POLYNOMIAL PARTITIONING ON VARIETIES OF CODIMENSION TWO AND POINT-HYPERSURFACE INCIDENCES IN FOUR DIMENSIONS

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ABSTRACT. We present a polynomial partitioning theorem for finite sets of points in the real locus of a complex algebraic variety of codimension at most two. This result generalizes the polynomial partitioning theorem on the Euclidean space of Guth and Katz, and its extension to hypersurfaces by Zahl and by Kaplan, Matoušek, Sharir and Safernová.

We also present a bound for the number of incidences between points and hypersurfaces in the four-dimensional Euclidean space. It is an application of our partitioning theorem together with the refined bounds for the number of connected components of a semi-algebraic set by Barone and Basu.

Contents

1. Introduction

The polynomial partitioning method was introduced by Guth and Katz in their seminal paper [GK10]. Applying it in conjunction with the Elekes' framework [ES11], they made a breakthrough in a long-standing problem of Erdős on the number of distinct distances between points in the plane, by nearly proving the distinct distances conjecture. Subsequently, this method has been applied to produce other new results and simpler proofs of known results in discrete geometry, see for instance [KMS12, ST12].

The Guth-Katz polynomial partitioning method gives a nonlinear decomposition of the Euclidean space, which plays a role analogous to cuttings or trapezoidal decompositions in the more classical Clarkson-Shor type divide-and-conquer arguments for such problems, see for instance [CEG+90].

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It can be summarized in the result below. For a polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$, we denote by $V(g)$ its set of zeros in $\mathbb{C}^d$ and, for a finite set $\mathcal{Q}$, we denote by $\mathrm{card}(\mathcal{Q})$ its cardinality.

Theorem 1.1 (Guth and Katz [GK10])**.** *Let $d \geq 1$ and $\mathcal{P} \subset \mathbb{R}^d$ be a finite subset. Given $\ell \geq 1$, there is a nonzero polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$ of degree bounded by ℓ such that, for each connected component C of $\mathbb{R}^d \setminus V(g)$,*

$$\mathrm{card}(\mathcal{P} \cap C) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\ell^d}\Big),$$

where the implicit constant in the O-notation depends only on d.

When applying this result in a concrete situation, one needs to couple it with a suitable bound for the number of connected components of the semi-algebraic set $\mathbb{R}^d \setminus V(g)$. This is provided by the classical works of Oleĭnik, Petrovskiĭ, Milnor and Thom on the Betti numbers of semi-algebraic varieties [PO49, Mil64, Tho65], which allow to treat the points in $\mathcal{P}$ outside the hypersurface $V(g)$.

However, it is possible that many, or even all, of the points in $\mathcal{P}$ are contained in this hypersurface. The points in $\mathcal{P} \cap V(g)$ are not partitioned, and a separate argument is needed for handling them. The natural approach would be to apply a polynomial partitioning theorem on $V(g)$ together with a suitable bound for the number of connected components of the resulting partition. After this step, it also possible that many of the points in $\mathcal{P} \cap V(g)$ are contained in the partitioning subvariety of codimension 2. Then one would like to apply a partitioning theorem on this subvariety, and so on.

To make this strategy work efficiently, one needs a polynomial partitioning theorem on varieties. For hypersurfaces, such a result has been achieved independently by Zahl [Zah13] and by Kaplan, Matoušek, Sharir and Safernová [KMSS12], and applied to incidence problems in $\mathbb{R}^3$. Extending it to varieties of arbitrary codimension has been identified as a major obstacle to apply the polynomial partitioning method to incidence problems in dimension $d \geq 4$, see for instance the discussion in [KMSS12, §3]. Our main objective in this paper is to present such a result for irreducible varieties of codimension two.

Given an irreducible algebraic variety $X \subset \mathbb{C}^d$ we denote by $\dim(X)$ and $\deg(X)$ its dimension and degree, respectively. We also denote by $\delta(X)$ the minimal integer $\delta \geq 1$ such that X is an irreducible component of the zero set of a family of polynomials of degree bounded by δ. These invariants are related by the inequalities in (2.1), namely

$$\delta(X) \leq \deg(X) \leq \delta(X)^{d-e}.$$

The following is a simplified version of our polynomial partitioning theorem (Theorem 3.1).

Theorem 1.2. *Let $d \geq 1$ and $X \subset \mathbb{C}^d$ an irreducible variety of codimension at most two. Let $\mathcal{P} \subset \mathbb{R}^d \cap X$ be a finite subset and $\ell \geq 6\, d\, \delta(X)$. Then there is a polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$ of degree bounded by ℓ with $\dim(X \cap V(g)) = \dim(X) - 1$ such that, for each connected component C of $\mathbb{R}^d \setminus V(g)$,*

$$\mathrm{card}(\mathcal{P} \cap C) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^{\dim(X)}}\Big).$$

When $X = \mathbb{C}^d$, the invariant $\delta(X)$ is equal to 1 whereas, when X is a hypersurface, it coincides with $\deg(X)$. Hence, Theorem 1.2 reduces in these cases to

Theorem 1.1 and to the polynomial partitioning theorems in [Zah13, KMSS12], respectively.

