BOUNDING THE NUMBER OF STABLE HOMOTOPY TYPES OF A PARAMETRIZED FAMILY OF SEMI-ALGEBRAIC SETS DEFINED BY QUADRATIC INEQUALITIES

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ABSTRACT. We prove a nearly optimal bound on the number of stable homotopy types occurring in a k-parameter semi-algebraic family of sets in R^ℓ, each defined in terms of m quadratic inequalities. Our bound is exponential in k and m, but polynomial in ℓ. More precisely, we prove the following. Let R be a real closed field and let

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k],$$

with $\deg_Y(P_i) \leq 2, \deg_X(P_i) \leq d, 1 \leq i \leq m$. Let $S \subset \mathrm{R}^{\ell+k}$ be a semi-algebraic set, defined by a Boolean formula without negations, whose atoms are of the form, $P \geq 0, P \leq 0, P \in \mathcal{P}$. Let $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ be the projection on the last k co-ordinates. Then the number of stable homotopy types amongst the fibers $S_{\mathbf{x}} = \pi^{-1}(\mathbf{x}) \cap S$ is bounded by $(2^m \ell k d)^{O(mk)}$.

1. INTRODUCTION

Let $S \subset \mathrm{R}^{\ell+k}$ be a semi-algebraic set over a real closed field R. Let $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ be the projection map on the last k co-ordinates, and for any $S \subset \mathrm{R}^{\ell+k}$ we will denote by π_S the restriction of π to S. Moreover, when the map π is clear from context, for any $\mathbf{x} \in \mathrm{R}^k$ we will denote by $S_{\mathbf{x}}$ the fiber $\pi_S^{-1}(\mathbf{x})$.

A fundamental theorem in semi-algebraic geometry states,

Theorem 1.1. *(Hardt's triviality theorem [22]) There exists a semi-algebraic partition of R^k, $\{T_i\}_{i \in I}$, such that the map π_S is definably trivial over each T_i.*

Theorem 1.1 implies that for each $i \in I$ and any point $\mathbf{y} \in T_i$, the pre-image $\pi^{-1}(T_i) \cap S$ is semi-algebraically homeomorphic to $S_{\mathbf{y}} \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $S_{\mathbf{y}}, \mathbf{y} \in T_i$, are semi-algebraically homeomorphic.

Hardt's theorem is a corollary of the existence of *cylindrical algebraic decompositions* (see [12]), which implies a double exponential (in k and ℓ) upper bound on the cardinality of I and hence on the number of homeomorphism types of the fibers of the map π_S. No better bounds than the double exponential bound are known, even though it seems reasonable to conjecture a single exponential upper bound on the number of homeomorphism types of the fibers of the map π_S.

In [9], the weaker problem of bounding the number of distinct *homotopy types*, occurring amongst the set of all fibers of π_S was considered, and a single exponential upper bound was proved on the number of homotopy types of such fibers.

Before stating this result more precisely we need to introduce a few notation. Let R be a real closed field, $\mathcal{P} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$, and let ϕ be a Boolean

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formula with atoms of the form $P = 0$, $P > 0$, or $P < 0$, where $P \in \mathcal{P}$. We call ϕ a $\mathcal{P}$-formula, and the semi-algebraic set $S \subset \mathrm{R}^{\ell+k}$ defined by ϕ, a $\mathcal{P}$-semi-algebraic set.

If the Boolean formula ϕ contains no negations, and its atoms are of the form $P = 0$, $P \geq 0$, or $P \leq 0$, with $P \in \mathcal{P}$, then we call ϕ a $\mathcal{P}$-closed formula, and the semi-algebraic set $S \subset \mathrm{R}^{\ell+k}$ defined by ϕ, a $\mathcal{P}$-closed semi-algebraic set.

The following theorem appears in [9].

Theorem 1.2. [9] *Let* $\mathcal{P} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$, *with* $\deg(P) \leq d$ *for each* $P \in \mathcal{P}$ *and cardinality* $\#\mathcal{P} = m$. *Then there exists a finite set* $A \subset \mathrm{R}^k$, *with*

$$\#A \leq (2^\ell mkd)^{O(k\ell)},$$

such that for every $\mathbf{x} \in \mathrm{R}^k$ *there exists* $\mathbf{z} \in A$ *such that for every* $\mathcal{P}$-*semi-algebraic set* $S \subset \mathrm{R}^{\ell+k}$, *the set* $S_{\mathbf{x}}$ *is semi-algebraically homotopy equivalent to* $S_{\mathbf{z}}$. *In particular, for any fixed* $\mathcal{P}$-*semi-algebraic set* S, *the number of different homotopy types of fibers* $S_{\mathbf{x}}$ *for various* $\mathbf{x} \in \pi(S)$ *is also bounded by*

$$(2^\ell mkd)^{O(k\ell)}.$$

A result similar to Theorem 1.2 has been proved for semi-Pfaffian sets as well in [9], and has been extended to arbitrary o-minimal structures in [6]. The bounds on the number of homotopy types proved in [9, 6] are all exponential in ℓ as well as k. The following example, which appears in [9], shows that in this generality the single exponential dependence on ℓ is unavoidable.

Example 1.3. Let $P \in \mathrm{R}[Y_1, \ldots, Y_\ell] \hookrightarrow \mathrm{R}[Y_1, \ldots, Y_\ell, X]$ be the polynomial defined by

$$P = \sum_{i=1}^{\ell} \prod_{j=0}^{d-1} (Y_i - j)^2.$$

The algebraic set defined by $P = 0$ in $\mathrm{R}^{\ell+1}$ with co-ordinates $Y_1, \ldots, Y_\ell, X$, consists of d^ℓ lines all parallel to the X axis. Consider now the semi-algebraic set $S \subset \mathrm{R}^{\ell+1}$ defined by

$$(P = 0) \ \wedge \ (0 \leq X \leq Y_1 + dY_2 + d^2Y_3 + \cdots + d^{\ell-1}Y_\ell).$$

It is easy to verify that, if $\pi : \mathrm{R}^{\ell+1} \to \mathrm{R}$ is the projection map on the X coordinate, then the fibers $S_{\mathbf{x}}$, for $\mathbf{x} \in \{0, 1, 2, \ldots, d^\ell - 1\} \subset \mathrm{R}$ are 0-dimensional and of different cardinality, and hence have different homotopy types.

1.1. **Semi-algebraic sets defined by quadratic inequalities.** One particularly interesting class of semi-algebraic sets is the class of semi-algebraic sets defined by quadratic inequalities. This class of sets has been investigated from an algorithmic standpoint [2, 20, 4, 5, 10], as well as from the point of view topological complexity, [1, 3, 8].

Semi-algebraic sets defined by quadratic inequalities are distinguished from arbitrary semi-algebraic sets by the fact that, if the number of inequalities is fixed, then the sum of their Betti numbers is bounded polynomially in the dimension. The following bound was proved by Barvinok [3].

Theorem 1.4. *Let* $S \subset \mathrm{R}^{\ell}$ *be a semi-algebraic set defined by the inequalities,* $P_1 \geq 0, \ldots, P_m \geq 0$, $\deg(P_i) \leq 2, 1 \leq i \leq m$. *Then,* $\sum_{i=0}^{\ell} b_i(S) \leq (m\ell)^{O(m)}$, *where* $b_i(S)$ *denotes the i-th Betti number of S.*

An extension of Barvinok's bound to arbitrary $\mathcal{P}$-closed (not just basic closed) semi-algebraic sets defined in terms of quadratic inequalities has been done recently in [11].

Now suppose that we have a parametrized family of sets, each defined in terms of m quadratic inequalities. More precisely, let

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_{\ell}, X_1, \ldots, X_k],$$

with $\deg_Y(P_i) \leq 2, \deg_X(P_i) \leq d, 1 \leq i \leq m$ ($X_1, \ldots, X_k$ are the *parameters*), and let $S \subset \mathrm{R}^{\ell+k}$ be a $\mathcal{P}$-closed semi-algebraic set. Let $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ denote the projection on the last k co-ordinates. Then for each $\mathbf{x} \in \mathrm{R}^k$ the semi-algebraic set $S_{\mathbf{x}}$ is defined by a Boolean formula involving at most m quadratic polynomials in $Y_1, \ldots, Y_{\ell}$.

Bounding the number of topological types amongst the fibers, $S_{\mathbf{x}}, \mathbf{x} \in R^k$, is an interesting special case of the more general problem mentioned in the last section. In view of the topological simplicity of semi-algebraic sets defined by few quadratic inequalities as opposed to general semi-algebraic sets (cf. Theorem 1.4), one might expect a much tighter bound on the number of topological types compared to the general case. However one should be cautious, since a tight bound on the Betti numbers of a class of semi-algebraic sets does not automatically imply a similar bound on the number of topological or even homotopy types occurring in that class.

In this paper we consider the problem of bounding the number of *stable homotopy types* (see Definition 3.4 below) of fibers $S_{\mathbf{x}}$, where π and S are as defined above. We prove a bound which for each fixed m, is polynomial in ℓ (the dimension of the fibers). In some special cases our bound can be extended to the number of homotopy types (see Theorem 4.17).

Our result can be seen as a follow-up to the recent work on bounding the number of homotopy types of fibers of general semi-algebraic maps studied in [9]. However, the bound in [9] applied to the special case of sets defined by quadratic inequalities would yield a bound exponential in both k and ℓ, as shown by Example 1.3, where the semi-algebraic set S is defined in terms of three polynomials.

Remark 1.5. Note that the notions of homeomorphism type, homotopy type and stable homotopy type are each strictly weaker than the previous one, since two semi-algebraic sets might be stable homotopy equivalent, without being homotopy equivalent (see [24], p. 462), and also homotopy equivalent without being homeomorphic. However, two closed and bounded semi-algebraic sets which are stable homotopy equivalent have isomorphic homology groups.

1.2. **Prior and Related Work.** Since sets defined by quadratic equalities and inequalities are the simplest class of topologically non-trivial semi-algebraic sets, the problem of classifying such sets topologically has attracted the attention of many researchers.

Motivated by problems related to stability of maps, Wall [27] considered the special case of real algebraic sets defined by two simultaneously diagonalizable quadratic forms in ℓ variables. He obtained a full topological classification of such varieties making use of Gale diagrams (from the theory of convex polytopes). In our notation, letting

$$Q_1 = \sum_{i=1}^{\ell} X_i Y_i^2,$$

$$Q_2 = \sum_{i=1}^{\ell} X_{i+\ell} Y_i^2,$$

and

$$S = \{(\mathbf{y}, \mathbf{x}) \in \mathbb{R}^{3\ell} \mid \ \| \mathbf{y} \| = 1, \quad Q_1(\mathbf{y}, \mathbf{x}) = Q_2(\mathbf{y}, \mathbf{x}) = 0\},$$

Wall obtains as a consequence of his classification theorem, that the number of different topological types of fibers $S_{\mathbf{x}}$ is bounded by $2^{\ell-1}$. Notice that in this case the number of parameters $(X_1, \ldots, X_{2\ell})$, as well as the number of variables $(Y_1, \ldots, Y_\ell)$, are both $O(\ell)$. Similar results were also obtained by López [23] using different techniques. Much more recently Briand [14] has obtained explicit characterization of the isotopy classes of real varieties defined by two general conics in $\mathbb{RP}^2$ in terms of the coefficients of the polynomials. His method also gives a decision algorithm for testing whether two such given varieties are isotopic.

In another direction Agrachev [1] studied the topology of semi-algebraic sets defined by quadratic inequalities, and he defined a certain spectral sequence converging to the homology groups of such sets. A parametrized version of Agrachev's construction is in fact a starting point of our proof of the main theorem in this paper.

2. Main Result

The main result of this paper is the following theorem.

Theorem 2.1. *Let* R *be a real closed field and let*

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k],$$

with $\deg_Y(P_i) \leq 2, \deg_X(P_i) \leq d, 1 \leq i \leq m$. *Let* $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ *be the projection on the last* k *co-ordinates. Then, for any* $\mathcal{P}$*-closed semi-algebraic set* $S \subset \mathrm{R}^{\ell+k}$, *the number of stable homotopy types amongst the fibers,* $S_{\mathbf{x}}$, *is bounded by* $(2^m \ell k d)^{O(mk)}$.

