# VANDERMONDE VARIETIES, MIRRORED SPACES, AND THE COHOMOLOGY OF SYMMETRIC SEMI-ALGEBRAIC SETS

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ABSTRACT. Let R be a real closed field, $d, k \in \mathbb{Z}_{>0}$, $\mathbf{y} = (y_1, \ldots, y_d) \in \mathrm{R}^d$, and let $V^{(k)}_{d,\mathbf{y}} \subset \mathrm{R}^k$ denote the Vandermonde variety defined by $p^{(k)}_1 = y_1, \ldots, p^{(k)}_d = y_d$, where $p^{(k)}_j = \sum_{i=1}^k X_i^j$. Then, the cohomology groups $\mathrm{H}^*(V^{(k)}_{d,\mathbf{y}}, \mathbb{Q})$ have the structure of $\mathfrak{S}_k$-modules. We prove that for all partitions $\lambda \vdash k$, and $d \geq 2$, the multiplicity of the Specht module $\mathbb{S}^\lambda$ in $\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}}, \mathbb{Q})$ is zero if $\mathrm{length}(\lambda) \geq i + 2d - 1$. This vanishing result allows us to prove a similar vanishing result for arbitrary symmetric semi-algebraic sets defined by symmetric polynomials of degrees bounded by $d$. These new results depend on results from the cohomological study of mirrored spaces due to Davis [23] and Solomon [37], as well as the fundamental results on Vandermonde varieties due to Arnold [1], Giventhal [26] and Kostov [28], and a careful topological analysis of certain regular cell complexes that arise in the process of combining these results.

A surprising outcome of the vanishing results stated above is a polynomial upper bound on the algorithmic complexity of the problem of computing certain (the first few) Betti numbers of semi-algebraic sets defined by symmetric polynomials of fixed degrees. The algorithmic problem of computing the Betti numbers of semi-algebraic sets (not necessarily symmetric) is of central importance in complexity theory (especially in the Blum-Shub-Smale model [17] where it plays the same role as that of 'counting' in discrete complexity theory [19, 16]), and has been the subject of many investigations (most recently by Bürgisser et al. [20]). No algorithm with complexity better than singly exponential is known for computing even the zero-th Betti number of a real algebraic variety defined by one polynomial of degree $\geq 3$. We prove that for each fixed $\ell, d \geq 0$, there exists an algorithm that takes as input a quantifier-free first order formula $\Phi$ with atoms $P = 0, P > 0, P < 0, P \in \mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$, where D is an ordered domain contained in R, and computes the isotypic decomposition, as well as the ranks of the first $(\ell + 1)$ cohomology groups, of the symmetric semi-algebraic set defined by $\Phi$. The complexity of this algorithm (measured by the number of arithmetic operations in D) is bounded by a *polynomial* in $k$ and $\mathrm{card}(\mathcal{P})$ (for fixed $d$ and $\ell$). This result contrasts with the **PSPACE**-hardness of the problem of computing just the zero-th Betti number (i.e. the number of semi-algebraically connected components) in the general case for $d \geq 2$ [34] (taking the ordered domain D to be equal to $\mathbb{Z}$).

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Contents



## 1. Introduction and Main Results

We fix a real closed field R. The intersections of the level sets of the first $d$ (weighted) Newton power sums in $\mathrm{R}^k$ for some $d \leq k$ have been called Vandermonde varieties by Arnold [1] and Giventhal [26], who studied their topological properties in detail. In fact, if one replaces the Newton power sums with any other set of generators of the ring of $\mathfrak{S}_k$-invariant polynomials (for example the elementary symmetric polynomials), the intersection of the level sets of the generators of degree at most $d$ give the same class of real varieties. (Indeed, Vandermonde varieties can be defined as level sets of the first $d$ generators of the invariant ring of any finite reflection group, and many results and techniques introduced in the current paper extend to more general reflection groups. However, the case of the symmetric group is the most important from the point of view of applications, and we restrict ourselves to this special case in this paper.) When the weights are all equal the Vandermonde varieties are also symmetric with respect to the standard action (by permuting coordinates) of the symmetric group $\mathfrak{S}_k$, and thus the cohomology groups of the Vandermonde varieties acquire the structure of finite dimensional $\mathfrak{S}_k$-modules (here and everywhere else in this paper without further mention we only consider cohomology with rational coefficients).

In their foundational work on the topic, Arnold [1], Giventhal [26] and Kostov [28], proved that the intersection of a symmetric Vandermonde variety with the

Weyl chamber in $\mathrm{R}^k$, defined by the inequalities $X_1 \leq \cdots \leq X_k$ is contractible if non-empty, which in turn implies that the quotient space of a symmetric Vandermonde variety is contractible if non-empty. In this paper, we study the $\mathfrak{S}_k$-module structure of the cohomology groups of symmetric Vandermonde varieties themselves (not just their quotient space).

1.1. **Main Results.** Our main representation-theoretic results concern the $\mathfrak{S}_k$-module structure of the cohomology groups of Vandermonde varieties, and more generally of symmetric semi-algebraic sets defined by symmetric polynomials of small degrees. We prove that the Specht modules corresponding to partitions having long lengths cannot occur with positive multiplicity in the isotypic decompositions of small dimensional cohomology modules of semi-algebraic sets defined by symmetric polynomials of small degree.

We then exploit these results to obtain the first algorithm with *polynomially bounded complexity* for computing the first few Betti numbers of such sets. This result is surprising because the analogous algorithmic problem of computing Betti numbers of general (not necessarily symmetric) semi-algebraic sets defined by polynomials of degree bounded by $d$ is a **PSPACE**-hard problem for $d \geq 2$, and thus unlikely to admit algorithms with polynomially bounded complexity.

1.1.1. *Representation-theoretic results.* We obtain restrictions on the Specht modules, $\mathbb{S}^\lambda, \lambda \vdash k$, that are allowed to appear depending on $d$ and $k$, as well as the dimension (or the degree) of the cohomology group under consideration. These restrictions are of two kinds. We prove that when $d$ is fixed, the Specht modules corresponding to partitions having long lengths cannot occur with positive multiplicity in the isotypic decompositions of small dimensional cohomology modules of Vandermonde varieties $V^{(k)}_{d,\mathbf{y}} \subset \mathrm{R}^k$, as well as a similar result for more general symmetric semi-algebraic sets defined by symmetric polynomials of degrees bounded by $d$. In the opposite direction, we prove that the Specht modules corresponding to partitions having short lengths cannot occur with positive multiplicity in the isotypic decompositions of the high dimensional cohomology modules of of Vandermonde varieties $V^{(k)}_{d,\mathbf{y}} \subset \mathrm{R}^k$, and a similar result for more general symmetric semi-algebraic sets as well.

**Notation 1.** For any symmetric semi-algebraic subset $S \subset \mathrm{R}^k$ and $i \geq 0$, we will denote by

$$\mathrm{Par}_i(S) = \{\lambda \vdash k \mid \mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(S)) \neq 0\}.$$

We prove the following theorems. (The notation used in the theorems in this section is mostly standard and/or self-explanatory; but readers unfamiliar with them should consult Section 1.3 below where we collect together some of the basic notation that we use throughout the paper).

**Theorem 1.** *Let $d, k \in \mathbb{Z}_{>0}, d \geq 2$, $\mathbf{y} = (y_1, \ldots, y_d) \in \mathrm{R}^d$, and let $V^{(k)}_{d,\mathbf{y}}$ denote the Vandermonde variety defined by $p^{(k)}_1 = y_1, \ldots, p^{(k)}_d = y_d$, where $p^{(k)}_j = \sum_{i=1}^k X_i^j$. Then, for all $\lambda \vdash k$:*

*(a)*

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \text{ for } i \leq \mathrm{length}(\lambda) - 2d + 1,$$

*or equivalently,*

$$\max_{\lambda \in \mathrm{Par}_i(V^{(k)}_{d,\mathbf{y}})} \mathrm{length}(\lambda) < i + 2d - 1; \tag{1.1}$$

*(b)*

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \text{ for } i \geq k - \mathrm{length}({}^t\lambda) + 1,$$

*or equivalently,*

$$\max_{\lambda \in \mathrm{Par}_i(V^{(k)}_{d,\mathbf{y}})} \mathrm{length}({}^t\lambda) < k - i + 1.$$

*Remark* 1 (Cases $d = 1, 2$). The case $d = 1$ is omitted in Theorem 1. Indeed, Part (a) is not true as stated in the case $d = 1$. In this case, $V^{(k)}_{d,\mathbf{y}}$ is the hyperplane defined by the equation

$$\sum_{i=1}^{k} X_i = y_1,$$

and is $\mathfrak{S}_k$-equivariantly contractible to the point $\frac{1}{k} \cdot (y_1, \ldots, y_1)$. Hence,

$$\begin{aligned} \mathrm{H}^i(V^{(k)}_{d,\mathbf{y}}) &\cong_{\mathfrak{S}_k} \mathbb{S}^{(k)}, \text{ if } i = 0, \\ &\cong_{\mathfrak{S}_k} 0, \text{ otherwise} \end{aligned}$$

(recall that the Specht module $\mathbb{S}^\lambda$ for $\lambda$ equal to the trivial partition $(k)$ is isomorphic to the one-dimensional trivial representation). It follows that for $i = 0$,

$$\mathrm{mult}_{\mathbb{S}^{(k)}}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 1 \neq 0,$$

but

$$\mathrm{length}((k)) = 1 \not< i + 2d - 1 = 0 + 2 - 1 = 1,$$

which violates (1.1).

On the other hand, the case $d = 2$ already indicates that the vanishing condition in Theorem 1 is sharp.

If $d = 2$ and $k \geq 3$, the Vandermonde variety $V^{(k)}_{d,\mathbf{y}}$ is the defined by the equation

$$\sum_{i=1}^{k} X_i = y_1, \sum_{i=1}^{k} X_i^2 = y_2,$$

and can be empty, a point, or semi-algebraically homeomorphic to a sphere of dimension $k - 2$ (depending on whether $y_1^2 - ky_2$ is $> 0, = 0$, or $< 0$, respectively). In the last case (i.e. when $y_1^2 - ky_2 < 0$):

$$\begin{aligned} \mathrm{H}^i(V^{(k)}_{2,\mathbf{y}}) &\cong_{\mathfrak{S}_k} \mathbb{S}^{(k)}, \text{ if } i = 0, \\ \mathrm{H}^i(V^{(k)}_{2,\mathbf{y}}) &\cong_{\mathfrak{S}_k} \mathbb{S}^{1^k}, \text{ if } i = k-2, \\ &\cong_{\mathfrak{S}_k} 0, \text{ otherwise} \end{aligned} \tag{1.2}$$

(see Subsection 2.4.1 below for a proof).

It follows that for $i = k - 2, k \geq 3$ and $y_2 > 0$,

$$\mathrm{mult}_{\mathbb{S}^{1^k}}(\mathrm{H}^{k-2}(V^{(k)}_{d,\mathbf{y}})) = 1 \neq 0 \Rightarrow 1^k \in \mathrm{Par}_{k-2}(V^k_{2,\mathbf{y}}),$$

and

$$\max_{\lambda \in \mathrm{Par}_{k-2}(V^{(k)}_{2,\mathbf{y}})} \mathrm{length}(\lambda) = \mathrm{length}(1^k) = k < k - 2 + 2 \cdot 2 - 1 = k + 1.$$

The restrictions on the $\mathfrak{S}_k$-module structure for Vandermonde varieties, produce via an application of an argument involving the (equivariant) Leray spectral sequence, similar (slightly looser) restrictions on the cohomology modules of arbitrary symmetric semi-algebraic sets defined by quantifier-free formula involving qualities and inequalities of symmetric polynomials of degrees bounded by $d \leq k$ (cf. Theorem 2).

**Theorem 2.** *Let $d, k \in \mathbb{Z}_{>0}$ $d \geq 2$, and $S \subset \mathrm{R}^k$ be a $\mathcal{P}$-semi-algebraic set with $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]_{\leq d}^{\mathfrak{S}_k}$. Then, for all $\lambda \vdash k$:*

*(a)*

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(S)) = 0, \text{ for } i \leq \mathrm{length}(\lambda) - 2d + 1,$$

*or equivalently,*

$$\max_{\lambda \in \mathrm{Par}_i(S)} \mathrm{length}(\lambda) < i + 2d - 1;$$

*(b)*

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(S)) = 0, \text{ for } i \geq k - \mathrm{length}({}^t\lambda) + d + 1,$$

*or equivalently,*

$$\max_{\lambda \in \mathrm{Par}_i(S)} \mathrm{length}({}^t\lambda) < k - i + d + 1.$$

Part (a) of Theorem 2 can be read as saying that for any fixed $i \geq 0$, and $S \subset \mathrm{R}^k$ a $\mathcal{P}$-semi-algebraic set with $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]_{\leq d}^{\mathfrak{S}_k}$,

$$\max_{\lambda \in \mathrm{Par}_i(S)} \mathrm{length}(\lambda) < i + 2d - 1 = O(d).$$

Similarly, Part (b) of Theorem 2 can be read as saying that

$$\max_{\lambda \in \mathrm{Par}_{k-i}(S))} \mathrm{length}({}^t\lambda) < i + d + 1 = O(d).$$

The following illustrative example shows that up to a multiplicative constant the above necessary conditions are tight.

**Example 1.** For $d, k \in \mathbb{Z}_{>0}$, let

$$F_{k,d,\varepsilon} = \sum_{i=1}^{k} \prod_{j=1}^{d} (X_i - j)^2 - \varepsilon \in \mathrm{R}[X_1, \ldots, X_k]_{\leq 2d}^{\mathfrak{S}_k},$$

and

$$V_{k,d,\varepsilon} = \mathrm{Z}(F_{k,d,\varepsilon}),$$

(where $\mathrm{Z}(P)$ denotes the real zeros of a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$). Note that $\deg(F_{k,d,\varepsilon}) = 2d$, and for $0 < \varepsilon \ll 1$, $V_{d,k,\varepsilon}$ consists of $d^k$ disjoint topological spheres, each sphere infinitesimally close (as a function of $\varepsilon$) to one of the $d^k$ points $\{1, \ldots, d\}^k \subset \mathrm{R}^k$.

Thus, for $0 < \varepsilon \ll 1$, $\dim_{\mathbb{Q}}(\mathrm{H}^0(V_{d,k,\varepsilon})) = \dim_{\mathbb{Q}}(\mathrm{H}^{k-1}(V_{d,k,\varepsilon})) = d^k$, and and $\mathrm{H}^i(V_{d,k,\varepsilon}) = 0, i \neq 0, k-1$.

We now describe the isotypic decomposition of $\mathrm{H}^i(V_{k,d,\varepsilon})$ for $0 < \varepsilon \ll 1$, and $i = 0, k-1$. It is easy to see that

$$\mathrm{H}^0(V_{k,d,\varepsilon}) \cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}_{\geq 0}^d \\ \sum_{i=1}^d \lambda_i = k}} \mathrm{H}^0(V_\lambda), \tag{1.3}$$

where $V_\lambda$ is the $\mathfrak{S}_k$-orbit of the connected component of $V_{k,d,\varepsilon}$ infinitesimally close (as a function of $\varepsilon$) to the point $\mathbf{x}^i = (\underbrace{1,\ldots,1}_{\lambda_1},\cdots,\underbrace{d,\ldots,d}_{\lambda_d})$.

For $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{Z}^d_{\geq 0}, \sum_{i=1}^d \lambda_i = k$, denote by $\widetilde{\lambda}$ the partition of $k$ obtained by permuting the $\lambda_i$'s so that they are in non-increasing order.

Then, for $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{Z}^d_{\geq 0}, \sum_{i=1}^d \lambda_i = k$,

$$\mathrm{H}^0(V_\lambda) \cong_{\mathfrak{S}_k} M^{\widetilde{\lambda}} \tag{1.4}$$

(where for any $\mu \vdash k$ we denote by $M^\mu$ the *Young module* associated to the partition $\mu$).

Moreover, as is well known, the isotypic decomposition of the Young module $M^\mu$ is given by the *Young's rule* ([22, Theorem 3.6.11])

$$M^\mu \cong_{\mathfrak{S}_k} S^\mu \oplus \bigoplus_{\mu' \trianglerighteq \mu, \mu' \neq \mu} K(\mu',\mu)\, \mathbb{S}^{\mu'}, \tag{1.5}$$

where $\trianglerighteq$ denotes the partial order often referred to as the *dominance order* on the set of partitions of $k$, and $K(\mu',\mu)$ are the *Kostka numbers* (see [22] for definitions). It follows from the definition of the partial order $\trianglerighteq$ that,

$$\mu' \trianglerighteq \mu \Rightarrow \mathrm{length}(\mu') \leq \mathrm{length}(\mu). \tag{1.6}$$

We can deduce from (1.3), (1.4) and (1.5), that

$$\mathrm{H}^0(V_{k,d,\varepsilon}) \cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{Z}^d_{\geq 0}\\ \sum_{i=1}^d \lambda_i = k}} \left( S^{\widetilde{\lambda}} \oplus \bigoplus_{\mu \trianglerighteq \widetilde{\lambda}, \mu \neq \widetilde{\lambda}} K(\mu,\widetilde{\lambda})\, \mathbb{S}^\mu \right). \tag{1.7}$$

This immediately implies using (1.7) and (1.6) that

$$\max_{\lambda \in \mathrm{Par}_0(V_{k,d,\varepsilon})} \mathrm{length}(\lambda) \leq d.$$

Moreover, it is clear that there exists $\lambda \vdash k$ with $\mathrm{length}(\lambda) = d$, such that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^0(V_{k,d,\varepsilon})) > 0,$$

which shows that the restriction, $\mathrm{length}(\lambda) = O(d)$ (in the case $i = 0$) in Part (a) of Theorem 2 is tight up to a multiplicative factor.

It follows from the $\mathfrak{S}_k$-equivariant Poincaré duality (see for example [14, Theorem 3.23]), that

$$\mathrm{H}^{k-1}(V_{k,d,\varepsilon}) \cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{Z}^d_{\geq 0}\\ \sum_{i=1}^d \lambda_i = k}} \left( S^{{}^t\widetilde{\lambda}} \oplus \bigoplus_{\mu \trianglerighteq \widetilde{\lambda}, \mu \neq \widetilde{\lambda}} K(\mu,\widetilde{\lambda})\, \mathbb{S}^{{}^t\mu} \right). \tag{1.8}$$

Together with (1.6), (1.8) implies that

$$\max_{\lambda \in \mathrm{Par}_{k-1}(V_{k,d,\varepsilon})} \mathrm{length}({}^t\lambda) \leq d.$$

It is also clear there exists $\lambda \vdash k$ with $\mathrm{length}({}^t\lambda) = d$, such that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^{k-1}(V_{k,d,\varepsilon})) > 0.$$

This shows that the restriction, $\mathrm{length}({}^t\lambda) = O(d)$ (in the case $i = 0$) in the Part (b) of Theorem 2 is also tight up to a multiplicative factor.

