R[X_1, \ldots, X_k]$ be a polynomial of degree at most d, and let $Z(Q) \subset R^k$ be a bounded smooth manifold of dimension $l \leq k$, such that the projection map onto the first co-ordinate, π, is a Morse function on $Z(Q)$. Then, the number of critical points of π restricted to $Z(Q)$ is bounded by $O(d)^k$.*

Proof: Consider the polynomial,

$$Q_1 = (1-\zeta)Q^2 - \zeta(X_1^{2d+2} + \cdots + X_k^{2d+2} + 1).$$

where ζ is an infinitesimal.

The algebraic set $Z(Q_1)$ is a smooth algebraic hypersurface of $R\langle\zeta\rangle^k$ on which π (the projection of $(x_1, \ldots, x_k) \in R\langle\zeta\rangle^k$ to $x_1 \in R\langle\zeta\rangle$) has $O(d)^k$ isolated critical points (see [5]).

Let S^ζ be the union of those connected components of the set defined by $Q_1 \leq 0$ which intersect $Z(Q)$.

Now, $Z(Q) \subset S^{\zeta}$. Moreover, $Z(Q)$ is a deformation retract of S^{ζ}.

Moreover, S^{ζ} is bounded by connected components of a smooth hypersurface $Z(Q_1)$, which has a finite number of critical points for the projection map onto the X_1 co-ordinate and these critical points are non-degenerate and have distinct X_1 co-ordinates.

We now claim that for every critical value of $Z(Q)$ there exists a critical value of $Z(Q_1)$ infinitesimally close to it. Since the number of critical values of $Z(Q_1)$ is bounded by $O(d)^k$ this would imply that the number of critical values and hence the number of critical points of $Z(Q)$ is bounded by $O(d)^k$.

We consider a sweep hyperplane orthogonal to the X_1 axis, and look at the sets $Z(Q)_{\leq x}, S^{\zeta}_{\leq x}$ as x varies from $-\infty$ to ∞. From classical Morse theory we know that the homotopy type of $Z(Q)_{\leq x}$ changes as x crosses a critical value. But, since $Z(Q)_{\leq x}$ and $S^{\zeta}_{\leq x}$ has the same homotopy type for every $x \in R$ this implies that the homotopy type of S^{ζ} has to change in an infinitesimal neighbourhood of this critical value. However, since the homotopy type of $S^{\zeta}_{\leq x}$ changes only if x crosses a critical value of $Z(Q_1)$, this proves that there exists a critical value of $Z(Q_1)$ in an infinitesimal neighbourhood of every critical value of $Z(Q)$. □

4 Proofs of the main Theorems

We first prove a technical result which will play an important role in the proof of theorem 3. Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X_1, \ldots, X_k]$ be a family of polynomials whose degrees are bounded by d.

Let C be a connected component of the basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$.

We assume that the zero sets $Z(P_i)$ are smooth hypersurfaces such that the sign invariant sets of the polynomials $\{P_1, \ldots, P_s\}$ form a Whitney stratification of R^n and that they satisfy conditions T1 and T2.

We prove the following proposition.

Proposition 1 *If C is defined as above and satisfies T1 and T2 then the number of vertices of C is bounded by $b^{-1}\binom{s}{k-1}O(d)^k$.*

Proof: We let x vary from $-\infty$ to $+\infty$ and study what happens to $C_{\leq x}$.

First observe that the number of connected components of $C_{\leq x}$ changes only at a finitely many points as x varies, and each such x is a critical value of a certain stratum of C, which is contained in an algebraic set of the form $P_{i_1} = \cdots = P_{i_l} = 0$. Also, note that every vertex, being a 0-dimensional stratum, is a critical point.

Let p be a vertex of C and let $N_1, \ldots, N_k$ be the outer normals at p. We call p a *good* vertex if and only if the the vector $(-1, 0, \ldots, 0)$ which is normal to the hyperplane $X_1 = 0$ lies in the positive cone generated by the vectors $N_1, \ldots, N_k$ translated to the origin (see figure 2).