Remark 2.2. Note that the bound in Theorem 2.1 (unlike that in Theorem 1.2) is polynomial in ℓ for fixed m and k. The exponential dependence on m is unavoidable, as can be seen from a slight modification of Example 1.3 above. Consider the semi-algebraic set $S \subset \mathrm{R}^{\ell+1}$ defined by

$$Y_i(Y_i - 1) = 0, \ 1 \leq i \leq m \leq \ell,$$
$$0 \leq X \leq Y_1 + 2 \cdot Y_2 + \ldots + 2^{m-1} \cdot Y_m.$$

Let $\pi : \mathrm{R}^{\ell+1} \to \mathrm{R}$ be the projection on the X-coordinate. Then the sets $S_{\mathbf{x}}$, $\mathbf{x} \in \{0, 1 \ldots, 2^{m-1}\}$, have different numbers of connected components, and hence have distinct (stable) homotopy types.

Remark 2.3. Note that the technique used to prove Theorem 1.2 in [9] does not directly produce better bounds in the quadratic case, and hence we need a new approach to prove a substantially better bound in this case. However, due to technical reasons, we only obtain a bound on the number of stable homotopy types, rather than homotopy types.

3. Mathematical Preliminaries

We first need to fix some notation and a few preliminary results needed later in the proof.

3.1. Some Notation.

Let R be a real closed field. For an element $a \in R$ introduce

$$\text{sign}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

If $\mathcal{P} \subset R[X_1, \ldots, X_k]$ is finite, we write the *set of zeros* of $\mathcal{P}$ in R^k as

$$Z(\mathcal{P}) = \Big\{ \mathbf{x} \in R^k \mid \bigwedge_{P \in \mathcal{P}} P(\mathbf{x}) = 0 \Big\}.$$

A *sign condition* σ on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization of the sign condition* σ is the basic semi-algebraic set

$$\mathcal{R}(\sigma) = \Big\{ \mathbf{x} \in R^k \mid \bigwedge_{P \in \mathcal{P}} \text{sign}(P(\mathbf{x})) = \sigma(P) \Big\}.$$

A sign condition σ is *realizable* if $\mathcal{R}(\sigma) \neq \emptyset$. We denote by $\text{Sign}(\mathcal{P})$ the set of realizable sign conditions on $\mathcal{P}$. For $\sigma \in \text{Sign}(\mathcal{P})$ we define the *level of* σ as the cardinality $\#\{P \in \mathcal{P} | \sigma(P) = 0\}$. For each level p, $0 \leq p \leq \#\mathcal{P}$, we denote by $\text{Sign}_p(\mathcal{P})$ the subset of $\text{Sign}(\mathcal{P})$ of elements of level p. Moreover, for a sign condition σ let

$$\mathcal{Z}(\sigma) = \Big\{ \mathbf{x} \in R^k \mid \bigwedge_{P \in \mathcal{P}, \, \sigma(P) = 0} P(\mathbf{x}) = 0 \Big\}.$$

3.2. Use of Infinitesimals.

Later in the paper, we will extend the ground field R by infinitesimal elements. We denote by $R\langle \zeta \rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in R (see [12] for more details). The sign of a Puiseux series in $R\langle \zeta \rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ. This induces a unique order on $R\langle \zeta \rangle$ which makes ζ infinitesimal: ζ is positive and smaller than any positive element of R. When $a \in R\langle \zeta \rangle$ is bounded from above and below by some elements of R, $\lim_\zeta(a)$ is the constant term of a, obtained by substituting 0 for ζ in a. Given a semi-algebraic set S in R^k, the *extension* of S to R', denoted $\text{Ext}(S, R')$, is the semi-algebraic subset of R'^k defined by the same quantifier free formula that defines S. The set $\text{Ext}(S, R')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the Tarski-Seidenberg transfer principle (see for instance [12]).

We will also need the following remark about extensions which is again a consequence of the Tarski-Seidenberg transfer principle.

Remark 3.1. Let S, T be two closed and bounded semi-algebraic subsets of R^k, and let R' be a real closed extension of R. Then S and T are semi-algebraically homotopy equivalent if and only if $\mathrm{Ext}(S, \mathrm{R}')$ and $\mathrm{Ext}(T, \mathrm{R}')$ are semi-algebraically homotopy equivalent.

We will need a few results from algebraic topology, which we state here without proof referring the reader to papers where the proofs appear.

The following inequalities are consequences of the Mayer-Vietoris exact sequence.

3.3. Betti numbers and Mayer-Vietoris Inequalities. We will use the following notation.

Notation 1. For each $m \in \mathbb{Z}_{\geq 0}$ we will denote by $[m]$ the set $\{1, \ldots, m\}$.

Proposition 3.2 (Mayer-Vietoris inequalities). *Let the subsets $W_1, \ldots, W_r \subset \mathrm{R}^n$ be all open or all closed. Then for each $i \geq 0$ we have,*

$$(3.1) \qquad b_i\left(\bigcup_{1 \leq j \leq r} W_j\right) \leq \sum_{J \subset [r]} b_{i - (\#J) + 1}\left(\bigcap_{j \in J} W_j\right)$$

and

$$(3.2) \qquad b_i\left(\bigcap_{1 \leq j \leq r} W_j\right) \leq \sum_{J \subset [r]} b_{i + (\#J) - 1}\left(\bigcup_{j \in J} W_j\right).$$

Proof. See [12]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following proposition gives a bound on the Betti numbers of the projection $\pi(V)$ of a closed and bounded semi-algebraic set V in terms of the number and degrees of polynomials defining V.

Proposition 3.3. [21] *Let R be a real closed field and let $\pi : \mathrm{R}^{m+k} \to \mathrm{R}^k$ be the projection map on to last k co-ordinates. Let $V \subset \mathrm{R}^{m+k}$ be a closed and bounded semi-algebraic set defined by a Boolean formula with s distinct polynomials of degrees not exceeding d. Then the n-th Betti number of the projection*

$$\mathrm{b}_n(\pi(V)) \leq (nsd)^{O(k + nm)}.$$

Proof. See [21]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.4. Stable homotopy equivalence. For any finite CW-complex X we will denote by $\mathbf{S}(X)$ the suspension of X.

Recall from [25] that for two finite CW-complexes X and Y, an element of

$$(3.3) \qquad\qquad \{X; Y\} = \varinjlim_i \, [\mathbf{S}^i(X), \mathbf{S}^i(Y)]$$

is called an *S-map* (or map in the *suspension category*). (When the context is clear we will sometime denote an S-map $f \in \{X; Y\}$ by $f : X \to Y$).

Definition 3.4. An S-map $f \in \{X; Y\}$ is an S-equivalence (also called a stable homotopy equivalence) if it admits an inverse $f^{-1} \in \{Y; X\}$. In this case we say that X and Y are stable homotopy equivalent.

If $f \in \{X; Y\}$ is an S-map, then f induces a homomorphism,

$$f_* : H_*(X) \to H_*(Y).$$

The following theorem characterizes stable homotopy equivalence in terms of homology.

Theorem 3.5. [24] *Let X and Y be two finite CW-complexes. Then X and Y are stable homotopy equivalent if and only if there exists an S-map $f \in \{X; Y\}$ which induces isomorphisms $f_* : H_i(X) \to H_i(Y)$ (see [17], pp. 604).*

3.4.1. *Spanier-Whitehead duality.* In order to compare the complements of closed and bounded semi-algebraic sets which are homotopy equivalent, we will use the duality theory due to Spanier and Whitehead [25]. We will need the following facts about Spanier-Whitehead duality (see [17], pp. 603 for more details). Let $X \subset \mathbf{S}^n$ be a finite CW-complex. Then there exists (up to stable homotopy equivalence) a dual complex, denoted $D_n X \subset \mathbf{S}^n \setminus X$. The dual complex $D_n X$ is defined only up to S-equivalence. In particular, any deformation retract of $\mathbf{S}^n \setminus X$ represents $D_n X$. Moreover, the functor D_n has the following property. If $Y \subset \mathbf{S}^n$ is another finite CW-complex, and the S-map represented by $\phi : X \to Y$ is a stable homotopy equivalence, then there exists a stable homotopy equivalence $D_n \phi$. Moreover, if the map $\phi : X \to Y$ is an inclusion, then the dual S-map $D_n \phi$ is also represented by a corresponding inclusion.

Remark 3.6. Note that, since Spanier-Whitehead duality theory deals only with finite polyhedra over $\mathbb{R}$, it extends without difficulty to general real closed fields using the Tarski-Seidenberg transfer principle.

3.5. **Homotopy colimits.** Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, where each A_i is a sub-complex of a finite CW-complex.

Let $\Delta_{[n]}$ denote the standard simplex of dimension $n-1$ with vertices in $[n]$. For $I \subset [n]$, we denote by Δ_I the $(\#I - 1)$-dimensional face of $\Delta_{[n]}$ corresponding to I, and by A_I the CW-complex $\bigcap_{i \in I} A_i$.

The homotopy colimit, $\mathrm{hocolim}(\mathcal{A})$, is a CW-complex defined as follows.

Definition 3.7.
$$\mathrm{hocolim}(\mathcal{A}) = \biguplus_{I \subset [n]} \Delta_I \times A_I / \sim$$
where the equivalence relation $\sim$ is defined as follows.

For $I \subset J \subset [n]$, let $s_{I,J} : \Delta_I \hookrightarrow \Delta_J$ denote the inclusion map of the face Δ_I in Δ_J, and let $i_{I,J} : A_J \hookrightarrow A_I$ denote the inclusion map of A_J in A_I.

Given $(\mathbf{s}, \mathbf{x}) \in \Delta_I \times A_I$ and $(\mathbf{t}, \mathbf{y}) \in \Delta_J \times A_J$ with $I \subset J$, then $(\mathbf{s}, \mathbf{x}) \sim (\mathbf{t}, \mathbf{y})$ if and only if $\mathbf{t} = s_{I,J}(\mathbf{s})$ and $\mathbf{x} = i_{I,J}(\mathbf{y})$.

We have a obvious map

$$f : \mathrm{hocolim}(\mathcal{A}) \longrightarrow \mathrm{colim}(\mathcal{A}) = \bigcup_{i \in [n]} A_i$$

sending $(\mathbf{s}, \mathbf{x}) \mapsto \mathbf{x}$. It is a consequence of the Smale-Vietoris theorem [26] that

Lemma 3.8. *The map*

$$f : \mathrm{hocolim}(\mathcal{A}) \longrightarrow \mathrm{colim}(\mathcal{A}) = \bigcup_{i \in [n]} A_i$$

is a homotopy equivalence.

Now let $\mathcal{A} = \{A_1, \ldots, A_n\}$ (resp. $\mathcal{B} = \{B_1, \ldots, B_n\}$) be a set of sub-complexes of a finite CW-complex. For each $I \subset [n]$ let $f_I \in \{A_I; B_I\}$ be a stable homotopy equivalence, having the property that for each $I \subset J \subset [n]$, $f_J = f_I|_{A_J}$. Then we have an induced S-map, $f \in \{\mathrm{hocolim}(\mathcal{A}); \mathrm{hocolim}(\mathcal{B})\}$, and we have that

Lemma 3.9. *The induced S-map $f \in \{\mathrm{hocolim}(\mathcal{A}); \mathrm{hocolim}(\mathcal{B})\}$ is a stable homotopy equivalence.*

Proof. Using the Mayer-Vietoris exact sequence it is easy to see that if the f_I's induce isomorphisms in homology, so does the map f. Now apply Theorem 3.5.   □

4. Proof of Theorem 2.1

4.1. **Proof Strategy.** The strategy underlying our proof of Theorem 2.1 is as follows. We first consider the special case of a semi-algebraic subset, $A \subset \mathbf{S}^\ell$, defined by a disjunction of m homogeneous quadratic inequalities restricted to the unit sphere in $\mathrm{R}^{\ell+1}$. We then show that there exists a closed and bounded semi-algebraic set C' (see (4.14) below for the precise definition of the semi-algebraic set C'), consisting of certain sphere bundles, glued along certain sub-sphere bundles, which is homotopy equivalent to A. The number of these sphere bundles, as well descriptions of their bases, are bounded polynomially in ℓ (for fixed m).