Theorems 1 and 2 are improvements over prior results in [14] (Theorem 2.5, Part (1)) having similar flavor in several different ways. Firstly, the restrictions on partitions given in [14, Theorem 2.5] are in terms of upper bounds on their *ranks* rather than their lengths. The rank of a partition $\mu$ is the length of the main diagonal in the Young diagram (cf. Definition 2) of $\mu$. While the length of a partition is an upper bound on its rank, a partition having small rank can be arbitrarily long. For example, the partition $1^k := (1, \ldots, 1)$ has rank 1, but its length is clearly the maximum possible, namely $k$. Secondly, the restrictions in [14, Theorem 2.5] do not take into consideration the dimension (or the degree) of the cohomology groups under consideration. In contrast, the restrictions on the partitions $\lambda$ given in Theorems 1 and 2 in the current paper, do depend in a strong manner on the dimension (or the degree) of the cohomology group. As a result in small dimensions, we obtain that only the partitions with a small length can appear unlike the restrictions obtained in [14], where there were no non-trivial restriction on the length. The restriction on the length is a key ingredient in the algorithmic result obtained in this paper.

The results of the current paper depend on:

(a) results from the cohomological study of mirrored spaces due to Davis [23] and Solomon [37],
(b) fundamental results on Vandermonde varieties due to Arnold [1], Giventhal [26] and Kostov [28], and
(c) a careful topological analysis of certain regular cell complexes that arise in the process of combining these results.

In contrast, the proofs of the results in [14] are based essentially on equivariant Morse theory which plays no role in the current paper. The reader who is curious about the interplay of results coming from different areas and how they combine together in the study of Vandermonde varieties, can skip forward to Examples 2.4.1 and 2.4.2 where the examples of Vandermonde varieties of degree 2 in $\mathrm{R}^k$, $k \geq 3$, and that of degree 3 in $\mathrm{R}^4$ are worked out in full detail.

1.1.2. *Algorithmic result.* The new result (Theorem 2) on the vanishing of the multiplicities of Specht modules (corresponding to partitions having long lengths) has an important algorithmic consequence. It paves the way for obtaining a new algorithm for computing the first $\ell + 1$ (for any fixed $\ell$) Betti numbers of any given semi-algebraic set $S \subset \mathrm{R}^k$ defined in terms of symmetric polynomials of small degrees with complexity which is polynomially bounded.

The algorithmic problem of computing Betti numbers of arbitrary semi-algebraic sets is a central and extremely well-studied problem in algorithmic semi-algebraic geometry. It has many ramifications, ranging from the applications in the theory of computational complexity where it plays the role of 'generalized counting' in real models of computation (see [19, 16]), to robot motion planning where the problem of computing the zero-th Betti umber, that is the number of connected components of the free space of a robot, which is usually a semi-algebraic set, is a central problem [36, 21]).

While many advances have been made in recent years [10, 3, 4, 20] (see also Remark 3 below) the best algorithm for computing *all* the Betti numbers of any given semi-algebraic set $S \subset \mathrm{R}^k$ still has doubly exponential (in $k$) complexity, even in the case where the degrees of the defining polynomials are assumed to be bounded by a constant ($\geq 2$) [36]. The existence of algorithms with *singly exponential complexity* for computing all the Betti numbers of a given semi-algebraic set is considered to be a major open question in algorithmic semi-algebraic geometry (see the survey [5]). One important reason why this problem is open is that while the Betti numbers of semi-algebraic sets are bounded by a singly exponential function [33, 39, 32], the best known algorithm for obtaining semi-algebraic triangulation has doubly exponential complexity [36].

As mentioned above some partial progress on this important problem has been made. Algorithms for computing the zero-th Betti number (i.e. the number of semi-algebraically connected components) of semi-algebraic sets have been investigated in depth, and nearly optimal algorithms are known for this problem [7, 15]. An algorithm with singly exponential complexity is known for computing the first Betti number of semi-algebraic sets is given in [10], and then extended to the first $\ell$ (for any fixed $\ell$) Betti numbers in [3]. The Euler-Poincaré characteristic, which is the alternating sum of the Betti numbers, is easier to compute, and a singly exponential algorithm for computing it is known [2, 8].

From the point of view of lower bounds, the problem of computing even the number of connected components (i.e. the zero-th Betti number) of general (not necessarily symmetric) semi-algebraic sets defined by polynomials of degrees bounded by any constant $d \geq 2$ is a **PSPACE**-hard problem [34], and thus unlikely to have algorithms with polynomially bounded complexity. In contrast to these results which are applicable to general semi-algebraic sets, we prove in this paper that there exists an algorithm with *polynomially bounded complexity*, for computing the first $\ell + 1$ Betti numbers of semi-algebraic sets defined by symmetric polynomials of degrees bounded by $d$, for every fixed $d$ and $\ell$. Before stating this theorem formally, we first make precise the notion of 'complexity' that we are going to use.

**Definition 1** (Definition of complexity)**.** In our algorithms we will usually take as input polynomials with coefficients belonging to an ordered domain (say D). By complexity of an algorithm we will mean the number of arithmetic operations and comparisons in the domain D. Since $\mathbb{Z}$ is always a subring of D, this will include operations involving integers. If $\mathrm{D} = \mathbb{R}$, then the complexity of our algorithm will agree with the Blum-Shub-Smale notion of real number complexity [17]. In case, $\mathrm{D} = \mathbb{Z}$, then we are able to deduce the bit-complexity of our algorithms in terms of the bit-sizes of the coefficients of the input polynomials, and this will agree with the classical (Turing) notion of complexity.

**Theorem 3.** *Let* D *be an ordered domain contained in a real closed field* R*, and let* $\ell, d \geq 0$*. There exists an algorithm with takes as input a finite set* $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$*, and a* $\mathcal{P}$*-formula* $\Phi$*, and computes:*

1. *For each* $i, 0 \leq i \leq \ell$*, a set* $M_i$ *of pairs* $(m_{i,\lambda} \in \mathbb{Z}_{>0}, \lambda \vdash k)$ *such that*

$$\mathrm{H}^i(S) \cong_{\mathfrak{S}_k} \bigoplus_{(m_{i,\lambda},\lambda) \in M_i} m_{i,\lambda}\ \mathbf{S}^{\lambda}.$$

2. *The tuple of integers*

$$(b_0(\mathcal{R}(\Phi)), \ldots, b_\ell(\mathcal{R}(\Phi))).$$

*Moreover, the complexity of the algorithm, measured by the number of arithmetic operations in* $\mathrm{D}$*, is bounded by* $(skd)^{2^{O(d+\ell)}}$.

*If* $\mathrm{D} = \mathbb{Z}$*, and the bit-sizes of the coefficients of the input is bounded by* $\tau$*, then the bit-complexity of our algorithm is bounded by*

$$(\tau skd)^{2^{O(d+\ell)}}.$$

*Remark* 2 (Polynomiality). Note that the complexity of the algorithm in Theorem 3 is bounded by a polynomial in $s$ and $k$ for every fixed $\ell, d$.

*Remark* 3 (Other models). We should also mention here that there has been recent work on the algorithmic problem of computing Betti numbers of semi-algebraic sets in which the authors have given algorithms with singly exponential complexity for computing all the Betti numbers of semi-algebraic sets [20]. Unlike, the algorithms described in the current paper which have uniform upper bounds on their complexity (i.e. independent of the coefficients of the input polynomials), the complexity of the algorithms in [20] depend on the 'condition number' of the input – and could be infinite if the given input is ill-conditioned. Thus, such algorithms will fail to produce any result on certain inputs. It is possible that the algorithmic insights from the current paper may have consequences for this different model, but we do not investigate this in this paper.

Several new ideas (compared to previous algorithms for computing Betti numbers of semi-algebraic sets) appear in the design of the algorithm cited in Theorem 3. The first key idea is of course to utilize the $\mathfrak{S}_k$-module structure of the cohomology of the given symmetric semi-algebraic sets. This reduces the problem of computing the dimensions of the cohomology groups, to computing the multiplicities of the various Specht modules appearing in them – the Betti numbers can then be recovered from these multiplicities, and the dimensions of the Specht modules for which there is an easily computable formula, namely the so called hook formula (see Eqn. (1.9) below).

The second key idea is to utilize the techniques underlying the proofs of Theorems 1 and 2. This helps us in two ways. Firstly, (in small dimensions) it guarantees that at most only a polynomial many of the multiplicities to be computed can be non-zero, and this restricts the set of partitions that enters into the computation. Secondly, it allows us to obtain a *dimension reduction*, reducing the problem of computing the multiplicities for any given symmetric semi-algebraic set $S \subset \mathrm{R}^k$ defined in terms of symmetric polynomials of degrees bounded by $d$, to the problem of computing the Betti numbers of pairs of semi-algebraic subsets, which are not symmetric any more but contained in a much smaller ($O(d + \ell)$) dimensional space. For the latter problem it suffices to use the standard algorithms mentioned previously. We refer the reader to Section 4.1 for a more detailed outline.

1.2. **Prior work.** The $\mathfrak{S}_k$-module properties of cohomology groups of symmetric semi-algebraic subsets of $\mathrm{R}^k$ defined by symmetric polynomials of degrees bounded by $d \leq k$ were studied in [11, 13, 14]. The main highlights of the results proved in the afore-mentioned papers are the following.

1. The $\mathfrak{S}_k$-equivariant cohomology groups, $\mathrm{H}^*_{\mathfrak{S}_k}(S)$, symmetric semi-algebraic subset $S \subset \mathrm{R}^k$ are isomorphic to $\mathrm{H}^*(S/\mathfrak{S}_k)$ (the cohomology groups of the quotient of $S$ by $\mathfrak{S}_k$). In [11], it was shown that unlike the ordinary Betti numbers, the equivariant Betti numbers of symmetric semi-algebraic sets defined in terms

of symmetric polynomials of degrees bounded by some fixed constant $d$, are bounded polynomially in the parameters $s, k$ (where $s$ is the number of polynomials appearing in the definition of $S$). This result was subsequently sharpened to a tight form, using different methods in [13].

2. The cohomology modules $\mathrm{H}^*(S)$ of the previous paragraph admit an isotypic decomposition into direct sums (as $\mathfrak{S}_k$-modules) of isotypic components, $m_\lambda \mathbb{S}^\lambda$, indexed by partitions $\lambda \vdash k$, and where $\mathbb{S}^\lambda$ is the Specht module indexed by $\lambda$, and $m_\lambda$ denotes the multiplicity of $\mathbb{S}^\lambda$ in $\mathrm{H}^*(S)$ (in other words, $m_\lambda = \dim_{\mathbb{Q}} \hom_{\mathbb{S}_k}(\mathrm{H}^*(S), \mathbb{S}^\lambda)$). In [14], the multiplicities $m_\lambda$'s were studied and several results were proved. In particular, it was shown that in the setting of the previous paragraph, $m_\lambda \neq 0$ implies that $\mathrm{rank}(\lambda) < 2d$. Moreover, for every fixed $d$, polynomial upper bounds were proved on the multiplicities $m_\lambda$. Note that unlike the results of the current paper, the restrictions on the partitions allowed to appear in the isotypic decomposition of the cohomology modules did not depend on the dimension (or the degree) of the cohomology. Moreover, the rank restriction allows partitions having both long rows, and long columns to appear (unlike in Theorems 1) and 2 . This improvement in the restriction (albeit only in small dimensions) is the key to the algorithmic result proved in the current paper.
3. The study of efficient algorithms for computing topological invariants of symmetric semi-algebraic sets has a shorter history than of such algorithms for arbitrary semi-algebraic set. Using the so called 'degree principle' proved by Timofte [40, 41, 42] and Riener [35], one can design an algorithm for deciding emptiness of symmetric algebraic sets in $\mathrm{R}^k$ defined by symmetric polynomials of degree $d$, having complexity $k^{O(d)}$ (i.e polynomial in $s, k$ for fixed $d$). The algorithmic questions of computing the equivariant Betti numbers and also the Euler-Poincaré characteristics were considered by the authors of the current papers. In [13], an algorithm with polynomially bounded complexity (polynomial in $k$ for fixed $d$) was described for computing all the equivariant Betti numbers of a closed symmetric semi-algebraic set $S \subset \mathrm{R}^k$ defined by a formula involving at most $s$ symmetric polynomials of degree bounded by $d$. Since we consider cohomology with rational coefficients and because $\mathfrak{S}_k$ is a finite group, there is isomorphism $\mathrm{H}^*(S/\mathfrak{S}_k) \cong \mathrm{H}^*_{\mathfrak{S}_k}(S)$, and hence this amounts to computing the Betti numbers of the quotient. In [12], an algorithm with polynomially bounded complexity (better than that of the algorithm mentioned above) was given for computing the equivariant as well as the ordinary Euler-Poincaré characteristics of symmetric semi-algebraic sets.

1.3. **Basic notation and definitions.** In this section we collect together some basic notation and definitions that we will use for the rest of the paper.

**Notation 2** (Zeros)**.** For $P \in \mathrm{R}[X_1, \ldots, X_k]$, we denote by $\mathrm{Z}(P, \mathrm{R}^k)$ the set of zeros of $P$ in $\mathrm{R}^k$. More generally, for any finite set $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we denote by $\mathrm{Z}(\mathcal{P}, \mathrm{R}^k)$ the set of common zeros of $\mathcal{P}$ in $\mathrm{R}^k$.

**Notation 3** (Realizations, $\mathcal{P}$- and $\mathcal{P}$-closed semi-algebraic sets)**.** For any finite family of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathrm{R}^k$, and a sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by $\mathcal{R}(\sigma, Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathcal{P}\},$$

and call it the *realization* of $\sigma$ on $Z$.

More generally, we call any Boolean formula $\Phi$ with atoms, $P = 0, P < 0, P > 0, P \in \mathcal{P}$, to be a $\mathcal{P}$*-formula*. We call the realization of $\Phi$, namely the semi-algebraic set

$$\mathcal{R}(\Phi) \quad := \quad \left\{\mathbf{x} \in \mathrm{R}^k \mid \Phi(\mathbf{x})\right\}$$

a $\mathcal{P}$*-semi-algebraic set*.

Finally, we call a Boolean formula without negations, and with atoms $P\{\geq, \leq\}0$, $P \in \mathcal{P}$, to be a $\mathcal{P}$*-closed formula*, and we call the realization, $\mathcal{R}(\Phi)$, a $\mathcal{P}$*-closed semi-algebraic set*.

**Notation 4** (Betti numbers)**.** Let $S \subset \mathrm{R}^k$ be any semi-algebraic set. We denote by $b_i(S) = \dim_{\mathbb{Q}} \mathrm{H}^i(S, \mathbb{Q})$. It is worth noting that the precise definition of the cohomology groups $\mathrm{H}^i(S, \mathbb{Q})$, requires some care if the semi-algebraic set $S$ is defined over an arbitrary (possibly non-archimedean) real closed field. For details we refer to [9, Chapter 6].

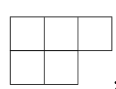
**Notation 5** (Symmetric polynomials of bounded degrees)**.** For all $d, k \geq 0$, we will denote by $\mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$ the subspace of the polynomial ring $\mathrm{R}[X_1, \ldots, X_k]$ consisting of symmetric polynomials of degree at most $d$.

**Notation 6** (Partitions and compositions)**.** We denote by $\mathrm{Par}(k)$ the set of *partitions* of $k$, where each partition $\lambda \in \mathrm{Par}(k)$ (also denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$, and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = k$. We call $\ell$ the length of the partition $\lambda$, and denote $\mathrm{length}(\lambda) = \ell$.

A tuple $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = k$ (but not necessarily non-increasing) will be called a *composition*, and we still call $\ell$ the length of the composition $\lambda$, and denote $\mathrm{length}(\lambda) = \ell$. The set of of all compositions of $k$ will be denoted by $\mathrm{Comp}(k)$.

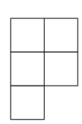
**Notation 7** (Transpose of a partition)**.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash k$, we will denote by ${}^t\lambda$ the *transpose* of $\lambda$. More precisely, ${}^t\lambda = ({}^t\lambda_1, \ldots, {}^t\lambda_{\tilde{\ell}})$, where ${}^t\lambda_j = \mathrm{card}(\{i \mid \lambda_i \geq j\})$.

**Definition 2** (Young diagrams)**.** Partitions are often identified with Young diagrams. We follow the English convention and associate the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with the Young diagram with its $i$-th row consisting of $\lambda_i$ boxes. Thus, the Young diagram corresponding to the partition $\lambda = (3, 2)$ is



,

the Young diagram associated to its transpose, ${}^t\lambda = (2, 2, 1)$, is



(note that the Young diagram of ${}^t\lambda$ is obtained by reflecting the Young diagram of $\lambda$ about its diagonal). Thus, for any partition $\lambda$, $\mathrm{length}(\lambda)$ (respectively $\mathrm{length}({}^t\lambda)$) equals the number of *rows* (respectively columns) of the Young diagram of $\lambda$ (respectively ${}^t\lambda$).

The representation theory of the symmetric groups, $\mathfrak{S}_k$, is a classical subject (see for example [22] for details) and it is well known that the irreducible representations (Specht modules) of $\mathfrak{S}_k$ are indexed by partitions of $k$.

**Notation 8** (Specht modules)**.** For $\lambda \vdash k$, we will denote by $\mathbb{S}^\lambda$ the corresponding Specht module. In particular, $\mathbb{S}^{(k)}$ is the one-dimensional trivial representation which we will also denote by $1_{\mathfrak{S}_k}$, and $\mathbb{S}^{(1^k)}$ is the one-dimensional sign representation which we will also denote by $\mathbf{sign}_k$.

**Definition 3** (Hook lengths)**.** Let $B(\lambda)$ denote the set of boxes in the Young diagram (cf. Definition 2) corresponding to a partition $\lambda \vdash k$. For a box $b \in B(\lambda)$, the length of the hook of $b$, denoted $h_b$ is the number of boxes strictly to the right and below $b$ plus 1.

The following classical formula (due to Frobenius) gives the dimensions of the representations $\mathbb{S}^\lambda$ in terms of the *hook lengths* of the partition $\lambda$ defined below.

$$\dim_{\mathbb{Q}} \mathbb{S}^\lambda \quad = \quad \frac{k!}{\prod_{b \in B(\lambda)} h_b}. \tag{1.9}$$

The rest of the paper is dedicated to the proofs of Theorems 1, 2, and 3. In Section 2 we give an outline of the proofs of Theorems 1 and 2, and also describe two key examples illustrating the main steps. In Section 3, we give the proofs of Theorems 1 and 2. In Section 4 we give the proof of Theorem 3, after introducing the necessary preliminary results.

## 2. Outline of our method and two key examples

2.1. **Outline of the proofs of Theorems 1 and 2.** We first observe that symmetric semi-algebraic subsets $S \subset \mathrm{R}^k$, defined in terms of equalities and inequalities of symmetric polynomials of degree at most $d$, admits a map to $\mathrm{R}^d$ (by the first $d$ Newton power sum polynomials restricted to $S$), whose fibers are Vandermonde varieties. Moreover the action of $\mathfrak{S}_k$ keeps the fibers stable, and thus the action of $\mathfrak{S}_k$ on $S$ also induces an action on the Leray spectral sequence of this map. As a result in order to prove the vanishing of certain irreducible $\mathfrak{S}_k$-modules, it suffices to prove this vanishing for Vandermonde varieties. The Vandermonde varieties are well studied and have nice topological and geometric properties. For us the most important property implicit in the work of Arnold, Giventhal and Kostov is that the intersection $Z$ of a Vandermonde variety $V$ with a Weyl chamber $\mathcal{W}^{(k)}$ in $\mathrm{R}^k$ is either a point or a regular cell of the dimension of the variety. Moreover, the structure of the boundary of $Z$ (in case $Z$ is a regular cell) is well understood in terms of the combinatorics of the faces of $\mathcal{W}^{(k)}$ with which $Z$ has a non-empty intersection.