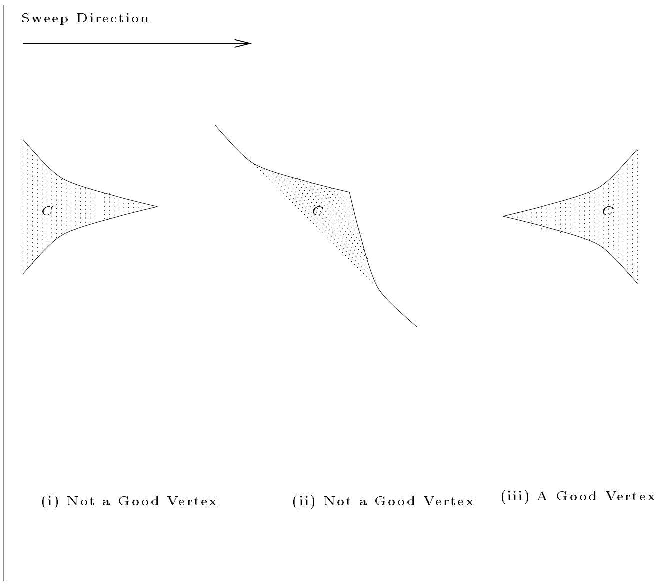


Figure 2: Different kinds of vertices.

We also observe that if p is a vertex of C and $c = \pi(p)$, then the number of connected components of $C_{\leq x}$ changes (increases by one) as we cross c from left to right if and only if p is a good vertex. This follows from the fact that in a small neighborhood of p, the set C can be very closely approximated by a convex cone bounded by the tangent hyperplanes at p to the k hypersurfaces incident on p. The total number of critical values coming from algebraic sets defined by less than k polynomials is bounded by $\sum_{0 \leq \ell < k} \binom{s}{\ell}(O(d))^k = \binom{s}{k-1}(O(d))^k$ using lemma 1. Also, for all large enough x , $C_{\leq x}$ is connected.

Thus, at every critical value corresponding to a good vertex a new connected component is born, no topological change occurs at critical values corresponding to vertices which are not good, and there are only $\binom{s}{k-1}(O(d))^k$ other critical values where the number of connected components can change. At these critical values only at most two connected components can join together. This allows us to bound the number of good vertices by $\binom{s}{k-1}(O(d))^k$.

Finally, since the outer normal cone at every vertex has (solid) angle bounded from below by b we can choose $O(b^{-1})$ directions for the normal of our sweep hyperplane such that every vertex is good for at least one direction. Thus, the total number of vertices is bounded by $b^{-1}\binom{s}{k-1}O(d)^k$.

$\square$

Remark 1: Note that if we consider the number of vertices in any m connected components, $C_1, \ldots, C_m$ instead of just C, the same argument bounds the number of good vertices by $m + \binom{s}{k-1}(O(d))^k$. This is so because at the end of the sweep we have m components left rather than only one.
Remark 2: The bound on the number of good vertices in a particular direction is independent of the bound on the angle of the normal cones and thus does not require robust transversality (condition T2).
Remark 3: The proof is valid not only for a single connected component of a basic semi-algebraic set, but for a connected component of any semi-algebraic set defined as an intersection of s semi-algebraic sets each of which is bounded by a semi-algebraic hypersurface of bounded degree.
Remark 4: The result holds with the weaker restriction that only the average of the normal cone angles over all vertices of C, be bounded from below by a constant b. If the average is bounded from below then at least a constant proportion of the angles has to be larger than (say) $b/2$. By the same argument as the proof outlined above we can bound the number of such vertices by $O(b^{-1}n^{k-1})$, and hence the total number of vertices is bounded by $O(b^{-1}n^{k-1})$.

We next establish a connection between the single cell problem and the problem of bounding the number of vertices in a connected component of a set defined as the intersection of n sets each of which is bounded by a smooth hypersurface of bounded degree. We will then be in a position to use proposition 1.

To establish this connection we show how to replace each surface patch S_i by a union of sets bounded by smooth hypersurfaces which contains S_i and is infinitesimally close to it. For this we use a classical perturbation technique originally due to Milnor [12].

Let $S \subset R^k$ be a surface patch. For simplicity, we first assume that S is a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_r \geq 0$, and let D be a connected component of S. We also assume that D is bounded.

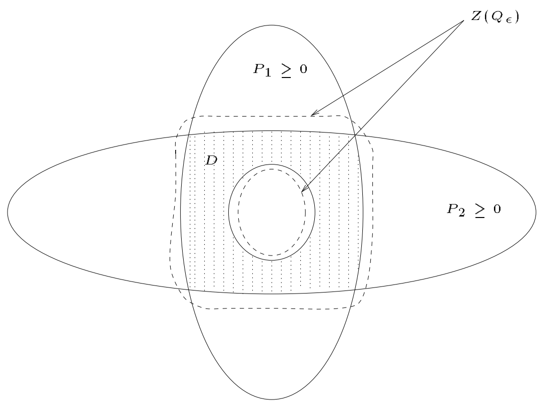
If the surface patch is contained in a variety $Z(Q)$ then we let $P_1 = Q$ and $P_2 = -Q$. Let ϵ be a positive infinitesimal.