In the presence of parameters $X_1, \ldots, X_k$, the set A, as well as C', will depend on the values of the parameters. However, using some basic homotopy properties of bundles, we show that the homotopy type of the set C' stays invariant under continuous deformation of the bases of the different sphere bundles which constitute C'. These bases also depend on the parameters, $X_1, \ldots, X_k$, but the degrees in $X_1, \ldots, X_k$ of the polynomials defining them are bounded by $O(\ell d)$. Now, using techniques similar to those used in [9], we are able to control the number of isotopy types of the bases which occur as the parameters vary over R^k. The bound on the number of isotopy types, also gives a bound on the number of possible homotopy types of the set C', and hence of A, for different values of the parameter.

In order to prove the results for semi-algebraic sets defined by more general formulas than disjunctions of weak inequalities, we first use Spanier-Whitehead duality to obtain a bound in the case of conjunctions, and then use the construction of homotopy colimits to prove the theorem for general $\mathcal{P}$-closed sets. Because of the use of Spanier-Whitehead duality we get bounds on the number of stable homotopy types, rather than homotopy types.

4.2. **Topology of sets defined by quadratic constraints.** One of the main ideas behind our proof of Theorem 2.1 is to parametrize a construction introduced by Agrachev in [1] while studying the topology of sets defined by (purely) quadratic inequalities (that is without the parameters $X_1, \ldots, X_k$ in our notation). However, we avoid construction of Leray spectral sequences as was done in [1]. For the rest of this section, we fix a set of polynomials

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$$

which are homogeneous of degree 2 in $Y_0, \ldots, Y_\ell$, and of degree at most d in $X_1, \ldots, X_k$.

We will denote by

$$Q = (Q_1, \ldots, Q_m) : \mathrm{R}^{\ell+1} \times \mathrm{R}^k \to \mathrm{R}^m,$$

the map defined by the polynomials $Q_1, \ldots, Q_m$, and generally, for $I \subset \{1, \ldots, m\}$, we denote by $Q_I : \mathrm{R}^{\ell+1} \times \mathrm{R}^k \to \mathrm{R}^I$, the map whose co-ordinates are given by Q_i, $i \in I$. When $I = [m]$, we will often drop the subscript I from our notation.

For any subset $I \subset [m]$, let $A_I \subset \mathbf{S}^\ell \times \mathrm{R}^k$ be the semi-algebraic set defined by

$$(4.1) \qquad A_I = \bigcup_{i \in I} \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \ \wedge \ Q_i(\mathbf{y}, \mathbf{x}) \leq 0\},$$

and let

$$(4.2) \qquad \Omega_I = \{\omega \in \mathrm{R}^m \mid |\omega| = 1, \omega_i = 0, i \notin I, \omega_i \leq 0, i \in I\}.$$

For $\omega \in \Omega_I$ we denote by $\omega Q \in \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ the polynomial defined by

$$(4.3) \qquad \omega Q = \sum_{i=0}^m \omega_i Q_i.$$

For $(\omega, \mathbf{x}) \in F_I = \Omega_I \times \mathrm{R}^k$, we will denote by $\omega Q(\cdot, \mathbf{x})$ the quadratic form in $Y_0, \ldots, Y_\ell$ obtained from ωQ by specializing $X_i = \mathbf{x}_i, 1 \leq i \leq k$.

Let $B_I \subset \Omega_I \times \mathbf{S}^\ell \times \mathrm{R}^k$ be the semi-algebraic set defined by

$$(4.4) \qquad B_I = \{(\omega, \mathbf{y}, \mathbf{x}) \mid \omega \in \Omega_I, \mathbf{y} \in \mathbf{S}^\ell, \mathbf{x} \in \mathrm{R}^k, \ \omega Q(\mathbf{y}, \mathbf{x}) \geq 0\}.$$

We denote by $\phi_1 : B_I \to F_I$ and $\phi_2 : B_I \to \mathbf{S}^\ell \times \mathrm{R}^k$ the two projection maps (see diagram below).

$$(4.5) \qquad \begin{array}{ccccc} & & B_I & & \\ & \overset{\phi_{I,1}}{\swarrow} & \vdots & \overset{\phi_{I,2}}{\searrow} & \\ F_I = \Omega_I \times \mathrm{R}^k & \longrightarrow & \mathrm{R}^k & \longleftarrow & \mathbf{S}^\ell \times \mathrm{R}^k \end{array}$$

The following key proposition was proved by Agrachev [1] in the unparametrized situation, but as we see below it works in the parametrized case as well.

Proposition 4.1. *The map ϕ_2 gives a homotopy equivalence between B_I and $\phi_2(B_I) = A_I$.*

Proof. In order to simplify notation we prove it in the case $I = [m]$, and the case for any other I would follow immediately. We first prove that $\phi_2(B) = A$. If $(\mathbf{y}, \mathbf{x}) \in A$, then there exists some $i, 1 \leq i \leq m$, such that $Q_i(\mathbf{y}, \mathbf{x}) \leq 0$. Then for $\omega = (-\delta_{1,i}, \ldots, -\delta_{m,i})$ (where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise), we see that $(\omega, \mathbf{y}, \mathbf{x}) \in B$. Conversely, if $(\mathbf{y}, \mathbf{x}) \in \phi_2(B)$, then there exists $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$ such that,

$$\sum_{i=1}^m \omega_i Q_i(\mathbf{y}, \mathbf{x}) \geq 0.$$

Since $\omega_i \leq 0, 1 \leq i \leq m$, and not all $\omega_i = 0$, this implies that $Q_i(\mathbf{y}, \mathbf{x}) \leq 0$ for some $i, 1 \leq i \leq m$. This shows that $(\mathbf{y}, \mathbf{x}) \in A$.

For $(\mathbf{y}, \mathbf{x}) \in \phi_2(B)$, the fiber

$$\phi_2^{-1}(\mathbf{y}, \mathbf{x}) = \{(\omega, \mathbf{y}, \mathbf{x}) \mid \omega \in \Omega \text{ such that } \omega Q(\mathbf{y}, \mathbf{x}) \geq 0\}$$

is a non-empty subset of Ω defined by a single linear inequality. Thus each non-empty fiber is an intersection of a convex cone with $\mathbf{S}^{m-1}$, and hence contractible.

The proposition now follows from the well-known Smale-Vietoris theorem [26]. $\square$

We will use the following notation.

Notation 2. For any quadratic form $Q \in \mathrm{R}[Y_0, \ldots, Y_\ell]$, we will denote by $\mathrm{index}(Q)$ the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form, that is of the matrix M_Q such that, $Q(\mathbf{y}) = \langle M_Q \mathbf{y}, \mathbf{y} \rangle$ for all $\mathbf{y} \in \mathrm{R}^{\ell+1}$ (here $\langle \cdot, \cdot \rangle$ denotes the usual inner product). We will also denote by $\lambda_i(Q), 0 \leq i \leq \ell$, the eigenvalues of Q, in non-decreasing order, i.e.,

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_\ell(Q).$$

For $I \subset [m]$, let

$$(4.6) \qquad F_{I,j} = \{(\omega, \mathbf{x}) \in \Omega_I \times \mathrm{R}^k \mid \mathrm{index}(\omega Q(\cdot, \mathbf{x})) \leq j\}.$$

It is clear that each $F_{I,j}$ is a closed semi-algebraic subset of F_I and that they induce a filtration of the space F_I given by

$$F_{I,0} \subset F_{I,1} \subset \cdots \subset F_{I,\ell+1} = F_I.$$

Lemma 4.2. *The fiber of the map $\phi_{I,1}$ over a point $(\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.*

Proof. As before, we prove the lemma only for $I = [m]$. The proof for a general I is identical. First notice that for $(\omega, \mathbf{x}) \in F_j \setminus F_{j-1}$, the first j eigenvalues of $\omega Q(\cdot, \mathbf{x})$

$$\lambda_0(\omega Q(\cdot, \mathbf{x})), \ldots, \lambda_{j-1}(\omega Q(\cdot, \mathbf{x})) < 0.$$

Moreover, letting $W_0(\omega Q(\cdot, \mathbf{x})), \ldots, W_\ell(\omega Q(\cdot, \mathbf{x}))$ be the co-ordinates with respect to an orthonormal basis $e_0(\omega Q(\cdot, \mathbf{x})), \ldots, e_\ell(\omega Q(\cdot, \mathbf{x}))$, consisting of eigenvectors of $\omega Q(\cdot, \mathbf{x})$, we have that $\phi_1^{-1}(\omega, \mathbf{x})$ is the subset of $\mathbf{S}^\ell = \{\omega\} \times \mathbf{S}^\ell \times \{\mathbf{x}\}$ defined by

$$\sum_{i=0}^\ell \lambda_i(\omega Q(\cdot, \mathbf{x})) W_i(\omega Q(\cdot, \mathbf{x}))^2 \geq 0,$$

$$\sum_{i=0}^\ell W_i(\omega Q(\cdot, \mathbf{x}))^2 = 1.$$

Since, $\lambda_i(\omega Q(\cdot, \mathbf{x})) < 0, 0 \leq i < j$, it follows that for $(\omega, \mathbf{x}) \in F_j \setminus F_{j-1}$, the fiber $\phi_1^{-1}(\omega, \mathbf{x})$ is homotopy equivalent to the $(\ell - j)$-dimensional sphere defined by setting

$$W_0(\omega Q(\cdot, \mathbf{x})) = \cdots = W_{j-1}(\omega Q(\cdot, \mathbf{x})) = 0$$

on the sphere defined by $\sum_{i=0}^\ell W_i(\omega Q(\cdot, \mathbf{x}))^2 = 1.$ $\square$

For each $(\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}$, let $L_j^+(\omega, \mathbf{x}) \subset \mathrm{R}^{\ell+1}$ denote the sum of the non-negative eigenspaces of $\omega Q(\cdot, \mathbf{x})$ (i.e. $L_j^+(\omega, \mathbf{x})$ is the largest linear subspace of $\mathrm{R}^{\ell+1}$ on which $\omega Q(\cdot, \mathbf{x})$ is positive semi-definite). Since $\mathrm{index}(\omega Q(\cdot, \mathbf{x})) = j$ stays invariant as $(\omega, \mathbf{x})$ varies over $F_{I,j} \setminus F_{I,j-1}$, $L_j^+(\omega, \mathbf{x})$ varies continuously with $(\omega, \mathbf{x})$.

We will denote by C_I the semi-algebraic set defined by

$$(4.7) \qquad C_I = \bigcup_{j=0}^{\ell+1} \{(\omega, \mathbf{y}, \mathbf{x}) \mid (\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}, \mathbf{y} \in L_j^+(\omega, \mathbf{x}), |\mathbf{y}| = 1\}.$$

The following proposition relates the homotopy type of B_I to that of C_I.