We recall that the symmetric group $\mathfrak{S}_k$ is generated by the transpositions $(i, i+1), 1 \leq i \leq k$, and we will denote the transposition $(i, i+1)$ by $s_i$, and we denote by $\mathrm{Cox}(k)$ the set $\{s_1, \ldots, s_{k-1}\}$. If we identify $\mathfrak{S}_k$ as the Weyl group of the root system $A_{k-1}$ in $V = \mathrm{R}^k$, the various $s_i$ are the root reflections corresponding to a set of fundamental roots, and $\mathcal{W}^{(k)}$ is a fundamental chamber. Each co-dimension one face of the $\mathcal{W}^{(k)}$ is the intersection of $\mathcal{W}^{(k)}$ with a hyperplane defined by $X_i = X_{i+1}$ for some $i, 1 \leq i \leq k$, and this thus labeled by the Coxeter element $s_i$, and we denote this face by $\mathcal{W}^{(k)}_{s_i}$.

In order to relate the cohomology of the symmetric Vandermonde variety $V$, with that of $Z = V \cap \mathcal{W}^{(k)}$, we make use of the notion of a *mirrored space*. Given a Coxeter system $(W, S)$, where $W$ is a Coxeter group and $S$ a set of reflections generating $W$, a space $Z$ with a family of closed subspaces $(Z_s)_{s\in S}$ is called a *mirror structure* on $Z$ [24, Chapter 5.1], and $Z$ along with the collection $(Z_s)_{s\in S}$ is called a *mirrored space* over $S$. Given a mirrored space, $Z, (Z_s)_{s\in S}$ over $S$, there is a classical construction of a space $\mathcal{U}(W, Z)$ with a $W$-action [29, 43, 44, 23].

For a general mirrored space $Z$, the cohomology groups $\mathrm{H}^*(\mathcal{U}(W, Z))$ gets a structure of a $W$-module from the $W$-action on $\mathcal{U}(W, Z)$, and $\mathrm{H}^*(\mathcal{U}(W, Z))$ can then be expressed as a direct sum of certain tensor products of $W$-modules, $\Psi_T^W$, and the cohomology groups of the pair $(Z, Z^T)$, where $T \subset \mathrm{Cox}(k)$, and $Z^T = \bigcup_{s\in S} Z_s$ ([24, Theorem 15.4.3]). In our situation, $(W, S) = (\mathfrak{S}_k, \mathrm{Cox}(k))$, $Z = V \cap \mathcal{W}^{(k)}$, $Z_s = Z \cap \mathcal{W}_s^{(k)}, s \in \mathrm{Cox}(k)$, and the space $\mathcal{U}(\mathfrak{S}_k, Z)$ is equivariantly homeomorphic to $V$ with the standard action of $\mathfrak{S}_k$. Applying [24, Theorem 15.4.3] to our situation we obtain that the cohomology group of $V$ are isomorphic to direct sums of tensor products of certain $\mathfrak{S}_k$-modules, $\Psi_T^{(k)}$, indexed by subsets $T \subset \mathrm{Cox}(k)$, and the cohomology groups of the pairs $(Z, Z^T), T \subset \mathrm{Cox}(k)$, where as before

$$Z^T = \bigcup_{s\in T} Z_s$$

(see Theorem 4 below).

The representations $\Psi_T^{(k)}$ may be understood as analogs of Specht modules, but defined in terms of MacMahon's tableau [30, Vol 1, Chapter 1, Sect IV, 129.] rather than Young's tableau (where the role of partitions is replaced by that of compositions). Unlike the Specht modules, the representations $\Psi_T^{(k)}$ need not be irreducible (see, for example (2.7) and (2.8) below). But we are able to obtain a necessary condition for a Specht module to appear with positive multiplicity in $\Psi_T^{(k)}$ using a recursive formula due to Solomon [37, Corollary 3.2] (cf. Eqn. (3.3) below). We show using an inductive argument (cf. Proposition 2) that only those Specht modules can appear in $\Psi_T^{(k)}$ whose number of rows is bounded by $\mathrm{card}(T) + 1$ (and a similar restriction in terms of the number of columns).

One final ingredient is the observation that in the case when $Z$ has the expected dimension $k - d$, then the intersection of $Z$ with the various faces of $\mathcal{W}^{(k)}$, induces a structure of a regular cell complex, and the boundary of $Z$ is then semi-algebraically homeomorphic to the $(k-d-1)$-dimensional sphere, and the intersection of $Z$ with the various $\mathcal{W}_s^{(k)}, s \in \mathrm{Cox}(k)$, gives an acyclic covering of the boundary of $Z$ having cardinality at most $k-1$. This implies via an argument using the nerve lemma and Alexander duality that the cohomology groups $\mathrm{H}^i(Z, Z^T)$ must vanish if $i$ is large compared to the cardinality of $T$ and also a dual statement (cf. Proposition 3).

Putting these together we obtain our theorem on the vanishing of certain multiplicities for Vandermonde varieties (cf. Theorem 1). Theorem 2 is then a consequence of Theorem 1 and an argument involving (an equivariant version of) the Leray spectral sequence.

Finally, the restriction result that we prove also allows us, via the Solomon-Davis formula alluded to above, and some additional ingredients (see the outline in Section 4.1) including certain standard algorithms from semi-algebraic geometry, to effectively compute the Betti numbers $b_i(S), 0 \leq i \leq \ell$, for any fixed $\ell$ with complexity which is polynomial in the number of variables and the number of polynomials. Here we are assuming that the degrees of the input polynomials are also bounded by a constant.

We will now proceed to describe two simple examples, whose analysis already exposes the central ideas behind the proofs of the main theorems.

But we first need to introduce a few relevant definitions and notation.

2.2. **Solomon decomposition of the symmetric group.** Recall that a Coxeter pair $(W, S)$, consists of a group $W$ and a set of generators, $S = \{s_i \mid i \in I\}$, of $W$ each having order 2, and numbers $(m_{i,j})_{i,j \in I}$ such that $(s_i s_j)_{m_{ij}} = e$. We consider the symmetric group $\mathfrak{S}_k$ as a Coxeter group with the set of Coxeter generators, $\mathrm{Cox}(k) = \{s_i = (i, i+1) \mid 1 \leq i \leq k-1\}$. Following the same notation as in [24], for $T \subset \mathrm{Cox}(k)$, we denote by $\mathfrak{S}_k^T$ the subgroup of $\mathfrak{S}_k$ generated by $T$. Let $A = \mathbb{Q}[\mathfrak{S}_k]$ denote the regular representation of $\mathfrak{S}_k$.

For $J \subset \mathrm{Cox}(k)$, let

$$
\begin{aligned}
\xi_J^{(k)} &= \mathrm{card}(\mathfrak{S}_k^J)^{-1} \sum_{w \in \mathfrak{S}_k^J} w, \\
\eta_J^{(k)} &= \mathrm{card}(\mathfrak{S}_k^J)^{-1} \sum_{w \in \mathfrak{S}_k^J} (-1)^{\ell(w)} w.
\end{aligned}
$$

For $P, Q \subset S^{(k)}, P \cap Q = \emptyset$, we denote (following [37]) by $\Psi_{P,Q}^{(k)}$ the sub-representation of the regular representation of $\mathfrak{S}_k$ defined by,

$$
\Psi_{P,Q}^{(k)} = A \xi_P^{(k)} \eta_Q^{(k)}. \tag{2.1}
$$

For ease of notation we will denote the representation $\Psi_{\mathrm{Cox}(k)-T,T}^{(k)}$ by $\Psi_T^{(k)}$. As remarked before the representations $\Psi_T^{(k)}$ (unlike the Specht modules) need not be irreducible in general. However, it is easy to see from (2.1) that in the following two special cases, they are indeed irreducible.

$$
\Psi_{\emptyset}^{(k)} \cong_{\mathfrak{S}_k} \mathbb{S}^{(k)} \cong_{\mathfrak{S}_k} 1_{\mathfrak{S}_k}, \tag{2.2}
$$

$$
\Psi_{\mathrm{Cox}(k)}^{(k)} \cong_{\mathfrak{S}_k} \mathbb{S}^{(1^k)} \cong_{\mathfrak{S}_k} \mathbf{sign}_k. \tag{2.3}
$$

Another easy consequence of (2.1) is

$$
\Psi_{\mathrm{Cox}(k)-T}^{(k)} \cong_{\mathfrak{S}_k} \Psi_T^{(k)} \otimes \mathbf{sign}_k. \tag{2.4}
$$

2.3. **Weyl chambers and mirrored spaces.** We now describe how the $\mathfrak{S}_k$-modules, $\Psi_T^{(k)}$ introduced in Section 2.2 can be used to decompose the cohomology groups of a symmetric semi-algebraic set $S$ as a direct sum of certain $\mathfrak{S}_k$-submodules. This decomposition (cf. Theorem 4 below) is a key ingredient in what follows.

**Notation 9.** We denote by $\mathcal{W}^{(k)} \subset \mathrm{R}^k$ the cone defined by $X_1 \leq X_2 \leq \cdots \leq X_k$, and by $\mathcal{W}^{(k),o}$ the interior of $\mathcal{W}^{(k)}$ (i.e. the cone defined by $X_1 < X_2 < \cdots < X_k$).

For every $m \geq 0$, and $\mathbf{w} = (w_1, \ldots, w_k) \in \mathrm{R}_{>0}^k$ we consider

$$
\begin{array}{cccc}
p_{\mathbf{w},m}^{(k)} : & \mathrm{R}^k & \longrightarrow & \mathrm{R} \\
& \mathbf{x} = (x_1, \ldots, x_k) & \longmapsto & \sum_{j=1}^k w_j x_j^m,
\end{array}
$$

and for every $d \geq 0$, and $\mathbf{w} \in \mathrm{R}^k_{>0}$ we denote by $\Phi^{(k)}_{\mathbf{w},d}$ the continuous map defined by

$$\begin{array}{rcccc} \Phi^{(k)}_{\mathbf{w},d} & : & \mathrm{R}^k & \longrightarrow & \mathrm{R}^{d'} \\ & & \mathbf{x} = (x_1, \ldots, x_k) & \longmapsto & (p^{(k)}_{\mathbf{w},1}(\mathbf{x}), \ldots, p^{(k)}_{\mathbf{w},d'}(\mathbf{x})), \end{array}$$

where $d' = \min(k, d)$.

Finally, we denote by

$$\Psi^{(k)}_{\mathbf{w},d} : \mathcal{W}^{(k)} \longrightarrow \mathrm{R}^{d'}$$

the restriction of $\Phi^{(k)}_{\mathbf{w},d}$ to $\mathcal{W}^{(k)}$.

If $\mathbf{w} = 1^k := (1, \ldots, 1)$, then we will denote by $p^{(k)}_m$ the polynomial $p^{(k)}_{\mathbf{w},m}$ (the $m$-th Newton sum polynomial), and by $\Phi^{(k)}_d$ (respectively, $\Psi^{(k)}_d$) the map $\Phi^{(k)}_{\mathbf{w},d}$ (respectively, $\Psi^{(k)}_{\mathbf{w},d}$).

For every $\mathbf{w} \in \mathrm{R}^k_{\geq 0}$, $d, k \geq 0, d \leq k$, and $\mathbf{y} \in \mathrm{R}^d$, we will denote by

$$V^{(k)}_{\mathbf{w},d,\mathbf{y}} := (\Phi^{(k)}_{\mathbf{w},d})^{-1}(\mathbf{y}), \text{ and } Z^{(k)}_{\mathbf{w},d,\mathbf{y}} := (\Psi^{(k)}_{\mathbf{w},d})^{-1}(\mathbf{y}).$$

If $\mathbf{w} = 1^k := (1, \ldots, 1)$, then we just denote $V^{(k)}_{\mathbf{w},d,\mathbf{y}}$ by $V^{(k)}_{d,\mathbf{y}}$, and $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ by $Z^{(k)}_{d,\mathbf{y}}$.

**Notation 10.** For $k \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{Comp}(k)$ the set of integer tuples

$$\lambda = (\lambda_1, \ldots, \lambda_\ell), \lambda_i > 0, |\lambda| := \sum_{i=1}^{\ell} \lambda_i = k.$$

**Definition 4.** For $k \in \mathbb{Z}_{\geq 0}$, and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathrm{Comp}(k)$, we denote by $\mathcal{W}_\lambda$ the subset of $\mathcal{W}^{(k)}$ defined by,

$$X_1 = \cdots = X_{\lambda_1} \leq X_{\lambda_1+1} = \cdots = X_{\lambda_1+\lambda_2} \leq \cdots \leq X_{\lambda_1+\cdots+\lambda_{\ell-1}+1} = \cdots = X_k,$$

and denote by $\mathcal{W}^o_\lambda$ the subset of $\mathcal{W}^{(k)}$ defined by

$$X_1 = \cdots = X_{\lambda_1} < X_{\lambda_1+1} = \cdots = X_{\lambda_1+\lambda_2} < \cdots < X_{\lambda_1+\cdots+\lambda_{\ell-1}+1} = \cdots = X_k.$$

We denote by $L_\lambda$ the subspace defined by

$$X_1 = \cdots = X_{\lambda_1}, X_{\lambda_1+1} = \cdots = X_{\lambda_1+\lambda_2}, \cdots, X_{\lambda_1+\cdots+\lambda_{\ell-1}+1} = \cdots = X_k,$$

which is the linear hull of $\mathcal{W}_\lambda$.

**Notation 11.** For $s = (i, i+1) \in \mathrm{Cox}(k)$, we denote by $\mathcal{W}^{(k)}_s$ the face of $\mathcal{W}^{(k)}$ defined by $X_i = X_{i+1}$. More generally, for $T \subset \mathrm{Cox}(k)$, we denote:

$$\begin{aligned} \mathcal{W}^{(k)}_T &= \bigcap_{s \in T} \mathcal{W}^{(k)}_s, \\ \mathcal{W}^{(k,T)} &= \bigcup_{s \in T} \mathcal{W}^{(k)}_s. \end{aligned}$$

We also define $\lambda(T) \in \mathrm{Comp}(k)$ implicitly by the equation

$$\mathcal{W}_{\lambda(T)} = \mathcal{W}^{(k)}_T. \tag{2.5}$$

**Notation 12.** Finally, for any semi-algebraic set $Z \subset \mathcal{W}^{(k)}$, $T \subset \mathrm{Cox}(k)$, we set

$$\begin{aligned} Z^T &= Z \cap \mathcal{W}^{(k,T)}, \\ Z_T &= Z \cap \mathcal{W}_T^{(k)}. \end{aligned}$$

For any semi-algebraic subset $S \subset \mathrm{R}^k$, we will denote

$$S_k = S \cap \mathcal{W}^{(k)},$$

and we will for convenience of notation write $S_{k,T}$ (respectively, $S_k^T$), in place of $(S_k)_T$ (respectively, $(S_k)^T$).

2.3.1. *Mirrored spaces and a key theorem.* We will also use the following theorem proved in a more general context of mirrored space in [24]. If $S$ is a closed symmetric semi-algebraic subset of $\mathrm{R}^k$, then (using Notation 12) $S_k \subset \mathcal{W}^{(k)}$. The tuple of closed subspaces $(S_{k,s} = S_k \cap \mathcal{W}_s^{(k)})_{s \in \mathrm{Cox}(k)}$ of $S_k$ (cf. Notation 11) is then an example of a mirror structure on $S_k$ over $\mathrm{Cox}(k)$, and $S$ is $\mathfrak{S}_k$-equivariantly homeomorphic to $\mathcal{U}(\mathfrak{S}_k, S_k)$ (using the language of [24, Chapter 5]). The following theorem is an adaptation of a more general theorem in [24] to the special case that we need.

**Theorem 4.** [24, Theorem 15.4.3] *Let $S$ be a closed symmetric semi-algebraic subset of $\mathrm{R}^k$. Then,*

$$\mathrm{H}_*(S) \cong_{\mathfrak{S}_k} \bigoplus_{T \subset \mathrm{Cox}(k)} \mathrm{H}_*(S_k, S_k^T) \otimes \Psi_T^{(k)}.$$

We are now ready to discuss the promised examples.

### 2.4. Examples.

2.4.1. *Key Example I.* We first consider the case $d = 2$ for $k \geq 3$, which has already being alluded to in Remark 1. Recall that in this case, the Vandermonde variety $V_{2,\mathbf{y}}^{(k)}$ is defined by the equation

$$\sum_{i=1}^k X_i = y_1, \sum_{i=1}^k X_i^2 = y_2,$$

and is empty, a point, or a semi-algebraically homeomorphic to a sphere of dimension $k-2$ (depending on whether $y_1^2 - ky_2$ is $> 0, = 0$, or $< 0$, respectively).

The first two cases are trivial. In the last case, $Z_{2,\mathbf{y}}^{(k)} = V_{2,\mathbf{y}}^{(k)} \cap \mathcal{W}^{(k)}$ is a closed disk of dimension $k-2$, and has a non-empty intersection with all the faces of the Weyl chamber $\mathcal{W}^{(k)}$. (See Figure 1 for the case $k = 4$, where $Z_{2,\mathbf{y}}^{(4)}$ is one of the triangles on the two-dimensional sphere equal to $V_{2,\mathbf{y}}^{4)}$. Notice that in this case $Z_{2,\mathbf{y}}^{(4)}$ meets all the three faces of the Weyl chamber $\mathcal{W}^{(4)}$.)

It follows that in this case

$$\begin{aligned} \mathrm{H}^i(Z_{2,\mathbf{y}}^{(k)}, Z_{2,\mathbf{y}}^{(k,T)}) &\cong \mathbb{Q} \text{ if } (i,T) = (0,\emptyset) \text{ or } (k-2, \mathrm{Cox}(k)), \\ &= 0 \text{ otherwise.} \end{aligned} \tag{2.6}$$

The $\mathfrak{S}_k$-module structure of $V_{2,\mathbf{y}}^{(k)}, y_1^2 - ky_2 < 0, k \geq 3$ stated in (1.2) in Remark 1 now follows from (2.6), (2.2), (2.3),and Theorem 4.

2.4.2. *Key example II.* We now study the cohomology of the symmetric Vandermonde varieties (curves) $V^{(4)}_{3,\mathbf{y}} \subset \mathrm{R}^4$, as $\mathfrak{S}_4$-modules, for various $\mathbf{y} = (y_1, y_2, y_3) \in \mathrm{R}^3$.

In this case the Weyl chamber $\mathcal{W}^{(4)} \subset \mathrm{R}^4$ has three faces corresponding to the compositions $(2,1,1)$, $(1,2,1)$ and $(1,1,2)$. In terms of the Coxeter elements $s_1 = (1,2)$, $s_2 = (2,3)$, and $s_3 = (3,4)$, these faces correspond to $s_1$, $s_2$, and $s_3$ respectively. In other words, using the notation introduced in (2.5),

$$\begin{aligned} \lambda(\{s_1\}) &= (2,1,1), \\ \lambda(\{s_2\}) &= (1,2,1), \\ \lambda(\{s_3\}) &= (1,1,2). \end{aligned}$$

Also, note that

$$\begin{aligned} \lambda(\{s_1, s_2\}) &= (3,1), \\ \lambda(\{s_1, s_3\}) &= (2,2), \\ \lambda(\{s_2, s_3\}) &= (1,3). \end{aligned}$$

We first need a preliminary calculation. Observe that

$$\begin{aligned} \mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_4} \Psi^{(3)}_{\emptyset} &\cong_{\mathfrak{S}_4} \mathbb{S}^{(4)} \oplus \mathbb{S}^{(3,1)} \\ &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\emptyset} \oplus \Psi^{(4)}_{\{s_1\}} \text{ (using Proposition 2).} \end{aligned}$$

From this we deduce that

$$\Psi^{(4)}_{\{s_1\}} \cong_{\mathfrak{S}_4} \mathbb{S}^{(3,1)}, \tag{2.7}$$

and using (2.4) that,

$$\Psi^{(4)}_{\mathrm{Cox}(4)-\{s_1\}} \cong_{\mathfrak{S}_4} \mathbb{S}^{(2,1,1)}. \tag{2.8}$$

Returning to the study of topology of the curve $V^{(4)}_{3,\mathbf{y}}$, there are five different cases possible depending on the configuration of the curve $V^{(4)}_{3,\mathbf{y}}$ inside $\mathcal{W}^{(4)}$. Recall (cf. Notation 9) that we denote $Z^{(k)}_{3,\mathbf{y}} = V^{(4)}_{3,\mathbf{y}} \cap \mathcal{W}^{(4)}$.