Consider the set S_ϵ defined by $Q_\epsilon \geq 0$, where

$$Q_\epsilon = \Pi_{1 \leq i \leq r}((1-\epsilon)P_i + \epsilon) - \epsilon^{r+1}(X_1^{2d} + \cdots + X_k^{2d} + 1),$$

where $2d > \sum_{1 \leq i \leq r} \deg(P_i)$.

The set S_ϵ contains S, and hence it has a connected component D_ϵ which contains D (see figure 3). Moreover, $D = \mathrm{eval}_\epsilon \mathrm{D}_\epsilon$, and D_ϵ is bounded by connected components of the smooth hypersurface $Z(Q_\epsilon)$. For a proof of these facts see [6].



D is a connected component of $P_1 \geq 0, P_2 \geq 0$.

Figure 3: Approximating D (the shaded region) by a connected component of $Q_\epsilon \geq 0$.

Thus, if all the surface patches $S_1, \ldots, S_n$ are basic semi-algebraic sets, then we can replace each connected component $D_{i,j}$ of S_i by the corresponding set $D_{i,j,\epsilon}$ defined as above. If C is a connected component of the complement of $\cup S_i$ then there exists a connected component, $C_\epsilon \subset R\langle\epsilon\rangle^k$, of the complement of $\cup_{i,j} D_{i,j,\epsilon}$ such that $C = C_\epsilon \cap R^k$. Moreover, C_ϵ is a connected component of a set defined as the intersection of sets bounded by smooth hypersurfaces.

The next lemma shows that near smooth points of S_i the unit normals to the approximating hypersurfaces $\partial D_{i,j,\epsilon}$ closely approximates the normals to the original surface.

Lemma 2 *For a smooth point p of a connected component $D_{i,j}$ of a surface patch S_i, let N_p be a unit normal at p. Then, inside any ball $B_p(\rho)$ around p of infinitesimal radius ρ, and any point $q \in B_p(\rho) \cap \partial D_{i,j,\epsilon}$, the unit normal N_q is such that, $1 - |N_p \cdot N_q|$ is infinitesimal.*

Proof: The proof uses Leibniz's rule to compute the partial derivatives of Q_ϵ and is omitted. $\square$

It is clear that the number of vertices of C_ϵ is at least the number of vertices of C. In order to see this, consider a vertex p in C which is the intersection of

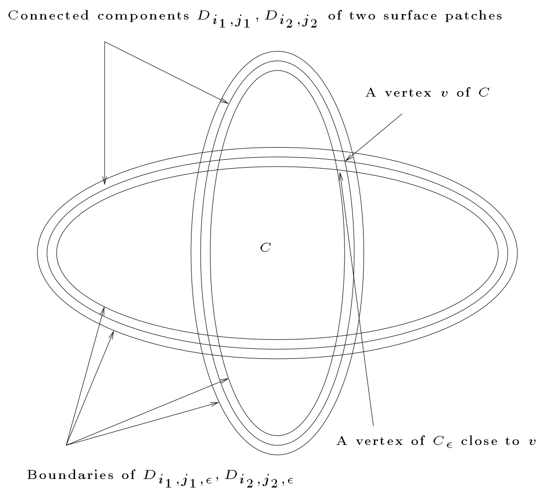


Figure 4: Surface patches replaced by sets bounded by smooth hypersurfaces.

k different surface patches, $S_{i_1}, \ldots, S_{i_k}$. Let the corresponding connected components be, $D_{i_1,j_1}, \ldots, D_{i_k,j_k}$. Then in an infinitesimal neighbourhood of p a surface patch S_{i_1} is sandwiched between two portions of the surface $\partial D_{i_1,j_1,\epsilon}$. Then there is at least one vertex of C_ϵ which is infinitesimally close to p and which is in the intersection of the k hypersurfaces, $\partial D_{i_1,j_1,\epsilon}, \ldots, \partial D_{i_k,j_k,\epsilon}$ (see figure 4).

Also, replacing the infinitesimal ϵ by a variable t we have that, $C = \cup_{t>0} C_t$. Moreover, using Hardt's triviality theorem we have that for all sufficiently small $t > 0$, the sets C_t have the same homotopy type. Using the fact that the singular homology functor commutes with direct limit, we deduce that that the homology groups of C are isomorphic to those of C_t for all sufficiently small $t > 0$, and hence to those of C_ϵ.

This shows that, we can replace the single cell C by another semi-algebraic set C_ϵ having the same homology groups, which is an intersection of sets bounded by smooth hypersurfaces, satisfying T1 and T2 and the number of vertices of C_ϵ bounds the number of vertices of C.