Proposition 4.3. *The semi-algebraic set C_I defined above is homotopy equivalent to B_I (see (4.4) for the definition of B_I).*

Proof. We give a deformation retraction of B_I to C_I constructed as follows. For each $(\omega, x) \in F_{I,\ell} \setminus F_{I,\ell-1}$, we can retract the fiber $\phi_1^{-1}(\omega, x)$ to the zero-dimensional sphere, $L_\ell^+(\omega, x) \cap \mathbf{S}^\ell$ by the following retraction. Let

$$W_0(\omega Q_I(\cdot, x)), \ldots, W_\ell(\omega Q_I(\cdot, x))$$

be the co-ordinates with respect to an orthonormal basis $e_0(\omega Q(\cdot, \mathbf{x})), \ldots, e_\ell(\omega Q(\cdot, \mathbf{x}))$, consisting of eigenvectors of $\omega Q_I(\cdot, x)$ corresponding to non-decreasing order of the eigenvalues of $\omega Q(\cdot, \mathbf{x})$. Then, $\phi_1^{-1}(\omega, x)$ is the subset of $\mathbf{S}^\ell$ defined by

$$\sum_{i=0}^{\ell} \lambda_i(\omega Q_I(\cdot, x)) W_i(\omega Q_I(\cdot, x))^2 \geq 0,$$

$$\sum_{i=0}^{\ell} W_i(\omega Q_I(\cdot, x))^2 = 1.$$

and $L_\ell^+(\omega, x)$ is defined by $W_0(\omega Q_I(\cdot, x)) = \cdots = W_{\ell-1}(\omega Q_I(\cdot, x)) = 0$. We retract $\phi_1^{-1}(\omega, x)$ to the zero-dimensional sphere, $L_\ell^+(\omega, x) \cap \mathbf{S}^\ell$ by the retraction sending, $(w_0, \ldots, w_\ell) \in \phi_1^{-1}(\omega, x)$, at time t to $((1-t)w_0, \ldots, (1-t)w_{\ell-1}, t'w_\ell)$, where $0 \leq t \leq 1$, and $t' = \left(\dfrac{1 - (1-t)^2 \sum_{i=0}^{\ell-1} w_i^2}{w_\ell^2} \right)^{1/2}$. Notice that even though the local co-ordinates $(W_0, \ldots, W_\ell)$ in $\mathrm{R}^{\ell+1}$ with respect to the orthonormal basis $(e_0, \ldots, e_\ell)$ may not be uniquely defined at the point (ω, x) (for instance, if the quadratic form $\omega Q_I(\cdot, x)$ has multiple eigen-values), the retraction is still well-defined since it only depends on the decomposition of $R^{\ell+1}$ into orthogonal complements $\mathrm{span}(e_0, \ldots, e_{\ell-1})$ and $\mathrm{span}(e_\ell)$ which is well defined. We can thus retract simultaneously all fibers over $F_{I\ell} \setminus F_{I,\ell-1}$ continuously, to obtain a semi-algebraic set $B_{I,\ell} \subset B_I$, which is moreover homotopy equivalent to B_I.
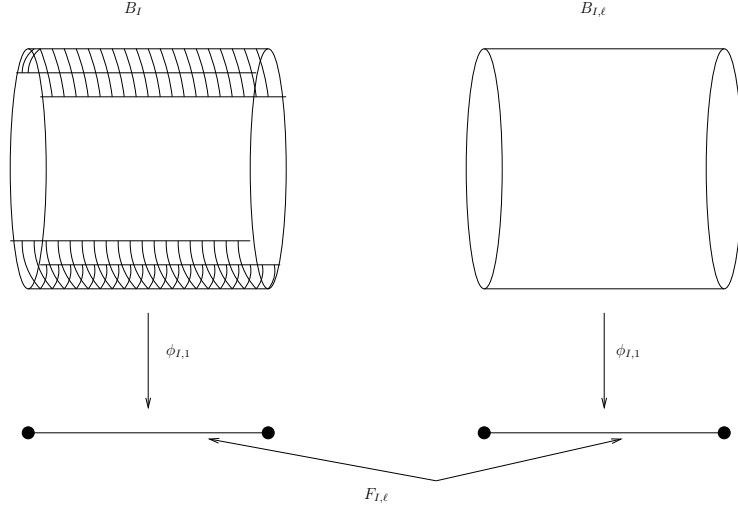
This retraction is schematically shown in Figure 1, where $F_{I,\ell}$ is the closed segment, and $F_{I,\ell-1}$ are its end points.

Now starting from $B_{I,\ell}$, retract all fibers over $F_{I,\ell-1} \setminus F_{I,\ell-2}$ to the corresponding one dimensional spheres, by the retraction sending

$$(w_0, \ldots, w_\ell) \in \phi_1^{-1}(\omega, x),$$

at time t to

$$((1-t)w_0, \ldots, (1-t)w_{\ell-2}, t'w_{\ell-1}, t'w_\ell),$$

B_I $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $B_{I,\ell}$

$\phi_{I,1}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\phi_{I,1}$

$F_{I,\ell}$

FIGURE 1. Schematic picture of the retraction of B_I to $B_{I,\ell}$.

where $0 \le t \le 1$, and

$$t' = \left(\frac{1 - (1-t)^2 \sum_{i=0}^{\ell-2} w_i^2}{\sum_{i=\ell-1}^{\ell} w_i^2} \right)^{1/2}$$

to obtain $B_{I,\ell-1}$, which is homotopy equivalent to $B_{I,\ell}$. Continuing this process we finally obtain $B_{I,0} = C_I$, which is clearly homotopy equivalent to B_I by construction. $\qquad\square$

Notice that the semi-algebraic set $\phi_1^{-1}(F_{I,j} \setminus F_{I,j-1}) \cap C_I$ is a $\mathbf{S}^{\ell-j}$-bundle over $F_{I,j} \setminus F_{I,j-1}$ under the map ϕ_1, and C_I is a union of these sphere bundles. We have good control over the bases, $F_{I,j} \setminus F_{I,j-1}$, of these bundles, that is, we have good bounds on the number as well as the degrees of polynomials used to define them. However, these bundles could be possibly glued to each other in complicated ways, and it is not immediate how to control this gluing data, since different types of gluing could give rise to different homotopy types of the underlying space. In order to get around this difficulty, we consider certain closed subsets, $F'_{I,j}$ of F_I, where each $F'_{I,j}$ is an infinitesimal deformation of $F_{I,j} \setminus F_{I,j-1}$, and form the base of a $\mathbf{S}^{\ell-j}$-bundle. Moreover, these new sphere bundles are glued to each other along sphere bundles over $F'_{I,j} \cap F'_{I,j-1}$, and their union, C'_I, is homotopy equivalent to C_I. Finally, the polynomials defining the sets $F'_{I,j}$ are in general position in a very strong sense, and this property is used later to bound the number of isotopy classes of the sets $F'_{I,j}$ in the parametrized situation.

We now make precise the argument outlined above. Let Λ_I be the polynomial in $\mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k, T]$ defined by

$$\begin{aligned} \Lambda_I &= \det(M_{Z_I \cdot Q} + T \,\mathrm{Id}_{\ell+1}), \\ &= T^{\ell+1} + H_{I,\ell} T^\ell + \cdots + H_{I,0}, \end{aligned}$$

where $Z_I \cdot Q = \sum_{i \in I} Z_i Q_i$, and each $H_{I,j} \in \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

Notice, that $H_{I,j}$ is obtained from $H_j = H_{[m],j}$ by setting the variable Z_i to 0 in the polynomial H_j for each $i \notin I$.

Note also that for $(\mathbf{z}, \mathbf{x}) \in \mathrm{R}^m \times \mathrm{R}^k$, the polynomial $\Lambda_I(\mathbf{z}, \mathbf{x}, T)$ being the characteristic polynomial of a real symmetric matrix has all its roots real. It then follows from Descartes' rule of signs (see for instance [12]), that for each $(\mathbf{z}, \mathbf{x}) \in \mathrm{R}^m \times \mathrm{R}^k$, where $\mathbf{z}_i = 0$ for all $i \notin I$, $\mathrm{index}(\mathbf{z}Q(\cdot, \mathbf{x}))$ is determined by the sign vector

$$(\mathrm{sign}(H_{I,\ell}(\mathbf{z}, \mathbf{x})), \dots, \mathrm{sign}(H_{I,0}(\mathbf{z}, \mathbf{x}))).$$

Hence, denoting by

$$(4.8) \qquad \mathcal{H}_I = \{H_{I,0}, \dots, H_{I,\ell}\} \subset \mathrm{R}[Z_1, \dots, Z_m, X_1, \dots, X_k],$$

we have

Lemma 4.4. *For each $j, 0 \le j \le \ell + 1$, $F_{I,j}$ is the intersection of F_I with a $\mathcal{H}_I$-closed semi-algebraic set $D_{I,j} \subset \mathrm{R}^{m+k}$.*

Notation 3. Let $D_{I,j}$ be defined by the formula

$$(4.9) \qquad D_{I,j} = \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma),$$

for some $\Sigma_{I,j} \subset \mathrm{Sign}(\mathcal{H}_I)$. Note that, $\mathrm{Sign}(\mathcal{H}_I) \subset \mathrm{Sign}(\mathcal{H})$ and $\Sigma_{I,j} \subset \Sigma_j$ for all $I \subset [m]$.

Now, let $\bar{\delta} = (\delta_\ell, \dots, \delta_0)$ and $\bar{\varepsilon} = (\varepsilon_{\ell+1}, \dots, \varepsilon_0)$ be infinitesimals such that

$$0 < \delta_0 \ll \cdots \ll \delta_\ell \ll \varepsilon_0 \ll \cdots \ll \varepsilon_{\ell+1} \ll 1,$$

and let

$$(4.10) \qquad\qquad\qquad \mathrm{R}' = \mathrm{R}\langle \bar{\varepsilon}, \bar{\delta} \rangle$$

Given $\sigma \in \mathrm{Sign}(\mathcal{H}_I)$, and $0 \le j \le \ell + 1$, we denote by $\mathcal{R}(\sigma_j^c) \subset \mathrm{R}'^{m+k}$ the set defined by the formula σ_j^c obtained by taking the conjunction of

$$-\varepsilon_j - \delta_i \le H_{I,i} \le \varepsilon_j + \delta_i \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 0,$$
$$H_{I,i} \ge -\varepsilon_j - \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 1,$$
$$H_{I,i} \le \varepsilon_j + \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = -1.$$

Similarly, we denote by $\mathcal{R}(\sigma_j^o) \subset \mathrm{R}'^{m+k}$ the set defined by the formula σ^o obtained by taking the conjunction of

$$-\varepsilon_j - \delta_i < H_{I,i} < \varepsilon_j + \delta_i \text{ for each } H_{i,I} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 0,$$
$$H_{I,i} > -\varepsilon_j - \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 1,$$
$$H_{I,i} < \varepsilon_j + \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = -1.$$

For each $j, 0 \le j \le \ell + 1$, let

$$D_{I,j}^o \;=\; \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma_j^o),$$
$$D_{I,j}^c \;=\; \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma_j^c),$$
$$D_{I,j}' \;=\; D_{I,j}^c \setminus D_{I,j-1}^o,$$
$$(4.11) \qquad\qquad F_{I,j}' \;=\; \mathrm{Ext}(F_I, \mathrm{R}') \cap D_{I,j}'.$$

where we denote by $D_{I,-1}^o = \emptyset$. We also denote by $F_I' = \mathrm{Ext}(F_I, \mathrm{R}')$.

We now note some extra properties of the sets $D'_{I,j}$'s.

Lemma 4.5. *For each* $j, 0 \leq j \leq \ell + 1$, $D'_{I,j}$ *is a* $\mathcal{H}'_I$*-closed semi-algebraic set, where*

$$(4.12) \qquad \mathcal{H}'_I = \bigcup_{i=0}^{\ell} \bigcup_{j=0}^{\ell+1} \{H_{I,i} + \varepsilon_j + \delta_i, H_{I,i} - \varepsilon_j - \delta_i\}.$$

Proof. Follows from the definition of the sets $D'_{I,j}$. $\qquad\square$

Lemma 4.6. *For* $0 \leq j + 1 < i \leq \ell + 1$,

$$D'_{I,i} \cap D'_{I,j} = \emptyset.$$

Proof. In order to keep notation simple we prove the proposition only for $I = [m]$. The proof for a general I is identical. The inclusions,

$$D_{j-1} \subset D_j \subset D_{i-1} \subset D_i,$$
$$D^o_{j-1} \subset D^c_j \subset D^o_{i-1} \subset D^c_i.$$

follow directly from the definitions of the sets

$$D_i, D_j, D_{j-1}, D^c_i, D^c_j, D^o_{i-1}, D^o_{j-1},$$

and the fact that,

$$\varepsilon_{j-1} \ll \varepsilon_j \ll \varepsilon_{i-1} \ll \varepsilon_i.$$

It follows immediately that,

$$D'_i = D^c_i \setminus D^o_{i-1}$$

is disjoint from D^c_j, and hence from D'_j. $\qquad\square$

We now associate to each $F'_{I,j}$ a $(\ell-j)$-dimensional sphere bundle as follows. For each $(\omega, \mathbf{x}) \in F''_{I,j} = \mathrm{Ext}(F_{I,j}, \mathrm{R}') \setminus F'_{I,j-1}$, let $L^+_j(\omega, \mathbf{x}) \subset \mathrm{R}'^{\ell+1}$ denote the sum of the non-negative eigenspaces of $\omega Q(\cdot, \mathbf{x})$ (i.e. $L^+_j(\omega, \mathbf{x})$ is the largest linear subspace of $\mathrm{R}^{\ell+1}$ on which $\omega Q(\cdot, \mathbf{x})$ is positive semi-definite). Since $\mathrm{index}(\omega Q(\cdot, \mathbf{x})) = j$ stays invariant as $(\omega, \mathbf{x})$ varies over $F''_{I,j}$, $L^+_j(\omega, \mathbf{x})$ varies continuously with $(\omega, \mathbf{x})$.

