Computing the Betti Numbers of Arrangements via Spectral Sequences

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Abstract

In this paper, we consider the problem of computing the Betti numbers of an arrangement of n compact semi-algebraic sets, $S_1, \ldots, S_n \subset \mathbb{R}^k$, where each S_i is described using a constant number of polynomials with degrees bounded by a constant. Such arrangements are ubiquitous in computational geometry. We give an algorithm for computing ℓ-th Betti number, $\beta_\ell(\cup_{i=1}^n S_i)$, $0 \leq \ell \leq k-1$, using $O(n^{\ell+2})$ algebraic operations. Additionally, one has to perform linear algebra on integer matrices of size bounded by $O(n^{\ell+2})$. All previous algorithms for computing the Betti numbers of arrangements, triangulated the whole arrangement giving rise to a complex of size $O(n^{2^k})$ in the worst case. Thus, the complexity of computing the Betti numbers (other than the zero-th one) for these algorithms was $O(n^{2^k})$. To our knowledge this is the first algorithm for computing $\beta_\ell(\cup_{i=1}^n S_i)$ that does not rely on such a global triangulation, and has a graded complexity which depends on ℓ.

Key words: Semi-algebraic Sets, Betti Numbers, Spectral Sequence

1 Introduction

The combinatorial, algebraic and topological analysis of arrangements of real algebraic hyper-surfaces in higher dimensions are active areas of research in computational geometry (see (1; 11; 23)). Arrangements of lines and hyperplanes have been studied quite extensively earlier. It was later realized that

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arrangements of curved surfaces are a significant generalization and have a wider range of applications. There has been substantial progress in analyzing the combinatorial complexity – that is the number of cells (appropriately defined) of various dimensions occurring in the boundary – of substructures in arrangements (23).

However, there is another source of geometric complexity in arrangements of hyper-surfaces – namely topological complexity. Arrangements of hyper-surfaces are distinguished from arrangements of hyperplanes by the fact that arrangements of hyper-surfaces are topologically more complicated than arrangements of hyperplanes. For instance a single hyper-surface or intersections of two or more hyper-surfaces, can have non-vanishing higher homology groups and thus sets defined in terms of such hyper-surfaces can be topologically more complicated in various non-intuitive ways. It is often necessary to estimate the topological complexity of arrangements (9) and sometimes these estimates even play a role in bounding the combinatorial complexity (see (3)).

An important measure of the topological complexity of a set S are the Betti numbers $\beta_i(S)$. Here and elsewhere in the paper the set S will always be semi-algebraic, (that is defined in terms of a finite number of real polynomial equalities and inequalities) and closed and $\beta_i(S)$ will denote the rank of the $H^i(S)$ (the i-th singular cohomology group with real coefficients). Intuitively, $\beta_i(S)$ measures the number of i-dimensional holes in S. The zero-th Betti number $\beta_0(S)$ is the number of connected components.

For example, if T is topologically a hollow torus, then $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1, \beta_i(T) = 0, i > 2$, confirming our intuition that the torus has two 1-dimensional holes and one 2-dimensional hole. Analogously, for the two dimensional sphere, S, $\beta_0(S) = 1, \beta_1(S) = 0, \beta_2(S) = 1, \beta_i(S) = 0, i > 2$.

1.1 Brief History

The basic result in bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovsky (19), Thom (24) and Milnor (16).

They proved:

Theorem 1 *(19; 24; 16) Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\sum_i \beta_i(S) = O(nd)^k.$$

However, the proof techniques used to prove theorem 1 does not allow us to prove bounds on the individual Betti numbers separately. Nor do they suggest an efficient algorithm for actually computing the Betti numbers.

The first bounds on the individual Betti numbers of semi-algebraic sets were proved in (2). The following two theorems were proved bounding the Betti numbers of semi-algebraic sets obtained as intersections or as unions of sets, each defined by a single polynomial inequality.

Theorem 2 *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

contained in a variety $Z(Q)$ of real dimension k', and

$$deg(Q) \leq d.$$

Then,

$$\beta_i(S) \leq \binom{n}{k'-i} O(d)^k.$$

Theorem 3 *Let $S \subset R^k$ be the set defined by the disjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq \binom{n}{i+1} O(d)^k.$$

Note that, a special case of the above theorem is the situation when S is the union of n sets each defined by a polynomial equation $P_i = 0$. We just replace the equalities by the inequalities, $-P_i^2 \geq 0$.

A crucial new ingredient in the proofs of these bounds is the use of a spectral sequence argument. In this paper, we extend this argument so as to be able to actually compute the Betti numbers of a union of semi-algebraic sets efficiently. We note that similar techniques would also work in the case of intersections.

In many applications in computational geometry one is often interested in understanding the topological complexity of the whole arrangement. For instance, unions of balls in R^3 has been studied by Edelsbrunner (9) from both combinatorial and topological view-point motivated by applications in molecular biology, and efficient algorithms for computing the various Betti numbers of such unions are currently being studied (10). There is also a whole body of mathematical literature studying the topology of arrangements of hyperplanes in complex as well as real spaces (see (20)).

1.2 Computing Topology via Global Triangulations

The standard technique of computing the Betti numbers of an arrangement is to associate a simplicial complex to the arrangement, and compute the simplicial homology groups of this complex (see (22)). Since compact semi-algebraic sets are triangulable (4), there always exists such a simplicial complex (corresponding to that of the triangulation). Thus, in order to compute the Betti numbers of an arrangement of n real algebraic hyper-surfaces in $\mathbb{R}^k$ it suffices to first triangulate the arrangement and then compute the Betti numbers of the corresponding simplicial complex. However, currently the most efficient way known to obtain such a triangulation is via the technique of cylindrical algebraic decomposition (8), and this produces $O(n^{2^k})$ simplices in the worst case. Moreover, if the sets S_i are defined by polynomials of degrees at most d, the size of the triangulation is bounded by $(O(nd))^{2^k}$. However, since the Betti numbers of such an arrangement is bounded by $O(n^k)$, it is reasonable to ask for algorithms whose complexity is bounded by $O(n^k)$. More efficient ways of decomposing arrangements into topological balls have been proposed. In (7), the authors provide a decomposition into $O^*(n^{2k-3})$ cells (see (13) for a recent improvement of this result). However, this decomposition does not produce a cell complex and is therefore not directly useful in computing the Betti numbers of the arrangement.