We finally consider the case when the surfaces patches are not necessarily basic, but the surface patch S_i is contained in $Z(Q_i)$ and is defined by some arbitrary Boolean formula with atoms of the form $P_{i,j}\{\leq, \geq\}0$. Since S_i is closed it can be expressed as the union of $O(r^k)$ basic semi-algebraic sets. Thus, $S_i = \cup S_{i,j}$ where each $S_{i,j}$ is basic.

We replace each connected component of each of the basic semi-algebraic sets $S_{i,j}$ by sets bounded by smooth hypersurfaces as earlier.

We have proved the following proposition.

Proposition 2 *Let C be a connected component of the complement of the surface patches $S_1, \ldots, S_n$, satisfying condition T1 (resp. T1 and T2). Then C can be replaced by another semi-algebraic set, C_ϵ, having the same homology groups as C, which is an intersection of $O(n)$ semi-algebraic sets each of which is bounded by a smooth hypersurface. Moreover, the hypersurfaces bounding these sets satisfy the conditions T1 (resp. T1 and T2). Further, the number of vertices of C is $O(M)$ where M is the number of vertices of C_ϵ.*

The above proposition together with proposition 1 (see remark 3) proves theorem 3.

4.1 Bound on the Sum of the Betti Numbers of a Connected Component of a Basic Semi-algebraic Set

Let C be a connected component of a basic semi-algebraic set S defined by $P_1 \geq 0, \ldots, P_s \geq 0$. By adding the additional polynomial inequality, $X_1^2 + \cdots + X_k^2 - \Omega \leq 0$, with sufficiently large $\Omega > 0$, we can assume that S is compact.

We now replace the set S by a new set, $S^+(\bar{\epsilon})$, defined by, $P_1 \geq -\epsilon\epsilon_1, \ldots, P_s \geq -\epsilon\epsilon_s$, where $\epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_s \gg \epsilon > 0$, are positive infinitesimals.

The following lemma appears in [6].

Lemma 3 *$S^+(\bar{\epsilon})$ has the same homology groups as S.*

We also need the following lemma.

Lemma 4 *The sets $Z(P_i + \epsilon\epsilon_i)$ are smooth hypersurfaces intersecting transversally. Moreover, the sign invariant sets of the family of polynomials, $\{P_1 + \epsilon\epsilon_1, \ldots, P_s + \epsilon\epsilon_s\}$ give rise to a Whitney stratification of R^k.*

Proof: It follows from Sard's lemma that for almost all choices of ϵ and ϵ_i (that is, outside a set of measure zero) $Z(P_i + \epsilon\epsilon_i)$ are smooth hypersurfaces intersecting transversally. Since the intersection of Whitney stratified sets intersecting transversally is also Whitney stratified the lemma follows. □

Consider the connected component, $C^+(\bar{\epsilon})$, of $S^+(\bar{\epsilon})$ which contains C. By Lemma 3, we have that $\beta_i(C) = \beta_i(C^+(\bar{\epsilon}))$. Also, by a suitable change of coordinates we can assume that the projection map π onto the first coordinate is a Morse function when restricted to each of the strata of $C^+(\bar{\epsilon})$.

We now look at the set $C^+(\bar{\epsilon})_{\leq x}$ as x goes from $-\infty$ to $+\infty$. Using theorem 6 from stratified Morse theory, it is easy to deduce that the sum of the Betti numbers of $C^+(\bar{\epsilon})_{\leq x}$ increases by at most 1 as x crosses a critical value corresponding to a critical point of a stratum of dimension > 0. Since, there are at most $\binom{s}{k-1}O(d)^k$ of such critical points and we already know that there are only $\binom{s}{k-1}O(d)^k$ good vertices, this proves that the sum of the Betti numbers of $C^+(\bar{\epsilon})$ and hence that of C is bounded by $\binom{s}{k-1}O(d)^k$. This proves theorem 1. Moreover, the above argument together with remark 1 after the proof of proposition 1 proves theorem 2.

5 Combinatorial Complexity of a Single Cell

As remarked before Halperin and Sharir [10] proved a bound of $O(n^{2+\epsilon})$ on the combinatorial complexity of a single cell in an arrangement of n surface patches in R^3. The basic ingredients of their method was to prove two preliminary results needed to "bootstrap" the recurrences appearing in the analysis. These are:

1. a sharp bound on the number of X_1-extreme vertices of the cell C (vertices whose X_1 co-ordinate is smallest or largest in a small neighbourhood of the vertex within the closure of C), and

2. a sharp bound on the number of vertices bounding "popular" faces of C (faces that are adjacent to C on all sides).