Case 1. The Vandermonde variety $V^{(4)}_{2,(y_1,y_2)}$ is empty: in this case $Z^{(4)}_{3,\mathbf{y}} = \emptyset$, and $\mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) = \mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) = 0$.

Case 2. The Vandermonde variety $V^{(4)}_{2,(y_1,y_2)}$ is singular and $V^{(4)}_{3,\mathbf{y}}$ is non-empty: in this case, $Z^{(4)}_{3,\mathbf{y}}$ is a point which must necessarily belong to the face labeled by (4) of $\mathcal{W}^{(4)}$. Thus, $Z^{(4)}_{3,\mathbf{y}}$ belongs to all non-zero faces of $\mathcal{W}^{(4)}$, and $y_2$ is a minimum value of $p^{(4)}_2$ on $V^{(4)}_{1,(y_1)}$. (This preceding fact follows from Theorem 5 stated later.)

In this case (using Notation 12)

$$\begin{aligned} \mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) &\cong \mathbb{Q}, \text{ if } T = \emptyset, \\ \mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) &= 0, \text{ otherwise.} \end{aligned}$$

This implies that

$$\begin{aligned} \mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\emptyset} \\ &\cong_{\mathfrak{S}_4} 1_{\mathfrak{S}_4} \text{ (using (2.2)).} \end{aligned}$$

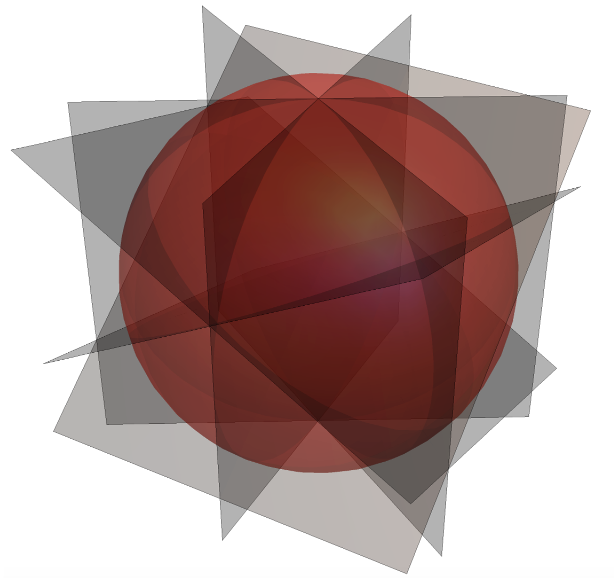


FIGURE 1. Example of the non-singular Vandermonde variety $V^{(4)}_{2,(y_1,y_2)}$.

It follows that $b_0(V^{(4)}_{3,\mathbf{y}}) = 1$ (using the Eqn. (1.9)). Clearly, $\mathrm{H}^1(V^{(4)}_{3,\mathbf{y}}) = 0$ in this case.

Case 3. The Vandermonde variety $V^{(4)}_{2,(y_1,y_2)}$ is non-empty and non-singular. Lets fix $y_1, y_2$ such that $V^{(4)}_{2,(y_1,y_2)}$ is non-empty and non-singular. In this case, $V^{(4)}_{2,(y_1,y_2)}$ is a sphere which is depicted in Figure 1.

The hyperplanes (shown in grey) in Figure 1 cutting out the $4! = 24$ triangles on the sphere are the walls of the various Weyl chambers. Notice that there are 14 vertices in the arrangement of great circles on the sphere, 8 of them incident on 3 circles and the remaining 6 incident on 2 circles. There are several sub-cases to consider. The (non-empty) sub-cases are depicted in Figures 2,3,4 and 5 ($V^{(4)}_{3,\mathbf{y}}$ is shown in blue).

It follows from Theorem 5 that there exist,

$$a(y_1, y_2) = \min_{\mathbf{x} \in V^{(4)}_{2,(y_1,y_2)}} p^{(4)}_3(\mathbf{x}) < b(y_1, y_2) < c(y_1, y_2) = \max_{\mathbf{x} \in V^{(4)}_{2,(y_1,y_2)}} p^{(4)}_3(\mathbf{x}),$$

giving a partition of R into points and open intervals (more precisely, three points and four open intervals) such that the Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ can be characterized topologically by which element of the partition $y_3$ belongs to.

3a. $y_3 \in (-\infty, a(y_1, y_2))$: In this case, $V^{(4)}_{3,\mathbf{y}} = \emptyset$;

3b. $y_3 = a(y_1, y_2)$: In this case, $V^{(4)}_{3,\mathbf{y}}$ is non-empty and singular, and coincides with 4 of the 8 vertices of degree 6, and $Z^{(4)}_{3,\mathbf{y}}$ is a point which must necessarily belong to the face labeled by $(3, 1)$ (cf. Theorem 5). In this case

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y},4}, Z^{(4,T)}_{3,\mathbf{y}}) = 0$$

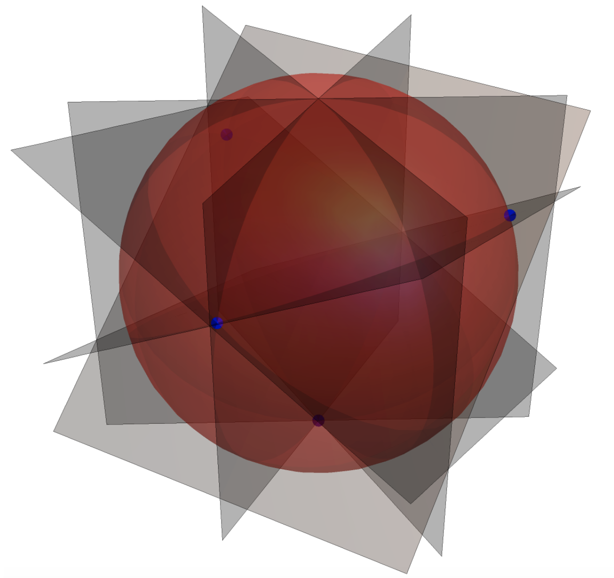


FIGURE 2. Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ in Case 3b.

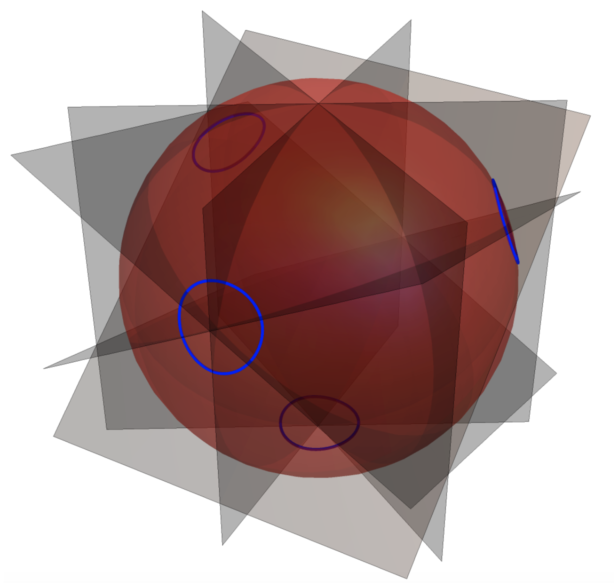


FIGURE 3. Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ in Case 3c.

if

$$T = \{s_2\}, \{s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}$$

(since in these cases $Z^{(4)}_{3,\mathbf{y}} = Z^{(4,T)}_{3,\mathbf{y}}$), and

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) \cong \mathbb{Q}$$

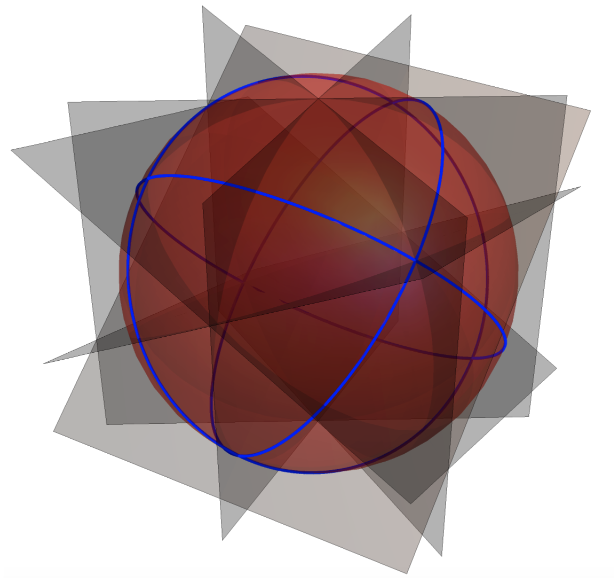


FIGURE 4. Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ in Case 3d.

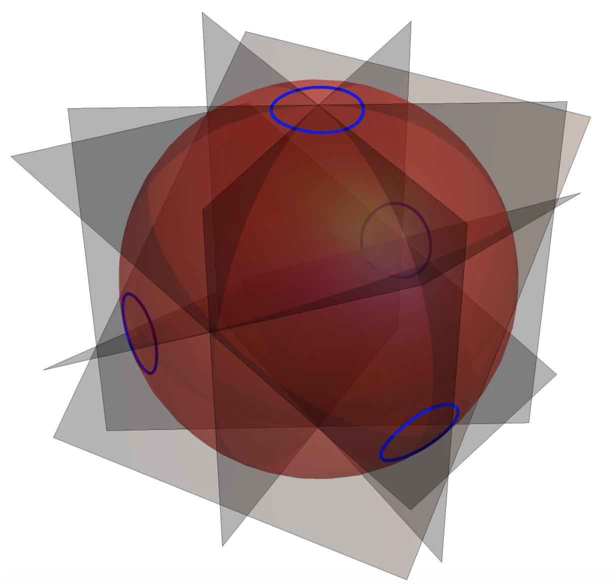


FIGURE 5. Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ in Case 3e.

in the case

$$T = \emptyset, \{s_1\}.$$

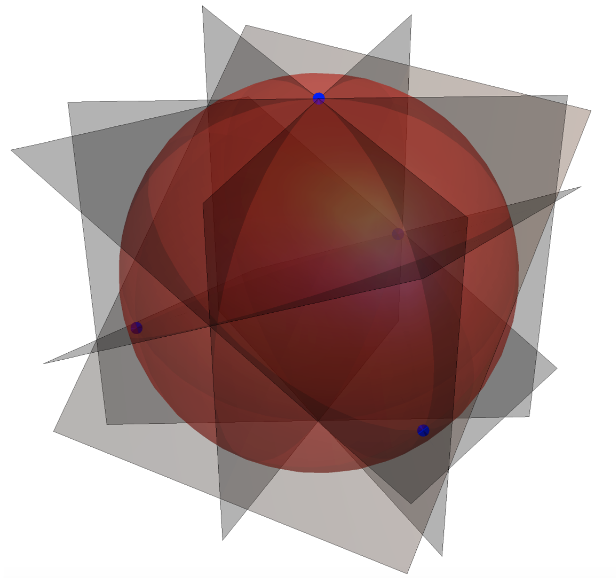


FIGURE 6. Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ in Case 3f.

This implies that

$$\begin{aligned} \mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\emptyset} \oplus \Psi^{(4)}_{\{s_1\}} \\ &\cong_{\mathfrak{S}_4} 1_{\mathfrak{S}_4} \oplus \mathbb{S}^{(3,1)} \text{ (using (2.2) and (2.7))}. \end{aligned}$$

It follows that

$$b_0(V^{(4)}_{3,\mathbf{y}}) = 1 + 3 = 4$$

(using (1.9) to derive $\dim_{\mathbb{Q}}(\mathbb{S}^{(3,1)}) = 3$). Clearly, $\mathrm{H}^1(V^{(4)}_{3,\mathbf{y}}) = 0$ in this case.

3c. $y_3 \in (a(y_1, y_2), b(y_1, y_2))$: In this case $V^{(4)}_{3,\mathbf{y}}$ is a non-singular curve, and $Z^{(4)}_{3,\mathbf{y}}$ intersects the faces labeled by $(1,1,2)$ and $(1,2,1)$ corresponding to Coxeter elements $s_3$ and $s_2$ respectively.
In this case,

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) = 0$$

if

$$T = \{s_2\}, \{s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}$$

and

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) \cong \mathbb{Q}$$

if

$$T = \emptyset, \{s_1\}.$$

This implies that

$$\begin{aligned} \mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\emptyset} \oplus \Psi^{(4)}_{\{s_1\}} \\ &\cong_{\mathfrak{S}_4} 1_{\mathfrak{S}_4} \oplus \mathbb{S}^{(3,1)} \text{ (using (2.2) and (2.7))}. \end{aligned}$$

In dimension one we have,

$$\mathrm{H}^1(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) = 0$$

if

$$T = \emptyset, \{s_1\}\{s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_1, s_2\}$$

and

$$\mathrm{H}^1(Z^{(4)}_{3,\mathbf{y}}, X^{(4,T)}_{3,\mathbf{y}}) \cong \mathbb{Q}$$

if

$$T = \{s_2, s_3\}, \{s_1, s_2, s_3\}.$$

This implies that

$$\begin{aligned} \mathrm{H}^1(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\{s_2,s_3\}} \oplus \Psi^{(4)}_{\{s_1,s_2,s_3\}} \\ &\cong_{\mathfrak{S}_4} \mathbb{S}^{2,1,1} \oplus \mathbf{sign}_4 \text{ (using (2.8) and (2.3)).} \end{aligned}$$

It follows that

$$b_0(V^{(4)}_{3,\mathbf{y}}) = 1 + 3 = 4,$$

and

$$b_1(V^{(4)}_{3,\mathbf{y}}) = 3 + 1 = 4.$$

3d. $y_3 = b(y_1, y_2)$: In this case, the Vandermonde variety $V^{(4)}_{3,\mathbf{y}}$ is of dimension 1 but has singularities, and $Z^{(4)}_{3,\mathbf{y}}$ intersects the faces labeled by $(2,2)$ and $(1,2,1)$ (the intersection with the face labeled $(1,2,1)$ are the singular points of $V^{(4)}_{3,\mathbf{y}}$). Thus, $Z^{(4)}_{3,\mathbf{y}}$ intersects the faces labeled by Coxeter elements $s_1, s_2$ and $s_3$.
In this case,

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) = 0$$

if

$$T = \{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_2, s_3\},$$

and

$$\mathrm{H}^0(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) \cong \mathbb{Q}$$

if

$$T = \emptyset.$$

This implies that

$$\begin{aligned} \mathrm{H}^0(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\emptyset} \\ &\cong_{\mathfrak{S}_4} 1_{\mathfrak{S}_4}. \end{aligned}$$

In dimension one we have,

$$\mathrm{H}^1(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) = 0,$$

if

$$T = \emptyset, \{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_3\},$$

and

$$\mathrm{H}^1(Z^{(4)}_{3,\mathbf{y}}, Z^{(4,T)}_{3,\mathbf{y}}) \cong \mathbb{Q},$$

if

$$T = \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}.$$

This implies that

$$\begin{aligned}\mathrm{H}^1(V^{(4)}_{3,\mathbf{y}}) &\cong_{\mathfrak{S}_4} \Psi^{(4)}_{\{s_1,s_2\}} \oplus \Psi^{(4)}_{\{s_2,s_3\}} \oplus \Psi^{(4)}_{\{s_1,s_2,s_3\}} \\ &\cong_{\mathfrak{S}_4} 2\mathbb{S}^{2,1,1} \oplus \mathbf{sign}_4 \text{ (using (2.8) and (2.3)).}\end{aligned}$$

It follows that

$$b_0(V^{(4)}_{3,\mathbf{y}}) = 1,$$

and

$$b_1(V^{(4)}_{3,\mathbf{y}}) = 2 \cdot 3 + 1 = 7.$$

This last equation can be verified directly by hand noting that $V^{(4)}_{3,\mathbf{y}}$ has the structure of a connected graph containing 6 vertices (the $\binom{4}{2}$ singular points consisting of the orbit of the point $Z^{(4)}_{3,\mathbf{y}} \cap \mathcal{W}_{(2,2)}$), and the degree of each vertex is 4. Thus the graph has 12 edges, and hence

$$\begin{aligned}\chi(V^{(4)}_{3,\mathbf{y}}) &= -6 \\ &= b_0(V^{(4)}_{3,\mathbf{y}}) - b_1(V^{(4)}_{3,\mathbf{y}}) \\ &= 1 - b_1(V^{(4)}_{3,\mathbf{y}}),\end{aligned}$$

and thus,

$$b_1(V^{(4)}_{3,\mathbf{y}}) = 7.$$

3e. $y_3 \in (b(y_1,y_2), c(y_1,y_2))$: In this case, $V^{(4)}_{3,\mathbf{y}}$ is a non-singular curve, and $Z^{(4)}_{3,\mathbf{y}}$ intersects the faces labeled by $(2,1,1)$ and $(1,2,1)$ corresponding to Coxeter elements $s_1$ and $s_2$ respectively. The isotypic decomposition of $\mathrm{H}^*(V^{(4)}_{3,\mathbf{y}})$ in this case is identical to the Case (3c) and is omitted.

3f. $y_3 = c(y_1,y_2)$: In this case, $V^{(4)}_{3,\mathbf{y}}$ is non-empty and singular, and coincides with other 4 (compared to Case (3b)) of the 8 vertices of degree 6. In this case, $Z^{(4)}_{3,\mathbf{y}}$ is a point which must necessarily belong to the face labeled by $(1,3)$. The isotypic decomposition of $\mathrm{H}^*(V^{(4)}_{3,\mathbf{y}})$ in this case is identical to the Case (3b) and is omitted.

3g. $y_3 \in (c(y_1,y_2), \infty)$: In this case, $V^{(4)}_{3,\mathbf{y}}$ is again empty.

