# Complexity of Computing Semi-algebraic Descriptions of the Connected Components of a Semi-algebraic Set

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**Abstract**

Given $Q \in \mathbf{R}[X_1, \ldots, X_k]$ with $deg(Q) \leq d$, we give an algorithm that outputs a semi-algebraic description for each of the semi-algebraically connected components of $Z(Q) \subset \mathbf{R}^k$. The complexity of the algorithm as well as the size of the output are bounded by $d^{O(k^3)}$.

More generally, given any semi-algebraic set $S$ defined by a quantifier-free formula involving a family of polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbf{R}[X_1, \ldots, X_k]$ whose degrees are at most $d$, we give an algorithm that outputs a semi-algebraic description for each of the semi-algebraically connected components of $S$. The complexity of the algorithm as well as the size of the output is bounded by $s^{k+1}d^{O(k^3)}$. This improves the previously best known bound of $(sd)^{k^{O(1)}}$ for this problem due to Canny, Grigor'ev, Vorobjov and Heintz, Roy and Solernò [9, 14].

## 1 Introduction

Let $\mathbf{R}$ be a real closed field. A semi-algebraic set in $\mathbf{R}^k$ is the set of points which satisfy a boolean combination of polynomial inequalities. The semi-algebraically connected components of a semi-algebraic set $S$ are themselves semi-algebraic [6].

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The problem we consider here is how to efficiently compute a semi-algebraic description of the semi-algebraically connected components of a semi-algebraic set. To fix notation, we let $S$ be a semi-algebraic set defined by a quantifier-free formula involving a family of polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbf{R}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq d$, for $1 \leq i \leq s$. All computations take place in the ring $\mathbf{D}$ generated by the coefficients of the polynomials in $\mathcal{P}$. By the *complexity* of our algorithms we mean the number of arithmetic operations (additions, multiplications and sign determinations) in the ring $\mathbf{D}$. Note that this ignores the cost of reading the input (see [2], section 1.3, page 1004, for further details).

For the past twenty years many researchers have sought and found algorithms to compute various topological properties of semi-algebraic sets. Based on Collins' Cylindrical Algebraic Decomposition (CAD) [10] it is possible to compute the semi-algebraically connected components, a decomposition of a semi-algebraic set into a finite number of smooth pieces (stratification) as well as the homology groups of $S$ in time polynomial in $s$ and $d$ and doubly exponential in $k$ (see [16, 11]).

In the past decade, single exponential bounds have been found for a number of topological invariants as well as for the complexity of algorithms to compute topological information about semi-algebraic sets. In particular, there are two algorithms in the literature which compute a semi-algebraic description of the semi-algebraically connected components of $S$. These are due to Canny, Grigor'ev, Vorobjov and Heintz, Roy, Solernò [9, 14].The complexity of their algorithms is $(sd)^{k^{O(1)}}$.

It is the purpose of this extended abstract to describe an algorithm with complexity $s^{k+1}d^{O(k^3)}$ for the same problem.

An important notion, described below, for the algorithmic study of connectivity properties of semi-algebraic sets which has made it possible to obtain algorithms with single exponential complexity is the notion of a roadmap which was introduced by Canny and studied by many others [7, 8, 13, 15, 12, 1, 3, 5].

A *roadmap* [7] of $S$ is a semi-algebraic set $R(S) \subset S$ which has dimension at most one and satisfies;

RM1 For every semi-algebraically connected component $C$ of $S$, $C \cap R(S)$ is nonempty and semi-algebraically connected,

RM2 for every $p \in R$ and $C$ a semi-algebraically connected component of (we use $\pi$ to denote the projection of $\mathbf{R}^k$ onto the first co-ordinate).

In [3, 4], we have presented algorithms to compute the road map of $S$ in time

$$\binom{O(s)}{k} s d^{O(k^2)}.$$

In the same papers, we have presented a *connecting* subroutine which takes as input any point of $S$ and outputs a semi-algebraic path in $S$ connecting the input point to the roadmap of $S$.

### 1.1 Our Result

We prove the following theorems.