Also, note that the problem of computing β_0 of an arrangement is easier and efficient algorithms whose complexity is $O(n^{k+1})$ is known for this problem. Moreover, the dependence on the degree is also single exponential in this case (see (6) for the best results in this direction).

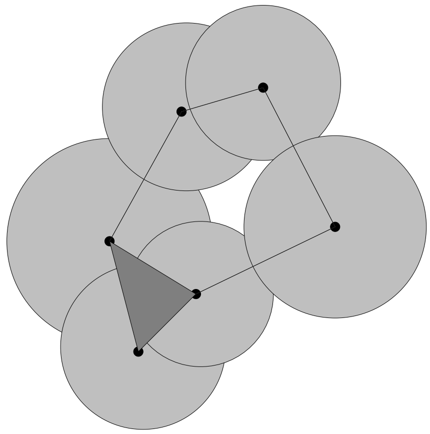


Fig. 1. The nerve complex of a union of disks.

1.3 Local Method

In certain simple situations, it is possible to compute the Betti numbers of an arrangement, without having to compute a triangulation. For instance, when the sets are compact and convex, a classical result of topology, the nerve lemma (21), gives us an efficient way of computing the Betti numbers of the union. The nerve lemma states that the homology groups of such a union is isomorphic to the homology groups of a combinatorially defined simplicial complex, the nerve complex. The nerve complex has n vertices, one vertex for each set in the union, and a simplex for each non-empty intersection among the sets. Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets. The ℓ-th Betti number of the union can then be computed from the $(\ell+1)$-skeleton using linear algebra. (Actually, the nerve lemma requires only that all finite intersections of the sets be topologically trivial and convex sets clearly satisfy this condition.) This technique would work, for instance, if one is interested in computing the Betti numbers of a union of balls in $\mathbb{R}^k$ (see (9)).

When the given sets are not necessarily convex, which would be the case in very many applications, the nerve lemma does not apply. However, even though one cannot directly associate a simplicial complex in general, it is possible to associate a more complicated combinatorial object – namely a *spectral sequence* of vector spaces – which converges to the cohomology groups of the union in finitely many steps. In fact, the nerve lemma is a particular case of this spectral sequence which degenerates in one step to yield the cohomology groups. Using the spectral sequence we will give an algorithm for computing the ℓ-th Betti number of an arrangement of n compact semi-algebraic sets, which needs to triangulate only $O(n^{\ell+2})$ sub-arrangements each of small (constant) size.

Spectral sequences were first introduced by Leray (14; 15) and are familiar objects in algebraic topology, and we refer the reader to (17) for a comprehensive survey. The spectral sequence method was used to prove the first graded bound on the individual Betti numbers of an arrangement of algebraic hyper-surfaces (2). In this paper, we show that using a spectral sequence, it is possible to compute the Betti numbers of an arrangement without having to compute a global triangulation of the whole arrangement. In order to compute the ℓ-th Betti number of the arrangement, it suffices to compute $O(n^{\ell+2})$ triangulations of all sub-arrangements consisting of at most $\ell+2$ of the given sets. Each such triangulation will be of constant size and description complexity and the cost of computing such a triangulation is $O(1)$. Thus, in order to compute all the non-zero Betti numbers, $\beta_0, \ldots, \beta_{k-1}$, we will need to produce $O(n^{k+1})$ different constant sized triangulations.

By complexity of our algorithm we will mean the number of arithmetic operations including comparisons on elements of the ring generated by the coefficients of the input polynomials (those describing the input sets). Thus, we are only counting the cost of computing the different triangulations, and not the cost of performing the linear algebra over $\mathbb{Q}$ in order to determine the Betti numbers. This is because the cost of the algebraic operations usually far outweigh the cost of integer arithmetic. However, we also provide bounds on the cost of doing the linear algebra separately.

We prove the following theorem.

Theorem 4 *Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity and let $S = \cup_{1 \leq i \leq n} S_i$, and $0 \leq \ell \leq k-1$. Then, there is an algorithm to compute $\beta_0(S), \ldots, \beta_\ell(S)$, whose complexity is $O(n^{\ell+2})$.*

The idea of using filtrations for computing Betti numbers has been used in (10) for incremental algorithms for computing the homology groups of certain complexes in low dimensions. However, our techniques in this paper are quite different.

Also note that, efficient decomposition of an arrangement of n algebraic surfaces of constant degree in $\mathbb{R}^k$, into simple cells remains one of the outstanding open problems in computational geometry. Here by simple we mean that the individual cells should be describable in terms of a fixed number of polynomials of fixed degree (independent of n). The dependence on the degrees of the input polynomials is allowed to be doubly exponential in k or even worse. The main conjecture in this area is that there exists such a decomposition of size $O(n^k)$, which is also a bound on the Betti numbers of such an arrangement. Such a decomposition would lead to more efficient algorithms for a host of different problems in computational geometry. Even though in this paper, we do not produce a decomposition of the whole arrangement of size $O(n^k)$, we prove that $O(n^{k+1})$ independent decompositions are enough to compute important topological information about the arrangement (namely the Betti numbers).

Finally, computing the Betti numbers of semi-algebraic sets in single exponential time is a major open question in algorithmic semi-algebraic geometry. The algorithm described in this paper does not answer this question, as we use triangulations whose sizes are doubly exponential in the dimension; their sizes are bounded by $d^{2^{O(k)}}$ where d is a bound on the degrees of the defining polynomials. However, as is usual in computational geometry we will assume that the degree d of the defining polynomials, as well as the dimension k are fixed constants and hence these triangulations are of sizes $O(1)$ for the purposes of this paper.

We assume that the reader is familiar with the notions of simplicial complexes, triangulations of semi-algebraic sets and simplicial co-homology theory. We refer the reader to (18) and (5) for a more detailed exposition of these topics.