Notice, that the Specht module $\mathbb{S}^{(2,2)}$ does not appear with positive multiplicity in $\mathrm{H}^*(V^{(4)}_{3,\mathbf{y}})$, $\mathbf{y} \in \mathrm{R}^3$ in the above analysis. Using an equivariant Leray spectral sequence argument (cf. proof of Theorem 2) we can deduce from this fact the following 'toy' theorem (which is not directly deducible from the statement of Theorem 2):

**Theorem.** *If $S \subset \mathrm{R}^4$ is a $\mathcal{P}$-semi-algebraic set, for $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_4]^{\mathfrak{S}_4}_{\leq 3}$, then*

$$\mathrm{mult}_{\mathbb{S}^{(2,2)}}(\mathrm{H}^*(S)) = 0.$$

*Proof.* See proof of Theorem 2 and the preceding remark. $\square$

*Remark* 4. Note that it follows from the analysis in Example 2.4.2 that

$$\begin{aligned}\max_{\mathbf{y}\in\mathrm{R}^3,\lambda\in\mathrm{Par}_0(V^{(4)}_{3,\mathbf{y}})} \mathrm{length}(\lambda) &= 2, \\ \max_{\mathbf{y}\in\mathrm{R}^3,\lambda\in\mathrm{Par}_1(V^{(4)}_{3,\mathbf{y}})} \mathrm{length}(\lambda) &= 4,\end{aligned}$$

while the Part (a) of Theorem 1 provides the upper bounds:

$$\begin{aligned}\max_{\mathbf{y}\in \mathrm{R}^3, \lambda\in \mathrm{Par}_0(V^{(4)}_{3,\mathbf{y}})} \mathrm{length}(\lambda) &< 0+2\cdot 3-1 = 5,\\ \max_{\mathbf{y}\in \mathrm{R}^3, \lambda\in \mathrm{Par}_1(V^{(4)}_{3,\mathbf{y}})} \mathrm{length}(\lambda) &< 1+2\cdot 3-1 = 6.\end{aligned}$$

We now return to the proofs of the main theorems.

## 3. Proofs of Theorems 1 and 2

We first need a few preliminary results.

### 3.1. Preliminary Results.

The following proposition which has been referred to before, and which describes the topological structure of the intersection of a general Vandermonde variety with a Weyl chamber, is a key topological ingredient in our proofs.

We start by recalling a standard definition.

**Definition 5.** We say that a semi-algebraic set $S \subset \mathrm{R}^k$ is a semi-algebraic regular cell of dimension $p$, if the pair $(\overline{S}, S)$ is semi-algebraically homeomorphic to $(\overline{B_p(\mathbf{0},1)}, B_p(\mathbf{0},1))$ where $B_p(\mathbf{0},1)$ denotes the unit ball in $\mathrm{R}^p$.

*Remark* 5 (Monotonicity and regularity of semi-algebraic sets). We will prove in Proposition 1 that the intersections of weighted Vandermonde varieties with the interior of $\mathcal{W}^{(k)}$ is a semi-algebraic regular cell of dimension $k-d$, if the dimension of the variety is equal to $k-d$, and this property will play an important role later in the paper (see Lemma 2 and Proposition 3). To prove that a given semi-algebraic set is a semi-algebraic regular cell is often not easy. In order to overcome this difficulty, a stronger notion, that of a *monotone cell*, was introduced in [6]. The property that a semi-algebraic set is a monotone cell is much easier to check. We do not reproduce the definition of a monotone cell here but refer the reader to [6, Theorem 9] for one of the several equivalent definitions which is the easiest to check for the sets $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$. Finally, the main result (Theorem 6) in [6] states that a semi-algebraic set which is a monotone cell is a semi-algebraic regular cell, which is what we will use in the proof of Proposition 1.

The following proposition which has been referred to before, and which describes the topological structure of the intersection of a general Vandermonde variety with a Weyl chamber, is a key topological ingredient in our proofs.

**Proposition 1.** *For every $\mathbf{w} \in \mathrm{R}^k_{>0}$, $d,k \geq 0, d \leq k$, and $\mathbf{y} \in \mathrm{R}^d$, $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is either empty, a point, or semi-algebraically homeomorphic to the closed ball of dimension $k-d$. In the last case, $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is semi-algebraically homeomorphic to the closure of a semi-algebraic regular cell of dimension $k-d$, and the boundary of $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is semi-algebraically homeomorphic to the sphere $\mathbf{S}^{k-d-1}$.*

*Proof.* Suppose that $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is not empty. Let $\mathbf{x} \in Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ and suppose that $\mathbf{x}$ is a regular point of the intersection of the Vandermonde variety $V_{\mathbf{w},d,\mathbf{y}}$ with the linear subspace $L_\lambda$ (i.e. the linear hull of the face $\mathcal{W}_\lambda$) for some $\lambda \in \mathrm{Comp}(k)$. Then, $\mathbf{x}$ is a regular point of $V_{\mathbf{w},d,\mathbf{y}}$, and $\mathbf{x} \in \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}$.

We next prove that if $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \neq \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}$, then $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ must be a point. Indeed, if $\mathbf{x} \in Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$, but $\mathbf{x} \notin \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}$, then by the above observation and [1, Theorem 5], $\mathbf{x} \in \mathcal{W}^o_\lambda$, with $\mathrm{length}(\lambda) < d$, and moreover $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^o_\lambda = \{\mathbf{x}\}$. Moreover, in this case $\mathbf{x}$ must be an isolated point of $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$, since any neighborhood of $\mathbf{x}$ in $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$, unless equal to just $\mathbf{x}$ itself, will contain some regular point $\mathbf{x}'$ of the intersection of $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ with $L_{\lambda'}$ with $\lambda \prec \lambda'$, and this would imply that $\mathbf{x} \in \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}$. But on the other hand we know that $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is contractible [28, Theorem 1.1]. This proves that in this case $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} = \{\mathbf{x}\}$, and hence if $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \neq \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}$, $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is a point.

So we might suppose that

$$Z^{(k)}_{\mathbf{w},d,\mathbf{y}} = \overline{Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}}. \tag{3.1}$$

In this case $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o} \neq \emptyset$, and using [1, Theorem 5] $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}$ is non-singular of dimension $k-d$. Now using [28, Corollary 2.2], and [6, Theorem 9] we deduce that $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}$ is a monotone cell (see [6] for the definition of a monotone cell). This implies using [6, Theorem 13] that $Z^{(k)}_{\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(k),o}$ is a regular cell. In conjunction with (3.1) this implies that $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is semi-algebraically homeomorphic to the closure of a regular cell, and the boundary of $Z^{(k)}_{\mathbf{w},d,\mathbf{y}}$ is semi-algebraically homeomorphic to the sphere $\mathbf{S}^{k-d-1}$. □

*Remark* 6. Using Proposition 1 again on the intersection of $Z_{\mathbf{w},d,k}$ with the faces of $\mathcal{W}^{(k)}$ we get that if $Z_{\mathbf{w},d,k}$ is not empty or a point, then its boundary is a regular cell complex (homeomorphic to $\mathbf{S}^{k-d-1}$).

We next give a necessary condition on partitions $\lambda$ such that $\mathbb{S}^\lambda$ can occur with positive multiplicity in the representation $\Psi^{(k)}_T$, in terms of $k$ and the cardinality of $T$.

**Notation 13.** For $\mu \vdash k-1$, we denote by $S(\mu)$ the set of all partitions of $k$ obtained by adding one to some part (row) of $\mu$.

**Example 2.** For example,

$$S((2,1)) = \{(3,1),(2,2),(2,1,1)\}.$$

The significance of the set $S(\mu)$ is encapsulated in the following lemma. With the same notation as in Notation 13:

**Lemma 1.**

$$\mathrm{Ind}^{\mathfrak{S}_k}_{\mathfrak{S}_{k-1}} \mathbb{S}^\mu = \sum_{\lambda \in S(\mu)} \mathbb{S}^\lambda.$$

*Proof.* This is just Pieri's rule. See for instance [31]. □

**Proposition 2.** *Let $k \geq 1$, $T \subset \mathrm{Cox}(k)$. Then, for $\lambda \vdash k$,*

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi^{(k)}_T) = 0 \text{ if } \mathrm{length}(\lambda) > \mathrm{card}(T)+1 \text{ or if } \mathrm{length}({}^t\lambda) > k - \mathrm{card}(T).$$

*Proof.* We first prove that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)}) \neq 0 \Rightarrow \mathrm{length}(\lambda) \leq \mathrm{card}(T) + 1. \tag{3.2}$$

The proof is by double induction on $k$, and on $t = \mathrm{card}(T)$. Clearly, (3.2) holds for $k = 1$ and for all $T$. Also, if $T = \emptyset$ (i.e $t = 0$)

$$\Psi_\emptyset^{(k)} \cong \mathbb{S}^{(k)},$$

and (3.2) holds for all $k \geq 1$.

Now suppose that the theorem is true for all $k' < k$, and for given $k$ for all $t' < t$ and suppose that $t > 0$. Let $s \in T$, and $T' = T - \{s\}$. Without loss of generality we can assume that $s = s_{k-1}$, and in this case we can identify $T' \subset \mathrm{Cox}(k)$ with the corresponding subset of $\mathrm{Cox}(k-1)$.

It follows from [37, Corollary 3.2] that

$$\mathrm{ind}_{\mathfrak{S}_{k-1}}^{\mathfrak{S}_k}(\Psi_{T'}^{(k-1)}) \cong \Psi_T^{(k)} \oplus \Psi_{T'}^{(k)}. \tag{3.3}$$

From the induction hypothesis it follows that for all $\mu \vdash k-1$,

$$\mathrm{mult}_{\mathbb{S}^\mu}(\Psi_{T'}^{(k-1)}) \neq 0 \Rightarrow \mathrm{length}(\mu) \leq \mathrm{card}(T') + 1 = \mathrm{card}(T). \tag{3.4}$$

Notice that it follows from Lemma 1 that for any partition $\mu \vdash k-1$,

$$\mathrm{ind}_{\mathfrak{S}_{k-1}}^{\mathfrak{S}_k}(\mathbb{S}^\mu) \cong \bigoplus_{\lambda \in S(\mu)} \mathbb{S}^\lambda \text{ (cf. Notation 13)}. \tag{3.5}$$

The inductive step follows from (3.3), (3.4) and (3.5), and the fact that for every $\lambda \in S(\mu)$, $\mathrm{length}(\mu) \leq \mathrm{length}(\lambda) \leq \mathrm{length}(\mu) + 1$.

This completes the proof of (3.2).

In order to deduce that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)}) \neq 0 \Rightarrow \mathrm{length}({}^t\lambda) \leq k - \mathrm{card}(T), \tag{3.6}$$

first observe that using (2.4)

$$\begin{aligned} \mathbb{S}^{{}^t\lambda} &\cong \mathbb{S}^\lambda \otimes \mathbb{S}^{1^k}, \\ \Psi_{\mathrm{Cox}(k)-T}^{(k)} &\cong \Psi_T^{(k)} \otimes \mathbb{S}^{1^k}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)}) \neq 0 &\Leftrightarrow \mathrm{mult}_{\mathbb{S}^{{}^t\lambda}}(\Psi_{\mathrm{Cox}(k)-T}^{(k)}) \neq 0 \\ &\Rightarrow \mathrm{length}({}^t\lambda) \leq \mathrm{card}(\mathrm{Cox}(k) - T) + 1 \text{ using (3.2)} \\ &\Rightarrow \mathrm{length}({}^t\lambda) \leq k - \mathrm{card}(T). \end{aligned}$$

□

We will need an elementary result concerning semi-algebraic regular cell complexes.

**Definition 6.** Let $X$ be a closed and bounded semi-algebraic set and $\mathcal{C}$ be a finite set of closed semi-algebraic subsets of $X$. We say that $\mathcal{C} = (C_i)_{i \in I}$, where $I$ is a finite set, is a *closed Leray cover* of $X$ if $\mathcal{C}$ satisfies:

(a) $X = \bigcup_{i \in I} C_i$;
(b) for each subset $J \subset I$, $\bigcap_{j \in J} C_j$ is empty or semi-algebraically contractible.

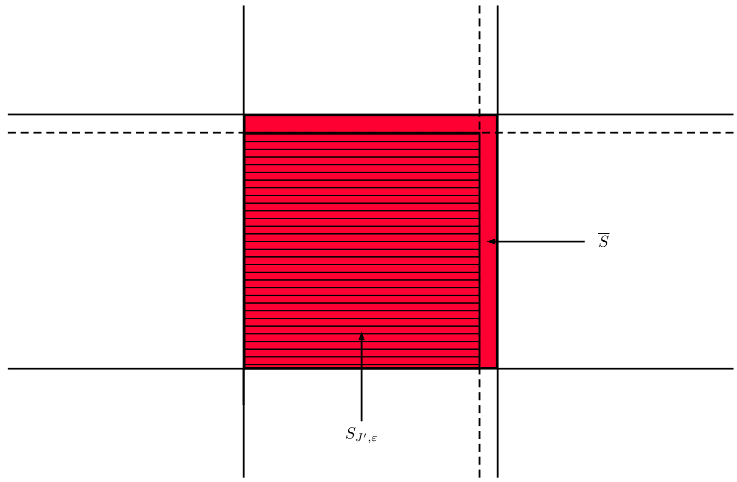



FIGURE 7. Schematic depiction of the sets $\overline{S}$ and $S_{J',\varepsilon}$

We say that $\mathcal{C}$ is a *regular* closed Leray cover if in addition for each subset $J \subset I$, $\bigcap_{j \in J} C_j$ is empty or the closure of a regular semi-algebraic cell.

**Notation 14** (Nerve complex associated to a closed Leray cover)**.** Given a closed Leray cover $\mathcal{C} = (C_i)_{i \in I}$ with $I = [1, N]$, we will denote by $\mathcal{N}(\mathcal{C})$ the simplicial complex, whose set of $p$-dimensional simplices are given by

$$\mathcal{N}_p(\mathcal{C}) = \{(\alpha_0, \ldots, \alpha_p) \mid 1 \leq \alpha_0 < \cdots < \alpha_i \leq N, C_{\alpha_0} \cap \cdots \cap C_{\alpha_p} \neq \emptyset\}.$$

We need the following technical lemma in the proof of Proposition 3 which plays an important role in the proof of Theorem 1.

**Lemma 2.** *Let $(P_i)_{i \in I}$, and $(Q_j)_{j \in J}$ be finite tuples of polynomials in $\mathrm{R}[X_1, \ldots, X_k]$, and $S \subset \mathrm{R}^k$ basic closed semi-algebraic set defined by*

$$\bigwedge_{i \in I} (P_i = 0) \wedge \bigwedge_{j \in J} (Q_j > 0),$$

*and the closure $\overline{S}$ of $S$ is defined by*

$$\bigwedge_{i \in I} (P_i = 0) \wedge \bigwedge_{j \in J} (Q_j \geq 0).$$

*Moreover, suppose that the pair $(\overline{S}, S)$ is semi-algebraically homeomorphic to*

$$(\overline{B_p(\mathbf{0},1)}, B_p(\mathbf{0},1))$$

*where $B_p(\mathbf{0},1)$ denotes the unit ball in $\mathrm{R}^p$.*

*Then for all $J' \subset J$, and all sufficiently small $\varepsilon > 0$, the semi-algebraic set $S_{J',\varepsilon}$ (see Figure 7) defined by*

$$\bigwedge_{i \in I} (P_i = 0) \wedge \bigwedge_{j \in J} (Q_j \geq \varepsilon) \bigwedge_{j \in J - J'} (Q_j \geq 0)$$

*is semi-algebraically contractible.*

*Proof.* Let $S', S''$ be the semi-algebraic subsets of $\overline{S}$ defined by

$$\bigwedge_{i \in I}(P_i = 0) \wedge \bigwedge_{j \in J}(Q_j > 0) \bigwedge_{j \in J-J'}(Q_j \geq 0),$$

and

$$\bigwedge_{i \in I}(P_i = 0) \wedge \bigwedge_{j' \in J'}(Q_{j'} = 0) \wedge \bigwedge_{j \in J-J'}(Q_j \geq 0),$$

respectively.

Observe that

$$S' = \overline{S} - S'',$$

and

$$S'' \subset \overline{S} - S.$$

Let $\phi : \overline{S} \times [0,1] \to \overline{S}$ be the homeomorphic image of the standard retraction of $\overline{B_p(\mathbf{0},1)}$ to $\mathbf{0}$ (i.e. $(\mathbf{x},t) \mapsto (1-t)\mathbf{x}$).

Since $S''$ is contained in the boundary of $S$, we can restrict the retraction $\phi$ to $S' = \overline{S} - S''$ and obtain that $S'$ is also semi-algebraically contractible. It now follows from the the local conic structure theorem for semi-algebraic sets [18, Theorem 9.3.6] that for all small enough $\varepsilon > 0$ that $S'$ and $S_{J',\varepsilon}$ are semi-algebraically homotopy equivalent, and hence $S_{J',\varepsilon}$ is also semi-algebraically contractible. □

**Proposition 3.** *Let $2 \leq d \leq k$, $\mathbf{y} \in \mathrm{R}^d$, $V = V^{(k)}_{d,\mathbf{y}}$, $\dim(V) = k-d$, $K = V \cap \bigcup_{s \in \mathrm{Cox}(k)} \mathcal{W}^{(k)}_s$, $I = \{s \in \mathrm{Cox}(k) \mid V \cap \mathcal{W}^{(k)}_s \neq \emptyset\}$. Let $J \subset I$, and $K^J = V \cap \bigcup_{s \in \mathrm{Cox}(k)} \mathcal{W}^{(k)}_s$. Then:*

1. *$K$ is semi-algebraically homeomorphic to the $\mathbf{S}^n$, where $n = k+d-1$.*
2. *The tuple $\mathcal{C} = (V_s = V \cap \mathcal{W}^{(k)}_s)_{s \in I}$ is a regular closed Leray cover of $K$.*
3. *$\mathrm{H}^i(K^J) = 0$ for $i \geq \mathrm{card}(J)$;*
4. *$\mathrm{H}^i(K^J) = 0$ for $0 < i \leq \mathrm{card}(J) - d - 1$;*
5. *$\mathrm{H}^0(K^J) \cong \mathbb{Q}$ if $\mathrm{card}(J) \geq d+1$.*

*Proof.* Parts (1) and (2) are immediate from Proposition 1, since each intersection of the various $V_s$ are semi-algebraically homeomorphic to some $Z^{(p)}_{\mathbf{w},\cdot,\mathbf{y}}$ for some $p$, $0 \leq p < k$, and $\mathbf{w} \in \mathbb{Z}^p_{>0}$ (using the notation from Proposition 1), and is thus empty, a point, or semi-algebraically homeomorphic to a regular cell of dimension $p$.

It follows from the nerve lemma that $\mathrm{H}^*(K^J) \cong \mathrm{H}^*(\mathcal{N}(\mathcal{C}^J))$, where $\mathcal{C}^J = (V_s)_{s \in J}$. Since $\mathcal{N}(\mathcal{C}^J)$ is a simplicial complex with $\mathrm{card}(J)$ vertices, $\mathrm{H}^i(\mathcal{N}(\mathcal{C}^J)) = 0$ for $i \geq \mathrm{card}(J)$. This proves Part (3).

We now prove Parts (4) and (5). We can assume that $J \neq \emptyset$ which implies that $K^J \neq \emptyset$, since otherwise the claim is obviously true.

For $s = (i, i+1) \in \mathrm{Cox}(k)$, let $P_s$ denote the polynomial $X_{i+1} - X_i$.

Then, for each $s \in I$, $V_s$ is the intersection with $V$ of the semi-algebraic set defined by

$$(P_s = 0) \wedge \bigwedge_{s' \in \mathrm{Cox}(k) - \{s\}} (P_{s'} \geq 0).$$

For $\varepsilon > 0$, denote by $K^J_\varepsilon$ denote the union of $V_{s,\varepsilon}, s \in J$, where $V_{s,\varepsilon}$ is the intersection with $K$ of the open semi-algebraic set defined by

$$(-\varepsilon < P_s < \varepsilon) \wedge \bigwedge_{s' \in \mathrm{Cox}(k)-\{s\}} (P_{s'} > -\varepsilon).$$

Then, using the local conic structure theorem for semi-algebraic sets [18, Theorem 9.3.6] [18], for all small enough $\varepsilon > 0$, $K^J_\varepsilon$ is semi-algebraically homotopy equivalent to $K^J$ and $K - K^J_\varepsilon$ is closed and semi-algebraically homotopy equivalent to $K - K^J$.