**Theorem 1** *If $Z(Q)$ is an algebraic set defined by a polynomial $Q \in \mathbf{D}[X_1, \ldots, X_k]$ with $\deg(Q) \leq d$ then there is an algorithm that outputs semi-algebraic descriptions of all the semi-algebraically connected components of $Z(Q)$. The complexity of the algorithm is bounded by $d^{O(k^3)}$ and the degrees of the polynomials that appear in the output are bounded by $d^{O(k^2)}$. Moreover, if the input polynomial has integer coefficients whose bit length is bounded by $\tau$ then the bit length of the coefficients output is $d^{O(k^2)}\tau$ and the total number of bit operations is bounded by $d^{O(k^3)}\tau^{O(1)}$.*

More generally, we prove:

**Theorem 2** *If $S$ is a semi-algebraic set defined by a quantifier-free formula involving a family of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbf{D}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq d$ for $1 \leq i \leq s$ then there is an algorithm that outputs semi-algebraic descriptions of all the semi-algebraically connected components of $S$. The complexity of the algorithm is bounded by $s^{k+1}d^{O(k^3)}$ and the degrees of the polynomials that appear in the output are bounded by $d^{O(k^2)}$. Moreover, if the input polynomials have integer coefficients whose bit length is bounded by $\tau$ then the bit length of the coefficients output is $d^{O(k^2)}\tau$ and the total number of bit operations is bounded by $s^{k+1}d^{O(k^3)}\tau^{O(1)}$.*

As is often the case in this subject, we in fact prove a uniform version of the theorem. Namely, we prove the following, which implies Theorem 2.

**Theorem 3** *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbf{D}[X_1, \ldots, X_k]$ with $deg(P_i) \leq d$ for $1 \leq i \leq s$. Then there exists an algorithm that outputs semi-algebraic descriptions of all the semi-algebraically connected components of every realizable sign condition of the family $\mathcal{P}$. The complexity of the algorithm is bounded by $s^{k+1}d^{O(k^3)}$. The degrees of the polynomials that appear in the output are bounded by $d^{O(k^2)}$. Moreover, if the input polynomials have integer coefficients whose bit length is bounded by $\tau$ the bit length of the coefficients output is $d^{O(k^2)}\tau$ and the total number of bit operations is bounded by $s^{k+1}d^{O(k^3)}\tau^{O(1)}$.*

The proof of this theorem is based on a parametrized version of the roadmap algorithm of an algebraic set and of the connecting routine as presented in [3, 5]. We do not need the roadmap algorithm for an arbitrary semi-algebraic set. As remarked on in [3, 5], the reason for the control on the bit length of the output and the arithmetic complexity in the case that the input polynomials have integer coefficients is due to the fact that all our techniques are based on linear algebra.

There is something a bit peculiar about the output of our algorithm. Each semi-algebraically connected component is described as a union of basic semi-algebraic sets (i.e. as a conjunction of polynomial equalities and inequalities). Many of these basic semi-algebraic sets may very well be empty. We can describe each semi-algebraically connected component as a union of *non-empty* basic semi-algebraic sets but this algorithm has complexity $s^{2k}d^{O(k^4)}$.

Notice too, that there is a separation between the combinatorial and algebraic part of the complexity. Moreover, the degree bound on the polynomials in the output is independent of the combinatorial parameter $s$.

## 2 Some Definitions and Notations

A *$k$-univariate representation* is a $k+2$-tuple

$$u = (f(T), g_0(T), g_1(T), \ldots, g_k(T)).$$

The point $p = (x_1, x_2, \ldots, x_k)$ in $\mathbf{R}[i]^k$ is *associated to $u$* if $x_i = g_i(t)/g_0(t)$, for $i = 1, \ldots, k$, with $f(t) = 0$.

For $x \in \mathbf{R}$ and $f \in \mathbf{R}[X]$ we write $\sigma_{f,x}$ for

$$(\mathrm{sign}(f(x)), \mathrm{sign}(f'(x)), \ldots, \mathrm{sign}(f^{(i)}(x)),$$

$$\ldots, \mathrm{sign}(f^{(\deg(f))}(x))).$$

It is a consequence of Thom's lemma [6] that if $f(x) = 0$ then $\sigma_{f,x}$ distinguishes $x$ from all the other roots of $f$ and we call $\sigma_{f,x}$ the *Thom encoding* of the root $x$.