The rest of the paper is organized as follows. In section 2, we state a result on triangulations of semi-algebraic sets which we will use heavily in the rest of the paper. We also describe an algebraic subroutine used to compute homomorphisms between the co-chain complexes of two triangulations, one of which is a refinement of the other. In section 3, we define double complexes of vector spaces and associated filtrations giving rise to two spectral sequences associated with any double complex. We state without proving some basic facts about spectral sequences, using (17) as our reference. In section 4, we describe the generalized Mayer-Vietoris exact sequence and the double complex associated to it. Finally, in section 5 we describe our algorithm for computing the Betti numbers of a union of semi-algebraic sets, each of constant description complexity.

2 Semi-algebraic Triangulations

Given a simplicial complex K, we will denote by $C^i(K)$ the $\mathbb{Q}$-vector space of i co-chains of K (that is the dual vector space to the vector space of formal $\mathbb{Q}$-linear combinations of the i-simplices in K), and denote by $C^*(K)$ the direct sum $\oplus_i C^i(K)$.

2.1 Triangulation of semi-algebraic sets

A triangulation of a compact semi-algebraic set S is a simplicial complex Δ together with a semi-algebraic homeomorphism from $|\Delta|$ to S. Given such a triangulation we will often identify the simplices in Δ with their images in S under the given homeomorphism, and will refer to the triangulation by Δ.

Given a triangulation Δ, the cohomology groups $H^i(S)$ are isomorphic to the simplicial cohomology groups $H^i(\Delta)$ of the simplicial complex Δ and are in fact independent of the triangulation Δ.

We call a triangulation $h_1 : |\Delta_1| \rightarrow S$ of a semi-algebraic set S, to be a *refinement* of a triangulation $h_2 : |\Delta_2| \rightarrow S$ if for every simplex $\sigma_1 \in \Delta_1$, there exists a simplex $\sigma_2 \in \Delta_2$ such that $h_1(\sigma_1) \subset h_2(\sigma_2)$.

If Δ_1, Δ_2 are two triangulations of a compact semi-algebraic set S, and Δ_1 is a refinement of Δ_2, then there exists a homomorphism $\lambda : C^*(\Delta_2) \rightarrow C^*(\Delta_1)$, such that the induced map $\lambda^* : H^*(\Delta_2) \rightarrow H^*(\Delta_1)$ is an isomorphism (18).

Identifying the simplices of Δ_i with their images under the homeomorphisms h_i, the homomorphism λ is obtained as follows.

We first choose a simplicial map, $\hat{\lambda} : C_*(\Delta_1) \rightarrow C_*(\Delta_2)$ which is a simplicial approximation to the identity (18). For any vertex $v \in \Delta_1$ we define $\hat{\lambda}(v) = v$ if v is also a vertex of Δ_2, else $\hat{\lambda}(v)$ is chosen to be any vertex of the simplex of Δ_2 containing v in its interior. Finally, for a simplex $\sigma = [v_0, \ldots, v_p]$ of Δ_1, $\hat{\lambda}([v_0, \ldots, v_p]) = [\hat{\lambda}(v_0), \ldots, \hat{\lambda}(v_p)]$, and $\hat{\lambda}$ is extended to $C_*(\Delta_1)$ by linearity. Clearly, $\hat{\lambda}$ is a simplicial approximation to the identity map and hence induces an isomorphism between the homology groups $H_*(\Delta_1)$ and $H_*(\Delta_2)$. Finally, λ is the homomorphism dual to $\hat{\lambda}$.

For future reference, we record the algorithm described above as a subroutine.

	Refinement Subroutine
Input:	Two semi-algebraic triangulations, Δ_1, Δ_2, of a compact semi-algebraic set $S \subset \mathbb{R}^k$, such that Δ_1 is a refinement of Δ_2.
Output:	For each q, $0 \le q \le k$, the matrix for homomorphisms, $\lambda_q : C^q(\Delta_2) \to C^q(\Delta_1)$, such that the induced map $\lambda_q^* : H^q(\Delta_2) \to H^q(\Delta_1)$ is an isomorphism.
Procedure:	Lexicographically order the vertices of Δ_2. For each vertex $v \in \Delta_1$, let $\hat{\lambda}(v) = v$, if v is also a vertex of Δ_2. Else, let $\hat{\lambda}(v)$ be the lexicographically smallest vertex of the simplex of Δ_2 containing v in its interior. For each simplex $\sigma = [v_0, \ldots, v_q]$ of Δ_1, let $\hat{\lambda}([v_0, \ldots, v_q]) = [\hat{\lambda}(v_0), \ldots, \hat{\lambda}(v_q)]$. For each $q, 0 \le q \le k$, compute the matrix M_q of the linear transformation $\hat{\lambda}_q$. Output the matrices, M_q^t, corresponding to the dual homomorphisms.

Let $S_1 \subset S_2$ be two compact semi-algebraic subsets of $\mathbb{R}^k$. We say that a semi-algebraic triangulation $h : |\Delta| \to S_2$ of S_2, *respects* S_1 if for every simplex $\sigma \in \Delta$, $h(\sigma) \cap S_1 = h(\sigma)$ or $\emptyset$. In this case, $h^{-1}(S_1)$ is naturally identified with a sub-complex of Δ and $h|_{h^{-1}(S_1)} : h^{-1}(S_1) \to S_1$ is a semi-algebraic triangulation of S_1. We will refer to this sub-complex by $\Delta|_{S_1}$.

We will need the following theorem which can be deduced from section 9.2 in (4) (see also (5)) and the Refinement Subroutine described above.

Theorem 5 *Let $S_1 \subset S_2 \subset \mathbb{R}^k$ be closed and bounded semi-algebraic sets, and let $h_i : \Delta_i \to S_i, i = 1, 2$ be semi-algebraic triangulations of S_1, S_2. Then, there exists a semi-algebraic triangulation $h : \Delta \to S_2$ of S_2, such that Δ respects S_1, Δ is a refinement of Δ_2, and $\Delta|_{S_1}$ is a refinement of Δ_1.*

Moreover, there exists an algorithm which computes such a triangulation and if the sets S_1, S_2, as well as the triangulations Δ_1, Δ_2 are of constant description complexity, then the triangulation Δ produced by the algorithm is also of constant description complexity.

