

COMPUTING THE BETTI NUMBERS OF SEMI-ALGEBRAIC SETS DEFINED BY PARTLY QUADRATIC SYSTEMS OF POLYNOMIALS

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ABSTRACT. Let \mathbb{R} be a real closed field, $\mathcal{Q} \subset \mathbb{R}[Y_1, \dots, Y_\ell, X_1, \dots, X_k]$, with $\deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in \mathcal{Q}, \#(\mathcal{Q}) = m$, and $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ with $\deg_X(P) \leq d, P \in \mathcal{P}, \#(\mathcal{P}) = s$. Let $S \subset \mathbb{R}^{\ell+k}$ be a semi-algebraic set defined by a Boolean formula without negations, with atoms $P = 0, P \geq 0, P \leq 0, P \in \mathcal{P} \cup \mathcal{Q}$. We describe an algorithm for computing the the Betti numbers of S generalizing a similar algorithm described in [6]. The complexity of the algorithm is bounded by $(\ell s m d)^{2^{O(m+k)}}$. The complexity of the algorithm interpolates between the doubly exponential time bounds for the known algorithms in the general case, and the polynomial complexity in case of semi-algebraic sets defined by few quadratic inequalities [6]. Moreover, for fixed m and k this algorithm has polynomial time complexity in the remaining parameters.

1. INTRODUCTION AND MAIN RESULTS

Let \mathbb{R} be a real closed field and $S \subset \mathbb{R}^k$ a semi-algebraic set defined by a Boolean formula with atoms of the form $P > 0, P < 0, P = 0$ for $P \in \mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$. We call S a \mathcal{P} -semi-algebraic set and the Boolean formula defining S a \mathcal{P} -formula. If, instead, the Boolean formula has atoms of the form $P = 0, P \geq 0, P \leq 0, P \in \mathcal{P}$, and additionally contains no negation, then we will call S a \mathcal{P} -closed semi-algebraic set, and the formula defining S a \mathcal{P} -closed formula. Moreover, we call a \mathcal{P} -closed semi-algebraic set S basic if the \mathcal{P} -closed formula defining S is a conjunction of atoms of the form $P = 0, P \geq 0, P \leq 0, P \in \mathcal{P}$.

For any closed semi-algebraic set $X \subset \mathbb{R}^k$, we denote by $b_i(X)$ the dimension of the \mathbb{Q} -vector space, $H_i(X, \mathbb{Q})$, which is the i -th homology group of X with coefficients in \mathbb{Q} . We refer to [10] for the definition of homology in the case of \mathbb{R} being an arbitrary real closed field, not necessarily the field of real numbers.

1.1. Brief History. Designing efficient algorithms of computing the Betti numbers of semi-algebraic sets is an important problem which has been considered by several authors. We give a brief description of the recent advances and direct the reader to the survey article [5] for a more detailed exposition.

For general semi-algebraic sets the best known algorithm for computing all the Betti numbers is via triangulation using cylindrical algebraic decomposition (see

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for example [10]) whose complexity is *doubly exponential* in the number of variables. There have been some small advances in obtaining singly exponential time algorithms for computing some of the Betti numbers, but we are still very far from having an algorithm for computing all the Betti numbers of a given semi-algebraic set in singly exponential time. Singly exponential time algorithms for computing the number of connected components of a semi-algebraic set S (i.e. $b_0(S)$) has been known for quite some time [18, 15, 17, 20, 13]. More recently, an algorithm with singly exponential complexity is given in [11] for computing the first Betti number of semi-algebraic sets. The above result is generalized in [12], where a singly exponential time algorithm is given for computing the first ℓ Betti numbers of semi-algebraic sets, where ℓ is allowed to be any constant. Finally, note that singly exponential time algorithm is also known for computing the Euler-Poincaré characteristic (which is the alternating sum of Betti numbers) of semi-algebraic sets [4].

In another direction, several researchers have considered a special class of semi-algebraic sets – namely, semi-algebraic sets defined using quadratic polynomials. While the topology of such sets can be arbitrarily complicated (since any semi-algebraic set can be defined as the image under a linear projection of a semi-algebraic defined by quadratic inequalities), it is possible to prove bounds on the Betti numbers of such sets which are polynomial in the number of variables and exponential in only the number of inequalities [3, 8]. (In contrast a semi-algebraic set defined by a single polynomial of degree > 2 can have exponentially large Betti numbers.) Polynomial time algorithms for testing emptiness of such sets (where the number of inequalities is fixed) were given in [2, 19]. A polynomial time algorithm (without any restriction on the number of inequalities) is given in [6](see also [7]) for computing a constant number of the top Betti numbers of semi-algebraic sets defined by quadratic inequalities. If moreover the number of inequalities is fixed then the algorithm computes all the Betti numbers in polynomial time. More precisely, an algorithm is described which takes as input a semi-algebraic set, S , defined by $Q_1 \geq 0, \dots, Q_m \geq 0$, where each $Q_i \in \mathbb{R}[Y_1, \dots, X_\ell]$ has degree ≤ 2 , and computes the top p Betti numbers of S , $b_{k-1}(S), \dots, b_{k-p}(S)$, in polynomial time. The complexity of the algorithm is $\sum_{i=0}^{p+2} \binom{m}{i} \ell^{2^{\mathcal{O}(\min(p,m))}}$. For fixed m , we obtain by letting $p = \ell$, an algorithm for computing all the Betti numbers of S whose complexity is $\ell^{2^{\mathcal{O}(m)}}$.

The goal of this paper is to design an algorithm for computing the Betti numbers of semi-algebraic sets defined in terms of *partly quadratic* systems of polynomials whose complexity interpolates between the doubly exponential time bounds for the known algorithms in the general case, and the polynomial complexity in case of semi-algebraic sets defined by few quadratic inequalities. Our algorithm is partly based on techniques developed in [9], where we prove a quantitative result *bounding* the Betti numbers of semi-algebraic sets defined by partly quadratic systems of polynomials. Before stating this result we introduce some notation that we are going to fix for the rest of the paper.

Notation 1. We denote by

- $\mathcal{Q} \subset \mathbb{R}[Y_1, \dots, Y_\ell, X_1, \dots, X_k]$, a family of polynomials with

$$\deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in \mathcal{Q}, \#(\mathcal{Q}) = m,$$

and by

- $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ a family of polynomials with

$$\deg_X(P) \leq d, P \in \mathcal{P}, \#(\mathcal{P}) = s.$$

The following theorem is proved in [9].

Theorem 1.1. *Let $S \subset \mathbb{R}^{\ell+k}$ be a $(\mathcal{P} \cup \mathcal{Q})$ -closed semi-algebraic set. Then*

$$b(S) \leq \ell^2(O(s + \ell + m)\ell d)^{k+2m}.$$

In particular, for $m \leq \ell$, we have $b(S) \leq \ell^2(O(s + \ell)\ell d)^{k+2m}$.

The above theorem interpolates previously known bounds on the Betti numbers of general semi-algebraic sets (which are exponential in the number of variables) [22, 24, 21, 4, 16], and bounds on Betti numbers of semi-algebraic sets defined by quadratic inequalities (which are exponential only in the number of inequalities and polynomial in the number of variables) [3, 8]. Indeed we recover these extreme cases by setting ℓ and m (respectively, s , d and k) to $O(1)$ in the above bound.

1.2. Main Results. The main result of this paper is algorithmic. We describe an algorithm (Algorithm 5 below) for computing all the Betti numbers of a closed semi-algebraic set defined by partly quadratic systems of polynomials. The complexity of this algorithm interpolates the complexity of the best known algorithms for computing the Betti numbers of general semi-algebraic sets on one hand, and those described by quadratic inequalities on the other.

Definition 1.2 (Complexity). By complexity of an algorithm we will mean the number of arithmetic operations (including comparisons) performed by the algorithm in \mathbb{R} . We refer the reader to [10, Chapter 8] for a full discussion about the various measures of complexity.

We prove the following theorem.

Theorem 1.3. *There exists an algorithm that takes as input the description of a $(\mathcal{P} \cup \mathcal{Q})$ -closed semi-algebraic set S (following the same notation as in Theorem 1.1) and outputs its Betti numbers $b_0(S), \dots, b_{\ell+k-1}(S)$. The complexity of this algorithm is bounded by $(\ell s m d)^{2^{O(m+k)}}$.*

The algorithm we describe is an adaptation of the algorithm in [6], to the case where there are parameters, and the degrees with respect to these parameters could be larger than two. In addition, in this paper we also treat the case of general $\mathcal{P} \cup \mathcal{Q}$ -closed sets, not just basic closed ones as was done in [6]. We also provide more details and analyze the complexity of the algorithm more carefully, in order to take into account the dependence on the additional parameters.

1.3. Significance from the point of view of computational complexity theory. The problem of computing the Betti numbers of semi-algebraic sets in general is a PSPACE-hard problem. We refer the reader to [6] and the references contained therein, for a detailed discussion of these hardness results. On the other hand, as shown in [6], the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of quadratic inequalities is solvable in polynomial time. This result depends critically on the quadratic dependence of the variables, as witnessed by the fact that the problem of computing the Betti numbers of a real algebraic variety defined by a single quartic equation is also PSPACE-hard.

We show in this paper that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of polynomial inequalities is solvable in polynomial time, even if we allow a small (constant sized) subset of the variables to occur with degrees larger than two in the polynomials defining the given set. Note that such a result is not obtainable directly from the results in [6] by the naive method of replacing the monomials having degrees larger than two by a larger set of quadratic ones (introducing new variables and equations in the process).

The rest of the paper is organized as follows. In Section 2 we describe some mathematical results concerning the topology of sets defined by quadratic inequalities. We often omit proofs if these appear elsewhere and just provide appropriate pointers to literature. In Section 3 we describe our algorithm for computing all the Betti numbers of semi-algebraic sets defined by partly quadratic systems of polynomials and prove its correctness and complexity bounds, thus proving Theorem 1.3.

2. TOPOLOGY OF SETS DEFINED BY PARTLY QUADRATIC SYSTEMS OF POLYNOMIALS

In this section we recall a construction described in [9] that will be important for the algorithm described later. We parametrize a construction introduced by Agrachev in [1] while studying the topology of sets defined by (purely) quadratic inequalities (that is without the parameters X_1, \dots, X_k in our notation). However, we do not make any non-degeneracy assumptions on our polynomials, and we also avoid construction of Leray spectral sequences as done in [1].