A *real $k$-univariate representation* is a pair $(u, \sigma)$ where $u$ is a $k$-univariate representation

$$u = (f(T), g_0(T), g_1(T), \ldots, g_k(T))$$

and $\sigma \in \{-1, 0, 1\}^{deg(f)}$ is the Thom encoding of a root $t_\sigma$ of $f(T)$ in $\mathbf{R}$. The point $p = (x_1, x_2, \ldots, x_k)$ in $\mathbf{R}^k$ is *associated to* $(u, \sigma)$ if $x_i = g_i(t_\sigma)/g_0(t_\sigma)$, for $i = 1, \ldots, k$.

A *parametrized $k$-univariate representation with parameter $Y$* is a $k+2$-tuple

$$u(Y) = (f(Y,T), g_0(Y,T), g_1(Y,T), \ldots, g_k(Y,T)),$$

When we specialize the parameter $Y$ to $y$, we get a $k$-univariate representation.

A *curve segment parametrized along the* $Y$ *axis* is specified by an open interval $D$ which is its domain of definition, a parametrized $k$-univariate representation

$$u(Y) = (f(Y,T), g_0(Y,T), g_1(Y,T), \ldots, g_k(Y,T)),$$

together with a $\sigma \in \{-1,0,1\}^{deg(f)}$ such that for every $y \in D$ there exists a real root $t_\sigma(y)$ of $f(y,T)$ with Thom encoding $\sigma$.

## 3 The case of an algebraic set

The roadmap algorithm in [3, 5] for an algebraic set $Z(Q)$ and a point $y \in Z(Q)$ takes as input a polynomial $Q \in \mathbf{R}[X_1,\ldots,X_k]$ together with a point $y = (y_1,\ldots,y_k) \in Z(Q)$ and outputs $R(Z(Q),y)$, which is a roadmap of $Z(Q)$ passing through $y$ and is stored as a graph $(V,E)$ whose vertices are labeled by the following;

1. a triangular system of equations in $X_1,\ldots,X_\ell$,
2. a collection of Thom encodings which specifies a real solution, $(x_1,\ldots,x_\ell) \in \mathbf{R}^\ell$ of the above triangular system ,
3. a univariate representation
$$(f(x_1,\ldots,x_\ell,T), g_0(x_1,\ldots,x_\ell,T),\ldots,$$
$$g_{\ell+1}(x_1,\ldots,x_\ell,T),\ldots,g_k(x_1,\ldots,x_\ell,T)),$$
where $g_i(x_1,\ldots,x_\ell,T) = x_i g_0(x_1,\ldots,x_\ell,T)$ for $1 \le i \le \ell$,
4. a Thom encoding of a real root of the univariate polynomial, $f(x_1,\ldots,x_\ell,T)$.

Each edge is labeled by a similar system except that it is parametrized along a co-ordinate direction (say $X_{\ell+1}$).

Any graph labeled with polynomial systems and Thom encodings (as described above) is the *type* of a roadmap. Note that the coefficients of the polynomials occurring in a type are allowed to come from any domain and a type need not correspond to an actual roadmap.

Let $Y = (Y_1,\ldots,Y_k)$ represent a parametric point of $Z(Q)$. The roadmap algorithm can be viewed as an algebraic computation tree. Let $\mathcal{L}$ consist of $Q$ together with the collection of polynomials in $\mathbf{D}[Y_1,\ldots,Y_k]$ which might (depending on the values of the parameters) occur at some node of the tree. Hence, for all points $y$ in a sign invariant set of $\mathcal{L}$ for which $Q(y) = 0$, the type of the roadmap $R(Z(Q),y)$ is the same and hence all such points belong to the same semi-algebraically connected component of $Z(Q)$.

A detailed examination of the roadmap algorithm reveals that the number of polynomials along any branch is at most $d^{O(k^2)}$ and their degrees are also at most $d^{O(k^2)}$. Moreover, this is the number of polynomials in $\mathcal{L}$.

We use the sample points subroutine of [2] (which has complexity $S^{k+1}D^{O(k)}$ where $S$ is the number of polynomials in $\mathcal{L}$ and $D$ a bound on their degrees) to compute a set of points meeting every semi-algebraically connected component of every non-empty sign condition on $\mathcal{L}$. For every such point $y$, we output the type of the roadmap $R(Z(Q),\{y\})$ (by computing the roadmap).