Also, all the homomorphisms in the following diagram,

$$\begin{array}{ccc} C^*(\Delta_2) & & C^*(\Delta_1) \\ \Big\downarrow \lambda_2 & & \Big\downarrow \lambda_1 \\ C^*(\Delta) & \xrightarrow{r} & C^*(\Delta|_{S_1}) \end{array}$$

are all computable in constant time. (Here the vertical homomorphisms λ_2, λ_1 are as described above and r is induced by restriction.)

Note that, if S_1 and S_2 are each defined by at most s_1 polynomials of degrees at most d_1, and the given triangulations h_1, h_2 are defined in terms of s_2 polynomials of degrees at most d_2, then the complexity of the algorithm in Theorem 5, the size of the triangulation Δ, as well as the number of polynomials defining Δ are all bounded by $(s_1 d_1 + s_2 d_2)^{2^{O(k)}}$. Moreover, the degrees of the polynomials appearing in the definition of Δ are bounded by $(d_1 + d_2)^{2^{O(k)}}$.

3 Double Complexes

In this section, we introduce the basic notions of a double complex of vector spaces and associated spectral sequences.

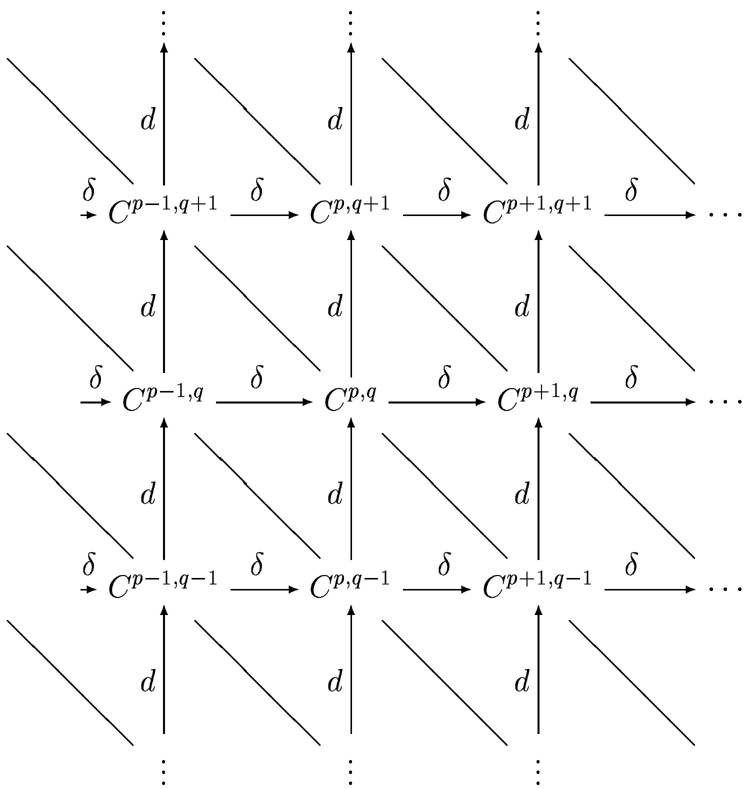
$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\Big\uparrow d & & \Big\uparrow d & & \Big\uparrow d & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
\Big\uparrow d & & \Big\uparrow d & & \Big\uparrow d & & \\
C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \cdots \\
\Big\uparrow d & & \Big\uparrow d & & \Big\uparrow d & & \\
C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \xrightarrow{\delta} & C^{2,0} & \xrightarrow{\delta} & \cdots
\end{array}$$

A *double complex* is a bi-graded vector space,

$$C = \oplus C^{p,q},$$

with co-boundary operators $d : C^{p,q} \to C^{p,q+1}$ and $\delta : C^{p,q} \to C^{p+1,q}$ and such that $d\delta + \delta d = 0$. In our case, the double complex would be a single quadrant double complex, which means that we can assume that $C^{p,q} = 0$ if either $p < 0$ or $q < 0$.

Out of a double complex we can form an ordinary complex of vector spaces, namely the *associated total complex*, which is a graded vector space, defined by $C^n = \oplus_{p+q=n} C^{p,q}$, with co-boundary operator $D = d + \delta : C^n \to C^{n+1}$.



There is a natural decreasing filtration that we can define on the associated total complex, by restricting p to be greater or equal k.

We denote by C^n_k the n-th graded piece of this complex. In other words,

$$C^n_k = \oplus_{p+q=n, p\geq k} C^{p,q}.$$

We denote

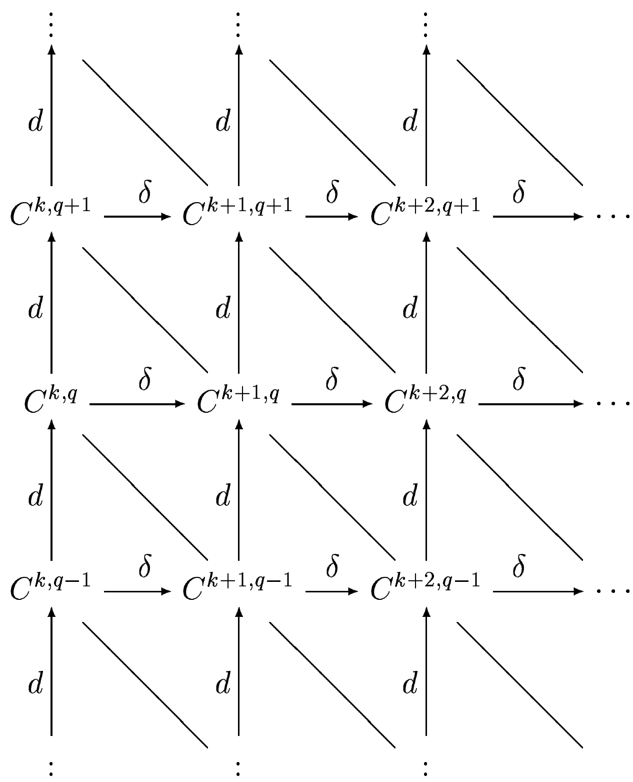
$$Z^n_k = \{z \in C^n_k | Dz = 0\},$$

$$B^n = DC_{n-1},$$

and

$$H_k^n = Z_k^n / Z_k^n \cap B^n.$$

We thus have a decreasing filtration, $\cdots \supset H_{k-1}^n \supset H_k^n \supset H_{k+1}^n \cdots$ of the cohomology group $H_D^n(C)$. We denote the successive quotients H_k^n / H_{k+1}^n by $H^{k,n-k}$.