We first need to fix some notation.

2.1. Mathematical Preliminaries.

2.1.1. *Some Notation.* For all $a \in R$ we define

$$\begin{aligned} \text{sign}(a) &= 0 \text{ if } a = 0, \\ &= 1 \text{ if } a > 0, \\ &= -1 \text{ if } a < 0. \end{aligned}$$

Let \mathcal{A} be a finite subset of $R[X_1, \dots, X_k]$. A *sign condition* on \mathcal{A} is an element of $\{0, 1, -1\}^{\mathcal{A}}$. The *realization of the sign condition* σ , $\mathcal{R}(\sigma, R^k)$, is the basic semi-algebraic set

$$\{x \in R^k \mid \bigwedge_{P \in \mathcal{A}} \text{sign}(P(x)) = \sigma(P)\}.$$

A *weak sign condition* on \mathcal{A} is an element of $\{\{0\}, \{0, 1\}, \{0, -1\}\}^{\mathcal{A}}$. The *realization of the weak sign condition* ρ , $\mathcal{R}(\rho, R^k)$, is the basic semi-algebraic set

$$\{x \in R^k \mid \bigwedge_{P \in \mathcal{A}} \text{sign}(P(x)) \in \rho(P)\}.$$

We often abbreviate $\mathcal{R}(\sigma, R^k)$ by $\mathcal{R}(\sigma)$, and we denote by $\text{Sign}(\mathcal{A})$ the set of realizable sign conditions $\text{Sign}(\mathcal{A}) = \{\sigma \in \{0, 1, -1\}^{\mathcal{A}} \mid \mathcal{R}(\sigma) \neq \emptyset\}$.

More generally, for any $\mathcal{A} \subset R[X_1, \dots, X_k]$ and a \mathcal{A} -formula Φ , we denote by $\mathcal{R}(\Phi, R^k)$, or simply $\mathcal{R}(\Phi)$, the semi-algebraic set defined by Φ in R^k .

2.1.2. *Use of Infinitesimals.* Later in the paper, we extend the ground field \mathbb{R} by infinitesimal elements. We denote by $\mathbb{R}\langle\zeta\rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in \mathbb{R} (see [10] for more details). The sign of a Puiseux series in $\mathbb{R}\langle\zeta\rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ . This induces a unique order on $\mathbb{R}\langle\zeta\rangle$ which makes ζ infinitesimal: ζ is positive and smaller than any positive element of \mathbb{R} . When $a \in \mathbb{R}\langle\zeta\rangle$ is bounded from above and below by some elements of \mathbb{R} , $\lim_{\zeta}(a)$ is the constant term of a , obtained by substituting 0 for ζ in a . We denote by $\mathbb{R}\langle\zeta_1, \dots, \zeta_n\rangle$ the field $\mathbb{R}\langle\zeta_1\rangle \cdots \langle\zeta_n\rangle$ and in this case ζ_1 is positive and infinitesimally small compared to 1, and for $1 \leq i \leq n-1$, ζ_{i+1} is positive and infinitesimally small compared to ζ_i , which we abbreviate by writing $0 < \zeta_n \ll \cdots \ll \zeta_1 \ll 1$.

Let \mathbb{R}' be a real closed field containing \mathbb{R} . Given a semi-algebraic set S in \mathbb{R}^k , the *extension* of S to \mathbb{R}' , denoted $\text{Ext}(S, \mathbb{R}')$, is the semi-algebraic subset of \mathbb{R}'^k defined by the same quantifier free formula that defines S . The set $\text{Ext}(S, \mathbb{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle (see for instance [10]).

2.2. Homogeneous Case.

Notation 2. We denote by

- \mathcal{Q}^h the family of polynomials obtained by homogenizing \mathcal{Q} with respect to the variables Y , i.e.

$$\mathcal{Q}^h = \{Q^h \mid Q \in \mathcal{Q}\} \subset \mathbb{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k],$$

where $Q^h = Y_0^2 Q(Y_1/Y_0, \dots, Y_\ell/Y_0, X_1, \dots, X_k)$,

- Φ a formula defining a \mathcal{P} -closed semi-algebraic set V ,
- A^h the semi-algebraic set

$$(2.1) \quad A^h = \bigcup_{Q \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\},$$

- W^h the semi-algebraic set

$$(2.2) \quad W^h = \bigcap_{Q \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\}.$$

Let

$$(2.3) \quad \Omega = \{\omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq m\}.$$

Let $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ and $\mathcal{Q}^h = \{Q_1^h, \dots, Q_m^h\}$. For $\omega \in \Omega$ we denote by $\langle\omega, \mathcal{Q}^h\rangle \in \mathbb{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k]$ the polynomial defined by

$$(2.4) \quad \langle\omega, \mathcal{Q}^h\rangle = \sum_{i=1}^m \omega_i Q_i^h.$$

For $(\omega, x) \in \Omega \times V$, we denote by $\langle\omega, \mathcal{Q}^h\rangle(\cdot, x)$ the quadratic form in Y_0, \dots, Y_ℓ obtained from $\langle\omega, \mathcal{Q}^h\rangle$ by specializing $X_i = x_i, 1 \leq i \leq k$.

Let $B \subset \Omega \times \mathbf{S}^\ell \times V$ be the semi-algebraic set defined by

$$(2.5) \quad B = \{(\omega, y, x) \mid \omega \in \Omega, y \in \mathbf{S}^\ell, x \in V, \langle\omega, \mathcal{Q}^h\rangle(y, x) \geq 0\}.$$

We denote by $\varphi_1 : B \rightarrow F$ and $\varphi_2 : B \rightarrow \mathbf{S}^\ell \times V$ the two projection maps (see diagram below).

$$\begin{array}{ccc}
& B & \\
\varphi_1 \swarrow & & \searrow \varphi_2 \\
F = \Omega \times V & & \mathbf{S}^\ell \times V \\
& \searrow & \swarrow \\
& V &
\end{array}$$

The following key proposition was proved by Agrachev [1] in the unparametrized situation, but as we see below it works in the parametrized case as well.

Proposition 2.1. *The semi-algebraic set B is homotopy equivalent to A^h .*

Proof. See [9]. □

We will use the following notation.

Notation 3. For a quadratic form $Q \in \mathbb{R}[Y_0, \dots, Y_\ell]$, we denote by $\text{index}(Q)$ the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form, i.e. of the matrix M such that $Q(y) = \langle My, y \rangle$ for all $y \in \mathbb{R}^{\ell+1}$ (here $\langle \cdot, \cdot \rangle$ denotes the usual inner product). We also denote by $\lambda_i(Q), 0 \leq i \leq \ell$ the eigenvalues of Q in non-decreasing order, i.e.

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \dots \leq \lambda_\ell(Q).$$

For $F = \Omega \times V$ as above we denote

$$F_j = \{(\omega, x) \in F \mid \text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) \leq j\}.$$

It is clear that each F_j is a closed semi-algebraic subset of F and we get a filtration of the space F given by

$$F_0 \subset F_1 \subset \dots \subset F_{\ell+1} = F.$$

Lemma 2.2. *The fibre of the map φ_1 over a point $(\omega, x) \in F_j \setminus F_{j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.*

Proof. See [9]. □

For each $(\omega, x) \in F_j \setminus F_{j-1}$, let $L_j^+(\omega, x) \subset \mathbb{R}^{\ell+1}$ denote the sum of the non-negative eigenspaces of $\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)$. Since $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) = j$ stays invariant as (ω, x) varies over $F_j \setminus F_{j-1}$, $L_j^+(\omega, x)$ varies continuously with (ω, x) .

We denote by C the semi-algebraic set defined by the following. We first define for $0 \leq j \leq \ell + 1$

$$(2.6) \quad C_j = \{(\omega, y, x) \mid (\omega, x) \in F_j \setminus F_{j-1}, y \in L_j^+(\omega, x), |y| = 1\},$$

and finally we define

$$(2.7) \quad C = \bigcup_{j=0}^{\ell+1} C_j.$$

The following proposition proved in [9] relates the homotopy type of B to that of C .

Proposition 2.3. *The semi-algebraic set C defined by (2.7) is homotopy equivalent to B .*

The following example which also appears in [9] illustrates Proposition 2.3.

Example 2.4. In this example $m = 2, \ell = 3, k = 0$, and $\mathcal{Q}^h = \{Q_1^h, Q_2^h\}$ with

$$Q_1^h = -Y_0^2 - Y_1^2 - Y_2^2,$$

$$Q_2^h = Y_0^2 + 2Y_1^2 + 3Y_2^2.$$

The set Ω is the part of the unit circle in the third quadrant of the plane, and $F = \Omega$ in this case (since $k = 0$). In the following Figure 1, we display the fibers of the map $\varphi_1^{-1}(\omega) \subset B$ for a sequence of values of ω starting from $(-1, 0)$ and ending at $(0, -1)$. We also show the spheres, $C \cap \varphi_1^{-1}(\omega)$, of dimensions 0, 1, and 2, that these fibers retract to. At $\omega = (-1, 0)$, it is easy to verify that $\text{index}(\langle \omega, \mathcal{Q}^h \rangle) = 3$, and the fiber $\varphi_1^{-1}(\omega) \subset B$ is empty. Starting from $\omega = (-\cos(\arctan(1)), -\sin(\arctan(1)))$ we have $\text{index}(\langle \omega, \mathcal{Q}^h \rangle) = 2$, and the fiber $\varphi_1^{-1}(\omega)$ consists of the union of two spherical caps, homotopy equivalent to \mathbf{S}^0 . Starting from $\omega = (-\cos(\arctan(1/2)), -\sin(\arctan(1/2)))$ we have $\text{index}(\langle \omega, \mathcal{Q}^h \rangle) = 1$, and the fiber $\varphi_1^{-1}(\omega)$ is homotopy equivalent to \mathbf{S}^1 . Finally, starting from $\omega = (-\cos(\arctan(1/3)), -\sin(\arctan(1/3)))$, $\text{index}(\langle \omega, \mathcal{Q}^h \rangle) = 0$, and the fiber $\varphi_1^{-1}(\omega)$ stays equal to \mathbf{S}^2 .