We first observe that from our bound on the number of good vertices we automatically derive an $O(n^{k-1})$ bound on the number of X_1-extreme vertices thus taking care of the first item. This is because when we replace the surface patches by sets containing them which are bounded by smooth hypersurfaces, we create a good vertex of $C\epsilon$ in an infinitesimal neighbourhood of every X_1 extreme vertex of C. (see figure 4).

Halperin and Sharir actually proves a bound on the number of incidences between vertices and *popular* faces. The cell itself (being a k-dimensional face) is popular. An ℓ-dimensional face is popular if locally the cell C lies in all its $2^{k-\ell}$ sides. We will give formal definitions later. One difference between our argument and that in [10] is that we have an extra dimension in the recurrence. We bound the number of popular faces of a given dimension in terms of the number of popular faces of one lower dimension. The number of popular edges can be bounded by $O(n^{k-1})$ using the bound on the number of X_1 extreme vertices, and this becomes the base case in the recurrence.

We also remark that in view of the fact that the set of surface patches changes in course of the inductive argument it is convenient to fix a point, say $Z \in R^k$, and always consider the single cell, $C_Z(\Sigma)$, containing this point, in whatever arrangement , Σ, we have in a particular step of the proof.

Following Halperin and Sharir, we call a vertex v of an arrangement $A(\Sigma)$ an *inner* vertex if it is formed by the intersection of the relative interiors of k distinct surfaces. In a small neighbourhood of v (assuming transversality of the intersection) the k surfaces divide up the neighbourhood into 2^k octants. A side R is any one of these octants and we call the pair (v, R) a 0-border. We say that (v, R) is a 0-border of C if the octant R intersects C.

We call a $(k-1)$-dimensional face popular if it borders C on both sides. Similarly, we will call any ℓ-dimensional face popular if it borders C on all its $2^{k-\ell}$ sides. We call a vertex popular if it is incident on a popular face.

We let $\pi^{(\ell)}(n)$ denote the maximum number of incidences of vertices on ℓ-dimensional popular faces. Notice, that $\pi^{(k)}(n)$ is a bound on the combinatorial complexity of a single cell.

For a vertex v which is the intersection of k surface patches $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ we let the index of v denoted by $j(v)$ to be the number of points of intersections of $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ to the right of v. Clearly, $0 \leq j(v) \leq N = O(d)^k$, where d is the bound on the degrees of the polynomials defining the surface patches.

We define $\pi^{(\ell,j)}(\Sigma)$ to be the number of incidences of vertices of index at most j on popular faces of dimension ℓ, and $\pi^{(\ell,j)}(n)$ the maximum of $\pi^{(\ell,j)}(\Sigma)$ over all possible arrangements ,Σ, of size n. Also, note that $\pi^{(\ell,N)}(n) = \pi^{(\ell)}(n)$.

We now obtain a recurrence on $\pi^{(\ell,j)}(n)$. Our method differs slightly from that of Halperin and Sharir, in that we have an extra index ℓ in our recurrence which is the dimension of the popular faces that we are bounding. However, the analysis is still very similar in spirit.

Let v be an inner vertex of a popular face f of dimension ℓ of the cell C. Assume that f lies in the intersection of $k-\ell$ surface patches $\sigma = \cap_{1 \leq i \leq k-\ell} S_i$ and let the vertex v be defined as the intersection of the k surface patches $S_1, \ldots, S_k$. (Note that if $\ell = k$ then the popular face is the cell C itself and $\sigma = R^k$.)

For $k - \ell < i \leq k$, let γ_i be the curve defined by $\cap_{1 \leq j \leq k, j \neq i} S_j$. Each γ_i contains an edge e_i incident on v and f. Moreover, since f is a popular face, and e_i is incident on f, $2^{k-\ell}$ of the 2^{k-1} sides of the edge e_i lie locally in C.

Consider the curve γ_i. If γ_i has another edge e_i' incident on another ℓ-dimensional popular face f' contained in σ, and if f and f' share a $(\ell-1)$-dimensional face g then g is a popular $(\ell-1)$-dimensional face of

C.

Each of the curves γ_i must contain a maximal relatively open X_1-monotone connected portion β_i such that at no point of β_i all the $2^{k-\ell}$ sides of β_i consistent with the $2^{k-\ell}$ sides of e_i lying locally in C , lie locally in C. Let z_i denote the other endpoint of β_i. Again we can assume that at least one of the β_i's emanate in the positive X_1 direction and one in the negative X_1 direction as otherwise v would be a X_1-extreme vertex in one of the $2^{k-\ell}$ sides of f.