The Leray spectral sequence is a sequence of complexes (E_r, d_r) such that $E_{r+1} = H_{d_r}(E_r)$.

Any element in $C^n = \oplus_{i+j=n} C^{i,j}$ will have a leading term at a position (p,q), where p denotes the smallest i such that the component at position $(i, n-i)$ does not vanish.

Let $Z^{p,q}$ denote the set of the (p,q) components of co-cycles whose leading term is at position (p', q'), with $p' \geq p$ and $p' + q' = p + q$. In other words, $Z^{p,q}$ denotes the set of all $a \in C^{p,q}$ such that the following system of equations has a solution.

$$da = 0 \tag{1}$$

$$
\begin{aligned}
\delta a &= -da^{(1)} \\
\delta a^{(1)} &= -da^{(2)} \\
\delta a^{(2)} &= -da^{(3)} \\
&\vdots
\end{aligned}
$$

Here, $a^{(i)} \in C^{p+i,q-i}$. Hence, the element $a \oplus a^{(1)} \oplus a^{(2)} \cdots$ lies in Z_p^{p+q} with $a \in Z^{p,q}$.

Similarly, let $B^{p,q} \subset C^{p,q}$ consist of all b with the property that the following system of equations admits a solution.

$$
\begin{aligned}
db^{(0)} + \delta b^{(-1)} &= b \\
db^{(-1)} + \delta b^{(-2)} &= 0 \\
db^{(-2)} + \delta b^{(-3)} &= 0 \\
&\vdots
\end{aligned}
\tag{2}
$$

Here, $b^{(-i)} \in C^{p-i,q+i-1}$.

It is easy to see that, $H^{p,q} \cong Z^{p,q}/B^{p,q}$.

Now, let

$$
\begin{aligned}
Z_r^{p,q} = \{a \in C^{p,q} | \exists (a^{(1)}, \ldots, a^{(r-1)}) | (a, a^{(1)}, \ldots, a^{(r-1)}) \\
\text{satisfies equations (1)}\}.
\end{aligned}
$$

Also, let

$$
\begin{aligned}
B_r^{p,q} = \{b \in C^{p,q} | \exists (b^{(0)}, b^{(-1)}, \ldots) | (b, b^{(0)}, b^{(-1)}, \ldots) \\
\text{satisfies equations (2)with } b^{-r} = b^{-r+1} = \cdots = 0\}.
\end{aligned}
$$

We thus have a sequence of vector subpaces of $C^{p,q}$, satisfying

$$
B_1^{p,q} \subset B_2^{p,q} \subset \cdots \subset B^{p,q} \subset Z^{p,q} \subset Z_1^{p,q} \subset \cdots \subset C^{p,q}.
$$

The (p,q)-th graded piece, $E_r^{p,q}$, of the r-th element, E_r, of the spectral sequence is defined by $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. It should be seen as an approximation to $H^{p,q} = Z^{p,q}/B^{p,q}$.

We will now define the differentials d_r. Let $[a] \in E_r^{p,q}$ for some $a \in Z_r^{p,q}$. Then, there exists $a^{(1)}, \ldots, a^{(r-1)}$ satisfying equations 1. It is a fact (see (17)) that the homomorphism, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, defined by

$$d_r[a] = [\delta a^{(r-1)}] \in E_r^{p+r,q-r+1}, \tag{3}$$

is well-defined (that is independent of the choice of the representative a).

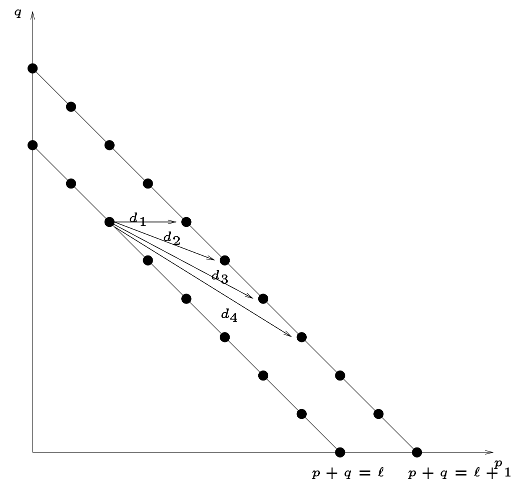


Fig. 2. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$

The sequence of graded complexes, (E_r, d_r), where the complex E_{r+1} is obtained from E_r by taking its homology with respect to d_r (that is $E_{r+1} = H_{d_r}(E_r)$) is called the *spectral sequence* associated to the double complex $C^{p,q}$ (with respect to the horizontal filtration).

Observe that the homomorphism d_r takes $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, and hence if the double complex $C^{p,q}$ is non-zero in only the first quadrant, then $d_r = 0$ for all $r > q + 1$. Thus, $E_{q+2}^{p,q} = E_\infty^{p,q}$ for such complexes.

4 The Mayer-Vietoris Double Complex

In this section, we describe the double complex of interest to us – namely, the one arising from the Mayer-Vietoris exact sequence.

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Note that the intersections of any number of the subcomplexes, A_i, is again a subcomplex of A. We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the sub-complex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

Let $C^i(A)$ denote the $\mathbb{Q}$-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.