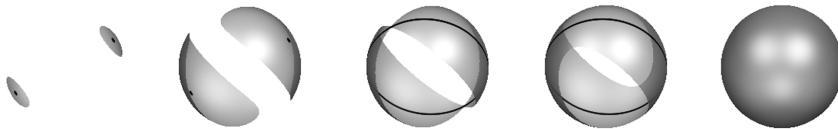


FIGURE 1. Type change: $\emptyset \rightarrow \mathbf{S}^0 \rightarrow \mathbf{S}^1 \rightarrow \mathbf{S}^2$. \emptyset is not shown.

Let $\Lambda \in \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k, T]$ be the polynomial defined by

$$\begin{aligned} \Lambda &= \det(T \cdot \text{Id}_{\ell+1} - M_{Z \cdot \mathcal{Q}^h}), \\ &= T^{\ell+1} + C_\ell T^\ell + \dots + C_0, \end{aligned}$$

where $Z \cdot \mathcal{Q}^h = \sum_{i=1}^m Z_i Q_i^h$, and each $C_i \in \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$.

Note that for $(\omega, x) \in \Omega \times \mathbb{R}^k$, the polynomial $\Lambda(\omega, x, T)$, being the characteristic polynomial of a real symmetric matrix, has all its roots real. It then follows from Descartes' rule of signs (see for instance [10]), that for each $(\omega, x) \in \Omega \times \mathbb{R}^k$, $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x))$ is determined by the sign vector

$$(\text{sign}(C_\ell(\omega, x)), \dots, \text{sign}(C_0(\omega, x))).$$

More precisely, the number of sign variations in the sequence

$$\text{sign}(C_0(\omega, x)), \dots, (-1)^i \text{sign}(C_i(\omega, x)), \dots, (-1)^\ell \text{sign}(C_\ell(\omega, x)), +1$$

is equal to $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x))$.

Hence, denoting

$$(2.8) \quad \mathcal{C} = \{C_0, \dots, C_\ell\} \subset \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k],$$

we have

Lemma 2.5. F_j is the intersection of F with a \mathcal{C} -closed semi-algebraic set $D_j \subset \mathbb{R}^{m+k}$, for each $0 \leq j \leq \ell + 1$. \square

3. COMPUTING THE BETTI NUMBERS

We now consider the algorithmic problem of computing all the Betti numbers of a semi-algebraic set defined by a partly quadratic system of polynomials.

3.1. Summary of the main idea. The main idea behind the algorithm can be summarized as follows.

By virtue of Proposition 2.1, in order to compute the Betti numbers of A^h , it suffices to construct a cell complex, $\mathcal{K}(B, V)$, whose associated space is homotopy equivalent to the set B defined by (2.5). In order to do so, we first compute a semi-algebraic triangulation, $h : \Delta \rightarrow F$, such that as (ω, x) varies over the image of any simplex $\sigma \in \Delta$, the $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x))$ stays fixed, and we have a continuous choice of an orthonormal basis,

$$\{e_0(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)\}$$

consisting of eigenvectors of the symmetric matrix associated to the quadratic form $\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)$.

Moreover, if $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) = j$ for $(\omega, x) \in h(\sigma)$, then $\varphi_1^{-1}(\omega, x)$ can be retracted to $\mathbf{S}^\ell \cap \text{span}(e_j(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x))$, and the flag of subspaces defined by the orthonormal basis, $\{e_0(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)\}$, gives an efficient regular cell decomposition of the sphere $\mathbf{S}^\ell \cap \text{span}(e_j(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x))$ into $2(\ell - j + 1)$ cells, having two cells of each dimension from 0 to $\ell - j$ (see Definition 3.3).

Now consider a pair of simplices, $\sigma, \tau \in \Delta$, with $\sigma \prec \tau$. The orthonormal basis $\{e_0(\tau, \omega, x), \dots, e_\ell(\tau, \omega, x)\}$, defined for $(\omega, x) \in h(\tau)$ might not have a continuous extension to $h(\sigma)$ on the boundary of $h(\tau)$. In particular, the cell decompositions of the fibers, $\mathbf{S}^\ell \cap \text{span}(e_j(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x))$, over points in $(\omega, x) \in h(\sigma)$ might not be compatible with those over neighboring points in $h(\tau)$. In order to obtain a proper cell complex we need to compute a common refinement of the cell decomposition of the sphere over each point in $(\omega, x) \in h(\sigma)$ induced by the basis $\{e_0(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)\}$, and the one obtained as a limit of those over certain points in $h(\tau)$ converging to (ω, x) . We need to further subdivide $h(\sigma)$ to ensure that over each cell of this sub-division the combinatorial type of the above refinements stays the same. Since a simplex $\sigma \in \Delta$ can be incident on many other simplices of Δ , we might in the above procedure need to simultaneously refine cell decompositions of the sphere coming from many different simplices. In order to ensure (for complexity reasons) that we do not have to simultaneously refine cell decompositions coming from too many simplices, we thicken the simplices infinitesimally and as a result only need to refine at most $m + k$ cell decompositions at a time.

Before describing the construction of $\mathcal{K}(B, V)$ in more detail, we need some preliminaries on triangulations.

3.2. Triangulations. We first need to recall a fact from semi-algebraic geometry about triangulations of semi-algebraic sets, and then we define the notion of an Index Invariant Triangulation and give an algorithm for computing it.

3.2.1. Triangulations of semi-algebraic sets. A triangulation of a closed and bounded semi-algebraic set S is a simplicial complex Δ together with a semi-algebraic homeomorphism from $|\Delta|$ to S . We always assume that the simplices in Δ are open.

Given such a triangulation we will often identify the simplices in Δ with their images in S under the given homeomorphism, and will refer to the triangulation by Δ .

Given a triangulation Δ , the cohomology groups $H^i(S)$ are isomorphic to the simplicial cohomology groups $H^i(\Delta)$ of the simplicial complex Δ and are in fact independent of the triangulation Δ (this fact is classical over \mathbb{R} ; see for instance [10] for a self-contained proof in the category of semi-algebraic sets).

We call a triangulation $h_1 : |\Delta_1| \rightarrow S$ of a semi-algebraic set S , to be a *refinement* of a triangulation $h_2 : |\Delta_2| \rightarrow S$ if for every simplex $\sigma_1 \in \Delta_1$, there exists a simplex $\sigma_2 \in \Delta_2$ such that $h_1(\sigma_1) \subset h_2(\sigma_2)$.

Let $S_1 \subset S_2$ be two compact semi-algebraic subsets of \mathbb{R}^k . We say that a semi-algebraic triangulation $h : |\Delta| \rightarrow S_2$ of S_2 , respects S_1 if for every simplex $\sigma \in \Delta$, $h(\sigma) \cap S_1 = h(\sigma)$ or \emptyset . In this case, $h^{-1}(S_1)$ is identified with a sub-complex of Δ and $h|_{h^{-1}(S_1)} : h^{-1}(S_1) \rightarrow S_1$ is a semi-algebraic triangulation of S_1 . We will refer to this sub-complex by $\Delta|_{S_1}$.

We will need the following theorem which can be deduced from Section 9.2 in [14] (see also [10]).

Theorem 3.1. *Let $S_1 \subset S_2 \subset \mathbb{R}^k$ be closed and bounded semi-algebraic sets, and let $h_i : \Delta_i \rightarrow S_i, i = 1, 2$ be semi-algebraic triangulations of S_1, S_2 . Then there exists a semi-algebraic triangulation $h : \Delta \rightarrow S_2$ of S_2 , such that Δ respects S_1 , Δ is a refinement of Δ_2 , and $\Delta|_{S_1}$ is a refinement of Δ_1 .*

Moreover, there exists an algorithm which computes such a triangulation with complexity bound $(sd)^{O(1)^k}$, where s is the number of polynomials used in the definition of S_1 and S_2 , and d is a bound on their degrees.

3.2.2. Parametrized eigenvector basis. Let $M(\omega, x)$ be the symmetric matrix associated to the quadratic form $\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)$ defined by (2.4). When $M(\omega, x)$ has simple eigenvalues for all possible choice of ω, x in some domain, there is a finite choice of orthonormal bases consisting of eigenvectors of $M(\omega, x)$. However, when $M(\omega, x)$ has multiple eigenvalues, the number of choices of orthonormal basis of eigenvectors is infinite. In order to avoid the problem caused by the latter situation we are going to use an infinitesimal deformation as follows.

Let $0 < \varepsilon \ll 1$ be an infinitesimal and

$$(3.1) \quad M_\varepsilon(\omega, x) = (1 - \varepsilon)M(\omega, x) + \varepsilon \operatorname{diag}(0, 1, 2, \dots, \ell).$$

Note that for every $(\omega, x) \in \Omega \times \mathbb{R}^k$ the eigenvalues of $M_\varepsilon(\omega, x)$ in $\mathbb{R}(\varepsilon)$ are distinct and nonzero. Indeed, replace ε by t in the definition of $M_\varepsilon(\omega, x)$ and obtain $M_t(\omega, x)$. Observe that the statement is true if $t = 1$, since the matrix $M_1(\omega, x)$ has distinct eigenvalues. Thus, the set of t 's in the algebraically closed field $\mathbb{R}[i]$ for which $M_t(\omega, x)$ has $\ell + 1$ distinct eigenvalues is non-empty, constructible and contains an open subset, since the condition of having distinct eigenvalues is a stable condition. Thus, there exists $\varepsilon_0 > 0$, such that for all $t \in (0, \varepsilon_0)$, $M_t(\omega, x)$ has $\ell + 1$ distinct eigenvalues, and hence it is also the case for the infinitesimal ε .