For every distinct non-empty sign-condition on $\mathcal{L}$ we use the roadmap algorithm to check whether or not a sample point from each of these conditions are in the same semi-algebraically connected component of $Z(Q)$.

It follows that a semi-algebraically connected component of $Z(Q)$ is described by the disjunction of the corresponding sign conditions.

It is not difficult to see that the complexity of this algorithm is $d^{O(k^3)}$.

This proves Theorem 1.

## 4 Parametrized connecting subroutine

Given a set of polynomials $\mathcal{P}$ we define the *combinatorial level* of the set $\mathcal{P}$ to be the minimum number $\ell$ satisfying:

1. No more than $\ell$ of the polynomials in $\mathcal{P}$ have a common real zero.
2. Any real zero common to $\ell$ polynomials of $\mathcal{P}$ is isolated.

Let $Y$ represent a parametric point of $\mathbf{R}^k$ and assume that the system $\mathcal{P}$ has combinatorial level at most $\ell$. For every $\mathcal{P}' \subset \mathcal{P}$, such that $0 \le |\mathcal{P}'| \le \ell$, we compute a set of pairs,

$$\Phi_{\mathcal{P}'}(Y) = \{(\phi_{\mathcal{P}',1}(Y), R_{\mathcal{P}',1}),\ldots,(\phi_{\mathcal{P}',N_{\mathcal{P}'}}(Y), R_{\mathcal{P}',N_{\mathcal{P}'}}\})$$

where each $\phi_{\mathcal{P}',i}(Y)$ is a first-order formula such that for every $y$ satisfying $\phi_{\mathcal{P}',i}(Y)$ the combinatorial type of the roadmap $R(Z(Q),y)$, is $R_{\mathcal{P}',i}$.

The algorithm we outline in this section is nothing more than a parametrized version of the connecting subroutine in [3, 5] which we now recall. The connecting subroutine takes as input a point $y \in \mathbf{R}^k$. The output is a semi-algebraic path which connects $y$ to some roadmap $R(Z(\mathcal{P}'))$ where $\mathcal{P}' \subset \mathcal{P}$.

Given a sign condition $\sigma \in \{0,1,-1\}^{\mathcal{P}}$ on $\mathcal{P}$, the corresponding *weak sign condition*, $\overline{\sigma}(P)$, is $= 0$ if $\sigma(P) = 0$, is $\ge 0$ if $\sigma(P) = 1$ and is $\le 0$ if $\sigma(P) = -1$. Denote by $\sigma_y$ the sign condition on $\mathcal{P}$ at $y$. Let

$$S_y = \{x \in \mathbf{R}^k \mid \forall\ P \in \mathcal{P},\ \ P(x)\overline{\sigma}_y(P)\}.$$

We say that $\overline{\sigma}_y$ is the weak sign condition defined on $\mathcal{P}$ at $y$ and denote by $\mathcal{P}_y$ the set of polynomials of $\mathcal{P}$ vanishing at $y$.

The connecting path $\Gamma_y$ connecting $y$ to the roadmap has the following properties. It consists of a sequence of semi-algebraic paths $\gamma_{y,1}$ joining $y = y_0$ to $y_1$, $\gamma_{y,2}$ joining $y_1$ to $y_2$, up to $\gamma_{y,j}$ joining $y_{j-1}$ to $y_j$ with $j \le \ell$. Each $\gamma_{y,i}$ is contained in $Z(\mathcal{P}_{y_{i-1}})$, and on $\gamma_{y,i}$ no polynomial of $\mathcal{P} \setminus \mathcal{P}_{y_{i-1}}$ vanishes. It is clear that $\Gamma_y$ is contained in $S_y$.

We first construct a roadmap $R(Z(\mathcal{P}_y)_{\pi(y)}, y)$ (where $\pi$ is the projection on the first coordinate and $Z(\mathcal{P}_y)_{\pi(y)}$ is the intersection of $Z(\mathcal{P}_y)$ with the hyperplane $X_1 = \pi(Y)$). If $y$ is connected to $R(Z(\mathcal{P}_y))$ by a path on which no polynomial of $\mathcal{P} \setminus \mathcal{P}_y$ vanishes the connection is done. Otherwise, let $y_1$ be a point such that there is a semi-algebraically open connected curve $C$ contained in $R(Z(\mathcal{P}_y)_{\pi(y)})$ going from $y$ to $y_1$ such that $\mathcal{P}_{y_1}$ strictly contains $\mathcal{P}_y$ and that no $P \in \mathcal{P} \setminus \mathcal{P}_y$ vanishes on $C$. Replace $y$ by $y_1$ and iterate. There can be at most $\ell$ steps of iteration.