We will denote by $d : C^q(A) \rightarrow C^{q+1}(A)$ the usual co-boundary homomorphism. More precisely, given $\omega \in C^q(A)$, and a $q+1$ simplex $[a_0, \ldots, a_{q+1}] \in A$,

$$d\omega([a_0, \ldots, a_{q+1}]) = \sum_{0 \leq i \leq q+1} (-1)^i \omega([a_0, \ldots, \hat{a}_i, \ldots, a_{q+1}]) \tag{4}$$

(here and everywhere else in the paper ˆ denotes omission). Now extend $d\omega$ to a linear form on all of $C_{q+1}(A)$ by linearity, to obtain an element of $C^{q+1}(A)$.

Recall that a sequence of vector space homomorphisms

$$\cdots \xrightarrow{d_{i-1}} V_i \xrightarrow{d_i} V_{i+1} \xrightarrow{d_{i+1}} \cdots$$

is said to be exact if $\ker(d_i) = \mathrm{Im}(d_{i-1})$ for each i.

The Mayer-Vietoris exact sequence is an exact sequence of vector spaces, each of the form $\oplus_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p})$. (Here and everywhere else in the paper $\oplus$ denotes the direct sum of vector spaces). The connecting homomorphisms are "generalized" restrictions and will be defined below.

Consider the following sequence of homomorphisms.

$$0 \longrightarrow C^*(A) \stackrel{r}{\longrightarrow} \oplus_{\alpha_0} C^*(A_{\alpha_0}) \stackrel{\delta}{\longrightarrow} \oplus_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1}) \stackrel{\delta}{\longrightarrow} \cdots \stackrel{\delta}{\longrightarrow}$$

$$\oplus_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \stackrel{\delta}{\longrightarrow} \oplus_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0, \ldots, \alpha_{p+1}}) \stackrel{\delta}{\longrightarrow} \cdots$$

where r is induced by restriction and the connecting homomorphisms δ are defined below.

Given an $\omega \in \oplus_{\alpha_0 < \cdots < \alpha_p} C^q(A_{\alpha_0, \ldots, \alpha_p})$ we define $\delta(\omega)$ as follows: First note that $\delta(\omega) \in \oplus_{\alpha_0 < \cdots < \alpha_{p+1}} C^q(A_{\alpha_0, \ldots, \alpha_{p+1}})$, and it suffices to define $\delta(\omega)_{\alpha_0, \ldots, \alpha_{p+1}}$ for each $p+2$-tuple $0 \leq \alpha_0 < \cdots < \alpha_{p+1} \leq n$. Note that, $\delta(\omega)_{\alpha_0, \ldots, \alpha_{p+1}}$ is a linear form on the vector space, $C_q(A_{\alpha_0, \ldots, \alpha_{p+1}})$, and hence is determined by its values on the q-simplices in the complex $A_{\alpha_0, \ldots, \alpha_{p+1}}$. Furthermore, each q-simplex, $s \in A_{\alpha_0, \ldots, \alpha_{p+1}}$ is automatically a simplex in each of the complexes $A_{\alpha_0, \ldots, \hat{\alpha}_i, \ldots \alpha_{p+1}}$, $0 \leq i \leq p+1$.

We define,

$$(\delta\omega)_{\alpha_0, \ldots, \alpha_{p+1}}(s) = \sum_{0 \leq i \leq p+1} (-1)^i \omega_{\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1}}(s).$$

It was shown in (2) that

Lemma 1 *The sequence defined above is exact.*

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \oplus_{1 \leq \alpha_0 < \cdots < \alpha_p \leq n} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \oplus_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \oplus_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \oplus_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \oplus_{\alpha_0} C^0(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} C^0(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} C^0(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Interchanging the roles of the horizontal and the vertical differentials, we obtain two spectral sequences, (E'_r, d'_r), (E_r, d_r), (corresponding to the vertical and horizontal filtrations respectively) associated with $\mathcal{M}^{p,q}$, both converging to $H^*_D(\mathcal{M})$. The first terms of these are $E'_1 = H_\delta \mathcal{M}, E'_2 = H_d H_\delta \mathcal{M}$, and $E_1 = H_d \mathcal{M}, E_2 = H_\delta H_d \mathcal{M}$. Because of the exactness of the generalized Mayer-Vietoris sequence, we have that, E'_1 is:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & C^3(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & C^2(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & C^1(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & C^0(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

and E'_2 is:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & 0\Big\uparrow & & 0\Big\uparrow & & 0\Big\uparrow & & \\
0 & \longrightarrow & H^3(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & 0\Big\uparrow & & 0\Big\uparrow & & 0\Big\uparrow & & \\
0 & \longrightarrow & H^2(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & 0\Big\uparrow & & 0\Big\uparrow & & 0\Big\uparrow & & \\
0 & \longrightarrow & H^1(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & 0\Big\uparrow & & 0\Big\uparrow & & 0\Big\uparrow & & \\
0 & \longrightarrow & H^0(A) & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & 0\Big\uparrow & & 0\Big\uparrow & & 0\Big\uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The degeneration of this sequence at E_2' shows that $H_D^*(\mathcal{M}) \cong H^*(A)$.

The initial term E_1 of the second spectral sequence is given by:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
0 & \longrightarrow & \oplus_{\alpha_0} H^3(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
0 & \longrightarrow & \oplus_{\alpha_0} H^2(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
0 & \longrightarrow & \oplus_{\alpha_0} H^1(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
0 & \longrightarrow & \oplus_{\alpha_0} H^0(A_{\alpha_0}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \oplus_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \cdots \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

5 Computing the Betti numbers

We define a new double complex $\bar{\mathcal{M}}$ and compute the terms $\bar{E}^{p,q}$ of its spectral sequence. We then show that the spectral sequences $E^{p,q}$ and $\bar{E}^{p,q}$ are isomorphic.

Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity. For $J \subset \{1, \ldots, n\}$, we denote by T_J (respectively S_J) the set $\cup_{j \in J} S_j$ (respectively $\cap_{j \in J} S_j$) .