Denote by $\Lambda(M_\varepsilon(\omega, x, T)) = \det(T \cdot \operatorname{Id}_{\ell+1} - M_\varepsilon(\omega, x))$ the characteristic polynomial of $M_\varepsilon(\omega, x)$. Let $\mathcal{A} \subset \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$ be a set of polynomials containing \mathcal{C} (see (2.8)) and such that for every sign condition $\rho \in \{0, 1, -1\}^{\mathcal{A}}$ and every $(\omega, x) \in \mathcal{R}(\rho, \Omega \times \mathbb{R}^k)$, the Thom encodings of the roots of $\Lambda(M_\varepsilon(\omega, x), T)$ stay fixed, as well as the list of the non-singular minors of size ℓ in $M_\varepsilon(\omega, x, T)$ at each root of $\Lambda(M_\varepsilon(\omega, x), T)$.

Then choosing a non-vanishing minor and using Cramer's rule, we find $(\ell + 1)^2$ rational functions in the variables u, ω, x, T which give for every $(u, \omega, x) \in \mathcal{R}(\rho, \Omega \times \mathbb{R}^{k+1})$ the coordinates of an eigenvector $v_\varepsilon(u, \omega, x, t_\varepsilon)$ associated to the eigenvalue t_ε (where u denotes the co-ordinate left out in the non-singular $\ell \times \ell$ minor chosen for this eigenvalue in the application of Cramer's rule). We denote by $e_\varepsilon(\omega, x, t_\varepsilon)$ the unit eigenvector $v_\varepsilon(1, \omega, x, t_\varepsilon) / \|v_\varepsilon(1, \omega, x, t_\varepsilon)\|$ when t_ε is an eigenvalue.

If the eigenvalues $\lambda_{\varepsilon,0} < \dots < \lambda_{\varepsilon,\ell}$ are in increasing order, we define

$$e_{\varepsilon,i}(\omega, x) = e_\varepsilon(\omega, x, \lambda_{\varepsilon,i}).$$

Note that for every $(\omega, x) \in \Omega \times \mathbb{R}^k$

$$(\lim_{\varepsilon} (e_{\varepsilon,0}(\omega, x)), \dots, \lim_{\varepsilon} (e_{\varepsilon,\ell}(\omega, x)))$$

is an orthonormal basis consisting of eigenvectors of $M(\omega, x)$.

3.2.3. Index Invariant Triangulations. We now define a certain special kind of semi-algebraic triangulation of F that will play an important role in our algorithm.

Definition 3.2. (Index Invariant Triangulation) An *index invariant triangulation* of F is a triangulation

$$h : \Delta \rightarrow F$$

of F , which respects all the realization of the weak sign conditions on \mathcal{P} and \mathcal{A} (see definition in 3.2.2). As a consequence, h respects the subsets F_I for every $I \subset \mathcal{Q}$. Moreover, $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x))$, stays invariant as (ω, x) varies over $h(\sigma)$, and the maps $e_{\varepsilon,0}(\sigma), \dots, e_{\varepsilon,\ell}$ sending $(\omega, x) \in h(\sigma)$ to the orthonormal basis $e_{\varepsilon,0}(\omega, x), \dots, e_{\varepsilon,\ell}(\omega, x)$, are uniformly defined. Note also that for every $(\omega, x) \in h(\sigma)$,

$$\{e_j(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)\} = \{\lim_{\varepsilon} (e_{\varepsilon,j}(\omega, x)), \dots, \lim_{\varepsilon} (e_{\varepsilon,\ell}(\omega, x))\}$$

is a basis for the linear subspace $L^+(\omega, x) \subset \mathbb{R}^{\ell+1}$, (which is the orthogonal complement to the sum of the eigenspaces corresponding to the first j eigenvalues of $\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)$).

We now describe an algorithm for computing index invariant triangulations.

ALGORITHM 1 (Index Invariant Triangulation).

INPUT

- A family of polynomials, $\mathcal{Q}^h = \{Q_1^h, \dots, Q_m^h\} \subset \mathbb{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k]$, where each Q_i^h is homogeneous of degree 2 in the variables Y_0, \dots, Y_ℓ , and of degree at most d in X_1, \dots, X_k ,
- another family of polynomials, $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$, with $\deg(P) \leq d, P \in \mathcal{P}, \#\mathcal{P} = s$,
- a \mathcal{P} -closed formula Φ defining a bounded \mathcal{P} -closed semi-algebraic set $V \subset \mathbb{R}^k$.

OUTPUT : an index invariant triangulation,

$$h : \Delta \rightarrow F$$

of F and for each simplex σ of Δ , the rational functions $e_{\varepsilon,0}(\sigma), \dots, e_{\varepsilon,\ell}(\sigma)$.

PROCEDURE

Step 1. Let $\varepsilon > 0$ be an infinitesimal and let $Z = (Z_1, \dots, Z_m)$. Let M_ε be the symmetric matrix corresponding to the quadratic form (in Y_0, \dots, Y_ℓ) defined by

$$M_\varepsilon(X, Z) = (1 - \varepsilon)(Z_1 Q_1^h + \dots + Z_m Q_m^h) + \varepsilon \bar{Q},$$

where $\bar{Q} = \sum_{i=0}^{\ell} i Y_i^2$. Compute the polynomials

$$(3.2) \quad \Lambda(Z, X, T) = \det(T \cdot \text{Id}_{\ell+1} - M_\varepsilon) = T^{\ell+1} + C_\ell T^\ell + \dots + C_0.$$

Step 2. Using Algorithm 11.19 in [10] (Restricted Elimination), compute a family of polynomials $\mathcal{A}' \subset \mathbb{R}[\varepsilon][Z_1, \dots, Z_m, X_1, \dots, X_k]$ such that for each $\rho \in \text{Sign}(\mathcal{A}')$, and $(\omega, x) \in \mathcal{R}(\rho, \Omega \times \mathbb{R}^k) \cap F$ the Thom encodings of the roots of $\Lambda(\omega, x, T)$ in $\mathbb{R}(\varepsilon)$ and the number of non-negative roots of $\Lambda(\omega, x, T)$ in $\mathbb{R}(\varepsilon)$ stay fixed, as well as the list of the non singular minors of size ℓ in $M_\varepsilon(\omega, x, T)$ at each root of $\Lambda(M_\varepsilon(\omega, x), T)$. Let $\mathcal{A} \subset \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$ be the set of all coefficients of the polynomials in \mathcal{A}' , when each of them is written as a polynomial in ε .

Step 3. Using the algorithm implicit in Theorem 3.1 (Triangulation), compute a semi-algebraic triangulation,

$$h : \Delta \rightarrow F,$$

respecting all the realizations of the weak sign conditions on $\mathcal{A} \cup \mathcal{P}$.

Step 4. For each simplex σ of Δ , output the maps $e_{\varepsilon,0}(\sigma), \dots, e_{\varepsilon,\ell}(\sigma)$.

COMPLEXITY ANALYSIS: The complexity of the algorithm is dominated by the complexity of Step 3, which is $(slmd)^{2^{O(m+k)}}$. \square

PROOF OF CORRECTNESS: It follows from the fact that the triangulation respects all weak sign conditions on \mathcal{A} that $\text{index}(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle)$ is constant for $(\omega, x) \in h(\sigma)$ for any simplex σ of Δ ,

Since $e_{\varepsilon,0}(\sigma, \omega, x), \dots, e_{\varepsilon,\ell}(\sigma, \omega, x)$ are orthonormal, so are $e_0(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)$ for every $(\omega, x) \in h(\sigma)$. Moreover, letting $j = \text{index}(\langle \omega, \mathcal{Q}^h(\cdot, x) \rangle)$ for $(\omega, x) \in h(\sigma)$, we have that $e_{\varepsilon,j}(\sigma, \omega, x), \dots, e_{\varepsilon,\ell}(\sigma, \omega, x)$ span the sum of the non-negative eigenspaces of $M_\varepsilon(\omega, x)$, their images under the \lim_ε map will span the sum of the non-negative eigenspaces of $M(\omega, x)$. \square

3.3. Computing Betti numbers in the homogeneous union case. Now that we obtained an Index Invariant Triangulation Δ , our next goal is to construct a cell complex $\mathcal{K}(B, V)$ homotopy equivalent to B (see Notation 2.5) that will be used to compute the Betti numbers of A^h (see Notation 2). The cell complex $\mathcal{K}(B, V)$ is obtained by glueing together certain regular cell complexes, $\mathcal{K}(\sigma)$, where $\sigma \in \Delta$.

3.3.1. *Definition of $\mathcal{C}(\Delta)$.* Let $1 \gg \varepsilon_0 \gg \varepsilon_1 \gg \dots \gg \varepsilon_{m+k} > 0$ be infinitesimals. For $\tau \in \Delta$, we denote by D_τ the subset of $\bar{\tau}$ defined by

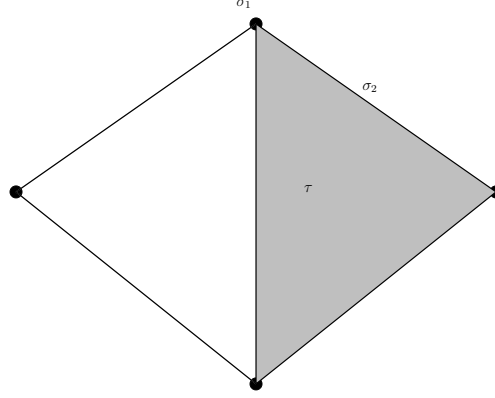
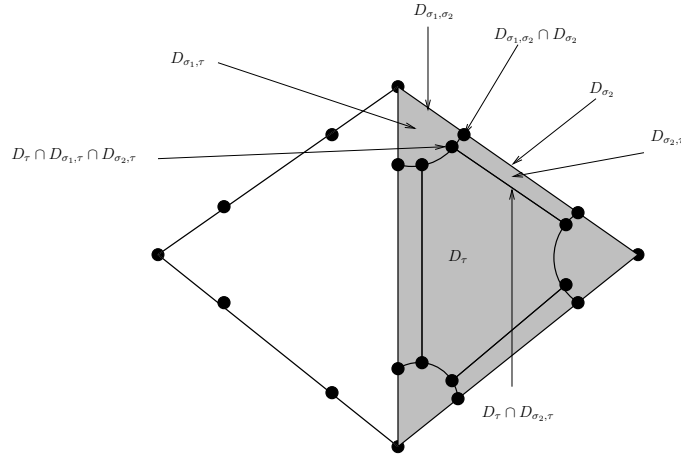
$$D_\tau = \{v \in \bar{\tau} \mid \text{dist}(v, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \tau\},$$

where dist refers to the ordinary Euclidean distance. Now, let $\sigma \prec \tau$ be two simplices of Δ . We denote by $D_{\sigma,\tau}$ the subset of $\bar{\tau}$ defined by

$$D_{\sigma,\tau} = \{v \in \bar{\tau} \mid \text{dist}(v, \sigma) \leq \varepsilon_{\dim(\sigma)}, \text{ and } \text{dist}(v, \theta) \geq \varepsilon_{\dim(\theta)} \text{ for all } \theta \prec \sigma\}.$$

Note that

$$|\Delta| = \bigcup_{\sigma \in \Delta} D_\sigma \cup \bigcup_{\sigma, \tau \in \Delta, \sigma \prec \tau} D_{\sigma,\tau}.$$

FIGURE 2. The complex Δ .FIGURE 3. The corresponding complex $\mathcal{C}(\Delta)$.