Let,

$$\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x}) < 0 \leq \lambda_j(\omega, \mathbf{x}) \leq \cdots \leq \lambda_\ell(\omega, \mathbf{x}),$$

be the eigenvalues of $\omega Q(\cdot, \mathbf{x})$ for $(\omega, \mathbf{x}) \in F''_{I,j}$. There is a continuous extension of the map sending $(\omega, \mathbf{x}) \mapsto L^+_j(\omega, \mathbf{x})$ to $(\omega, \mathbf{x}) \in F'_{I,j}$.

To see this observe that for $(\omega, \mathbf{x}) \in F''_{I,j}$ the block of the first j (negative) eigenvalues, $\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x})$, and hence the sum of the eigenspaces corresponding to them can be extended continuously to any infinitesimal neighborhood of $F''_{I,j}$, and in particular to $F'_{I,j}$. Now $L^+_j(\omega, \mathbf{x})$ is the orthogonal complement of the sum of the eigenspaces corresponding to the block of negative eigenvalues, $\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x})$.

We will denote by $C'_{I,j} \subset F'_{I,j} \times \mathrm{R}'^{\ell+1}$ the semi-algebraic set defined by

$$(4.13) \qquad C'_{I,j} = \{(\omega, \mathbf{y}, \mathbf{x}) \mid (\omega, \mathbf{x}) \in F'_{I,j}, \mathbf{y} \in L^+_j(\omega, \mathbf{x}), |\mathbf{y}| = 1\}.$$

Note that the projection $\pi_{I,j} : C'_{I,j} \to F'_{I,j}$, makes $C'_{I,j}$ the total space of a $(\ell-j)$-dimensional sphere bundle over $F'_{I,j}$.

Now observe that

$$C'_{I,j-1} \cap C'_{I,j} = \pi_{I,j}^{-1}(F'_{I,j} \cap F'_{I,j-1}),$$

and

$$\pi_{I,j}|_{C'_{I,j-1} \cap C'_{I,j}} : C'_{I,j-1} \cap C'_{I,j} \to F'_{I,j} \cap F'_{I,j-1}$$

is also a $(\ell - j)$ dimensional sphere bundle over $F'_{I,j} \cap F'_{I,j-1}$.

Let

(4.14)
$$C'_I = \bigcup_{j=0}^{\ell+1} C'_{I,j}.$$

We have that

Proposition 4.7. C'_I *is homotopy equivalent to* $\mathrm{Ext}(C_I, \mathrm{R}')$, *where* C_I *and* R' *are defined in (4.7) and (4.10) respectively.*

Proof. Let $\bar{\varepsilon} = (\varepsilon_{\ell+1}, \ldots, \varepsilon_0)$ and let

$$R_i = \begin{cases} R\langle \bar{\varepsilon}, \delta_\ell, \ldots, \delta_i \rangle, \ 0 \le i \le \ell, \\ R\langle \varepsilon_{\ell+1}, \ldots, \varepsilon_{i-\ell-1} \rangle, \ \ell + 1 \le i \le 2\ell + 2, \\ R, \ i = 2\ell + 3. \end{cases}$$

First observe that $C_I = \lim_{\varepsilon_{\ell+1}} C'_I$ where C_I is the semi-algebraic set defined in (4.7) above.

Now let,

$$\begin{aligned} C_{I,-1} &= C'_I, \\ C_{I,0} &= \lim_{\delta_0} C'_I, \\ C_{I,i} &= \lim_{\delta_i} C_{I,i-1}, 1 \le i \le \ell, \\ C_{I,\ell+1} &= \lim_{\varepsilon_0} C_{I,\ell}, \\ C_{I,i} &= \lim_{\varepsilon_{i-\ell-2}} C_{I,i-1}, \ell + 2 \le i \le 2\ell + 3. \end{aligned}$$

Notice that each $C_{I,i}$ is a closed and bounded semi-algebraic set. Also, for $i \ge 0$, let $C_{I,i-1,t} \subset \mathrm{R}_i^{m+\ell+k}$ be the semi-algebraic set obtained by replacing δ_i (resp., ε_i) in the definition of $C_{I,i-1}$ by the variable t. Then there exists $t_0 > 0$, such that for all $0 < t_1 < t_2 \le t_0$, $C_{I,i-1,t_1} \subset C_{I,i-1,t_2}$.

It follows (see Lemma 16.17 in [12]) that for each i, $0 \le i \le 2\ell + 3$, $\mathrm{Ext}(C_{I,i}, \mathrm{R}_i)$ is homotopy equivalent to $C_{I,i-1}$. $\square$

4.2.1. *Partitioning the parameter space.* The goal of this section is to prove the following proposition (Proposition 4.8). The techniques used in the proof are similar to those used in [9] for proving a similar result. We go through the proof in detail in order to extract the right bound in terms of the parameters d, k, ℓ and m.

Proposition 4.8. *There exists a finite set of points* $T \subset \mathrm{R}^k$, *with*

$$\#T \le (2^m \ell k d)^{O(mk)},$$

such that for any $\mathbf{x} \in \mathrm{R}^k$, *there exists* $\mathbf{z} \in T$, *with the following property.*

There is a semi-algebraic path, $\gamma : [0,1] \to \mathrm{R}'^k$ and a continuous semi-algebraic map, $\phi : \Omega \times [0,1] \to \Omega$ (see (4.2) and (4.10) for the definition of Ω and R'), with $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{z}$, and for each $I \subset [m]$,

$$\phi(\cdot, t)|_{F'_{I,j,\mathbf{x}}} : F'_{I,j,\mathbf{x}} \to F'_{I,j,\gamma(t)}$$

is a homeomorphism for each $0 \le t \le 1$.

Before proving Proposition 4.8 we need a few preliminary results. Let

$$(4.15) \qquad \mathcal{H}'' = \mathcal{H}' \cup \{Z_1, \ldots, Z_m, Z_1^2 + \cdots + Z_m^2 - 1\},$$

where $\mathcal{H}' = \mathcal{H}'_{[m]}$ is defined in (4.12) above.

Note that for each j, $0 \le j \le \ell + 1$, $F'_{I,j}$ is a $\mathcal{H}''$-closed semi-algebraic set. Moreover, let $\psi : \mathrm{R}'^{m+k} \to \mathrm{R}'^k$ be the projection onto the last k co-ordinates.

Notation 4. We fix a finite set of points $T \subset \mathrm{R}^k$ such that for every $\mathbf{x} \in \mathrm{R}^k$ there exists $\mathbf{z} \in T$ such that for every $\mathcal{H}''$-semi-algebraic set V, the set $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{z}) \cap V$.

The existence of a finite set T with this property follows from Hardt's triviality theorem (Theorem 1.1) and the Tarski-Seidenberg transfer principle, as well as the fact that the number of $\mathcal{H}''$-semi-algebraic sets is finite.

Now, we note some extra properties of the family $\mathcal{H}''$.

Lemma 4.9. *If $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$, then $p \le k + m$ and $\mathcal{R}(\sigma) \subset \mathrm{R}'^{m+k}$ is a non-singular $(m + k - p)$-dimensional manifold such that at every point $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$, the $(p \times (m + k))$-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial Z_i}, \frac{\partial P}{\partial Y_j} \right)_{P \in \mathcal{H}'', \; \sigma(P)=0, \; 1 \le i \le m, \; 1 \le j \le k}$$

has maximal rank p.

Proof. Let $\mathrm{Ext}(\mathbf{S}^{m-1}, \mathrm{R}')$ be the unit sphere in R'^m. Suppose without loss of generality that

$$\{P \in \mathcal{H}'' \mid \sigma(P) = 0\} = \{H_{i_1} - \varepsilon_{j_1} - \delta_{i_1}, \ldots, H_{i_{p-1}} - \varepsilon_{j_{p-1}} - \delta_{i_{p-1}}, \sum_{i=1}^m Z_i^2 - 1\}$$

since the equation $Z_i = 0$ eliminates the variable Z_i from the polynomials. It follows that it suffices to show that the algebraic set

$$(4.16) \qquad V = \bigcap_{r=1}^{p-1} \{(\mathbf{z}, \mathbf{x}) \in \mathrm{Ext}(\mathbf{S}^{m-1}, \mathrm{R}') \times \mathrm{R}'^k \mid H_{i_r}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_r} + \delta_{i_r}\}$$

is a smooth $((m-1) + k - (p-1))$-dimensional manifold such that at every point on it the $(p \times (m + k))$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}, \frac{\partial P}{\partial Y_j} \right)_{P \in \mathcal{H}'', \; \sigma(P)=0, \; 1 \le i \le m, \; 1 \le j \le k}$$

has maximal rank p.

Let $p \le m + k$. Consider the semi-algebraic map $P_{i_1,\ldots,i_{p-1}} : \mathbf{S}^{m-1} \times \mathrm{R}^k \to \mathrm{R}^{p-1}$ defined by

$$(\mathbf{z}, \mathbf{x}) \mapsto (H_{i_1}(\mathbf{z}, \mathbf{x}), \ldots, H_{i_{p-1}}(\mathbf{z}, \mathbf{x})).$$

By the semi-algebraic version of Sard's theorem (see [13]), the set of critical values of $P_{i_1,\ldots,i_{p-1}}$ is a semi-algebraic subset C of R^{p-1} of dimension strictly less than $p-1$. Since $\bar\delta$ and $\bar\varepsilon$ are infinitesimals, it follows that

$$(\varepsilon_{j_1} + \delta_{i_1}, \ldots, \varepsilon_{j_{p-1}} + \delta_{i_{p-1}}) \notin \mathrm{Ext}(C, R').$$

Hence, the algebraic set V defined in (4.16) has the desired properties, and the same is true for the basic semi-algebraic set $\mathcal{R}(\sigma)$.

We now prove that $p \le m+k$. Suppose that $p > m+k$. As we have just proved,

$$\{H_{i_1}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_1} + \delta_{i_1}, \ldots, H_{i_{m+k-1}}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_{m+k-1}} + \delta_{i_{m+k-1}}\}$$

is a finite set of points. But the polynomial $H_{i_{p-1}} - \varepsilon_{j_{p-1}} - \delta_{i_{p-1}}$ cannot vanish on each of these points as $\bar\delta$ and $\bar\varepsilon$ are infinitesimals. $\qquad\square$

Lemma 4.10. *For every* $\mathbf{x} \in \mathrm{R}^k$, *and* $\sigma \in \mathrm{Sign}_p(\mathcal{H}''_{\mathbf{x}})$, *where*

$$\mathcal{H}''_{\mathbf{x}} = \{P(Z_1, \ldots, Z_m, \mathbf{x}) \mid P \in \mathcal{H}''\},$$

the following holds.

    (1) $0 \le p \le m$, *and*
    (2) $\mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$ *is a non-singular* $(m-p)$*-dimensional manifold such that at every point* $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$, *the* $(p \times m)$*-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}''_{\mathbf{x}}, \sigma(P)=0, 1 \le i \le m}$$

    has maximal rank p.

Proof. Note that $P_{\mathbf{x}} = P(Z_1, \ldots, Z_m, \mathbf{x}) \in \mathrm{R}'[Z_1, \ldots, Z_m]$ for each $P \in \mathcal{H}''$ and $\mathbf{x} \in \mathrm{R}^k$. The proof is now identical to the proof of Lemma 4.9. $\qquad\square$

Lemma 4.11. *For any bounded* $\mathcal{H}''$*-semi-algebraic set* V *defined by*

$$V = \bigcup_{\sigma \in \Sigma_V \subset \mathrm{Sign}(\mathcal{H}'')} \mathcal{R}(\sigma),$$

the partitions

$$\mathrm{R}'^{m+k} = \bigcup_{\sigma \in \mathrm{Sign}(\mathcal{H}'')} \mathcal{R}(\sigma),$$

$$V = \bigcup_{\sigma \in \Sigma_V} \mathcal{R}(\sigma),$$

are compatible Whitney stratifications of R'^{m+k} *and* V *respectively.*

Proof. Follows directly from the definition of Whitney stratification (see [19, 16]), and Lemma 4.9. $\qquad\square$

Fix some sign condition $\sigma \in \mathrm{Sign}(\mathcal{H}'')$. Recall that $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$ is a *critical point* of the map $\psi_{\mathcal{R}(\sigma)}$ if the Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}'', \sigma(P)=0,\ 1 \le i \le m}$$

at $(\mathbf{z}, \mathbf{x})$ is not of the maximal possible rank. The projection $\psi(\mathbf{z}, \mathbf{x})$ of a critical point is a *critical value* of $\psi_{\mathcal{R}(\sigma)}$.