For each j, $0 \leq j \leq \ell + 1$, and for each $(j+1)$-tuple $(i_0, \ldots, i_j)$ with $1 \leq i_0 < i_1 < \cdots < i_j \leq n$, we compute a semi-algebraic triangulation $\Delta^j_{i_0,\ldots,i_j}$ of the set $T_{\{i_0,\ldots,i_j\}} = S_{i_0} \cup \ldots \cup S_{i_j}$, such that for each $J \subset \{i_0, \ldots, i_j\}$, $J \neq \emptyset$, $\Delta^j_{i_0,\ldots,i_j}$ respects T_J and $\Delta^j_{i_0,\ldots,i_j}|_{T_J}$ is a refinement of the triangulation $\Delta^{|J|-1}_J$.

We denote by λ the homomorphisms,

$$\lambda : C^*(\Delta_J^{|J|-1}|_{S_J}) \to C^*(\Delta^j_{i_0,\ldots,i_j}|_{S_J}).$$

Let Δ be a semi-algebraic triangulation (which we do not compute) of $S = \cup_{1\leq i\leq n} S_i$, which respects the sets $T_{\{i_0,\ldots,i_j\}}$ and such that $\Delta|_{T_{\{i_0,\ldots,i_j\}}}$ is a refinement of $\Delta^j_{i_0,\ldots,i_j}$.

Let A be the simplicial complex corresponding to the triangulation Δ, and $A_1, \ldots, A_n$ the sub-complexes corresponding to the sets $S_1, \ldots, S_n$. Since the triangulation Δ respects the sets S_i, it is clear that each set S_i gives rise to a sub-complex of A. Note that the intersections of any number of the sub-complexes, A_i, is again a subcomplex of A. We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the sub-complex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$. Let $\mathcal{M}$ be the Mayer-Vietoris double complex corresponding to $A_1, \ldots, A_n$ (see Section 4) and E the spectral sequence associated to it corresponding to the horizontal filtration.

We now define the double complex $\bar{\mathcal{M}}$. Let $\bar{A}_{i_0,\ldots,i_p}$ be the simplicial complex corresponding to the triangulation, $\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}}$. We define the bigraded double complex $\bar{\mathcal{M}}^{p,q}$, with a total differential $\bar{D} = \bar{\delta} + (-1)^p \bar{d}$ as follows:

$$\bar{\mathcal{M}}^{p,q} = \oplus_{1\leq \alpha_0 < \cdots < \alpha_p \leq n} C^q(\bar{A}_{\alpha_0,\ldots,\alpha_p}).$$

The vertical differentials $\bar{d}$ are direct sums of the usual co-boundary homomorphisms:

$$\bar{d} : C^q(\bar{A}_{i_0,\ldots,i_p}) \to C^{q+1}(\bar{A}_{i_0,\ldots,i_p}).$$

For $\bar{a} \in \oplus_{1\leq i_0 < \cdots < i_p \leq n} C^q(\bar{A}_{i_0,\ldots,i_p})$, let $\bar{a}_{i_0,\ldots,i_p}$ be the component of $\bar{a}$ in $C^q(\bar{A}_{i_0,\ldots,i_p})$ and let $\bar{a}'_{i_0,\ldots,i_p;i_{p+1}} = \lambda(a_{i_0,\ldots,i_p}) \in C^q(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_p}})$.

Let, $r(\bar{a}'_{i_0,\ldots,i_p;i_{p+1}}) \in C^q(\bar{A}_{i_0,\ldots,i_{p+1}})$ be the image of $\bar{a}'_{i_0,\ldots,i_p;i_{p+1}}$ under the restriction homomorphism.

Let

$$\eta_{i_0,\ldots,i_p;i_{p+1}} : C^q(\bar{A}_{i_0,\ldots,i_p}) \to C^q(\bar{A}_{i_0,\ldots,i_{p+1}})$$

denote the homomorphism which takes

$$\bar{a}_{i_0,\ldots,i_p} \mapsto r(\bar{a}'_{i_0,\ldots,i_p;i_{p+1}}).$$

Then,

$$\bar{\delta}(\bar{a})_{i_0,\ldots,i_{p+1}} = \sum_{0 \le j \le p+1} (-1)^j \eta_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1};i_j}(\bar{a}_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}).$$

We have the following commutative diagram:

$$\begin{array}{ccc}
C^*(A_{i_0,\ldots,i_p}) & \xrightarrow{\ r\ } & C^*(A_{i_0,\ldots,i_p,i_{p+1}}) \\
\uparrow \lambda & & \uparrow \lambda \\
C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_p}}) & \xrightarrow{\ r\ } & C^*(\bar{A}_{i_0,\ldots,i_{p+1}}) \\
\uparrow \lambda & & \uparrow Id \\
C^*(\bar{A}_{i_0,\ldots,i_p}) & & C^*(\bar{A}_{i_0,\ldots,i_p,i_{p+1}})
\end{array}$$

Moreover, the homomorphisms λ can be chosen such that for all $0 < i_0 < \cdots < i_p \le n$ and $i_{p+1}, i'_{p+1} \notin \{i_0, \ldots, i_p\}$ the following diagram is commutative.

$$\begin{array}{ccc}
C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i'_{p+1}}|_{S_{i_0,\ldots,i_p}}) & \xrightarrow{\ \lambda\ } & C^*(A_{i_0,\ldots,i_p}) \\
\uparrow \lambda & & \uparrow \lambda \\
C^*(\bar{A}_{i_0,\ldots,i_p}) & \xrightarrow{\ \lambda\ } & C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_p}})
\end{array}$$

The commutativity of the above diagram implies that the homomorphisms λ induce a homomorphism between the spectral sequences E_r and $\bar{E}_r$. Moreover, it is clear that the induced homomorphisms

$$\bar{E}_1^{p,q} \to E_1^{p,q} \cong \oplus_{1 \le i_0 < \cdots < i_p \le n} H^q(S_{i_0,\ldots,i_p}),$$

are isomorphisms. It follows that (see (17), page 66, Theorem 3.4) $E_r^{p,q}$ and $\bar{E}_r^{p,q}$ are isomorphic for all $r \ge 1$.

We next show how to compute the ranks of the different $\bar{E}_2^{p,q}$. We compute a basis for the vector spaces $\bar{Z}_2^{p,q}$, $\bar{B}_2^{p,q}$ as follows.