Also, observe that the various D_τ 's and $D_{\sigma,\tau}$'s are all homeomorphic to closed balls, and moreover all non-empty intersections between them also have the same property. Thus, the union of the D_τ 's and $D_{\sigma,\tau}$'s together with the non-empty intersections between them form a regular cell complex, $\mathcal{C}(\Delta)$, whose underlying topological space is $|\Delta|$ (see Figures 2 and 3).

3.3.2. *Definition of $\mathcal{K}(\sigma)$ and $\mathcal{K}(\sigma,\tau)$ where σ,τ are simplices of Δ .* We now associate to each D_σ (respectively, $D_{\sigma,\tau}$) a regular cell complex, $\mathcal{K}(\sigma)$, (respectively, $\mathcal{K}(\sigma,\tau)$) homotopy equivalent to $\varphi_1^{-1}(h(D_\sigma))$ (respectively, $\varphi_1^{-1}(h(D_{\sigma,\tau}))$).

For each $\sigma \in \Delta$, and $(\omega, x) \in h(\sigma)$, the orthonormal basis

$$\{e_0(\sigma, \omega, x), \dots, e_\ell(\sigma, \omega, x)\}$$

determines a complete flag of subspaces, $\mathcal{F}(\sigma, \omega, x)$, consisting of

$$\begin{aligned} F^0(\sigma, \omega, x) &= 0, \\ F^1(\sigma, \omega, x) &= \text{span}(e_\ell(\sigma, \omega, x)), \\ F^2(\sigma, \omega, x) &= \text{span}(e_\ell(\sigma, \omega, x), e_{\ell-1}(\sigma, \omega, x)), \\ &\vdots \\ F^{\ell+1}(\sigma, \omega, x) &= \mathbf{R}^{\ell+1}. \end{aligned}$$

Definition 3.3. For $0 \leq j \leq \ell$, let $c_j^+(\sigma, \omega, x)$ (respectively, $c_j^-(\sigma, \omega, x)$) denote the $(\ell - j)$ -dimensional cell consisting of the intersection of the $F^{\ell-j+1}(\sigma, \omega, x)$ with the unit hemisphere in $\mathbf{R}^{\ell+1}$ defined by

$$\begin{aligned} &\{y \in \mathbf{S}^\ell \mid \langle y, e_j(\sigma, \omega, x) \rangle \geq 0\} \\ \text{(respectively, } &\{y \in \mathbf{S}^\ell \mid \langle y, e_j(\sigma, \omega, x) \rangle \leq 0\} \text{)}. \end{aligned}$$

The regular cell complex $\mathcal{K}(\sigma)$ (as well as $\mathcal{K}(\sigma, \tau)$) is defined as follows.

For each $v \in |\Delta|$ and $\sigma \in \Delta$, let $v(\sigma) \in |\sigma|$ denote the point of $|\sigma|$ closest to v .

The cells of $\mathcal{K}(\sigma)$ are

$$\{(y, \omega, x) \mid y \in c_j^\pm(\sigma, \omega, x), (\omega, x) \in h(c)\},$$

where $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) \leq j \leq \ell$, and $c \in \mathcal{C}(\Delta)$ is either D_σ itself, or a cell contained in the boundary of D_σ .

Similarly, the cells of $\mathcal{K}(\sigma, \tau)$ are

$$\{(y, \omega, x) \mid y \in c_j^\pm(\sigma, h(v(\sigma))), v = h^{-1}(\omega, x) \in c\},$$

where $\text{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) \leq j \leq \ell$, $c \in \mathcal{C}(\Delta)$ is either $D_{\sigma, \tau}$ itself, or a cell contained in the boundary of $D_{\sigma, \tau}$.

3.3.3. Definition of $\mathcal{K}(D)$, where D is a cell of $\mathcal{C}(\Delta)$. Our next step is to obtain cellular subdivisions of each non-empty intersection amongst the spaces associated to the complexes constructed above, and thus obtain a regular cell complex, $\mathcal{K}(B, V)$, whose associated space, $|\mathcal{K}(B, V)|$, will be shown to be homotopy equivalent to B (Proposition 3.6 below).

First notice that $|\mathcal{K}(\sigma', \tau')|$ (respectively, $|\mathcal{K}(\sigma)|$) has a non-empty intersection with $|\mathcal{K}(\sigma, \tau)|$ only if $D_{\sigma', \tau'}$ (respectively, $D_{\sigma'}$) intersects $D_{\sigma, \tau}$.

Let D be some non-empty intersection amongst the D_σ 's and $D_{\sigma, \tau}$'s, that is D is a cell of $\mathcal{C}(\Delta)$. Then $D \subset |\tau|$ for a unique simplex $\tau \in \Delta$, and

$$D = D_{\sigma_1, \tau} \cap \cdots \cap D_{\sigma_p, \tau} \cap D_\tau,$$

with $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_p \prec \sigma_{p+1} = \tau$ and $p \leq m + k$.

For each $i, 1 \leq i \leq p + 1$, let $\{f_0(\sigma_i, v), \dots, f_\ell(\sigma_i, v)\}$ denote a orthonormal basis of $\mathbf{R}^{\ell+1}$ where

$$f_j(\sigma_i, v) = \lim_{t \rightarrow 0} e_j(\sigma_i, h(tv(\sigma_i) + (1-t)v(\sigma_1))), 0 \leq j \leq \ell,$$

and let $\mathcal{F}(\sigma_i, v)$ denote the corresponding flag, consisting of

$$\begin{aligned} F^0(\sigma_i, v) &= 0, \\ F^1(\sigma_i, v) &= \text{span}(f_\ell(\sigma_i, v)), \\ F^2(\sigma_i, v) &= \text{span}(f_\ell(\sigma_i, v), f_{\ell-1}(\sigma_i, v)), \\ &\vdots \\ F^{\ell+1}(\sigma_i, v) &= \mathbf{R}^{\ell+1}. \end{aligned}$$

We thus have $p + 1$ different flags,

$$\mathcal{F}(\sigma_1, v), \dots, \mathcal{F}(\sigma_{p+1}, v),$$

and these give rise to $p + 1$ different regular cell decompositions of \mathbf{S}^ℓ .

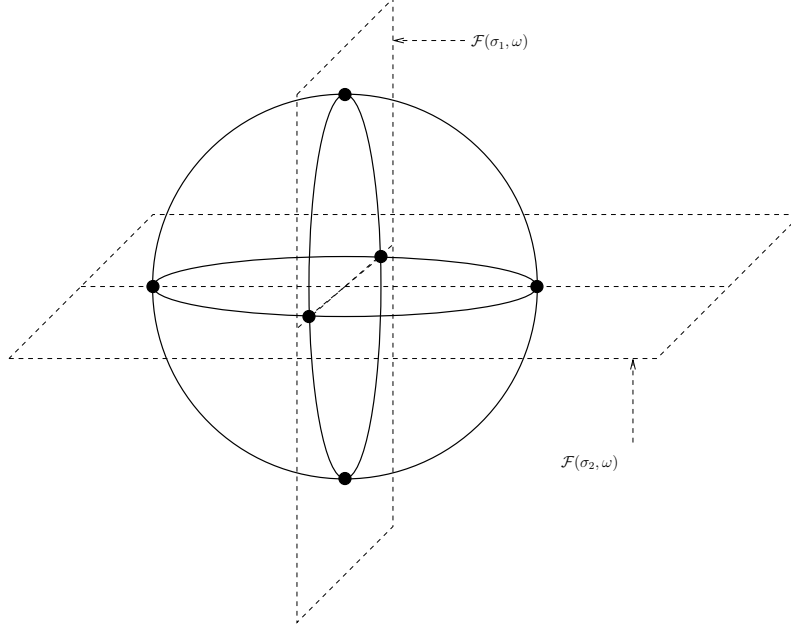


FIGURE 4. The cell complex $\mathcal{K}'(D, v)$.

There is a unique smallest regular cell complex, $\mathcal{K}'(D, v)$, that refines all these cell decompositions, whose cells are the following. Let $L \subset \mathbf{R}^{\ell+1}$ be any j -dimensional linear subspace, $0 \leq j \leq \ell + 1$, which is an intersection of linear subspaces L_1, \dots, L_{p+1} , where $L_i \in \mathcal{F}(\sigma_i, v)$, $1 \leq i \leq p + 1 \leq m + k + 1$. The elements of the flags, $\mathcal{F}(\sigma_1, v), \dots, \mathcal{F}(\sigma_{p+1}, v)$ of dimensions $j + 1$, partition L into polyhedral cones of various dimensions. The intersections of these cones with \mathbf{S}^ℓ , over all such subspaces $L \subset \mathbf{R}^{\ell+1}$, are the cells of $\mathcal{K}'(D, v)$. Figure 4 illustrates the refinement described above in case of two flags in \mathbf{R}^3 . We denote by $\mathcal{K}(D, v)$ the sub-complex of $\mathcal{K}'(D, v)$ consisting of only those cells included in $L^+(\sigma_1, h(v(\sigma_1))) \cap \mathbf{S}^\ell$.

We now triangulate $h(D)$ using the algorithm implicit in Theorem 3.1 (Triangulation), so that the combinatorial type of the arrangement of flags,

$$\mathcal{F}(\sigma_1, v), \dots, \mathcal{F}(\sigma_{p+1}, v)$$

and hence the combinatorial type of the cell decomposition $\mathcal{K}'(D, v)$, stays invariant over the image, $h_D(\theta)$, of each simplex, θ , of this triangulation.