We now claim that for all small enough $\varepsilon > 0$, $(V_s - K^J_\varepsilon)_{s \in I-J}$ is a closed Leray cover of $K - K^J_\varepsilon$. Let $J' \subset I - J$, and consider $\bigcap_{s\in J'}(V_s - K^J_\varepsilon)$. Then, there exists $J'' \subset J$ such that $\bigcap_{s\in J'}(V_s - K^J_\varepsilon)$ is the intersection with $V$ of the semi-algebraic set defined by

$$\bigwedge_{s\in J'}(P_s = 0) \wedge \bigwedge_{s \in J''}(P_s \geq \varepsilon) \wedge \bigwedge_{s\in \mathrm{Cox}(k)-(J'\cup J'')} (P_s \geq 0).$$

It follows from from Lemma 2 and the above description that for all $\varepsilon > 0$ small enough, $\bigcap_{s\in J'}(V_s - K^J_\varepsilon)$ is either empty or semi-algebraically contractible, and hence $(V_s - K^J_\varepsilon)_{s\in I-J}$ is a closed Leray cover of $K - K^J_\varepsilon$. Using the same argument involving the nerve complex as in the previous paragraph we obtain that

$$\mathrm{H}^i(K - K^J_\varepsilon) = 0$$

for $i \geq \mathrm{card}(I) - \mathrm{card}(J)$. However, by Alexander duality (see for example [38, page 296]) we have that

$$\tilde{\mathrm{H}}^i(K^J_\varepsilon) \cong \tilde{\mathrm{H}}^i(K^J) \cong \tilde{\mathrm{H}}_{n-i-1}(K - K^J_\varepsilon). \tag{3.7}$$

It follows from Part (3) and (3.7) that $\tilde{\mathrm{H}}^i(K^J) = 0$ for $n-i-1 \geq \mathrm{card}(I)-\mathrm{card}(J)$ or equivalently for $i \leq n - \mathrm{card}(I) + \mathrm{card}(J) - 1$.

Since, $\mathrm{card}(I) \leq n + d$, it follows that $\tilde{\mathrm{H}}^i(K^J) = 0$ for $0 \leq i \leq \mathrm{card}(J) - d - 1$. Parts (4) and (5) of the proposition follows. □

### 3.2. **Proofs of Theorems 1 and 2.**

*Proof of Theorem 1.* Let $V = V^{(k)}_{d,\mathbf{y}}$. We first prove Part (a). From Proposition 1 we have that $V$ is either empty, or a finite union of points, or of dimension $k - d$. If $V$ is empty there is nothing to prove. Suppose that $V$ is not empty.

Using Theorem 4 we have that

$$\mathrm{H}^i(V) \cong \bigoplus_{T\subset \mathrm{Cox}(k)} \mathrm{H}^i(V_k, V_k^T) \otimes_{\mathbb{Q}} \Psi^{(k)}_T. \tag{3.8}$$

Since we have from Proposition 2 that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi^{(k)}_T) = 0 \text{ if } \mathrm{length}(\lambda) > \mathrm{card}(T) + 1,$$

we might as well also assume that

$$\mathrm{length}(\lambda) \leq \mathrm{card}(T) + 1,$$

or that

$$\mathrm{card}(T) \geq \mathrm{length}(\lambda) - 1.$$

It thus suffices to prove that $\mathrm{H}^i(V_k, V_k^T) = 0$, for all pairs $(i, T)$ satisfying:

$$\begin{array}{rcl} i & \leq & \mathrm{length}(\lambda) - 2d + 1, \\ \mathrm{card}(T) & \geq & \mathrm{length}(\lambda) - 1, \end{array}$$

for which it suffices to prove that $\mathrm{H}^i(V_k, V_k^T) = 0$ for all $(i, T)$ satisfying

$$i \leq \mathrm{card}(T) - 2d + 2 \Leftrightarrow \mathrm{card}(T) \geq i + 2d - 2. \tag{3.9}$$

We now fix the pair $(i, T)$ satisfying (3.9), and treat the cases $i = 0$, $i = 1$, and $i > 1$ separately.

Case $i = 0$: In this case, if $V_k \neq \emptyset$, $\mathrm{H}^0(V_k, V_k^T) \neq 0$ if and only if $V_k^T = \emptyset$. If $V_k \neq \emptyset$, it must meet a $d$-dimensional face of the $\mathcal{W}^{(k)}$, which is incident on $k - d$ of the $k - 1$ codimension one faces, $\mathcal{W}_s^{(k)}, s \in \mathrm{Cox}(k)$, of $\mathcal{W}^{(k)}$. This implies that

$$V_k^T = \emptyset \Rightarrow \mathrm{card}(T) \leq d - 1.$$

Since, for $d > 1$, $2d - 2 > d - 1$, it follows that

$$\mathrm{card}(T) \geq i + 2d - 2 = 2d - 2 \Rightarrow \mathrm{card}(T) > d - 1 \Rightarrow V_k^T \neq \emptyset \Rightarrow \mathrm{H}^0(V_k, V_k^T) = 0.$$

This completes the proof of Part (a) in the case $i = 0$.

We now consider the cases $i = 1, i > 1$. Let for $s \in \mathrm{Cox}(k)$, $V_s = V \cap \mathcal{W}_s^{(k)}$. We denote (following the notation in Proposition 3)

$$\begin{array}{rcl} I & = & \{s \in \mathrm{Cox}(k) \mid V_s \neq \emptyset\}, \\ J_T & = & T \cap I, \\ K & = & \displaystyle\bigcup_{s \in I} V_s, \\ K^{J_T} & = & \displaystyle\bigcup_{s \in J_T} V_s \\ & (= & V_k^T). \end{array}$$

Using Parts (1) and (2) of Proposition 3, $K$ is semi-algebraically homeomorphic to $\mathbf{S}^n$, with $n = k - d - 1$, $\mathcal{C} = (V_s)_{s \in I}$, is a regular closed Leray cover of $K$ (cf. Definition 6).

It follows from [1, Theorem 7] that the maximum and minimum of $p_{d+1}^{(k)}$ is obtained on $V_k$ in two distinct $d$-dimensional faces of $\mathcal{W}^{(k)}$. Moreover, each of these two distinct $d$-dimensional faces are incident on exactly $k - d$ co-dimension one faces, $\mathcal{W}_s^{(k)}, s \in \mathrm{Cox}(k)$, of $\mathcal{W}^{(k)}$. This implies that $\mathrm{card}(I) \geq k - d + 1$. We thus have

$$k - d + 1 \leq \mathrm{card}(I) \leq k - 1 = n + d. \tag{3.10}$$

Clearly, $\mathrm{card}(J_T) = \mathrm{card}(T \cap I) \leq \mathrm{card}(T)$.

On the other hand,

$$
\begin{aligned}
\mathrm{card}(J_T) &= \mathrm{card}(T\cap I)\\
&= \mathrm{card}(T)+\mathrm{card}(I)-\mathrm{card}(T\cup I)\\
&\geq \mathrm{card}(T)+\mathrm{card}(I)-\mathrm{card}(\mathrm{Cox}(k))\\
&\geq \mathrm{card}(T)+\mathrm{card}(I)-(k-1)\\
&\geq \mathrm{card}(T)+(k-d+1)-(k-1) \text{ (using inequality (3.10))}\\
&= \mathrm{card}(T)-d+2. \qquad (3.11)
\end{aligned}
$$

Case $i=1$: We only need to consider the case $i=1\leq \mathrm{card}(T)-2d+2$. We distinguish the following two cases:

- If $T=\emptyset$, then since $d>1$, the inequality $i=1\leq \mathrm{card}(T)-2d+2$ cannot hold.
- If $T\neq\emptyset$, and $i=1\leq \mathrm{card}(T)-2d+2$, then

  $$\mathrm{card}(J_T)\geq \mathrm{card}(T)-d+2\geq 2d-1-d+2=d+1,$$

  and it follows from Part (5) of Proposition 3 that $\mathrm{H}^0(V_k^T)=\mathrm{H}^(K^{J_T})\cong\mathbb{Q}$. In this case the restriction homomorphism $\mathrm{H}^0(V_k)\to\mathrm{H}^0(V_k^T)$ is an isomorphism which implies that $\mathrm{H}^1(V_k,V_k^T)=0$.

Case $i>1$: In this case, we can assume that $\dim(V)=k-d$. Otherwise, $V$ is zero-dimensional and $\mathrm{H}^i(V)=0$ for $i>0$.

From the exactness of the long exact sequence,

$$\cdots\to\mathrm{H}^{i-1}(V_k^T)\to\mathrm{H}^i(V_k,V_k^T)\to\mathrm{H}^i(V_k)\to\cdots$$

of the pair $(V_k,V_k^T)$ and the fact that $\mathrm{H}^i(V_k)=0$ for $i\geq 1$, it suffices to prove that $\mathrm{H}^{i-1}(V_k^T)=0$ for $1<i\leq\mathrm{card}(T)-2d+2$ or equivalently $\mathrm{H}^j(V_k^T)=0$ for $1\leq j\leq \mathrm{card}(T)-2d+1$.

Applying Parts (3) and (4) of Proposition 3, noting that $K^T=V_k^T$, we obtain that

$$\mathrm{H}^j(K^{J_T})=\mathrm{H}^j(V_k^T)=0$$

for $0<j\leq\mathrm{card}(T)-2d+1$. This completes the proof for the case $i>1$.
This completes the proof of Part (a).

We now prove Part (b). First assume that $\dim(V)=k-d$.

Since we have from Proposition 2 that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)})=0 \text{ if } \mathrm{length}({}^t\lambda)>k-\mathrm{card}(T),$$

we might as well also assume that

$$\mathrm{length}({}^t\lambda)\leq k-\mathrm{card}(T),$$

or that

$$\mathrm{card}(T)\leq k-\mathrm{length}({}^t\lambda).$$

It thus suffices to prove that $\mathrm{H}^i(V_k,V_k^T)=0$, for all pairs $(i,T)$ satisfying:

$$
\begin{aligned}
i &\geq k-\mathrm{length}({}^t\lambda)+1,\\
\mathrm{card}(T) &\leq k-\mathrm{length}({}^t\lambda),
\end{aligned}
$$

for which it suffices to prove that $\mathrm{H}^i(V_k,V_k^T)=0$ for all $(i,T)$ satisfying

$$i\geq\mathrm{card}(T)+1.$$

From the exactness of the long exact sequence,

$$\cdots \to \mathrm{H}^{i-1}(V_k^T) \to \mathrm{H}^i(V_k, V_k^T) \to \mathrm{H}^i(V_k) \to \cdots$$

of the pair $(V_k, V_k^T)$ and the fact that $\mathrm{H}^i(V_k) = 0$ for $i \geq 1$, it suffices to prove that $\mathrm{H}^{i-1}(V_k^T) = 0$ for $i \geq \mathrm{card}(T) + 1$ or equivalently $\mathrm{H}^j(V_k^T) = 0$ for $j \geq \mathrm{card}(T)$.

It follows from Part (3) of Proposition 3, that $\mathrm{H}^j(V_k^T) = \mathrm{H}^j(K^{J_T}) = 0$ for $j \geq \mathrm{card}(T)$.

If $\dim(V) = 0$, we only need to consider the case $i = 0$. In this case, we need to show that for $\lambda \vdash k$ satisfying

$$\mathrm{length}({}^t\lambda) \geq k + 1,$$

$\mathrm{mult}_{\mathbb{S}^{{}^t\lambda}}(\mathrm{H}^0(V)) = 0$. But since $\mathrm{length}({}^t\lambda) \leq k$, this case does not occur. This completes the proof of Part (b). □

3.3. **Replacing an arbitrary semi-algebraic set by a closed and bounded one.** Before proving Theorem 2 we first recall a fundamental construction due to Gabrielov and Vorobjov [25] which allows us to reduce to the case when the given symmetric semi-algebraic set is closed and bounded.

We first need some preliminaries. In this section we recall some basic facts about real closed fields and real closed extensions.

3.3.1. *Real closed extensions and Puiseux series.* We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [9] for further details.

**Notation 15.** For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in $\varepsilon$ with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\cdots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$.

We refer the reader to [9, Chapter 6] for the definitions of cohomology of semi-algebraic sets over arbitrary real closed fields.

Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, $S$ be a $\mathcal{P}$-semi-algebraic set defined by a $\mathcal{P}$-formula $\Phi$. Without loss of generality we can suppose that

$$\Phi = \Phi_1 \vee \cdots \vee \Phi_N,$$

where for $1 \leq i \leq N$,

$$\Phi_i = \bigwedge_{P \in \mathcal{P}_{i,0}} (P = 0) \wedge \bigwedge_{P \in \mathcal{P}_{i,1}} (P > 0) \wedge \bigwedge_{P \in \mathcal{P}_{i,-1}} (P < 0),$$

where $\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \mathcal{P}_{i,-1}$ is a partition of the set $\mathcal{P}$.

For $\varepsilon, \delta > 0$ we denote

$$\Phi_{i,\varepsilon,\delta} = \bigwedge_{P \in \mathcal{P}_{i,0}} ((P - \varepsilon \leq 0) \wedge (P + \varepsilon \geq 0)) \wedge \bigwedge_{P \in \mathcal{P}_{i,1}} (P - \delta \geq 0) \wedge \bigwedge_{P \in \mathcal{P}_{i,-1}} (P + \delta \leq 0),$$

and

$$\Phi_{\varepsilon,\delta} = \bigwedge_{i=1}^{N} \Phi_{i,\varepsilon,\delta}.$$

Gabrielov and Vorobjov [25] proved the following theorem. [1]

[1]The theorem in [25] is not stated using the language of non-archimedean extensions and Puiseux series but it is easy to translate it into the form stated here.

**Theorem.** [25, Theorem 1.10] *Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ and $S = \mathcal{R}(\Phi)$, where $\Phi$ is a $\mathcal{P}$-formula. For $0 \leq m \leq k$, let*

$$\widetilde{\Phi}_m = \left( \bigvee_{0 \leq j \leq m} \Phi_{\varepsilon_j, \delta_j} \right) \wedge (\varepsilon(X_1^2 + \cdots + X_k^2 - 1 \leq 0)), \tag{3.12}$$

*and let $\widetilde{S}_m = \mathcal{R}(\widetilde{\Phi}_m) \subset \mathrm{R}\langle \varepsilon, \varepsilon_0, \delta_0, \cdots, \varepsilon_m, \delta_m \rangle^k$. Then,*

$$\mathrm{H}^i(S) \cong \mathrm{H}^i(\widetilde{S}_m)$$

*for $0 \leq i < m$.*

*Remark* 7. Observe that $\widetilde{S}_m$ is a bounded a $\widetilde{\mathcal{P}}_m$-closed semi-algebraic set, where

$$\widetilde{\mathcal{P}}_m = \bigcup_{P \in \mathcal{P}} \bigcup_{0 \leq i \leq m} \{P \pm \varepsilon_i, P \pm \delta_i\} \cup \{\varepsilon \sum_i X_i^2 - 1\}.$$

Moreover, if $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$, $d \geq 2$, then

$$\widetilde{\mathcal{P}}_m \subset \mathrm{R}\langle \varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_m, \delta_m \rangle [X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d},$$

and $\mathrm{card}(\widetilde{\mathcal{P}}_m) = 4m \cdot \mathrm{card}(\mathcal{P}) + 1$.

Furthermore, it is easy to verify (by following closely the proof of the theorem in [25]) that if $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}$, and hence $S, \widetilde{S}_m$ are both symmetric, then the isomorphisms $\mathrm{H}^i(S) \cong \mathrm{H}^i(\widetilde{S}_m)$, $0 \leq i < m$, in the theorem are in fact $\mathfrak{S}_k$-equivariant.

In our algorithmic application (cf. Algorithm 3 below) we will replace the given semi-algebraic set $S \subset \mathrm{R}^k$ by the closed and bounded semi-algebraic set $\widetilde{S}_{\ell+1} \subset \mathrm{R}\langle \varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1} \rangle^k$. By the preceding theorem the first $\ell + 1$ Betti numbers of $S$ and $\widetilde{S}_{\ell+1}$ are equal. Moreover, the number of infinitesimals appearing in the definition of $\widetilde{S}_{\ell+1}$ is bounded by $O(\ell)$. The number of infinitesimals used to make the deformation from $S$ to $\widetilde{S}_{\ell+1}$ is important for analyzing the complexity of our algorithms. In our algorithms, we will extend the given ring of coefficients to a polynomial ring in these infinitesimals. As a result each arithmetic operation in this larger ring needs several operations to be performed in the original ring – and this added cost enters as a multiplicative factor in the complexity upper bounds (see proof of Proposition 8).

*Proof of Theorem 2.* In view of the Remark 7 (replacing $S$ by $\widetilde{S}_k$) we can assume that the given semi-algebraic set $S$ is closed and bounded. Since $S$ is a $\mathcal{P}$-semi-algebraic set, and $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$, it follows from the fundamental theorem of symmetric polynomials, that

$$S = (\Phi_d^{(k)})^{-1}(\Phi_d^{(k)}(S)).$$

Let $f = \Phi_d^{(k)}|_S$ and observe that $f$ is a proper map. We have a spectral sequence (the Leray spectral sequence of the map $f$), converging to $\mathrm{H}^{p+q}(S)$, whose $E_2$-term is given by

$$E_2^{p,q} = \mathrm{H}^p(T, R^q f_*(\mathbb{Q}_S)),$$

where $T = f(S)$, and $\mathbb{Q}_S$ denotes the constant sheaf on $S$.

We also have using the proper base change theorem (see for example [27, §3, Theorem 6.2] that for $\mathbf{y} \in T$,

$$R^q f_*(\mathbb{Q}_S)_{\mathbf{y}} \cong \mathrm{H}^q(V^{(k)}_{d,\mathbf{y}}, \mathbb{Q}), \tag{3.13}$$

and this gives $R^q f_*(\mathbb{Q}_S)$ the structure of a sheaf of $\mathfrak{S}_k$-modules. Moreover, since the action of $\mathfrak{S}_k$ on $S$ leaves the fibers of the map $f : S \to T$ invariant, the action of $\mathfrak{S}_k$ on $E_2^{p,q}$ is given by its action on the sheaf $R^q f_*(\mathbb{Q}_S)$.

Now, $\mathrm{H}^n(S)$ is isomorphic as an $\mathfrak{S}_k$-module to a ($\mathfrak{S}_k$-equivariant) subquotient of

$$\bigoplus_{p+q=n} E_2^{p,q}.$$

Using Theorem 1, we have that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \text{ for } i \le \mathrm{length}(\lambda) - 2d + 1.$$

This implies using (3.13) that,

$$\begin{aligned} \mathrm{mult}_{\mathbb{S}^\lambda}(E_2^{p,n-p}) \quad &= \quad 0, \text{ for } n - p \le \mathrm{length}(\lambda) - 2d + 1, \\ &\qquad \text{or equivalently for } n \le \mathrm{length}(\lambda) - 2d + p + 1. \end{aligned} \tag{3.14}$$

From the fact that $\mathrm{H}^{p+q}(S)$ is a ($\mathfrak{S}_k$-equivariant) subquotient of $\bigoplus_{p+q} E_2^{p,q}$, and (3.14), we obtain that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^n(S)) \quad = \quad 0, \text{ for } n \le \mathrm{length}(\lambda) - 2d + 1.$$

This proves Part (a).