Let $C_1 \subset \mathrm{R}'^{m+k}$ be the set of critical points of $\psi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{p \leq m} \mathrm{Sign}_p(\mathcal{H}''),$$

(i.e., over all $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$ with $\dim(\mathcal{R}(\sigma)) \geq k$). For a bounded $\mathcal{H}''$-semi-algebraic set V, let $C_1(V) \subset V$ be the set of critical points of $\psi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{p \leq m} \mathrm{Sign}_p(\mathcal{H}'') \cap \Sigma_V$$

(i.e., over all $\sigma \in \Sigma_V$ with $\dim(\mathcal{R}(\sigma)) \geq k$).

Let $C_2 \subset \mathrm{R}'^{m+k}$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'')$$

(i.e., over all $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$ with $\dim(\mathcal{R}(\sigma)) < k$). For a bounded $\mathcal{H}''$-semi-algebraic set V, let $C_2(V) \subset V$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'') \cap \Sigma_V$$

(i.e., over all $\sigma \in \Sigma_V$ with $\dim(\mathcal{R}(\sigma)) < k$).

Denote $C = C_1 \cup C_2$, and $C(V) = C_1(V) \cup C_2(V)$.

Lemma 4.12. *For each bounded $\mathcal{H}''$-semi-algebraic V, the set $C(V)$ is closed and bounded.*

Proof. The set $C(V)$ is bounded since V is bounded. The union $C_2(V)$ of strata of dimensions less than k is closed since V is closed.

Let $\sigma_1 \in \mathrm{Sign}_{p_1}(\mathcal{H}'') \cap \Sigma_V$, $\sigma_2 \in \mathrm{Sign}_{p_2}(\mathcal{H}'') \cap \Sigma_V$, where $p_1 \leq m$, $p_1 < p_2$, and if $\sigma_1(P) = 0$, then $\sigma_2(P) = 0$ for any $P \in \mathcal{H}''$. It follows that stratum $\mathcal{R}(\sigma_2)$ lies in the closure of the stratum $\mathcal{R}(\sigma_1)$. Let $\mathcal{J}$ be the finite family of $(p_1 \times p_1)$-minors such that $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ is the set of all critical points of $\pi_{\mathcal{R}(\sigma_1)}$. Then $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_2)$ is either contained in $C_2(V)$ (when $\dim(\mathcal{R}(\sigma_2)) < k$), or is contained in the set of all critical points of $\pi_{\mathcal{R}(\sigma_2)}$ (when $\dim(\mathcal{R}(\sigma_2)) \geq k$). It follows that the closure of $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ lies in the union of the following sets:

  (1) $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$,
  (2) sets of critical points of some strata of dimensions less than $m + k - p_1$,
  (3) some strata of dimension less than k.

Using induction on descending dimensions in case (2), we conclude that the closure of $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ is contained in $C(V)$. Hence, $C(V)$ is closed. $\qquad\square$

Definition 4.13. We denote by $G_i = \psi(C_i), i = 1, 2$, and $G = G_1 \cup G_2$. Similarly, for each bounded $\mathcal{H}''$-semi-algebraic set V, we denote by $G_i(V) = \psi(C_i(V))$, $i = 1, 2$, and $G(V) = G_1(V) \cup G_2(V)$.

Lemma 4.14. *We have $T \cap G = \emptyset$. In particular, $T \cap G(V) = \emptyset$ for every bounded $\mathcal{H}''$-semi-algebraic set V.*

Proof. By Lemma 4.10, for all $\mathbf{x} \in T$, and $\sigma \in \mathrm{Sign}_p(\mathcal{H}''_{\mathbf{x}})$,

  (1) $0 \leq p \leq m$, and

(2) $\mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$ is a non-singular $(m-p)$-dimensional manifold such that at every point $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$, the $(p \times m)$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}_{\mathbf{x}}'', \sigma(P)=0, 1 \leq i \leq m}$$

has the maximal rank p.

If a point $\mathbf{x} \in T \cap G_1 = T \cap \psi(C_1)$, then there exists $\mathbf{z} \in \mathrm{R}'^m$ such that $(\mathbf{z}, \mathbf{x})$ is a critical point of $\psi_{\mathcal{R}(\sigma)}$ for some $\sigma \in \bigcup_{p \leq m} \mathrm{Sign}_p(\mathcal{H}'')$, and this is impossible by (2).

Similarly, $\mathbf{x} \in T \cap G_2 = T \cap \psi(C_2)$, implies that there exists $\mathbf{z} \in \mathrm{R}'^m$ such that $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$ for some $\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'')$, and this is impossible by (1). $\qquad\square$

Let D be a connected component of $\mathrm{R}'^k \setminus G$, and for a bounded $\mathcal{H}''$-semi-algebraic set V, let $D(V)$ be a connected component of $\psi(V) \setminus G(V)$.

Lemma 4.15. *For every bounded $\mathcal{H}''$-semi-algebraic set V, all fibers $\psi^{-1}(\mathbf{x}) \cap V$, $\mathbf{x} \in D$ are homeomorphic.*

Proof. Lemma 4.10 and Lemma 4.11 imply that $\widehat{V} = \psi^{-1}(\psi(V) \setminus G(V)) \cap V$ is a Whitney stratified set having strata of dimensions at least k. Moreover, $\psi|_{\widehat{V}}$ is a proper stratified submersion. By Thom's first isotopy lemma (in the semi-algebraic version, over real closed fields [16]) the map $\psi|_{\widehat{V}}$ is a locally trivial fibration. In particular, all fibers $\psi^{-1}(\mathbf{x}) \cap V$, $\mathbf{x} \in D(V)$ are homeomorphic for every connected component $D(V)$. The lemma follows, since the inclusion $G(V) \subset G$ implies that either $D \subset D(V)$ for some connected component $D(V)$, or $D \cap \psi(V) = \emptyset$. $\qquad\square$

Lemma 4.16. *For each $\mathbf{x} \in T$, there exists a connected component D of $\mathrm{R}'^k \setminus G$, such that $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{x}_1) \cap V$ for every bounded $\mathcal{H}''$-semi-algebraic set V and for every $\mathbf{x}_1 \in D$.*

Proof. Let V be a bounded $\mathcal{H}''$-semi-algebraic set and $\mathbf{x} \in T$. By Lemma 4.14, $\mathbf{x}$ belongs to some connected component D of $\mathrm{R}'^k \setminus G$. Lemma 4.15 implies that $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{x}_1) \cap V$ for every $\mathbf{x}_1 \in D$. $\qquad\square$

We now are able to prove Proposition 4.8.

Proof of Proposition 4.8. Recall that $G = G_1 \cup G_2$, where G_1 is the union of sets of critical values of $\psi_{\mathcal{R}(\sigma)}$ over all strata $\mathcal{R}(\sigma)$ of dimensions at least k, and G_2 is the union of projections of all strata of dimensions less than k.

By Lemma 4.16 it suffices to bound the number of connected components of the set $\mathrm{R}'^k \setminus G$. Denote by $\mathcal{E}_1$ the family of closed sets of critical points of $\psi_{\mathcal{Z}(\sigma)}$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimensions at least k (the notation $\mathcal{Z}(\sigma)$ was introduced in Section 3.1). Let $\mathcal{E}_2$ be the family of closed sets $\mathcal{Z}(\sigma)$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimensions equal to $k-1$. Let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. Denote by E the image under the projection ψ of the union of all sets in the family $\mathcal{E}$.

Because of the transversality condition, every stratum of the stratification of V, having dimension less than $m+k$, lies in the closure of a stratum having the next higher dimension. In particular, this is true for strata of dimensions less than $k-1$. It follows that $G \subset E$, and thus every connected component of the complement $\mathrm{R}'^k \setminus E$ is contained in a connected component of $\mathrm{R}'^k \setminus G$. Since $\dim(E) < k$, every connected component of $\mathrm{R}'^k \setminus G$ contains a connected component of $\mathrm{R}'^k \setminus E$.

Therefore, it is sufficient to estimate from above the Betti number $b_0(R'^k \setminus E)$ which is equal to $b_{k-1}(E)$ by the Alexander's duality.

The total number of sets $\mathcal{Z}(\sigma)$, such that $\sigma \in \mathrm{Sign}(\mathcal{H}'')$ and $\dim(\mathcal{Z}(\sigma)) \geq k-1$, is $O(\ell^{2(m+1)})$ because each $\mathcal{Z}(\sigma)$ is defined by a conjunction of at most $m+1$ of possible $O(\ell^2 + m)$ polynomial equations.

Thus the cardinality $\#\mathcal{E}$, as well as the number of images under the projection π of sets in $\mathcal{E}$ is $O(\ell^{2(m+1)})$. According to (3.1) in Proposition 3.2, $b_{k-1}(E)$ does not exceed the sum of certain Betti numbers of sets of the type

$$\Phi = \bigcap_{1 \leq i \leq p} \pi(U_i),$$

where every $U_i \in \mathcal{E}$ and $1 \leq p \leq k$. More precisely, we have

$$b_{k-1}(E) \leq \sum_{1 \leq p \leq k} \sum_{\{U_1,\ldots,U_p\} \subset \mathcal{E}} b_{k-p}\left(\bigcap_{1 \leq i \leq p} \pi(U_i)\right).$$

Obviously, there are $O(\ell^{2(m+1)k})$ sets of the kind Φ.

Using inequality (3.2) in Proposition 3.2, we have that for each Φ as above, the Betti number $b_{k-p}(\Phi)$ does not exceed the sum of certain Betti numbers of unions of the kind,

$$\Psi = \bigcup_{1 \leq j \leq q} \pi(U_{i_j}) = \pi\left(\bigcup_{1 \leq j \leq q} U_{i_j}\right),$$

with $1 \leq q \leq p$. More precisely,

$$b_{k-p}(\Phi) \leq \sum_{1 \leq q \leq p} \sum_{1 \leq i_1 < \cdots < i_q \leq p} b_{k-p+q-1}\left(\pi\left(\bigcup_{1 \leq j \leq q} U_{i_j}\right)\right).$$

It is clear that there are at most $2^p \leq 2^k$ sets of the kind Ψ.

If a set $U \in \mathcal{E}_1$, then it is defined by m polynomials of degrees at most $O(\ell d)$. If a set $U \in \mathcal{E}_2$, then it is defined by $O(2^m)$ polynomials of degrees $O(m\ell d)$, since the critical points on strata of dimensions at least k are defined by $O(2^m)$ determinantal equations, the corresponding matrices have orders $O(m)$, and the entries of these matrices are polynomials of degrees at most $O(\ell d)$.

It follows that the closed and bounded set

$$\bigcup_{1 \leq j \leq q} U_{i_j}$$

is defined by $O(k2^m))$ polynomials of degrees $O(\ell d)$.

By Proposition 3.3, $b_{k-p+q-1}(\Psi) \leq (2^m k \ell d)^{O(mk)}$ for all $1 \leq p \leq k$, $1 \leq q \leq p$. Then $b_{k-p}(\Phi) \leq (2^m k \ell d)^{O(mk)}$ for every $1 \leq p \leq k$. Since there are $O(\ell^{2(m+1)k})$ sets of the kind Φ, we get the claimed bound

$$b_{k-1}(E) \leq (2^m k \ell d)^{O(mk)}.$$

The rest of the proof follows from Proposition 4.16. $\square$

4.3. The Homogeneous Case. We first consider the case where all the polynomials in $\mathcal{Q}$ are homogeneous in variables $Y_0, \ldots, Y_\ell$ and we bound the number of homotopy types among the fibers $S_{\mathbf{x}}$, defined by the $\mathcal{Q}$-closed semi-algebraic subsets S of $\mathbf{S}^\ell \times \mathbf{R}^k$. We first the prove the following theorems for the special cases of unions and intersections.