Note that in case the eigenvalues of $M(h(v))$ are all distinct, we have that $\mathcal{F}(\sigma_1, v) = \cdots = \mathcal{F}(\sigma_{p+1}, v)$ since the vectors

$$f_0(\sigma_i, v), \dots, f_\ell(\sigma_i, v)$$

is an orthonormal basis of eigen-vectors with each $f_j(\sigma_i, v)$ uniquely defined upto sign. Thus the cell decompositions of \mathbf{S}^ℓ induced by the flags $\mathcal{F}(\sigma_1, v) = \cdots = \mathcal{F}(\sigma_{p+1}, v)$ are identical to each other and hence also to $\mathcal{K}'(D, v)$.

However, if the eigenvalues of $M(h(v))$ are not all distinct then the refinement $\mathcal{K}'(D, v)$ can be non-trivial. For example suppose we have that $\lambda_\alpha(h(v)) = \cdots = \lambda_\beta(h(v))$, $0 \leq \alpha < \beta \leq \ell$. Then in general the sub-flags consisting of subspaces $F_{\ell+1-\beta}(\sigma_i, v) \subset \cdots \subset F_{\ell+1-\alpha}(\sigma_i, v)$ and $F_{\ell+1-\beta}(\sigma_j, v) \subset \cdots \subset F_{\ell+1-\alpha}(\sigma_j, v)$ will in general not coincide for $i \neq j$.

In this case the combinatorial type of the refinement $\mathcal{K}'(D, v)$ is determined by the dimensions of the intersections amongst the subspaces

$$F_{\ell+1-\beta}(\sigma_i, v), \dots, F_{\ell+1-\alpha}(\sigma_i, v), 1 \leq i \leq p+1.$$

The dimensions of these intersections are determined by the minimal linear dependencies amongst the vectors $f_i(\sigma_j, v)$, $\alpha \leq i \leq \beta$, $1 \leq j \leq p+1$, and these are in turn determined by the ranks of matrices with at most $\ell+1$ rows of the following form. The rows of the matrix consists of at most $p+1$ blocks, with the j -th block of the shape $f_\alpha(\sigma_j, v), \dots, f_\beta(\sigma_j, v)$, where $\alpha \leq \alpha_j \leq \beta$. (Note that every row of the above matrix consists of rational functions evaluated at a single root $\lambda_\alpha(h(v))$ of $\Lambda(M(h(v)), T)$, and this root is common to all the rows. This fact is important since in order to perform algebraic computations on the entries of the matrix we need to eliminate just one variable corresponding to this single root.)

Introducing an infinitesimal δ such that $1 \gg \delta \gg \varepsilon > 0$, we note that for each $0 \leq j \leq \ell$,

$$\begin{aligned} f_j(\sigma_i, h^{-1}(\omega, x)) &= \lim_{\delta} e_{\varepsilon, j}(h(\delta v(\sigma_i) + (1 - \delta)v(\sigma_1))) \\ &= \lim_{t \rightarrow 0} e_j(\sigma_i, h(tv(\sigma_i) + (1 - t)v(\sigma_1))). \end{aligned}$$

We consider all matrices with at most $\ell+1$ rows consisting of blocks of the shape, $f_\alpha(\sigma_j, v), \dots, f_\beta(\sigma_j, v)$, with $0 \leq \alpha \leq \alpha_j \leq \beta \leq \ell$, $0 \leq j \leq m+k$, and $\lambda_\alpha(h(v)) = \cdots = \lambda_\beta(h(v))$. The number of such matrices is clearly bounded by $\ell^{\mathcal{O}(m+k)}$.

Using the uniform formula defining $e_{\varepsilon, j}(\sigma_i)$ and Proposition 14.7 of [10], and Algorithm 8.16 in [10] (for computing determinants over an arbitrary domain), we compute a family of polynomials in $\mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$ such that over each sign condition of this family the rank of the given matrix stays fixed.

Let

$$\mathcal{A}_D \subset \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$$

be the union of all these sets of polynomials.

The combinatorial type of the cell decomposition $\mathcal{K}'(D, v)$ will stay invariant as (ω, x) varies over each connected component of any realizable sign condition on $\mathcal{A}_D \subset \mathbb{R}[Z_1, \dots, Z_m, X_1, \dots, X_k]$.

Given the degree bounds on the rational functions defining $\{e_{\varepsilon,0}(\sigma), \dots, e_{\varepsilon,\ell}(\sigma)\}$, $(\omega, x) \in h(\sigma)$, and the complexity of Algorithm 8.16 in [10], it is clear that the number and degrees of the polynomials in the family \mathcal{A}_D are bounded by $(s\ell md)^{2^{O(m+k)}}$. We then use the algorithm implicit in Theorem 3.1 (Triangulation), with \mathcal{A}_D as input, to obtain the required triangulation.

The closures of the sets

$$\{(y, \omega, x) \mid y \in c \in \mathcal{K}(D, h^{-1}(\omega, x)), (\omega, x) \in h(h_D(\theta))\}$$

form a regular cell complex which we denote by $\mathcal{K}(D)$.

The following proposition gives an upper bound on the size of the complex $\mathcal{K}(D)$. We use the notation introduced in the previous paragraph.

Proposition 3.4. *For each $(\omega, x) \in h(D)$, the number of cells in $\mathcal{K}(D, h^{-1}(\omega, x))$ is bounded by $\ell^{O(m+k)}$. Moreover, the number of cells in the complex $\mathcal{K}(D)$ is bounded by $(s\ell md)^{2^{O(m+k)}}$.*

Proof. The first part of the proposition follows from the fact that there are at most $(\ell + 1)^{m+k+1}$ choices for the linear space L and the number of $(j - 1)$ dimensional cells contained in L is bounded by 2^{m+k} (which is an upper bound on the number of full dimensional cells in an arrangement of at most $m + k$ hyperplanes). The second part is a consequence of the complexity estimate in Theorem 3.1 (Triangulation) and the bounds on number and degrees of polynomials in the family \mathcal{A}_D stated above. \square

3.3.4. *Definition of $\mathcal{K}(B, V)$.* Note that there is a homeomorphism

$$i_{D, \sigma_i} : |\mathcal{K}(\sigma_i, \tau)| \cap \varphi_1^{-1}(h(D)) \rightarrow |\mathcal{K}(D)|$$

which takes each cell of $|\mathcal{K}(\sigma_i, \tau)| \cap \varphi_1^{-1}(h(D))$ to a union of cells in $\mathcal{K}(D)$. We use these homeomorphisms to glue the cell complexes $\mathcal{K}(\sigma_i, \tau)$ together to form the cell complex $\mathcal{K}(B, V)$.

Definition 3.5. The complex $\mathcal{K}(B, V)$ is the union of all the complexes $\mathcal{K}(D)$ constructed above, where we use the maps i_{D, σ_i} to make the obvious identifications. It is clear that $\mathcal{K}(B, V)$ so defined is a regular cell complex.

We have

Proposition 3.6. *$|\mathcal{K}(B, V)|$ is homotopy equivalent to B .*

Proof. We have from Proposition 2.3 that the semi-algebraic set $C \subset B$ (see (2.7) for definition) is homotopy equivalent to B . We now prove that $|\mathcal{K}(B, V)|$ is homotopy equivalent to C which will prove the proposition.

Let $X_{m+k} = |\mathcal{K}(B, V)|$ and for $0 \leq j \leq m + k - 1$, let $X_j = \lim_{\varepsilon_j} X_{j+1}$.

It follows from an application of the Vietoris-Smale theorem [23] that for each j , $0 \leq j \leq m + k - 1$, $\text{Ext}(X_j, \mathbb{R}\langle \varepsilon_0, \dots, \varepsilon_j \rangle)$ is homotopy equivalent to X_{j+1} . Also, by construction of $\mathcal{K}(B, V)$, we have that $X_0 = \lim_{\varepsilon_0} |\mathcal{K}(B, V)| = C$, which proves the proposition. \square

We also have

Proposition 3.7. *The number of cells in the cell complex $\mathcal{K}(B, V)$ is bounded by $(s\ell md)^{2^{O(m+k)}}$.*

Proof. The proposition is a consequence of Proposition 3.4 and the fact that the number of cells in the complex $\mathcal{C}(\Delta)$ is bounded by $(s\ell md)^{2^{O(m+k)}}$. \square

3.3.5. *Algorithm for computing the Betti numbers in the homogeneous union case.* We now describe formally an algorithm for computing the Betti numbers of A^h using the complex $\mathcal{K}(B, V)$ described above.

ALGORITHM 2 (Betti numbers, homogeneous union case).

INPUT

- A family of polynomials $\mathcal{Q}^h \subset \mathbb{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k]$, homogeneous of degree 2 in the variables Y_0, \dots, Y_ℓ , $\deg_X(Q^h) \leq d$, $Q^h \in \mathcal{Q}^h$, $\#(\mathcal{Q}^h) = m$,
- another family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$, with $\deg_X(P) \leq d$, $P \in \mathcal{P}$, $\#(\mathcal{P}) = s$,
- a \mathcal{P} -closed formula $\Phi(x)$ defining a bounded \mathcal{P} -closed semi-algebraic set $V \subset \mathbb{R}^k$.

OUTPUT

- a description of the cell complex $\mathcal{K}(B, V)$,
- the Betti numbers of A^h where the semi-algebraic set A^h is defined by

$$A^h = \bigcup_{Q^h \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\}.$$

PROCEDURE

- Step 1. Call Algorithm 1 (Index Invariant Triangulation) with input $\mathcal{Q}^h, \mathcal{P}$ and Φ and compute h and Δ .
- Step 2. Construct the cell complex $\mathcal{C}(\Delta)$ (following its definition given in Section 3.3).
- Step 3. For each cell $D \in \mathcal{C}(\Delta)$, compute, using the algorithm implicit in Theorem 3.1 (Triangulation), the cell complex $\mathcal{K}(D)$.
- Step 4. Compute a description of $\mathcal{K}(B, V)$, including the matrices corresponding to the differentials in the complex $\mathbf{C}_\bullet(\mathcal{K}(B, V))$.
- Step 5. Compute the Betti numbers of the complex $\mathbf{C}_\bullet(\mathcal{K}(B, V))$ using linear algebra.