In order to prove Part (b), recall first that Theorem 1 implies that

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \text{ for } i \ge k - \mathrm{length}({}^t\lambda) + 1. \tag{3.15}$$

Using (3.13) and (3.15) we obtain that,

$$\begin{aligned} \mathrm{mult}_{\mathbb{S}^\lambda}(E_2^{p,n-p}) \quad &= \quad 0, \text{ for } n - p \ge k - \mathrm{length}({}^t\lambda) + 1 \\ &\qquad \text{or equivalently for } n \ge p + k - \mathrm{length}({}^t\lambda) + 1. \end{aligned} \tag{3.16}$$

Now observe that since $\dim(T) \le d$, $E_2^{p,q} = 0$ for $p \ge d$. Applying this to (3.16), we get that

$$\begin{aligned} \mathrm{mult}_{\mathbb{S}^\lambda}(E_2^{p,n-p}) \quad &= \quad 0, \text{ for } n - p \ge k - \mathrm{length}({}^t\lambda) + 1, \\ &\qquad \text{or equivalently for } n \ge k + d - \mathrm{length}({}^t\lambda) + 1. \end{aligned}$$

This completes the proof of Part (b). □

## 4. Proof of Theorem 3

In this section we prove Theorem 3 by describing an algorithm for efficiently computing the first $\ell + 1$ Betti numbers of any given symmetric semi-algebraic subset of $\mathrm{R}^k$ defined by symmetric polynomials of degrees bounded by $d$, having complexity bounded by a polynomial in $k$ (for fixed $d$ and $\ell$).

We first outline our method.

4.1. **Outline of the proof of Theorem 3.** We use Theorem 4 to decompose the task of computing $b_i(S) = \dim_{\mathbb{Q}} \mathrm{H}^i(S)$ into two parts:

(A) computing the dimensions of $\mathrm{H}^i(S_k, S_k^T)$;
(B) computing the isotypic decompositions of the modules $\Psi_T^{(k)}$ for various subsets $T \subset \mathrm{Cox}(k)$. Notice that using Theorem 2, in order to compute $b_i(S)$ for $i \leq \ell$, we need to compute isotypic decompositions of $\Psi_T^{(k)}$ with $\mathrm{card}(T) < \ell + 2d - 1$.

We first describe an algorithm (cf. Algorithm 1) for computing the isotypic decomposition of $\Psi_T^{(k)}$, which has complexity polynomially bounded in $k$ if $\mathrm{card}(T)$ is bounded by $\ell + 2d - 1$ (considering $\ell$ and $d$ to be fixed). The key ingredient for this algorithm is Proposition 2 which allows a recursive scheme to be used for computing the decomposition. The fact that we need to consider only subsets $T$ of small cardinality (using Theorem 2) is key in keeping the complexity bounded by a polynomial. This accomplishes task (B).

We next address task (A). We first prove that that the cohomology groups of the pair $(S_k, S_k^T)$ are isomorphic to those of another semi-algebraic pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ (cf. Proposition 6). Proposition 6 is the key mathematical result behind our algorithm. The advantage of the pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ over the original pair $(S_k, S_k^T)$ is that $\widetilde{S_k^{(T)}}, \widetilde{S_k^T}$ are subsets of an $O(d+\ell)$-dimensional space (unlike $S_k, S_k^T$ which are subsets of $\mathcal{W}^{(k)} \subset \mathrm{R}^k$). Moreover, a semi-algebraic description of $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ can be computed efficiently (i.e. with polynomially bounded complexity) from that of the pair $(S_k, S_k^T)$ using a slightly modified version of efficient quantifier elimination algorithm over reals (cf. Algorithm 2). The number and the degrees of the polynomials appearing in the description of $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ are bounded by a polynomial in $k$ (for fixed $d$ and $\ell$). Finally, we compute the Betti numbers of the pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ using effective algorithms for computing semi-algebraic triangulations (cf. Algorithm 3). We exploit the fact that this is now a constant (i.e. $O(d+\ell)$) dimensional problem, and we can use algorithms which have doubly exponential complexity in the number of variables without affecting the overall polynomial complexity of our algorithm.

4.2. **Computing the isotypic decomposition of $\Psi_T^{(k)}$.** We now describe more precisely our algorithm for computing the multiplicities of various Specht modules in the representations $\Psi_T^{(k)}$.

---

**Algorithm 1** (Computing isotypic decomposition of $\Psi_T^{(k)}$)

---

**Input:**
An integer $k \in \mathbb{Z}_{>0}$, and $T \subset \mathrm{Cox}(k)$.
**Output:**
(A) The set $\mathrm{Par}(k, T) = \{\lambda \vdash k \mid \mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)}) \neq 0\}$;
(B) $\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)})$ for each $\lambda \in \mathrm{Par}(k, T)$.

---

**Procedure:**

1: **if** $T = \emptyset$ **then**
2: Output $\mathrm{Par}(k, T) = \{(k)\}$, and $\mathrm{mult}_{\mathbb{S}^{(k)}}(\Psi_T^{(k)}) = 1$ and terminate.
3: **else**
4: **if** $k = 2$ **then**
5: output $\mathrm{Par}(k, T) = \{(1, 1)\}$, and $\mathrm{mult}_{\mathbb{S}^{(1,1)}}(\Psi_T^{(k)}) = 1$ and terminate.
6: **end if**
7: **for** $\lambda \vdash k$ **do**
8: $m_\lambda \leftarrow 0$.
9: **end for**
10: $P_T \leftarrow \emptyset$.
11: $T \leftarrow \{s_{i+(k-1-j_0)} \in \mathrm{Cox}(k) \mid s_i \in T\}$, where $j_0 = \max\{j \mid s_j \in T\}$.
12: **end if**
13: $T' \leftarrow T \setminus \{s_{k-1}\}$.
14: Using a recursive call to Algorithm 1 with input $k-1$ and $T'$, compute $\mathrm{Par}(k-1, T')$ and $\mathrm{mult}_{\mathbb{S}^\mu}(\Psi_{T'}^{(k-1)})$ for each $\mu \in \mathrm{Par}(k-1, T')$.
15: **for** $\mu \in \mathrm{Par}(k-1, T')$ **do**
16: **for** $\lambda \in S(\mu)$ **do** (cf. Notation 13)
17: $P_T \leftarrow P_T \cup \{\lambda\}$.
18: $m_\lambda \leftarrow m_\lambda + \mathrm{mult}_{\mathbb{S}^\mu}(\Psi_{T'}^{(k-1)})$.
19: **end for**
20: **end for**
21: Using a recursive call to Algorithm 1 with input $k$ and $T'$, compute $\mathrm{Par}(k, T')$ and $\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_{T'}^{(k)})$ for each $\lambda \in \mathrm{Par}(k, T')$.
22: **for** $\lambda \in \mathrm{Par}(k, T)$ **do**
23: $m_\lambda \leftarrow m_\lambda - \mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_{T'}^{(k)})$.
24: **if** $m_\lambda = 0$ **then**
25: $P_T \leftarrow P_T \setminus \{\lambda\}$.
26: **else**
27: $P_T \leftarrow P_T \cup \{\lambda\}$.
28: **end if**
29: **end for**
30: Output $\mathrm{Par}(k, T) = P_T$, and for each $\lambda \in \mathrm{Par}(k, T)$, output $\mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)}) = m_\lambda$.

**Proposition 4.** *Algorithm 1 is correct and has complexity, measured by the number of arithmetic operations in $\mathbb{Z}$, bounded by $k(O(n))^n$. Moreover, the cardinality of the set $\mathrm{Par}(k, T)$ output is also bounded by $k(O(n))^n$.*

*Proof.* Let $F(k, n)$ denote the maximum complexity of the algorithm over all inputs $(k, T)$, where $\mathrm{card}(T) = n$. We can assume that $F(k, n)$ is also an upper bound on the cardinality of the set $\mathrm{Par}(k, T)$ produced in the output of the algorithm. First consider the recursive call to the algorithm in Line 14. Using (3.2), we have that for each $\mu$ belonging to the output $\mathrm{Par}(k-1, T')$ of this recursive call is bounded by, $\mathrm{length}(\mu) \le \mathrm{card}(T') + 1 = n$. Thus the total cost of the '**for**' loop in Line 15 is bounded by $CnF(k-1, n-1)$ for all large enough constant $C > 0$. The cost of the recursive call in Line 21 is bounded by $F(k, n-1)$, and the cost of the '**for**' loop in Line 22 is bounded by $CF(k, n-1)$ for all large enough constant $C > 0$. Thus the function $F(k,n)$ satisfies the following inequalities for some large enough constant $C > 0$:

$$\begin{array}{rcl} F(k,0) & \leq & C \cdot k, \\ F(2,\cdot) & \leq & C, \\ F(k,n) & \leq & C \cdot (nF(k-1,n-1) + F(k,n-1) + k) \\ & \leq & C \cdot (n+1)(F(k,n-1)+k). \end{array}$$

It follows from the above inequalities that $F(k,n)$ that there exists some constant $C'$ such that

$$F(k,n) \leq k \cdot (C'n)^n.$$

□

4.3. **The pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ and its properties.** In this section we define the pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$, and prove its key property.

**Notation 16.** For any finite set $T$ and $s \in T$, we denote by $\Delta_T \subset \mathrm{R}^T$, the standard simplex in $\mathrm{R}^T$. In other words, $\Delta_T$ is the convex hull of the points $(e_s)_{s\in T}$, where $e_s$ is defined by $\pi_t(e_s) = \delta_{s,t}$ where for each $t \in T$, $\pi_t : \mathrm{R}^T \to \mathrm{R}$ is the projection map on to the $t$-th coordinate. For $T' \subset T$, we denote by $\Delta_{T'}$, th convex hull of the points $(e_s)_{s\in T'}$, and call $\Delta_{T'}$ the face of $\Delta_T$ corresponding to the subset $T'$.

**Definition 7.** Let $k \in \mathbb{Z}_{\geq 0}$, and $\lambda, \mu \in \mathrm{Comp}(k)$. We denote, $\lambda \prec \mu$, if $\mathcal{W}_\lambda \subset \mathcal{W}_\mu$. It is clear that $\prec$ is a partial order on $\mathrm{Comp}(k)$ making $\mathrm{Comp}(k)$ into a poset.

**Notation 17.** For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathrm{Comp}(k)$, we denote $\mathrm{length}(\lambda) = \ell$, and for $k, d \in \mathbb{Z}_{\geq 0}$, we denote

$$\begin{array}{rcl} \mathrm{CompMax}(k,d) & = & \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathrm{Comp}(k) \mid \lambda_{2i+1} = 1, 0 \leq i < d/2\}, \\ \mathrm{Comp}(k,d) & = & \displaystyle\bigcup_{\lambda \in \mathrm{CompMax}(k,d)} \{\lambda' \in \mathrm{Comp}(k) \mid \lambda' \prec \lambda\} \text{ if } d \leq k, \\ & = & \mathrm{Comp}(k), \text{ if } d > k. \end{array}$$

We denote by

$$\mathcal{W}_d^{(k)} = \bigcup_{\lambda \in \mathrm{Comp}(k,d)} \mathcal{W}_\lambda.$$

We state the following important theorem due to Arnold [1] which has been referred to in Example 2.4.2. It plays a key role in the proof of Proposition 5 below. Since we refer the reader to [13] for the proof of Proposition 5, we do not use Theorem 5 subsequently in this paper.

**Theorem 5.** [1, Theorem 7]

*For every $\mathbf{w} \in \mathrm{R}^k_{\geq 0}$, $d, k \geq 0$, $d' = \min(k,d)$, and $\mathbf{y} \in \mathrm{R}^{d'}$ the function $p^{(k)}_{\mathbf{w},d+1}$ has exactly one local maximum on $(\Psi^{(k)}_{\mathbf{w},d})^{-1}(\mathbf{y})$, which furthermore depends continuously on $\mathbf{y}$.*

*Moreover, a point $\mathbf{x} \in V_{\mathbf{w},\mathbf{y}} \cap \mathcal{W}^{(k)}$ is a local maximum if and only if $\mathbf{x} \in \mathcal{W}^{(k)}_\lambda$ for some $\lambda \in \mathrm{Comp}(k,d')$.*

We need some more notation.

**Notation 18.** For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathrm{Comp}(k)$, we denote by $\iota_\lambda : \mathcal{W}^{(\ell)} \to \mathcal{W}^{(k)}$ the embedding that takes $(y_1, \ldots, y_\ell) \in \mathcal{W}^{(\ell)}$ to the point $(\underbrace{y_1, \ldots, y_1}_{\lambda_1}, \ldots, \underbrace{y_\ell, \ldots, y_\ell}_{\lambda_\ell})$.

**Notation 19.** For $T \subset \mathrm{Cox}(k)$ and $d \geq 0$, we denote:

$$\mathcal{W}^{(k)}_{T,d} \quad = \quad \iota_{\lambda(T)}(\mathcal{W}_d^{(\mathrm{length}(\lambda(T)))}).$$

**Definition 8.** For any semi-algebraic set $S \subset \mathrm{R}^k$, $T', T \subset \mathrm{Cox}(k), T' \subset T$, and $d \geq 0$, we set

$$\begin{aligned}
S_k &= S \cap \mathcal{W}^{(k)}, \\
S_{k,d} &= S \cap \mathcal{W}^{(k)}_d, \\
S_k^T &= \mathcal{W}^{(k,T)} \cap S, \\
S_{k,T} &= \mathcal{W}^{(k)}_T \cap S, \\
S_{k,T,d} &= S \cap \mathcal{W}^{(k)}_{T,d}.
\end{aligned}$$

**Proposition 5.** *Let $1 < d$, and $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$, $S \subset \mathrm{R}^k$, a $\mathcal{P}$-closed and bounded semi-algebraic set, and $\mathbf{w} \in \mathrm{R}^k_{>0}$. Then the following holds.*

1. *The map $\Psi^{(k)}_{\mathbf{w},d}$ restricted to $S_{k,d}$ is a semi-algebraic homeomorphism on to its image, and*
2. *$\Psi^{(k)}_{\mathbf{w},d}(S_{k,d}) = \Psi^{(k)}_{\mathbf{w},d}(S_k)$.*

*Proof.* Both parts follow from the weighted version of Part (1) of Proposition 9 in [13]. □

We have the following corollary of Proposition 5 that we will need. With the same hypothesis as in Proposition 5:

**Corollary 1.** *For each subset $T \subset \mathrm{Cox}(k)$, $\Psi^{(k)}_d$ restricted to $S_{k,T,d}$ is a semi-algebraic homeomorphism on to its image, and*

$$\Psi^{(k)}_d(S_{k,T}) = \Psi^{(k)}_d(S_{k,T,d}).$$

*Proof.* Let $\ell = \mathrm{length}(\lambda(T))$, and $S'_\ell = \iota^{-1}_{\lambda(T)}(S_{k,T})$ (cf. Notation 18). Then, by Definition 8

$$S_{k,T,d} = \iota_{\lambda(T)}(S'_{\ell,d}),$$

and

$$\Psi^{(k)}_d|_{S_{k,T}} = \Psi^{(\ell)}_{\lambda(T),d} \circ \iota^{-1}_{\lambda(T)}.$$

The corollary now follows from Proposition 5, and the fact that $\iota_{\lambda(T)}$ is a semi-algebraic homeomorphism on to its image. □

Now, let $1 < d$, and $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$, $S \subset \mathrm{R}^k$, a $\mathcal{P}$-closed and bounded semi-algebraic set, and $T \subset \mathrm{Cox}(k)$.

We define:

**Definition 9.**

$$\widetilde{S_k^{(T)}} = \Psi^{(k)}_d(S_k) \times \Delta_T \subset \mathrm{R}^d \times \mathrm{R}^T,$$

and

$$\widetilde{S_k^T} = \bigcup_{T' \subset T} \Psi_d^{(k)}(S_{k,T'}) \times \Delta_{T'} \subset \widetilde{S_k^{(T)}}.$$

The key property of the pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$ defined above that will be used later is the following.

Using the definitions given above we have:

**Proposition 6.**

$$\mathrm{H}^*(\widetilde{S_k^{(T)}}, \widetilde{S_k^T}) \cong \mathrm{H}^*(S_k, S_k^T).$$

Before proving Proposition 6 we recall the notion of the *blow-up complex* of a collection of closed and bounded semi-algebraic subsets of $\mathrm{R}^N$.

**Definition 10** (Blow-up complex)**.** Given a finite family $\mathcal{A} = (A_\alpha)_{\alpha \in I}$ of closed and bounded semi-algebraic subsets of $\mathrm{R}^N$, we denote

$$\mathrm{Bl}(\mathcal{A}) = \coprod_{J \subset I} A_J \times \Delta_J / \sim,$$

where for $J \subset I$, $A_J = \bigcap_{\alpha \in J} A_\alpha$, and $\Delta_J$ is the face of the standard simplex $\Delta_I \subset \mathrm{R}^I$ (i.e. $\Delta_J = \{(x_\alpha)_{\alpha \in I} \in \Delta_I \mid \forall(\alpha \notin J) x_\alpha = 0\}$, and $\sim$ is the obvious identification.

It is an easy consequence of the Vietoris-Begle theorem that (using the same notation as in Definition 10) the map

$$\pi : \mathrm{Bl}(\mathcal{A}) \to A = \bigcup_\alpha A_\alpha, \pi(x;t) = x,$$

is a homotopy equivalence.

Moreover, if $\mathcal{B} = (B_\alpha)_{\alpha \in I}$ is another family of closed and bounded semi-algebraic sets, such that for each $\alpha \in I$, $A_\alpha \subset B_\alpha$, then there is an obvious inclusion $\mathrm{Bl}(\mathcal{A}) \hookrightarrow \mathrm{Bl}(\mathcal{A})$, and we have a commutative diagram,

$$\begin{array}{ccc} \mathrm{Bl}(\mathcal{A}) & \hookrightarrow & \mathrm{Bl}(\mathcal{B}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ A = \bigcup_\alpha A_\alpha & \hookrightarrow & B = \bigcup_\alpha B_\alpha \end{array},$$

where the horizontal arrows are inclusions. This gives a map between the pairs $(\mathrm{Bl}(\mathcal{B}), \mathrm{Bl}(\mathcal{A})) \to (B, A)$. (In particular, note that if $B_\alpha = B$ for all $\alpha \in I$, $\mathrm{Bl}(\mathcal{B}) = B \times \Delta_I$.)

**Lemma 3.** *The induced homomorphism*

$$\mathrm{H}^*(B, A) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{B}), \mathrm{Bl}(\mathcal{A}))$$

*is an isomorphism.*

*Proof.* The lemma is a consequence of the 'five-lemma', and the fact that the induced homomorphisms, $\pi^* : \mathrm{H}^*(A) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{A})), \mathrm{H}^*(B) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{B}))$ are isomorphisms. $\square$

*Proof of Proposition 6.* Let $\mathcal{A} = (S_{k,\{s\}})_{s\in T}$, and $\mathcal{B} = (S_k)_{s\in T}$. Then, using Lemma 3 and noting that $S_k^T = \bigcup_{s\in T} S_{k,\{s\}}$, we have that

$$\mathrm{H}^*(S_k, S_k^T) \cong \mathrm{H}^*(\mathrm{Bl}(\mathcal{B}), \mathrm{Bl}(\mathcal{A})).$$

Moreover, observe that for $T'' \subset T' \subset T$, we have a commutative diagram

$$\begin{tikzcd} S_{k,T'} \arrow[r, hook] \arrow[d, "\Psi_d^{(k)}"] & S_{k,T''} \arrow[d, "\Psi_d^{(k)}"] \\ \Psi_d^{(k)}(S_{k,T'}) \arrow[r, hook] & \Psi_d^{(k)}(S_{k,T''}) \end{tikzcd}$$

where the horizontal arrows are inclusions.