Theorem 4.17. *Let* R *be a real closed field and let*

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathbf{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k],$$

where each Q_i is homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$, and of degree at most d in $X_1, \ldots, X_k$.

For $i \in [m]$, let $A_i \subset \mathbf{S}^\ell \times \mathbf{R}^k$ be semi-algebraic sets defined by

$$A_i = \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \wedge Q_i(\mathbf{y}, \mathbf{x}) \leq 0)\},$$

Let $\pi : \mathbf{S}^\ell \times \mathbf{R}^k \to \mathbf{R}^k$ be the projection on the last k co-ordinates.

Then the number of homotopy types amongst the fibers $\bigcup\limits_{i=1}^{m} A_{i,\mathbf{x}}$ is bounded by

$$(2^m \ell k d)^{O(mk)}.$$

With the same assumptions as in Theorem 4.17 we have

Theorem 4.18. *The number of stable homotopy types amongst the fibers $\bigcap\limits_{i=1}^{m} A_{i,\mathbf{x}}$ is bounded by*

$$(2^m \ell k d)^{O(mk)}.$$

Before proving Theorems 4.17 and 4.18 we first prove two preliminary lemmas.

Lemma 4.19. *There exists a finite set $T \subset \mathbf{R}^k$, with*

$$\#T \leq (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathbf{R}^k$ there exists $\mathbf{z} \in T$, a semi-algebraic set $D_{\mathbf{x},\mathbf{z}} \subset \mathbf{R}'^{m+\ell}$, and semi-algebraic maps $f_{\mathbf{x}}, f_{\mathbf{z}}$, as shown in the diagram below, such that $f_{\mathbf{x}}, f_{\mathbf{z}}$ are both homotopy equivalences.

(4.17)

$$
\begin{array}{ccc}
& D_{\mathbf{x},\mathbf{z}} & \\
& \swarrow^{f_{\mathbf{x}}} \quad {}_{\sim} \quad {}_{\sim} \searrow^{f_{\mathbf{z}}} & \\
\mathrm{Ext}(\bigcup\limits_{i \in [m]} A_{i,\mathbf{x}}, \mathbf{R}') & & \mathrm{Ext}(\bigcup\limits_{i \in [m]} A_{i,\mathbf{z}}, \mathbf{R}')
\end{array}
$$

Moreover, for each $I \subset [m]$, there exists a subset $D_{I,\mathbf{x},\mathbf{z}} \subset D_{\mathbf{x},\mathbf{z}}$, such that the restrictions, $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$, of $f_{\mathbf{x}}, f_{\mathbf{z}}$ to $D_{I,\mathbf{x},\mathbf{z}}$ give rise to the following diagram in which all maps are again homotopy equivalences.

(4.18)

$$
\begin{array}{ccc}
& D_{I,\mathbf{x},\mathbf{z}} & \\
& \swarrow^{f_{I,\mathbf{x}}} \quad {}_{\sim} \quad {}_{\sim} \searrow^{f_{I,\mathbf{z}}} & \\
\mathrm{Ext}(\bigcup\limits_{i \in I} A_{i,\mathbf{x}}, \mathbf{R}') & & \mathrm{Ext}(\bigcup\limits_{i \in I} A_{i,\mathbf{z}}, \mathbf{R}')
\end{array}
$$

For each $I \subset J \subset [m]$, $D_{I,\mathbf{x},\mathbf{z}} \subset D_{J,\mathbf{x},\mathbf{z}}$ and the maps $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ are restrictions of $f_{J,\mathbf{x}}, f_{J,\mathbf{z}}$.

Proof of Lemma 4.19. By Proposition 4.8, there exists $T \subset \mathrm{R}^k$ with

$$\#T \leq (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$, with the following property.

There is a semi-algebraic path, $\gamma : [0,1] \to \mathrm{R}'^k$ and a continuous semi-algebraic map, $\phi : \Omega \times [0,1] \to \Omega$, with $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{z}$, and for each $I \subset [m]$,

$$\phi(\cdot, t)|_{F'_{I,j,\mathbf{x}}} : F'_{I,j,\mathbf{x}} \to F'_{I,j,\gamma(t)},$$

is a homeomorphism for each $0 \leq t \leq 1$ (see (4.2), (4.10) and (4.11) for the definition of Ω, R' and $F'_{I,j}$).

Now, observe that $C'_{I,j,\mathbf{x}}$ (resp. $C'_{I,j,\mathbf{z}}$) is a sphere bundle over $F'_{I,j,\mathbf{x}}$ (resp. $F'_{I,j,\mathbf{z}}$). Moreover

$$C'_{I,j,\mathbf{x}} = \{(\omega, \mathbf{y}) \mid \omega \in F'_{I,j,\mathbf{x}}, \mathbf{y} \in L_j^+(\omega, \mathbf{x}), |\mathbf{y}| = 1\},$$

and, for $\omega \in F'_{I,j,\mathbf{x}} \cap F'_{I,j-1,\mathbf{x}}$, we have $L_j^+(\omega, \mathbf{x}) \subset L_{j-1}^+(\omega, \mathbf{x})$.

We now prove that the map ϕ induces a homeomorphism $\tilde{\phi} : C'_{\mathbf{x}} \to C'_{\mathbf{z}}$, which for each $I \subset [m]$ and $0 \leq j \leq \ell$ restricts to a homeomorphism $\tilde{\phi}_{I,j} : C'_{I,j,\mathbf{x}} \to C'_{I,j,\mathbf{z}}$.

First recall that by a standard result in the theory of bundles (see for instance, [18], p. 313, Lemma 5), the isomorphism class of the sphere bundle $C'_{I,j,\mathbf{x}} \to F'_{I,j,\mathbf{x}}$, is determined by the homotopy class of the map,

$$
\begin{aligned}
F'_{I,j,\mathbf{x}} &\to Gr(\ell + 1 - j, \ell + 1) \\
\omega &\mapsto L_j^+(\omega, \mathbf{x}),
\end{aligned}
$$

where $Gr(m,n)$ denotes the Grassmannian variety of m dimensional subspaces of R'^n.

The map ϕ induces for each $j, 0 \leq j \leq \ell$, a homotopy between the maps

$$
\begin{aligned}
f_0 : F'_{I,j,\mathbf{x}} &\to Gr(\ell + 1 - j, \ell + 1) \\
\omega &\mapsto L_j^+(\omega, \mathbf{x})
\end{aligned}
$$

and

$$
\begin{aligned}
f_1 : F'_{I,j,\mathbf{z}} &\to Gr(\ell + 1 - j, \ell + 1) \\
\omega &\mapsto L_j^+(\omega, \mathbf{z})
\end{aligned}
$$

(after indentifying the sets $F'_{I,j,\mathbf{x}}$ and $F'_{I,j,\mathbf{z}}$ since they are homeomorphic) which respects the inclusions $L_j^+(\omega, \mathbf{x}) \subset L_{j-1}^+(\omega, \mathbf{x})$, and $L_j^+(\omega, \mathbf{z}) \subset L_{j-1}^+(\omega, \mathbf{z})$.

The above observation in conjunction with Lemma 5 in [18] is sufficient to prove the equivalence of the sphere bundles $C'_{I,j,\mathbf{x}}$ and $C'_{I,j,\mathbf{z}}$. But we need to prove a more general equivalence, involving all the sphere bundles $C'_{I,j,\mathbf{x}}$ simultaneously, for $0 \leq j \leq \ell$.

However, note that the proof of Lemma 5 in [18] proceeds by induction on the skeleton of the CW-complex of the base of the bundle. After choosing a sufficiently fine triangulation of the set $F'_{I,\mathbf{x}} \cong F'_{I,\mathbf{z}}$ compatible with the closed subsets

$F'_{I,j,\mathbf{x}} \cong F'_{I,j,\mathbf{z}}$, the same proof extends without difficulty to this slightly more general situation to give a fiber preserving homeomorphism, $\tilde{\phi} : C'_{\mathbf{x}} \to C'_{\mathbf{z}}$, which restricts to an isomorphism of sphere bundles, $\tilde{\phi}_{I,j} : C'_{I,j,\mathbf{x}} \to C'_{I,j,\mathbf{z}}$, for each $I \subset [m]$ and $0 \le j \le \ell$.

We have the following maps.

$$(4.19)$$

$$
\begin{array}{ccccccc}
\mathrm{Ext}(A_{\mathbf{x}}, \mathrm{R}') & \xleftarrow{\phi_2} & \mathrm{Ext}(B_{\mathbf{x}}, \mathrm{R}') & \xleftarrow{i} & \mathrm{Ext}(C_{\mathbf{x}}, \mathrm{R}') & \xleftarrow{r} & C'_{\mathbf{x}} \\
& & & & & & \downarrow{\tilde{\phi}} \\
\mathrm{Ext}(A_{\mathbf{z}}, \mathrm{R}') & \xleftarrow{\phi_2} & \mathrm{Ext}(B_{\mathbf{z}}, \mathrm{R}') & \xleftarrow{i} & \mathrm{Ext}(C_{\mathbf{z}}, \mathrm{R}') & \xleftarrow{r} & C'_{\mathbf{z}}
\end{array}
$$

The map i is the inclusion map, and r is a retraction shown to exist by Proposition 4.7.

Since all the maps ϕ_2, i, r have been shown to be homotopy equivalences, by Propositions 4.1, 4.3, and 4.7 respectively, their composition is also a homotopy equivalence.

Moreover, for each $I \subset [m]$, the maps in the above diagram restrict properly to give a corresponding diagram:

$$(4.20)$$

$$
\begin{array}{ccccccc}
\mathrm{Ext}(A_{I,\mathbf{x}}, \mathrm{R}') & \xleftarrow{\phi_2} & \mathrm{Ext}(B_{I,\mathbf{x}}, \mathrm{R}') & \xleftarrow{i} & \mathrm{Ext}(C_{I,\mathbf{x}}, \mathrm{R}') & \xleftarrow{r} & C'_{I,\mathbf{x}} \\
& & & & & & \downarrow{\tilde{\phi}} \\
\mathrm{Ext}(A_{I,\mathbf{z}}, \mathrm{R}') & \xleftarrow{\phi_2} & \mathrm{Ext}(B_{I,\mathbf{z}}, \mathrm{R}') & \xleftarrow{i} & \mathrm{Ext}(C_{I,\mathbf{z}}, \mathrm{R}') & \xleftarrow{r} & C'_{I,\mathbf{z}}
\end{array}
$$

Now let $D_{\mathbf{x},\mathbf{z}} = C'_{\mathbf{x}}$, and $f_{\mathbf{x}} = \phi_2 \circ i \circ r$ and $f_{\mathbf{z}} = \phi_2 \circ i \circ r \circ \tilde{\phi}$. Finally, for each $I \subset [m]$, let $D_{I,\mathbf{x},\mathbf{z}} = C'_{I,\mathbf{x}}$ and the maps $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ the restrictions of $f_{\mathbf{x}}$ and $f_{\mathbf{z}}$ respectively to $D_{I,\mathbf{x},\mathbf{z}}$. The collection of sets $D_{I,\mathbf{x},\mathbf{z}}$ and the maps $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ clearly satisfy the conditions of the lemma. This completes the proof of the lemma. $\qquad\square$

Remark 4.20. Note that if R_1 is a real closed sub-field of R, then Lemma 4.19 continues to hold after we substitute "$T \subset \mathrm{R}_1^{k}$" and "for all $\mathbf{x} \in \mathrm{R}_1^{k}$" in place of "$T \subset \mathrm{R}^{k}$" and "for all $\mathbf{x} \in \mathrm{R}^{k}$" in the statement of the lemma. This is a consequence of the Tarski-Seidenberg transfer principle.