The analysis is divided into several cases.

case (a): At least one of the curves β_i ends at a point z_i which is either an endpoint of the original intersection curve γ_i or a point of local X_1-extremum on that curve. In either case we can charge v to the point z_i and note that each z_i gets charged only a constant number of times in this process and the total number of such z_i is $O(n^{k-1})$. In what follows we assume that this situation does not hold, and hence z_i is a vertex of a ℓ-dimensional popular face of C.

case (b) In this case we assume that $\beta_i \cup \{z_i\}$ is not intersected by the surface S_i. We define the *popularity level* of a point $w \in \sigma$ to be the smallest number of surfaces in Σ whose removal makes w lie in a popular ℓ-dimensional cell contained in σ in the resulting subarrangement. If w is a vertex of $A(\Sigma)$ and incident to some face $f \subset \sigma$ we say that (w,f) lies at a *restricted popularity level* $p((w,f)) = m$ if by removing m surfaces from Σ , none of which is incident to w, the face f becomes a popular face of the cell containing Z and if m is the smallest number with this property.

Let t denote the number of distinct surfaces of Σ that intersect $\beta_i \cup \{z_i\}$. We fix some threshold parameter $\xi = \xi_j$ to be defined later and consider the following two subcases:

case (b.i) $t \geq \xi$: In this case we charge (v,f) to a block of ξ points of intersection between $\beta_i \cup \{z_i\}$ and the surfaces of Σ as follows. For each surface S intersecting $\beta_i \cup \{z_i\}$ choose its point of intersection that lies nearest to v along β_i. We obtain at least ξ such designated points and we charge (v,f) to the block of the first ξ points. It is clear that each of these vertices can be charged no more than a constant number of times. Moreover, each charged vertex lie at a restricted popularity level $\leq \xi$. We bound the number M of pairs (w,f') of inner vertices of $A(\Sigma)$ and incident ℓ-dimensional popular faces f', that lie at restricted popularity level $\leq \xi$; the number of pairs (v,f) in the present subcase being $O(M/\xi)$.

We choose a random sample $\mathcal{R}$ of $r = n/\xi$ surfaces of Σ. Let w be an inner vertex of $A(\Sigma)$ at restricted level $p \leq \xi$, and let $\mathcal{Q}$ be a specific collection of p surfaces, none incident to w, whose removal makes w a vertex of a popular ℓ-dimensional face. The probability that w shows up as a vertex of $C_Z(\mathcal{R})$, is at least $\binom{n-p-k}{r-k}/\binom{n}{r}$. Thus, we have that,

$$\sum_{0 \leq p \leq \xi} \frac{\binom{n-p-k}{r-k}}{\binom{n}{r}} F_p \leq E[\pi^{(l)}(\mathcal{R})] \leq \pi^{(l)}(r),$$

where F_p is the number of pairs (w,f') of $A(\Sigma)$ at restricted popularity level p and $E[\cdot]$ denotes expectation. One can now easily verify that,

$$\sum_{0 \leq p \leq \xi} F_p = O(\xi^k \pi^{(l)}(n/\xi)),$$

and so the number of pairs in this subcase is $O(\xi^{k-1}\pi^{(l)}(n/\xi))$.

case (b.ii) $t < \xi$: In this case if these t surfaces are removed from the arrangement then v becomes a vertex of a popular $(\ell-1)$-dimensional face of C. We can again use a random sampling argument by drawing a random sample $\mathcal{R}$ of $r = n/\xi$ surfaces of Σ.

Let $E'[\mathcal{R}]$ be the expected number of vertices in $A(\mathcal{R})$ of index $\leq j$ which are incident to popular $(\ell-1)$-dimensional faces of $C_Z(\mathcal{R})$. The probability that the vertex v will show up as a vertex of such a popular face of $C_Z(\mathcal{R})$ is $\geq \frac{\binom{n-t-k}{r-k}}{\binom{n}{r}}$. If G_t is the number of pairs in the full arrangement that falls into the present subcase, we have that,

$$\sum_{0 \leq t \leq \xi} \frac{\binom{n-t-k}{r-k}}{\binom{n}{r}} G_t \leq E'[\mathcal{R}] \leq \pi^{(\ell-1,j)}(r).$$