This allows us to define a map, $\mathrm{Bl}(\mathcal{B}) \to \widetilde{S_k^{(T)}}$, by

$$((x;t),(x';t') \mapsto (\Psi_d^{(k)}(x);t), \Psi_d^{(k)}(x');t')),$$

which restricts to a map $\mathrm{Bl}(\mathcal{A}) \to \widetilde{S_k^T}$.

Hence, we have a induced map of pairs

$$(\mathrm{Bl}(\mathcal{B}), \mathrm{Bl}(\mathcal{A})) \to (\widetilde{S_k^{(T)}}, \widetilde{S_k^T}). \tag{4.1}$$

The fibers of the maps $\mathrm{Bl}(\mathcal{B}) \to \widetilde{S_k^{(T)}}$, $\mathrm{Bl}(\mathcal{A}) \to \widetilde{S_k^T}$, defined above are weighted Vandermonde varieties inside Weyl chambers and are thus contractible using Proposition 1. Hence, the induced homomorphisms, $\mathrm{H}^*(\widetilde{S_k^{(T)}}) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{B}))$, $\mathrm{H}^*(\widetilde{S_k^T}) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{A})$ are isomorphisms.

Using the 'five lemma' we obtain that the homomorphism,

$$\mathrm{H}^*(\widetilde{S_k^{(T)}}, \widetilde{S_k^T}) \to \mathrm{H}^*(\mathrm{Bl}(\mathcal{B}), \mathrm{Bl}(\mathcal{A}))$$

induced by the map in (4.1) is an isomorphism. This proves the proposition. □

4.4. **Algorithm for computing a semi-algebraic description of the pair $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$.** We now describe an efficient algorithm which takes as input the semi-algebraic description of a symmetric semi-algebraic subset $S \subset \mathrm{R}^k$, which uses symmetric polynomials of degree at most $d$, and produces semi-algebraic descriptions of $\widetilde{S_k^{(T)}}$ and $\widetilde{S_k^T}$.

---

**Algorithm 2** (Computing semi-algebraic descriptions of $(\widetilde{S_k^{(T)}}, \widetilde{S_k^T})$)

---

**Input:**

(A) Integers $k, d \geq 0$, $d \leq k$;
(B) a finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]_{\leq d}^{\mathfrak{S}_k}$;
(C) a $\mathcal{P}$-closed formula, $\Phi$ such that $\mathcal{R}(\Phi) = S$;
(D) $T \subset \mathrm{Cox}(k)$.

**Output:**

(A) An ordered domain D contained in a real closed field R;
(B) A finite family of polynomials $\widetilde{\mathcal{Q}} \subset \mathrm{D}[(Y_s)_{s\in T}, Z_1, \ldots, Z_d]$;
(C) $\widetilde{\mathcal{Q}}$ formulas, $\widetilde{\Phi_k^{(T)}}$ and $\widetilde{\Phi_k^T}$, such that $\mathcal{R}(\widetilde{\Phi_k^{(T)}}) = \widetilde{S_k^{(T)}}$ and $\mathcal{R}(\widetilde{\Phi_k^T}) = \widetilde{S_k^T}$.

---

**Procedure:**

1: **for** $\lambda \in \mathrm{CompMax}(k,d)$ **do**

2: Using the algorithm from [13, Corollary 6] applied to the family $\mathcal{P}$, the formula $\Phi \wedge \bigwedge_{1 \leq i \leq k-1}(X_i \leq X_{i+1})$, and the linear equations defining the subspace $L_\lambda$ containing the face $\mathcal{W}_\lambda$, and the polynomial map $\Phi_d^{(k)}$, obtain a family of polynomials formula $\mathcal{Q}_\lambda \subset \mathrm{R}[Z_1,\ldots,Z_d]$, and $\mathcal{Q}_\lambda$-formula $\Phi_\lambda$, such that $\mathcal{R}(\Phi_\lambda) = \Psi_d^{(k)}(S \cap \mathcal{W}_\lambda)$.

3: **end for**

4: $\Theta \leftarrow (\sum_{s\in T} Y_s - 1 = 0) \wedge \bigwedge_{s \in T}(Y_s \geq 0)$.

5: $\widetilde{\mathcal{Q}} \leftarrow \{\sum_{s\in T} Y_s - 1\} \cup \bigcup_{s\in T}\{Y_s\} \cup \bigcup_{\lambda \in \mathrm{CompMax}(k,d)} \mathcal{Q}_\lambda$.

6: $\widetilde{\Phi_k^{(T)}} \leftarrow \Theta \wedge \bigvee_{\lambda\in\mathrm{CompMax}(k,d)} \Phi_\lambda$.

7: **for** $T' \subset T$ **do**

8: **for** $\mu \in \mathrm{CompMax}(\mathrm{length}(\lambda(T')),d)$ **do**

9: Using the Algorithm from [13, Corollary 6] applied to the family $\mathcal{P}$, the formula $\Phi \wedge \bigwedge_{1 \leq i \leq k-1}(X_i \leq X_{i+1})$, the linear equations defining the subspace the face $\iota_\mu(\mathcal{W}_\lambda^{(\mathrm{length}(T'))})$, and the polynomial map $\Phi_d^{(k)}$, obtain a family of polynomials formula $\mathcal{Q}_{T',\mu} \subset \mathrm{R}[Z_1,\ldots,Z_d]$, and $\mathcal{Q}_{T',\mu}$-formula $\Phi_{T',\mu}$, such that $\mathcal{R}(\Phi_{T',\mu}) = \Phi_d^{(k)}(S \cap \iota_\mu(\mathcal{W}_\mu^{(\mathrm{length}(T'))})$.

10: **end for**

11: $\Phi_{k,T'} = \bigvee_{\mu\in\mathrm{CompMax}(\mathrm{length}(\lambda(T')),d)} \Phi_{T',\mu} \wedge (\sum_{s\in T'} Y_s - 1 = 0) \wedge \bigwedge_{s\in T-T'}(Y_s = 0)$.

12: **end for**

13:
$$\widetilde{\mathcal{Q}} \leftarrow \widetilde{\mathcal{Q}} \cup \bigcup_{T'\subset T} \bigcup_{\mu\in\mathrm{CompMax}(\mathrm{length}(\lambda(T')),d)} \mathcal{Q}_{T',\mu}.$$

14: $\widetilde{\Phi_k^T} \leftarrow \bigvee_{T'\subset T} \Phi_{k,T'}$.

**Proposition 7.** *Algorithm 2 is correct and its complexity, measured by the number of arithmetic operations in the domain* D*, is bounded by*
$$(skd)^{O(d+\mathrm{card}(T))}.$$
*Moreover,* $\mathrm{card}(\widetilde{\mathcal{Q}}) \leq (skd)^{O(d+\mathrm{card}(T))}$*, and the degrees of the polynomials in* $\widetilde{\mathcal{Q}}$ *are bounded by* $d^{O(d+\mathrm{card}(T))}$.

*Proof.* It follow from Proposition 5 and [13, Corollary 6], that the first order formulas $\Phi_\lambda, \lambda \in \mathrm{CompMax}(k,d)$, computed in Line 2 of Algorithm 2 have the property that
$$\mathcal{R}\left(\bigvee_{\lambda\in\mathrm{CompMax}(k,d)} \Phi_\lambda\right) = \Phi_d^{(k)}(S_k).$$
It now follows from the definition of $\widetilde{S_k^{(T)}}$ (cf. Definition 9), that the formula $\widetilde{\Phi_k^{(T)}}$ computed in Line 6 in Algorithm 2 satisfies
$$\mathcal{R}(\widetilde{\Phi_k^{(T)}}) = \widetilde{S_k^{(T)}}.$$

Similarly, it follows from Corollary 1, and [13, Corollary 6], that the first order formulas $\Phi_{T',\mu}, \mu \in \mathrm{CompMax}(\mathrm{length}(\lambda(T')), d)$ computed in Line 9 of Algorithm 2 have the property that,

$$\mathcal{R}\left(\bigvee_{\mu \in \mathrm{CompMax}(\mathrm{length}(\lambda(T')),d)} \Phi_{T',\mu}\right) = \Phi_d^{(k)}(S_{k,T',d}).$$

It now follows from the definition of $\widetilde{S_{k,T}}$ (cf. Definition 9), that the formula $\widetilde{\Phi_k^T}$ computed in Line 14 of Algorithm 2 satisfies

$$\mathcal{R}(\widetilde{\Phi_k^T}) = \widetilde{S_k^T}.$$

This completes the proof of the correctness of Algorithm 2. The complexity upper bound is a consequence of the complexity bound in [13, Corollary 6], and the following:

(i) the number of iterations of the '**for**' loop in Line 1 is bounded by

$$\mathrm{card}(\mathrm{CompMax}(k,d)) \leq k^{O(d)};$$

(ii) the number of iterations of the '**for**' loop in Line 7 bounded by

$$2^{\mathrm{card}(T)};$$

and,

(iii) the number of iterations of the '**for**' loop in Line 8is bounded by

$$\mathrm{card}(\mathrm{CompMax}(\mathrm{length}(T'),d)) \leq k^{O(d)}.$$

□

4.5. **Algorithm for computing the isotypic decomposition of cohomology groups and the Betti numbers of symmetric semi-algebraic sets.** We are now in a position to describe our algorithm for computing the isotypic decomposition and the Betti numbers of symmetric semi-algebraic sets (in dimensions $\leq \ell+1$), which will finally prove Theorem 3.

---

**Algorithm 3** (Computing the isotypic decomposition and the dimensions of the first $\ell+1$ cohomology groups of a symmetric semi-algebraic set)

---

**Input:**

(A) An ordered domain D contained in a real closed field R;
(B) Integers $k, d, \ell \geq 0$, $\ell, d \leq k$;
(C) a finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]_{\leq d}^{\mathfrak{S}_k}$;
(D) a $\mathcal{P}$-formula $\Phi$, such that $\mathcal{R}(\Phi) = S \subset \mathrm{R}^k$.

**Output:**

(A) For each $i, 0 \leq i \leq \ell$, a set $M_i$ of pairs $(m_{i,\lambda} \in \mathbb{Z}_{>0}, \lambda \vdash k)$ such that

$$\mathrm{H}^i(S) \cong_{\mathfrak{S}_k} \bigoplus_{(m_{i,\lambda},\lambda) \in M_i} m_{i,\lambda}\, \mathbf{S}^\lambda.$$

(B) The integers $b_0(S), \ldots, b_\ell(S)$.

---

**Procedure:**

1: $\Phi \leftarrow \widetilde{\Phi}_{\ell+1}$ (cf. Eqn. (3.12)).
2: $\mathrm{D} \leftarrow \mathrm{D}' = \mathrm{D}[\varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1}]$.
3: $\mathrm{R} \leftarrow \mathrm{R}' = \mathrm{R}\langle \varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1} \rangle$.
4: **for** $T \subset \mathrm{Cox}(k), \mathrm{card}(T) < \ell + 2d - 1$ **do**
5: Compute using Algorithm 2, the family of polynomials $\widetilde{\mathcal{Q}}$ and the formulas $\widetilde{\Phi_k^{(T)}}$ and $\widetilde{\Phi_k^T}$.
6: Compute a semi-algebraic triangulation $h_T : |K_T| \to \mathcal{R}(\widetilde{\Phi_k^{(T)}})$, such that $h_T^{-1}(\mathcal{R}(\widetilde{\Phi_k^{(T)}}) = |K'_T|$, $K'_T$ is a sub-complex of $K_T$, as in the proof of Theorem 5.43 [9].
7: Compute $b_i(\mathcal{R}(\widetilde{\Phi_k^{(T)}}), \widetilde{\Phi_k^T}) = b_i(K_T, K'_T)$ for $0 \le i \le \ell$ (using for example the Gauss-Jordan elimination algorithm from elementary linear algebra).
8: Compute using Algorithm 1, the set $\mathrm{Par}(k, T)$.
9: **for** $\lambda \in \mathrm{Par}(k, T)$ **do**
10: $m_{\lambda,T} \leftarrow \mathrm{mult}_{\mathbb{S}^\lambda}(\Psi_T^{(k)})$.
11: **end for**
12: **end for**
13: **for** $0 \le i \le \ell$ **do**
14: $M_i \leftarrow \emptyset$.
15: **for** $\lambda \in \mathrm{Par}(k), \mathrm{length}(\lambda) \le i + 2d - 1$ **do**
16: $m_{i,\lambda} \leftarrow 0$.
17: **end for**
18: **for** $T \subset \mathrm{Cox}(k), \mathrm{card}(T) < i + 2d - 1$ **do**
19: **for** $\lambda \in \mathrm{Par}(k, T)$ **do**
20: $m_{i,\lambda} \leftarrow m_{i,\lambda} + b_i(\mathcal{R}(\widetilde{\Phi_k^{(T)}}), \mathcal{R}(\widetilde{\Phi_k^T})) \cdot m_{\lambda,T}$.
21: **end for**
22: **end for**
23: **for** $\lambda \in \mathrm{Par}(k), \mathrm{length}(\lambda) \le i + 2d - 1$ **do**
24: **if** $m_{i,\lambda} \ne 0$ **then**
25: $M_i \leftarrow M_i \cup \{(m_{i,\lambda}, \lambda)\}$.
26: **end if**
27: **end for**
28: Output $M_i$ and
$$b_i(S) = \sum_{\lambda \in \mathrm{Par}(k), \mathrm{length}(\lambda) \le i+2d-1} m_{i,\lambda} \cdot \dim_{\mathbb{Q}} \mathbb{S}^\lambda,$$
calculating $\dim_{\mathbb{Q}} \mathbb{S}^\lambda$ using Eqn. (1.9).
29: **end for**

**Proposition 8.** *Algorithm 3 is correct and has complexity, measured by the number of arithmetic operations in the domain* D*, bounded by* $(skd)^{2^{O(d+\ell)}}$.

*Proof.* First observe that the formula $\widetilde{\Phi}_{\ell+1}$ in Line 1 is a $\widetilde{P}_{\ell+1}$-closed formula, where
$$\widetilde{P}_{\ell+1} \subset \mathrm{D}[\varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1}]^{\mathfrak{S}_k}_{\le d},$$
and $\mathcal{R}(\widetilde{\Phi}_{\ell+1})$ is closed and bounded.

It follows from Proposition 7, that the pair of formulas $(\widetilde{\Phi_k^{(T)}}, \widetilde{\Phi_k^T})$ computed in Line 5 of Algorithm 3 has the property that,

$$(\mathcal{R}(\widetilde{\Phi_k^{(T)}}), \mathcal{R}(\widetilde{\Phi_k^T})) = (\widetilde{S_k^{(T)}}, \widetilde{S_k^T}).$$

It follows from Proposition 6, that

$$\mathrm{H}^*(\widetilde{S_k^{(T)}}, \widetilde{S_k^T}) \cong \mathrm{H}^*(S_k, S_k^T),$$

and it follows from Theorem 5.43 in [9], that the numbers $b_i(\widetilde{S_k^{(T)}}, \widetilde{S_k^T}) = b_i(S_k, S_k^T)$ are computed correctly in Line 7 of Algorithm 3 (for $0 \leq i \leq \ell$).

It follows from Theorem 4 that,

$$b_i(S) = \sum_{T \subset \mathrm{Cox}(k)} b_i(S_k, S_k^T) \cdot \dim \Psi_T^{(k)}. \tag{4.2}$$

It follows from (3.9) that the sum on the right hand side of Eqn. (4.2) needs to be taken only over those $T \subset \mathrm{Cox}(k)$, satisfying $\mathrm{card}(T) < i + 2d - 1$, i.e.

$$b_i(S) = \sum_{T \subset \mathrm{Cox}(k), \mathrm{card}(T) < i+2d-2} b_i(S_k, S_k^T) \cdot \dim \Psi_T^{(k)}.$$

The correctness of the algorithm now follows from Proposition 4.

In order to analyze the complexity, first notice that in Line 2, the ordered domain $\mathrm{D}$ is replaced by the ordered domain $\mathrm{D}' = \mathrm{D}[\varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1}]$. Each subsequent arithmetic operation takes place in the larger domain

$$\mathrm{D}' = \mathrm{D}[\varepsilon, \varepsilon_0, \delta_0, \ldots, \varepsilon_{\ell+1}, \delta_{\ell+1}].$$

Since the number of arithmetic operations in $\mathrm{D}$ needed for computing the sum and the product of two polynomials in $\mathrm{D}'$ of degrees bounded by $D$ is at most $D^{O(\ell)}$, and the degrees of the polynomials in $\mathrm{D}'$ that show up in the intermediate computations are well controlled, it suffices to bound the number of arithmetic operations in the new ring $\mathrm{D}'$.

The number of iterations of the '**for**' loop in Line 4 is bounded by $\binom{k-1}{\ell+2d-2} = k^{O(d+\ell)}$. In each iteration, notice that the semi-algebraic sets $\widetilde{S_k^{(T)}}, \widetilde{S_k^T} \subset \mathrm{R}^{\mathrm{card}(T)} \times \mathrm{R}^d$, and thus the number of variables in the calls to the triangulation algorithm in Line 6 equals $\mathrm{card}(T) + d \leq (\ell + 2d - 1) + d = O(\ell + d)$. The number of arithmetic operations in $\mathrm{D}'$ in each iteration is thus bounded by

$$(\ell s d k)^{2^{O(d+\ell)}}$$

from the complexity bounds in Propositions 4, 7, and the complexity of the triangulation algorithm.

Since, the degrees of the polynomials appearing in the computations are bounded by $d^{2^{O(d+\ell)}}$, it follows that the number of arithmetic operations in $\mathrm{D}$ is also bounded by

$$(\ell s d k)^{2^{O(d+\ell)}}.$$

It follows from Proposition 4, that the number of iterations of the '**for**' loop in Line 9 is bounded by $k(d+\ell)^{O(d+\ell)}$. Also, the number of iterations of the '**for**' loop in Line 15 is bounded by $k^{O(d+\ell)}$ using the trivial upper bound on the number of partitions of $k$ of length bounded by $\ell + 2d - 1$ and the number of iterations of the '**for**' loop in Line 18 is bounded by $\binom{k-1}{\ell+2d-2} = k^{O(d+\ell)}$. Thus, the complexity of the whole algorithm is bounded by

$$(\ell s d k)^{2^{O(d+\ell)}}.$$

□

*Proof of Theorem 3.* The theorem now follows directly from Proposition 8. □

*Remark* 8. We note that using the more sophisticated algorithm for computing the first $\ell + 1$ Betti numbers of semi-algebraic sets given in [3], it is possible to improve the dependence on $d$ in the complexity upper bound in Theorem 3 from doubly exponential to singly exponential. However, since our focus is on obtaining an algorithm with polynomially bounded complexity for fixed $d$ and $\ell$, we chose not to introduce the technical modifications that would be required to achieve this.

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