We now redo this process in a parametrized way. Thus, in the connecting subroutine we first compute parametrically the roadmap of the algebraic set corresponding to $\mathcal{P}_Y$ in the slice $X_1 = \pi(Y)$, containing $Y$ as well as the set of intersection points $M_{\mathcal{P}_Y}$, of the roadmap $R(\mathcal{P}_Y)$ with this slice.

We then intersect all the paths of this roadmap rooted at $Y$ with the zero set of all the polynomials in $\mathcal{P}$. Either we are able to reach a point of $M_{\mathcal{P}_Y}$ along one such path, staying inside the weak sign condition defined by $Y$, or we consider the first point of intersection, $Y_1$ of such a path with $Z(\mathcal{P}')$ for some $\mathcal{P}'$ strictly containing $\mathcal{P}_Y$ and we repeat the process replacing $Y$ by $Y_1$ and the $\mathcal{P}_Y$ by the set $\mathcal{P}_{Y_1}$.

Of course, when we do this in a parametrized way we do not know the order in which we meet the polynomials in the iteration described above. Instead, we consider all possible orders. Thus for every permutation of every $j$ tuple ($j \leq \ell$) of polynomials in $\mathcal{P}$, we mimic the computation of the connecting subroutine as if this is the order in which the polynomials occur in the computation.

Once we fix an ordered $j$-tuple of polynomials (say) $P_1, \ldots, P_j$ we consider all the possible types of connecting curve that might occur.

We claim that there exists a family of polynomials in $Y$ of size $d^{O(k^2)}$ and degrees bounded by $d^{O(k^2)}$ such that the signs of these polynomials determine this type. In order to see this consider the connecting subroutine with an oracle which tells you the polynomials which vanish at the intersection points $y_1, y_2, \ldots$, in the at most $\ell$ iterations. The complexity of the connecting subroutine (not counting the number of calls to this oracle) in this case is $d^{O(k^2)}$. Moreover, in the parametrized situation the total number of polynomials in $Y$ which are possibly encountered in the sign determinations during the course of the algorithm is bounded by $d^{O(k^2)}$.

We then use the sample points subroutine to generate a set of points representing every sign condition of this family of polynomials. For each such sample point $y$ we compute the type of the connecting curve. More precisely, we compute the parametrized representations of curves, the points of intersections $y_1, \ldots, y_j$, as well as the various Thom encodings.

The total number of sample points is bounded by $d^{O(k^3)}$. For a sample point $y$ let $\Gamma_y$ be the type of connecting curve. We also get parametrized (by $Y$) descriptions of the points $y_1, \ldots, y_j$.

We now construct a formula which expresses the predicate that no polynomial in the family $\mathcal{P}$ changes sign on any path $\gamma_{y,i}$ other than at their endpoints.

Since each $\gamma_{y,i}$ is a sequence of curves given by parametrized univariate representations of degrees at most $d^{O(k)}$, and the end points defined by polynomials of degrees at most $d^{O(k^2)}$, we can construct such a formula for each of them having complexity bounded by $sd^{O(k^2)}$. Note that each such predicate is a conjunction of $s$ independent predicates expressing that the polynomial $P_j$ maintains its signs at all points of $\gamma_i$ other than its endpoints.

The conjunction of all these formulas is $\phi_y(Y)$ which has size at most $sd^{O(k^2)}$. We now distinguish between the various segments of the roadmap of the algebraic set $Z(P_1^2 + \cdots + P_j^2)$ on which the end point $y_j$ of the connecting path can possibly lie.

For $j < \ell$, the roadmap on the algebraic set $Z(P_1^2 + \cdots + P_j^2)$ is cut up into $sd^{O(k^2)}$ segments by the zero sets of the family of polynomials $\mathcal{P}$, and the end point of the connecting path may lie on any one of these segments. For each such segment we adjoin the corresponding condition to the predicate $\phi_y(Y)$ giving rise to $sd^{O(k^2)}$ different predicates.