With the same hypothesis as in Lemma 4.19 we also have,

Lemma 4.21. *There exists a finite set* $T \subset \mathrm{R}^{k}$, *with*

$$\#T \le (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathrm{R}^{k}$ *there exists* $\mathbf{z} \in T$, *for each* $I \subset [m]$, *a semi-algebraic set* $E_{I,\mathbf{x},\mathbf{z}}$ *defined over* R'', *where* $\mathrm{R}'' = \mathrm{R}\langle \varepsilon, \bar{\varepsilon}, \bar{\delta} \rangle$ *(see (4.10 for the definition of* $\bar{\varepsilon}$ *and* $\bar{\delta}$*), and S-maps* $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ *as shown in the diagram below such that* $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ *are*

both stable homotopy equivalences.

$$(4.21) \qquad \begin{array}{ccc} & E_{I,\mathbf{x},\mathbf{z}} & \\ & {}^{g_{\mathbf{x}}}\nearrow \quad \nwarrow{}^{g_{\mathbf{z}}} & \\ & \sim \qquad \sim & \\ \mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}'') & & \mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{z}}, \mathrm{R}'') \end{array}$$

For each $I \subset J \subset [m]$, $E_{J,\mathbf{x},\mathbf{z}} \subset E_{I,\mathbf{x},\mathbf{z}}$ *and the maps* $g_{J,\mathbf{x}}, g_{J,\mathbf{z}}$ *are restrictions of of* $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$.

Proof. Let $1 \gg \varepsilon > 0$ be an infinitesimal. For $1 \leq i \leq m$, we define

$$(4.22) \qquad \tilde{Q}_i = Q_i + \varepsilon(Y_0^2 + \cdots + Y_\ell^2),$$

$$(4.23) \qquad \tilde{A}_i = \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \ \wedge \ \tilde{Q}_i(\mathbf{y}, \mathbf{x}) \leq 0)\}.$$

Note that the set $\bigcap_{i \in I} \tilde{A}_{i,\mathbf{x}}$ is homotopy equivalent to $\mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}\langle\varepsilon\rangle)$ for each $I \subset [m]$ and $\mathbf{x} \in \mathrm{R}^k$. Applying Lemma 4.19 (see Remark 4.20) to the family $\tilde{\mathcal{Q}} = \{-\tilde{Q}_1, \ldots, -\tilde{Q}_m\}$, we have that there exists a finite set $T \subset \mathrm{R}^k$ with

$$\#T \leq (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$ such that for each $I \subset [m]$, the following diagram

$$(4.24) \qquad \begin{array}{ccc} & \tilde{D}_{I,\mathbf{x},\mathbf{z}} & \\ {}^{\tilde{f}_{I,\mathbf{x}}}\swarrow \quad & & \quad \searrow{}^{\tilde{f}_{I,\mathbf{z}}} \\ \sim \qquad & & \qquad \sim \\ \mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'') & & \mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{z}}, \mathrm{R}'') \end{array}$$

where for each $\mathbf{x} \in \mathrm{R}^k$ we denote

$$\tilde{A}_{i,\mathbf{x}} = \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \ \wedge \ -\tilde{Q}_i(\mathbf{y}, \mathbf{x}) \leq 0)\},$$

$\tilde{f}_{I,\mathbf{x}}, \tilde{f}_{I,\mathbf{z}}$ are homotopy equivalences.

Note that for each $\mathbf{x} \in \mathrm{R}^k$, the set $\mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}'')$ is a deformation retract of the complement of $\mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'')$ and hence is Spanier-Whitehead dual to $\mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'')$. The lemma now follows by taking the Spanier-Whitehead dual of diagram (4.24) above for each $I \subset [m]$. $\qquad\square$

Proof of Theorem 4.17. Follows directly from Lemma 4.19. $\qquad\square$

Proof of Theorem 4.18. Follows directly from Lemma 4.21. $\qquad\square$

We now prove a homogenous version of Theorem 2.1

Theorem 4.22. *Let* R *be a real closed field and let*

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k],$$

where each Q_i is homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$, and of degree at most d in $X_1, \ldots, X_k$.

Let $\pi : \mathbf{S}^\ell \times \mathrm{R}^k \to \mathrm{R}^k$ be the projection on the last k co-ordinates. Then, for any $\mathcal{Q}$-closed semi-algebraic set $S \subset \mathbf{S}^\ell \times \mathrm{R}^k$, the number of stable homotopy types amongst the fibers $S_{\mathbf{x}}$ is bounded by

$$(2^m \ell k d)^{O(mk)}.$$

Proof. We first replace the family $\mathcal{Q}$ by the family,

$$\mathcal{Q}' = \{Q_1, \ldots, Q_{2m}\} = \{Q, -Q \mid Q \in \mathcal{Q}\}.$$

Note that the cardinality of $\mathcal{Q}'$ is $2m$. Let

$$A_i = \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \wedge Q_i(\mathbf{y}, \mathbf{x}) \le 0)\}.$$

It follows from Lemma 4.21 that there exists a set $T \subset \mathrm{R}^k$ with

$$\#T \le (2^m \ell k d)^{O(mk)}$$

such that for every $I \subset [2m]$ and $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$ and a semi-algebraic set $E_{I,\mathbf{x},\mathbf{z}}$ defined over $\mathrm{R}'' = \mathrm{R}\langle \varepsilon, \bar{\varepsilon}, \bar{\delta} \rangle$ and S-maps $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ as shown in the diagram below such that $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ are both stable homotopy equivalences.

(4.25)

$$E_{I,\mathbf{x},\mathbf{z}}$$

$$g_{I,\mathbf{x}} \nearrow \qquad \nwarrow g_{I,\mathbf{z}}$$
$$\sim \qquad \qquad \sim$$

$$\mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{x}}, \mathrm{R}'') \qquad\qquad \mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{z}}, \mathrm{R}'')$$

Now notice that each $\mathcal{Q}$-closed set S is a union of sets of the form $\bigcap_{i \in I} A_i$ with $I \subset [2m]$. Let

$$S = \bigcup_{I \in \Sigma \subset 2^{[2m]}} \bigcap_{i \in I} A_i.$$

Moreover, the intersection of any sub-collection of sets of the kind, $\bigcap_{i \in I} A_i$ with $I \subset [2m]$, is also a set of the same kind. More precisely, for any $\Sigma' \subset \Sigma$ there exists $I_{\Sigma'} \in 2^{[2m]}$ such that

$$\bigcap_{I \in \Sigma'} \bigcap_{i \in I} A_i = \bigcap_{i \in I_{\Sigma'}} A_i.$$

We are not able to show directly a stable homotopy equivalence between $S_{\mathbf{x}}$ and $S_{\mathbf{z}}$. Instead, we note that the S-maps $g_{I,\mathbf{x}}$ and $g_{I,\mathbf{z}}$ induce S-maps (cf. Definition 3.7)

$$\tilde{g}_{\mathbf{x}} : \mathrm{hocolim}(\{\mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{x}}, \mathrm{R}'') \mid I \in \Sigma\}) \longrightarrow \mathrm{hocolim}(\{E_{I,\mathbf{x},\mathbf{z}} \mid I \in \Sigma\})$$

$$\tilde{g}_{\mathbf{z}} : \mathrm{hocolim}(\{\mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{z}}, \mathrm{R}'') \mid I \in \Sigma\}) \longrightarrow \mathrm{hocolim}(\{E_{I,\mathbf{x},\mathbf{z}} \mid I \in \Sigma\})$$

which are stable homotopy equivalences by Lemma 3.9 since each $g_{I,\mathbf{x}}$ and $g_{I,\mathbf{z}}$ is a stable homotopy equivalence.

Since $\mathrm{hocolim}(\{\bigcap_{i\in I} A_{i,\mathbf{x}} \mid I \in \Sigma\})$ (resp. $\mathrm{hocolim}(\{\bigcap_{i\in I} A_{i,\mathbf{z}} \mid I \in \Sigma\})$) is homotopy equivalent by Lemma 3.8 to $\bigcup_{I\in\Sigma}\bigcap_{i\in I} A_{i,\mathbf{x}}$ (resp. $\bigcup_{I\in\Sigma}\bigcap_{i\in I} A_{i,\mathbf{z}}$), it follows (see Remark 3.1) that $S_{\mathbf{x}} = \bigcup_{I\in\Sigma}\bigcap_{i\in I} A_{i,\mathbf{x}}$ is stable homotopy equivalent to $S_{\mathbf{z}} = \bigcup_{I\in\Sigma}\bigcap_{i\in I} A_{i,\mathbf{z}}$. This proves the theorem. $\square$

4.4. Inhomogeneous case. We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let ϕ be a $\mathcal{P}$-closed formula defining the $\mathcal{P}$-closed semi-algebraic set $S \subset \mathrm{R}^{\ell+k}$. Let $1 \gg \varepsilon > 0$ be an infinitesimal, and let

$$P_0 = \varepsilon^2 \left(\sum_{i=1}^{\ell} Y_i^2 + \sum_{i=1}^{k} X_i^2 \right) - 1.$$

Let $\tilde{\mathcal{P}} = \mathcal{P} \cup \{P_0\}$, and let $\tilde{\phi}$ be the $\tilde{\mathcal{P}}$-closed formula defined by

$$\tilde{\phi} = \phi \wedge \{P_0 \leq 0\},$$

defining the $\tilde{\mathcal{P}}$-closed semi-algebraic set $S_b \subset \mathrm{R}\langle\varepsilon\rangle^{\ell+k}$. Note that the set S_b is bounded.

It follows from the local conical structure of semi-algebraic sets at infinity [13] that the semi-algebraic set S_b has the same homotopy type as $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$.

Considering each P_i as a polynomial in the variables $Y_1, \ldots, Y_\ell$ with coefficients in $\mathrm{R}[X_1, \ldots, X_k]$, and let P_i^h denote the homogenization of P_i. Thus the polynomials $P_i^h \in \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ and are homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$.

Let $S_b^h \subset \mathbf{S}^\ell \times \mathrm{R}\langle\varepsilon\rangle^k$ be the semi-algebraic set defined by the $\tilde{\mathcal{P}}^h$-closed formula $\tilde{\phi}^h$ (replacing P_i by P_i^h in $\tilde{\phi}$). It is clear that S_b^h is a union of two disjoint, closed and bounded semi-algebraic sets each homeomorphic to S_b, which has the same homotopy type as $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$.

The theorem is now proven by applying Theorem 4.22 to the family $\tilde{\mathcal{P}}^h$ and the semi-algebraic set S_b^h. Note that two fibers $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ are stable homotopy equivalent if and only if $\mathrm{Ext}(S_{\mathbf{x}}, \mathrm{R}\langle\varepsilon\rangle)$ and $\mathrm{Ext}(S_{\mathbf{y}}, \mathrm{R}\langle\varepsilon\rangle)$ are stable homotopy equivalent (see Remark 3.1). $\square$

5. METRIC UPPER BOUNDS

In [9] certain metric upper bounds related to homotopy types were proven as applications of the main result. Similar results hold in the quadratic case, except now the bounds have a better dependence on ℓ. We state these results without proof.

We first recall the following results from [9]. Let $V \subset \mathrm{R}^\ell$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} \subset \mathbb{Z}[Y_1, \ldots, Y_\ell]$. Suppose for each $P \in \mathcal{P}$, $\deg(P) < d$, and the maximum of the absolute values of coefficients in P is less than some constant M, $0 < M \in \mathbb{Z}$. For $a > 0$ we denote by $B_\ell(0, a)$ the open ball of radius a in R^ℓ centered at the origin.

Theorem 5.1. *There exists a constant $c > 0$, such that for any $r_1 > r_2 > M^{d^{c\ell}}$ we have*

(1) *$V \cap B_\ell(0, r_1)$ and $V \cap B_\ell(0, r_2)$ are homotopy equivalent, and*

(2) $V \setminus B_\ell(0, r_1)$ and $V \setminus B_\ell(0, r_2)$ are homotopy equivalent.

In the special case of quadratic polynomials we get the following improvement of Theorem 5.1.

Theorem 5.2. *Let* R *be a real closed field. Let* $V \subset \mathrm{R}^\ell$ *be a* $\mathcal{P}$*-semi-algebraic set, where*

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell],$$

with $\deg(P_i) \leq 2$, $1 \leq i \leq m$ *and the maximum of the absolute values of coefficients in* $\mathcal{P}$ *is less than some constant* M, $0 < M \in \mathbb{Z}$.

There exists a constant $c > 0$, *such that for any* $r_1 > r_2 > M^{\ell^{cm}}$ *we have,*

(1) $V \cap B_\ell(0, r_1)$ *and* $V \cap B_\ell(0, r_2)$ *are stable homotopy equivalent, and*
(2) $V \setminus B_\ell(0, r_1)$ *and* $V \setminus B_\ell(0, r_2)$ *are stable homotopy equivalent.*

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