Again, this implies that,

$$\sum_{0 \leq t \leq \xi} G_t = O(\xi^k \pi^{(\ell-1,j)}(n/\xi).$$

case (c): In the remaining case (which can only happen if $j(v) > 0$), the surface S_i intersects $\beta_i \cup \{z_i\}$ in at least one point w (if there are several such points take w to be the farthest one from v along β_i.) Again, let t denote the number of distinct surfaces of Σ, excluding S_i, that intersect $\beta_i \cup \{z_i\}$. We have two subcases:

case (c.i) $t \geq \xi$: This case is similar to case (b.i) earlier and we charge (v,f) to a block of ξ vertices as in (b.i). Using a similar argument one can show that the number of pairs in this subcase is restricted to $O(\xi^{k-1}\pi^{(l)}(n/\xi))$.

case (c.ii) $t < \xi$: In this case if we remove these t surfaces (without removing S_i) the point w becomes a vertex of an ℓ-dimensional popular face f' of the resulting subarrangement. In this case, we charge (v, f) to the pair (w, f'). Clearly, each pair (w, f') is charged only a constant number of times in this way.

We next estimate the number of such pairs (w, f'). We again draw a random sample $\mathcal{R}$ of $r = n/\xi$ surfaces of Σ. Let $E''[\mathcal{R}]$ be the expected number of pairs (w, f') of $C_Z(\mathcal{R})$ such that w has index $\leq j-1$. If H_t is the number of pairs (v, f) that fall in the present subcase, using similar arguments as before we get,

$$\sum_{0\leq t\leq \xi} \frac{\binom{n-t-k}{r-k}}{\binom{n}{r}} H_t \leq E''[\mathcal{R}] \leq \pi^{(\ell,j-1)}(r).$$

This implies that,

$$\sum_{0\leq t\leq \xi} H_t = O(\xi^k \pi^{(\ell,j-1)}(n/\xi).$$

Summing all the cases we obtain the following recurrence on $\pi^{(\ell,j)}(n)$.

$$\pi^{(\ell,j)}(n) = O(\xi^{k-1}\pi^{(l)}(n/\xi) + \xi^k\pi^{(\ell-1,j)}(n/\xi)$$
$$+\xi^k\pi^{(\ell,j-1)}(n/\xi) + n^{k-1}).$$

Moreover,

$$\pi^{(0,0)} = O(n^{k-1}).$$

These recurrences yield the desired bound of $O(n^{k-1+\epsilon})$ on $\pi^{(k)}(n)$ and hence on the combinatorial complexity of a single cell.

5.1 Solving the recurrence

We follow a similar scheme as in [10] with minor modifications. Fix some $\epsilon > 0$. We claim that $\pi^{(\ell,j)}(n) \leq B_{\ell,j} n^{k-1+\epsilon}$, for $0 \leq \ell \leq k, 0 \leq j \leq N$, where the constants $B_{\ell,j}$ depend on ϵ, ℓ, j, and on the parameters r, d, k (see section 1) used in the definition of the surface patches.

We prove this by induction on n. We use a different parameter $\xi_{\ell,j}$ for each pair ℓ, j in the recurrence. Thus, rewriting the recurrence equations we have,

$$\pi^{(\ell,j)}(n) = c(\xi_{\ell,j}^{k-1}\pi^{(l)}(n/\xi_{\ell,j}) + \xi_{\ell,j}^k\pi^{(\ell-1,j)}(n/\xi_{\ell,j})$$
$$+\xi_{\ell,j}^k\pi^{(\ell,j-1)}(n/\xi_{\ell,j}) + n^{k-1}),$$

for appropriate constants $c, \xi_{\ell,j}$. We assume that $c > 1$, $\xi_{0,0}$ to be sufficiently large, and put $\xi_{\ell,j} = \xi_{0,0}^{\epsilon^{\ell+j}}$.

Also, by choosing the $B_{\ell,j}$'s sufficiently large we can assume that the claimed bounds hold for all $n \leq \xi_{0,0}$.

We choose, $B_{0,0} > 2c\xi_{0,0}^{1-\epsilon}$ and $B_{\ell,j} = \frac{(\ell+j+1)(2c)^{\ell+j}}{\xi^{\epsilon}_{\ell+j}}\xi_{0,0}^{\epsilon} B_{0,0}$, and require that $\xi_{0,0}$ be sufficiently large so that $\xi_{0,0}^{\epsilon^{k+N}} > 2(k+N)(2c)^{k+N}$.

Applying the inductive hypothesis on the right side of the above recurrence we see that the claimed bounds will hold if the following inequalities are satisfied:

$$\frac{cB_{k,N}}{\xi^{\epsilon}_{\ell,j}} + c(B_{\ell-1,j} + B_{\ell,j-1})\xi_{\ell,j}^{1-\epsilon} + \frac{c}{\xi_{0,0}^{\epsilon}} \leq B^{\ell,j},$$

for $0 \leq \ell \leq k, 0 \leq j, \leq N$.