COMPLEXITY ANALYSIS: The complexity of the algorithm is $(slmd)^{2^{O(m+k)}}$, using the complexity of Algorithm 1. \square

PROOF OF CORRECTNESS: The correctness of the algorithm is a consequence of the correctness of Algorithm 1 and Proposition 3.6. \square

3.4. Computing Betti numbers in the homogeneous intersection case.

3.4.1. *Definition of $\mathcal{K}(B_I, V)$.* We now define a subcomplex of $\mathcal{K}(B, V)$ corresponding to a subset $I \subset [m]$.

We first extend a few definitions from Section 2.

For each subset $I \subset [m]$, we denote by \mathcal{Q}_I^h the subset of \mathcal{Q}^h of polynomials with indices in I and by Ω_I the subset of

$$\Omega = \{\omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq m\},$$

obtained by setting the coordinates corresponding to the elements of $[m] \setminus I$ to 0. More precisely,

$$\Omega_I = \{\omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, \text{ for } i \in I, \text{ and } \omega_i = 0 \text{ for } i \in [m] \setminus I\}.$$

Note that we have a natural inclusion $\Omega_I \hookrightarrow \Omega_{[m]} = \Omega$.

Similarly, we denote by $F_I \subset F = F_{[m]}$, the set $\Omega_I \times V$, and denote by $B_I \subset \Omega_I \times \mathbf{S}^\ell \times V$ the semi-algebraic set defined by

$$B_I = \{(\omega, y, x) \mid \omega \in \Omega_I, y \in \mathbf{S}^\ell, x \in V, \langle \omega, \mathcal{Q}^h \rangle(y, x) \geq 0\}.$$

We denote by $\varphi_{1,I} : B_I \rightarrow F_I$ and $\varphi_{2,I} : B_I \rightarrow \mathbf{S}^\ell \times V$ the two projection maps. Now we define $\mathcal{K}(B_I, V)$ for every $I \subset [m]$.

Definition 3.8. The complex $\mathcal{K}(B_I, V)$ is the union of all the complexes $\mathcal{K}(D)$ in $\mathcal{C}(\Delta_I)$, where $\mathcal{C}(\Delta_I)$ is the subcomplex of $\mathcal{C}(\Delta)$ consisting of cells contained in $\Delta_I = h^{-1}(F_I)$.

Using proofs similar to the ones give for $\mathcal{K}(B, V)$, we have

Proposition 3.9. $|\mathcal{K}(B_I, V)|$ is homotopy equivalent to B_I . □

ALGORITHM 3 (Computing the collection of $\mathcal{K}(B_I, V)$, $I \subset [1 \dots, m]$).

INPUT

- $\mathcal{Q}^h = \{Q_1^h, \dots, Q_m^h\} \subset \mathbb{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k]$, where each Q_i^h is homogeneous of degree 2 in the variables Y_0, \dots, Y_ℓ , and of degree at most d in X_1, \dots, X_k ,
- $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$, with $\deg(P) \leq d, P \in \mathcal{P}$,
- a \mathcal{P} -closed formula $\Phi(x)$ defining a bounded \mathcal{P} -closed semi-algebraic set $V \subset \mathbb{R}^k$.

OUTPUT

- For each subset $I \subset [m]$, a description of the cell complex $\mathcal{K}(B_I, V)$.
- For each $I \subset J \subset [m]$, a homomorphism

$$i_{I,J} : C_\bullet(B_I, V) \rightarrow C_\bullet(B_J, V)$$

inducing the inclusion homomorphism $i_{I,J*} : H_*(B_I, V) \rightarrow H_*(B_J, V)$.

PROCEDURE

Step 1. Call Algorithm 2 to compute $\mathcal{K}(B, V)$.

Step 2. Give a description of $\mathcal{K}(B_I, V)$ for each $I \subset [m]$ and compute the matrices corresponding to the differentials in the complex $C_\bullet(\mathcal{K}(B_I, V))$.

Step 3. For $I \subset J \subset [m]$ with compute the matrices for the homomorphisms of complexes,

$$i_{I,J} : C_\bullet(\mathcal{K}(B_I, V)) \rightarrow C_\bullet(\mathcal{K}(B_J, V))$$

in the following way.

The complex $\mathcal{K}(B_I, V)$ is a subcomplex of $\mathcal{K}(B_J, V)$ by construction. Compute the matrix for the inclusion homomorphism,

$$i_{I,J} : C_\bullet(\mathcal{K}(B_I, V)) \rightarrow C_\bullet(\mathcal{K}(B_J, V))$$

and output the matrix for the homomorphism.

COMPLEXITY ANALYSIS: The complexity of the algorithm is $(slmd)^{2^{O(m+k)}}$, using the complexity of Algorithm 1. □

PROOF OF CORRECTNESS: The correctness of the algorithm is a consequence of the correctness of Algorithm 1 and Proposition 3.6. □

3.4.2. *Algorithm for computing the Betti numbers in the homogeneous intersection case.* Let $W^h \subset \mathbf{S}^\ell \times \mathbf{R}^k$ be the semi-algebraic set defined by

$$W^h = \bigcap_{Q \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\},$$

using Notation 2.

Then

$$(3.3) \quad \mathbb{H}_*(W^h) \cong \mathbb{H}^*(\text{Tot}_\bullet(\mathcal{N}_{\bullet, \bullet}(\mathcal{K}(B, V)))),$$

where $\mathcal{N}_{\bullet, \bullet}(\mathcal{K}(B, V))$ is the bi-complex

$$(3.4) \quad \mathcal{N}_{p, q}(\mathcal{K}(B, V)) = \bigoplus_{J \subset [m], \#(J)=p+1} \mathbb{C}_q(\mathcal{K}(B_J, V)),$$

with the horizontal and vertical differentials defined as follows. The vertical differentials

$$(3.5) \quad d_{p, q} : \mathcal{N}_{p, q}(\mathcal{K}(B, V)) \rightarrow \mathcal{N}_{p, q-1}(\mathcal{K}(B, V)),$$

are induced by the boundary homomorphisms,

$$\partial_q : \mathbb{C}_q(\mathcal{K}(B_I, V)) \rightarrow \mathbb{C}_{q-1}(\mathcal{K}(B_J, V)),$$

and the horizontal differentials

$$(3.6) \quad \delta_{p, q} : \mathcal{N}_{p, q}(\mathcal{K}(B, V)) \rightarrow \mathcal{N}_{p+1, q}(\mathcal{K}(B, V))$$

are defined by

$$(\delta_{p, q}(\varphi))_J = \sum_{j \in J} i_{J \setminus \{j\}, J}(\varphi_{J \setminus \{j\}}),$$

where $J \subset [m], \#(J) = p + 1$,

$$\varphi \in \mathcal{N}_{p, q}(\mathcal{K}(B, V)) = \bigoplus_{J \subset [m], \#(J)=p+1} \mathbb{C}_\bullet(\mathcal{K}(B_J, V)),$$

and for $I \subset J \subset [m]$

$$i_{I, J} : \mathbb{C}_\bullet(\mathcal{K}(B_I, V)) \rightarrow \mathbb{C}_\bullet(\mathcal{K}(B_J, V))$$

denotes the homomorphism induced by inclusion.

For a proof of (3.3) see [6].

Using (3.3), we are able to compute the Betti numbers of W^h using only linear algebra, once we have computed the various complexes $\mathcal{K}(B_I, V)$, as well as the homomorphisms $i_{I, J}$ for all $I \subset J \subset [m]$ using Algorithm 3. Moreover, the complexity of this algorithm is asymptotically the same as that of Algorithm 3.

We now formally describe this algorithm.

ALGORITHM 4 (Betti numbers, homogeneous intersection case).

INPUT

- A family of polynomials, $\mathcal{Q}^h = \{Q_1^h, \dots, Q_m^h\} \subset \mathbf{R}[Y_0, \dots, Y_\ell, X_1, \dots, X_k]$, homogeneous of degree 2 with respect to Y_0, \dots, Y_ℓ , $\deg_X(Q^h) \leq d, Q^h \in \mathcal{Q}^h$,
- another family, $\mathcal{P} \subset \mathbf{R}[X_1, \dots, X_k]$ with $\deg_X(P) \leq d, P \in \mathcal{P}, \#(\mathcal{P}) = s$,
- a formula Φ defining a bounded \mathcal{P} -closed semi-algebraic set V .

OUTPUT the Betti numbers $b_i(W^h)$, where W^h is the semi-algebraic set defined by

$$W^h = \bigcap_{Q^h \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q^h(y, x) \leq 0 \wedge \Phi(x)\}.$$

PROCEDURE

Step 1. Call Algorithm 3 (Computing the collection of $\mathcal{K}(B_I, V)$) to compute for each $I \subset J \subset [m]$, the complex $C_\bullet(\mathcal{K}(B_I, V))$ using the natural basis consisting of the cells of $\mathcal{K}(B_I, V)$ of various dimensions, as well as the matrices in this basis for the inclusion homomorphisms

$$i_{I,J} : C_\bullet(\mathcal{K}(B_I, V)) \rightarrow C_\bullet(\mathcal{K}(B_J, V)).$$

Step 2. Using the data from the previous step, compute matrices corresponding to the differentials in the complex, $\text{Tot}_\bullet(\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B, V)))$, where $\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B, V))$ is the bi-complex described by (3.4)-(3.6).

Step 3. Compute, using linear algebra subroutines

$$b_i(W^h) = H_i(\text{Tot}_\bullet(\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B, V)))).$$

COMPLEXITY ANALYSIS: The complexity of the algorithm is dominated by the first step, whose complexity is $(s\ell md)^{2^{O(m+k)}}$, using the complexity of Algorithm 3. \square

PROOF OF CORRECTNESS: The correctness of the algorithm is a consequence of the correctness of Algorithm 3 and (3.3). \square

3.5. Computing Betti numbers of general $\mathcal{P} \cup \mathcal{Q}$ -closed sets. Let $S \subset \mathbb{R}^{\ell+k}$ be a semi-algebraic set defined by a $\mathcal{P} \cup \mathcal{Q}$ closed formula Φ .