The vector space $\bar{Z}_2^{p,q}$ is the set of all $\bar{a} \in \oplus_{1 \leq i_0 < \cdots < i_p \leq n} C^q(\bar{A}_{i_0,\ldots,i_p})$ such that the following system of equations has a solution.

$$
\begin{aligned}
\bar{d}\bar{a} &= 0, \\
\bar{\delta}(\bar{a}) &= -\bar{d}\bar{a}^{(1)}.
\end{aligned}
$$

Similarly, $\bar{B}_2^{p,q}$ is the set of all $\bar{a} \in \oplus_{i_0,\ldots,i_p} C^q(\bar{A}_{i_0,\ldots,i_p})$ such that the following system of equations has a solution.

$$
\begin{aligned}
\bar{d}\,\bar{b}^{(0)} + \bar{\delta}(\bar{b}^{(-1)}) &= \bar{b}, \\
\bar{b}^{(-i)} &= 0, \; i \geq 2.
\end{aligned}
$$

We compute the ranks of the vector spaces $\bar{Z}_2^{p,q}$ and $\bar{B}_2^{p,q}$ using Gaussian elimination on the matrices corresponding to the above systems of equations, and $\mathrm{rank}(\bar{E}_2^{p,q}) = \mathrm{rank}(\bar{Z}_2^{p,q}) - \mathrm{rank}(\bar{B}_2^{p,q})$.

Similarly, we can compute $\bar{E}_r$ for $1 \leq r \leq \ell + 2$, by computing the sequence $\bar{a}, \bar{a}^{(1)}, \bar{a}^{(2)}, \ldots, \bar{a}^{(r-1)}$, as well as the sequence $\bar{b}, \bar{b}^{(0)}, \bar{b}^{(-1)}, \ldots, \bar{b}^{(-r+1)}$.

The crucial observation here is that $\bar{E}_r^{p,q}$ can be computed just from the data of the various local triangulations, $\Delta_{i_0,\ldots,i_{p+r-1}}^{p+r-1}$.

The rank of ℓ-th cohomology group of S is equal to

$$
\sum_{p+q=\ell} \mathrm{rank}(E_{q+2}^{p,q}) = \sum_{p+q=\ell} \mathrm{rank}(\bar{E}_{q+2}^{p,q}).
$$

Thus, in order to compute $\beta_\ell(S)$ it suffices to compute the ranks of $\bar{E}_{\ell-p+2}^{p,\ell-p}, 0 \leq p \leq \ell$, and hence we do not have to consider intersections of more than $\ell + 2$ sets at a time.

We now give a formal description of the algorithm.

	Algorithm Betti
Input:	A number $\ell, 0 \leq \ell \leq k-1$, and the various triangulations, $\Delta^p_{i_0,\ldots,i_p}$, $0 \leq p \leq \ell+1$.
Output:	$\beta_\ell(S)$.
Procedure:	For each $p, 0 \leq p \leq \ell$, let $q = \ell - p$ and $r = \ell - p + 2$.

Compute bases for vector subspaces $\bar{Z}^{p,q}_r, \bar{B}^{p,q}_r$ of $\oplus_{i_0,\ldots,i_p} C^q(\bar{A}_{i_0,\ldots,i_p})$ as follows.

The vector space $\bar{Z}^{p,q}_r$ is the set of all $\bar{a} \in \oplus_{i_0,\ldots,i_p} C^q(\bar{A}_{i_0,\ldots,i_p})$ such that the following system of equations has a solution.

$$\begin{aligned} \bar{d}\bar{a} &= 0 \\ \bar{\delta}(\bar{a}) &= -\bar{d}\bar{a}^{(1)} \\ &\vdots \\ \bar{\delta}(\bar{a}^{(r)}) &= -\bar{d}\bar{a}^{(r-1)} \end{aligned}$$

Similarly, $\bar{B}^{p,q}_r$ is the set of all $\bar{b} \in \oplus_{i_0,\ldots,i_p} C^q(\bar{A}_{i_0,\ldots,i_p})$ such that the following system of equations has a solution.

$$\begin{aligned} \bar{d}\,\bar{b}^{(0)} + \bar{\delta}(\bar{b}^{(-1)}) &= \bar{b} \\ &\vdots \\ \bar{d}\,\bar{b}^{(-r+2)} + \bar{\delta}(\bar{b}^{(-r+1)}) &= 0 \\ \bar{b}^{(-i)} &= 0, \ i \geq r. \end{aligned}$$

Compute the ranks of the vector spaces $\bar{Z}^{p,q}_r$ and $\bar{B}^{p,q}_r$ using Gaussian elimination and output

$$\sum_{0 \leq p \leq \ell} (\text{rank}(\bar{Z}^{p,\ell-p}_{\ell-p+2}) - \text{rank}(\bar{B}^{p,\ell-p}_{\ell-p+2})).$$

5.1 Complexity

We only count the number of arithmetic operations in the ring of the coefficients. Since each triangulation takes $O(1)$ time, it suffices to count the number of distinct triangulations we need to compute the ranks of $\bar{E}^{p,\ell-p}_{\ell-p+2}$.

The triangulations needed are computed at the beginning, and there are $\sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ of these. Thus, the total number of triangulations needed is $O(n^{\ell+2})$.

The dimensions of the matrices whose ranks are computed in the computation of $\bar{E}^{p,\ell-p}_{\ell-p+2}$ are bounded by $O(n^{p+1} + n^{p+2} + \cdots + n^{p+1+\ell-p+1})$. Thus, the Gaussian elimination is applied to matrices of size at most $O(n^{\ell+2})$.

Also, note that if each of the sets S_i are defined by at most s polynomials of degree at most d, then the complexity of computing each of the different triangulations $\Delta^j_{i_0,\ldots,i_j}$ is bounded by $(sd)^{2^{O(k^2)}}$. This is a consequence of the complexity estimate following Theorem 5. Thus, the constant hidden in the big-Oh notation has a dependence on s, d of the form $(sd)^{2^{O(k^2)}}$.

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