For $j = \ell$, the algebraic set $Z(P_1^2 + \cdots + P_j^2)$ consists of at most $d^{O(k)}$ isolated points and we specify which one of them is reached by the connecting curve. Thus, we get $d^{O(k)}$ new predicates in this case.

We now take the union over every permutation of every $j$ tuple of polynomials, for all $j \leq \ell$, and over all the sample points $y$ (described above) of the formulas $\phi_y(Y)$, and over all possible segments of the end points as described above.

Each semi-algebraically connected component of a set defined by a weak sign condition associated to a point of $\mathbf{R}^k$ can now be described by a disjunction of some subset of these formulas. Moreover, we can easily identify these subsets by looking at the portion of the roadmap to which the point $Y$ gets connected.

There are $j!\binom{s}{j}$ different permutations of $j$-tuples of polynomials. For $j < \ell$ we obtain for each of these a set of $sd^{O(k^3)}$ different formulas each of length at most $sd^{O(k^2)}$, and for $j = \ell$, $d^{O(k^3)}$ different formulas each of length at most $sd^{O(k^2)}$.

Thus the complexity of obtaining the semi-algebraic descriptions of all the semi-algebraically connected components as well as the size of the descriptions obtained are bounded by $s^{\ell+1}d^{O(k^3)}$.

The degrees of the polynomials appearing in these descriptions are bounded by $d^{O(k^2)}$.

We have proved the following theorem.

**Theorem 4** *If $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbf{D}[X_1, \ldots, X_k]$ with $deg(P_i) \leq d$ for $1 \leq i \leq s$ has combinatorial level $\ell$, then there exists an algorithm that outputs semi-algebraic descriptions of all the semi-algebraically connected components of every realizable weak sign condition of the family $\mathcal{P}$. The complexity of the algorithm is bounded by $s^{\ell+1}d^{O(k^3)}$. The degrees of the polynomials that appear in the output are bounded by $d^{O(k^2)}$. Moreover, if the input polynomials have integer coefficients whose bit length is bounded by $\tau$ the bit length of the coefficients output is $d^{O(k^2)}\tau$ and the total number of bit operations is bounded by $s^{\ell+1}d^{O(k^3)}\tau^{O(1)}$.*

## 5 Computing the Connected Components of a General Semi-algebraic Set

Let $S$ be defined by a quantifier-free first order formula involving the family of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$.

We use two infinitesimals $\epsilon \gg \epsilon'$ to replace the set $\mathcal{P}$ by a new set of polynomials $\mathcal{P}^*$ which has combinatorial level $k$ (as was done in [5, 4]).

More precisely, $\mathcal{P}^*$ consists of the following $4s$ polynomials:

$$\mathcal{P}^* = \cup_{i=1,\ldots,s}\{(1-\epsilon')P_i - \epsilon H_{4i-3}, (1-\epsilon')P_i + \epsilon H_{4i-2},$$

$$(1-\epsilon')P_i - \epsilon'\epsilon H_{4i-1}, (1-\epsilon')P_i + \epsilon'\epsilon H_{4i}\},$$

where $H_i = (1 + \sum_{1\leq j\leq k} i^j x_j^{d'})$ and $d'$ is an even number greater than the degree of any $P_i$.

According to [5], the set $\mathcal{P}^*$ has combinatorial level $k$ and there is a 1-1 correspondence between the semi-algebraically connected components of realizable sign conditions of $\mathcal{P}$ and the semi-algebraically connected components of a subset $\Sigma$ of the realizable weak sign conditions on $\mathcal{P}^*$.

We now follow the previous algorithm and obtain descriptions of the semi-algebraically connected components of all weak sign conditions in $\Sigma$. By [5] we know that every semi-algebraically connected component (say $C$) of every strict sign condition of the original family $\mathcal{P}$ corresponds to a semi-algebraically connected component (say $C'$) of a weak sign condition on $\mathcal{P}^*$. Moreover, $C = C' \cap \mathbf{R}^k$. Thus, we can now get rid of the infinitesimals using the same technique as in [2] (section 6.2, page 1043).

The complexity of the algorithm is easily seen to be bounded by $s^{k+1}d^{O(k^3)}$.

This proves Theorem 3.

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