Let $\bar{\Sigma}_{\mathcal{Q}}$ denote the set of all possible weak sign conditions on the family \mathcal{Q} , i.e.

$$\bar{\Sigma}_{\mathcal{Q}} = \{0, \{0, 1\}, \{0, -1\}\}^{\mathcal{Q}}.$$

In the last section we defined a bi-complex $\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B, V))$ whose total complex has homology groups isomorphic to those of the semi-algebraic set $W^h = \mathcal{R}(\rho^h \cap \phi)$, where $\rho \in \bar{\Sigma}_{\mathcal{Q}}$ is given by $\rho(Q_i) = \{0, -1\}$ for each $i, 1 \leq i \leq m$, and ρ^h is obtained from ρ by replacing each $Q_i \in \mathcal{Q}$ by Q_i^h . We now generalize this definition to the case of multiple weak sign conditions. More precisely, given a set $\Sigma = \{\rho_1, \dots, \rho_N\} \subset \bar{\Sigma}_{\mathcal{Q}}$, we define a corresponding bi-complex having properties similar to that of $\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B, V))$, but now with respect to Σ instead of a single weak sign condition ρ .

Without loss of generality we can write Φ in the form

$$\Phi = \bigvee_{\rho \in \bar{\Sigma}_{\mathcal{Q}}} \rho \wedge \phi_\rho,$$

where each ϕ_ρ is a \mathcal{P} -closed formula.

Let

$$\begin{aligned} W_\rho &= \mathcal{R}(\rho \wedge \phi_\rho, \mathbb{R}^{\ell+k}), \\ V_\rho &= \mathcal{R}(\phi_\rho, \mathbb{R}^k). \end{aligned}$$

Let $1 \gg \varepsilon > 0$ be an infinitesimal, and let

$$Q_0 = \varepsilon^2(Y_1^2 + \cdots + Y_\ell^2) - 1,$$

$$P_0 = \varepsilon^2(X_1^2 + \cdots + X_k^2) - 1,$$

and $S_b \subset \mathbf{R}\langle \varepsilon \rangle^{\ell+k}$ be the semi-algebraic set defined by

$$S_b = \bigcap_{i=0}^m \{(y, x) \mid Q_0(y) \leq 0 \wedge P_0(x) \leq 0 \wedge \Phi(x)\}.$$

We denote by Φ_b (resp. $\phi_{\rho,b}$) the formula $(Q_0(y) \leq 0) \wedge (P_0(x) \leq 0) \wedge \Phi$ (resp. $(P_0(x) \leq 0) \wedge \phi_\rho$).

Let $S_b^h, W_{\rho,b}^h \subset \mathbf{S}^\ell \times \mathbf{R}\langle \varepsilon \rangle^k$ be the sets defined by Φ_b and $\rho \wedge (Q_0^h \leq 0) \wedge \phi_{\rho,b}$ respectively on $\mathbf{S}^\ell \times \mathbf{R}\langle \varepsilon \rangle^k$ after replacing each $Q_i \in \mathcal{Q}$ by Q_i^h in the formulas Φ_b and ρ .

Let

$$V_{\rho,b} = \mathcal{R}(\phi_{\rho,b}, \mathbf{R}\langle \varepsilon \rangle^k).$$

Let $\mathcal{Q}_\pm^h = \{\pm Q^h \mid Q^h \in \mathcal{Q}^h\}$, and let $\mathcal{K}(B, V)$ denote the complex constructed by Algorithm 3, with input the families of polynomials \mathcal{Q}_\pm^h , $\mathcal{P}_b = \mathcal{P} \cup \{P_0\}$, and the semi-algebraic subset $V = B_k(0, 1/\varepsilon)$.

It follows from the correctness of Algorithm 4 that for each $\rho \in \bar{\Sigma}_\mathcal{Q}$, there exists $J_\rho \subset \mathcal{Q}_\pm^h$, and a subcomplex, $\mathcal{K}(B_{J_\rho}, V_{\rho,b}) \subset \mathcal{K}(B, V)$, such that

$$\mathrm{H}_*(\mathrm{Tot}_\bullet(\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B_{J_\rho}, V_{\rho,b})))) \cong \mathrm{H}_*(W_{\rho,b}^h).$$

More generally, for any $\Sigma = \{\rho_1, \dots, \rho_N\} \subset \bar{\Sigma}_\mathcal{Q}$, there exists a subcomplex, $\mathcal{K}(B_{J_\rho}, V_{\rho_1,b} \cap \cdots \cap V_{\rho_N,b}) \subset \mathcal{K}(B, V)$, such that the homology groups of the complex

$$(3.7) \quad \mathrm{C}_{\Sigma,\bullet} = \mathrm{Tot}_\bullet(\mathcal{N}_{\bullet,\bullet}(\mathcal{K}(B_{J_\rho}, V_{\rho_1,b} \cap \cdots \cap V_{\rho_N,b})))$$

are naturally isomorphic to those of $W_{\rho,b}^h$, where ρ is the common refinement of ρ_1, \dots, ρ_N defined by

$$(3.8) \quad \rho(P) = \bigcap_{i=0}^N \rho_i(P)$$

for each $P \in \mathcal{A}$. Moreover, for $\Sigma \subset \Sigma' \subset \bar{\Sigma}_\mathcal{Q}$, there exists a natural homomorphism,

$$i_{\Sigma,\Sigma'} : \mathrm{C}_{\Sigma',\bullet} \rightarrow \mathrm{C}_{\Sigma,\bullet}$$

such that the induced homomorphism,

$$i_{\Sigma,\Sigma',*} : \mathrm{H}_*(\mathrm{C}_{\Sigma',\bullet}) \rightarrow \mathrm{H}_*(\mathrm{C}_{\Sigma,\bullet})$$

is the one induced by the inclusion

$$\bigcap_{\rho \in \Sigma'} W_{\rho,b}^h \hookrightarrow \bigcap_{\rho \in \Sigma} W_{\rho,b}^h.$$

Definition 3.10. Let $\mathrm{C}_\bullet(\Phi)$ denote the complex defined by

$$(3.9) \quad \mathrm{C}_\bullet(\Phi) = \mathrm{Tot}_\bullet(\mathcal{N}_{\bullet,\bullet}(\Phi)),$$

where

$$(3.10) \quad \mathcal{N}_{p,q}(\Phi) = \bigoplus_{\Sigma \subset \bar{\Sigma}_\mathcal{Q}, \#(\Sigma)=p+1} \mathrm{C}_{\Sigma,q}.$$

The vertical and horizontal homomorphisms in the complex $\mathcal{N}_{\bullet, \bullet}(\Phi)$ are induced by the differentials in the individual complexes $C_{\Sigma, \bullet}$ and the inclusion homomorphisms $i_{\Sigma, \Sigma'}$ respectively.

By the properties of the complexes $C_{\Sigma, \bullet}$ stated above and the exactness of the generalized Mayer-Vietoris sequence, we obtain

Theorem 3.11.

$$H_*(S_b^h) \cong H_*(C_{\bullet}(\Phi)).$$

We are now in a position to describe formally the algorithm for computing all the Betti numbers of a given $\mathcal{P} \cup \mathcal{Q}$ -closed set S .

3.5.1. *Description of the algorithm in the general case.*

ALGORITHM 5 (Betti numbers, general case).

INPUT

- A family of polynomials $\mathcal{Q} = \{Q_1, \dots, Q_m\} \subset \mathbb{R}[Y_1, \dots, Y_\ell, X_1, \dots, X_k]$, with $\deg_Y(Q_i) \leq 2, \deg_X(Q_i) \leq d, 1 \leq i \leq m$,
- another family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ with $\deg(P) \leq d, P \in \mathcal{P}$,
- a $\mathcal{Q} \cup \mathcal{P}$ -closed semi-algebraic set S defined by a $\mathcal{Q} \cup \mathcal{P}$ -closed formula Φ .

OUTPUT the Betti numbers $b_0(S), \dots, b_{k+\ell-1}(S)$.

PROCEDURE

Step 1. Define $Q_0 = \varepsilon_0^2(Y_1^2 + \dots + Y_\ell^2) - 1, P_0 = \varepsilon_0^2(X_1^2 + \dots + X_k^2) - 1$. Replace S by $\mathcal{R}(S, \mathbb{R}\langle\varepsilon\rangle) \cap (\mathcal{R}(P_0 \leq 0) \times \mathcal{R}(Q_0 \leq 0))$.

Step 2. Define $\mathcal{Q}_{\pm}^h = \{\pm Q^h \mid Q^h \in \mathcal{Q}^h\} \cup \{Q_0^h\}$, and let $\mathcal{K}(B, V)$ denote the complex constructed by Algorithm 3, with input the families of polynomials $\mathcal{Q}_{\pm}^h, \mathcal{P}_b$, and the semi-algebraic set $V = B_k(0, 1/\varepsilon) \subset \mathbb{R}\langle\varepsilon\rangle^k$.

Step 3. Compute, using the definitions given above, the matrices corresponding to the differentials in the complex $C_{\bullet}(\Phi)$.

Step 4. Compute, using linear algebra subroutines, for each $i, 0 \leq i \leq k + \ell - 1$

$$b_i(S_b^h) = H_i(C_{\bullet}(\Phi)).$$

Step 5. Output for each $i, 0 \leq i \leq k + \ell - 1$,

$$b_i(S) = \frac{1}{2} b_i(S_b^h).$$

PROOF OF CORRECTNESS: The correctness of the algorithm follows from Theorem 3.11 and the correctness of Algorithm 3. \square

COMPLEXITY ANALYSIS: Since $\#\bar{\Sigma}_{\mathcal{Q}} = 3^m$, the number of subsets that enters in the definition of $\mathcal{N}_{\bullet, \bullet}(\Phi)$ (cf. (3.10)) is at most 2^{3^m} . The complexity of the algorithm is now seen to be $(slmd)^{2^{O(m+k)}}$, using the complexity of Algorithm 3. \square

Proof of Theorem 1.3. The correctness and complexity analysis of Algorithm 5 also proves Theorem 1.3. \square

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