This is equivalent to,

$$\frac{c(k+N+1)(2c)^{k+N}\xi_{0,0}^{\epsilon}B_{0,0}}{\xi_{0,0}^{\epsilon^{k+N+1}}} + \frac{2c(\ell+j)(2c)^{\ell+j-1}\xi_{0,0}^{\epsilon}B_{0,0}}{\xi_{0,0}^{\epsilon^{\ell+j}}}$$

$$+\frac{c}{\xi_{0,0}^{\epsilon}} \leq \frac{(\ell+j+1)(2c)^{\ell+j}\xi_{0,0}^{\epsilon}B_{0,0}}{\xi_{0,0}^{\epsilon^{\ell+j+1}}}.$$

Using the fact that, $\xi_{0,0}^{\epsilon^{k+N}} > 2(k+N)(2c)^{k+N}$, the last inequality is seen to be implied by,

$$\frac{c}{\epsilon_{0,0}^{\epsilon}} \leq \frac{(c^{\ell+j} - \frac{1}{2})\xi_{0,0}^{\epsilon}B_{0,0}}{\xi_{0,0}^{\epsilon^{\ell+j+1}}},$$

or by

$$\frac{c\xi_{\ell,j}^{1+\epsilon}}{\xi_{0,0}^{\epsilon}} \leq (c^{\ell+j} - \frac{1}{2})\xi_{0,0}^{\epsilon}B_{0,0}.$$

However, when $\ell + j$ increases the left hand side decreases while the right hand side increases. Hence, it suffices to verify for $\ell = 0, j = 0$, which trivially holds by our choice of $B_{0,0}$. This complete the induction.

References

[1] P.K. Agarwal, M. Sharir Arrangements of surfaces in higher dimensions, Chapter in *Handbook of Computational Geometry*, J.R. Sack (Ed.), North-Holland, to appear.

[2] B. Aronov, M. Sharir Castles in the air revisited, *Discrete and Computational Geometry, 12 (1994), 119-150.*

[3] J. Bochnak, M. Coste, M.-F. Roy Géométrie algébrique réelle. Springer-Verlag (1987).

[4] S. Basu, R. Pollack, M.-F. Roy On the combinatorial and algebraic complexity of Quantifier Elimination, *JACM, 43, 1002–1045, (1996).*

[5] S. Basu, R. Pollack, M.-F. Roy Computing Roadmaps of Semi-algebraic Sets on a Variety, submitted, preprint available at www.math.nyu.edu/faculty/pollack/index.html.

[6] S. Basu On Bounding the Betti Numbers and Computing the Euler Characteristics of Semi-algebraic Sets, to appear in *Discrete and Computational Geometry.*

[7] M. Goresky, R. MacPherson *Stratified Morse Theory,* Springer-Verlag (1987).

[8] L.J. Guibas, M. Sharir, S. Sifrony On the general motion planning problem with two degrees of freedom, *Discrete and Computational Geometry, 4 (1989), 491-521.*

[9] D. Halperin Arrangements, *Handbook of Discrete and Computational Geometry,* J. O'Rourke, J.E. Goodman editors, CRC Press, 1997.

[10] D. Halperin, M. Sharir Almost tight upper bounds for the single cell and zone problems in three dimensions, *Discrete and Computational Geometry, 14 (1995), 385-410.*

[11] P. McMullen The maximum number of faces of a convex polytope, *Mathematika, 17: 179-184, 1970.*

[12] J. Milnor On the Betti numbers of real varieties, *Proc. AMS 15, 275-280 (1964).*

[13] O. A. Oleinik, I. B. Petrovskii On the topology of real algebraic surfaces, *Izv. Akad. Nauk SSSR 13, 389-402 (1949).*

[14] J. Pach, M. Sharir The upper envelope of piecewise linear functions and the boundary of a region enclosed by convex plates: combinatorial analysis, *Discrete and Computational Geometry, 4 (1989), 291-309.*

[15] B. Tagansky A new technique for analyzing substructures in arrangements of piecewise linear surfaces, *Discrete and Computational Geometry, 16 (1996), 455 -479.*

[16] R. Thom Sur l'homologie des varietes algebriques reelles, Differential and Combinatorial Topology, Ed. S.S. Cairns, Princeton Univ. Press, 255-265 (1965).

[17] A. Wiernik, M. Sharir Planar realization of nonlinear Davenport-Schinzel sequences by segments, *Discrete and Computational Geometry, 3 (1988), 15-47.*