As for the Guth-Katz theorem, the proof of this result is based on the ham sandwich theorem obtained by Stone and Tukey from the Borsuk-Ulam theorem. The new key ingredient is the systematic use of the upper and lower bounds for Hilbert functions due to Chardin [Cha89] and Chardin and Philippon [CP99].

Remark 1.3. The polynomial partition method also applies to problems in computational geometry, in particular to range searching with semi-algebraic sets. Concurrently with this paper, Matoušek and Safernová have also obtained a polynomial partitioning theorem on varieties [MS14, Theorem 1.1], focused on obtaining efficient range searching algorithms. For varieties of codimension two, their result is weaker than ours. Nevertheless, it is strong enough for their application to range searching.

As a test case for Theorem 1.2, we consider the problem of bounding the number of point-hypersurface incidences. Given a set $\mathcal{P}$ of points of $\mathbb{R}^d$ and a set $\mathcal{V}$ of subvarieties of $\mathbb{R}^d$ or of $\mathbb{C}^d$, we denote by $I(\mathcal{P},\mathcal{V})$ their number of incidences, that is, the number of pairs $(p,V)\in\mathcal{P}\times\mathcal{V}$ with $p\in V$.

The following fundamental result was proved by Szemerédi and Trotter in 1983, in response to a problem of Erdős.

Theorem 1.4 (Szemerédi and Trotter [ST83])**.** *Let $\mathcal{P}$ be a set of m points of $\mathbb{R}^2$ and $\mathcal{L}$ a set of n lines in $\mathbb{R}^2$. Then*

$$I(\mathcal{P},\mathcal{L}) = O(m^{\frac{2}{3}}n^{\frac{2}{3}} + m + n).$$

This theorem has led to an extensive study of incidences of points and curves in the plane, and of points and varieties in higher dimensions. In particular, it was extended by Pach and Sharir to incidences between points in the plane and curves having a bounded degree of freedom [PS98]. Later on, Zahl obtained an analogous result for the incidences between points in $\mathbb{R}^3$ and algebraic surfaces having a bounded degree of freedom [Zah13]. A similar result was independently obtained by Kaplan, Matoušek, Sharir and Safernová for the incidences between points in $\mathbb{R}^3$ and unit spheres [KMSS12].

We present the following bound for the number of incidences between points in $\mathbb{R}^4$ and threefolds.

Theorem 1.5. *Given $k,c\geq 1$, let $\mathcal{P}$ be a finite set of points of $\mathbb{R}^4$ and $\mathcal{H}$ a finite set of hypersurfaces of $\mathbb{C}^4$ satisfying the following conditions:*

(a) *the degrees of the hypersurfaces in $\mathcal{H}$ are bounded by c;*
(b) *the intersection of any family of four distinct hypersurfaces in $\mathcal{H}$ is finite;*
(c) *for any subset of k distinct points in $\mathcal{P}$, the number of hypersurfaces in $\mathcal{H}$ containing them is bounded by c.*

Set $m=\operatorname{card}(\mathcal{P})$ and $n=\operatorname{card}(\mathcal{H})$. Then

$$I(\mathcal{P},\mathcal{H}) = O_{h,k}(m^{1-\frac{k-1}{4k-1}}n^{1-\frac{3}{4k-1}} + m + n).$$

This result is an application of Theorem 1.2 together with the refined bounds for the number of connected components of a semi-algebraic set due to Barone and Basu [BB12, BB13]. Our whole approach is strongly inspired in the treatment of the unit distance problem in three dimensions in [Zah13, KMSS12].

Theorem 1.5 is a particular case of a conjectural bound for the number of point-hypersurface incidences in $\mathbb{R}^d$ (Conjecture 4.1). Related with this, we propose two further conjectures: a generalization of our polynomial partitioning theorem to varieties of arbitrary codimension (Conjecture 3.4) and a bound for the number of connected components of a semi-algebraic set depending on the degree of that variety, instead of the Bézout number of a set of defining equations (Conjecture 2.8). If one can show that these two conjecture are true, it would be an important step in proving Conjecture 4.1 *via* the polynomial partitioning method.

Remark 1.6. The results of this paper were announced in the talk [Som14] at the IPAM workshop "Tools from algebraic geometry". Shortly afterwards, a proof by Fox, Pach, Suk, Sheffer and Zahl of a weaker version of Conjecture 4.1 with an extra factor m^ε was announced in Sheffer's blog [She14] and eventually appeared in [FPS+14].

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2. Preliminaries on Hilbert functions and semi-algebraic geometry

Throughout this paper, we denote by $\mathbb{N}$ the set of nonnegative integers. Bold letters denote finite sets or sequences of objects, where the type and number should be clear from the context: for instance, $\boldsymbol{x}$ might denote the group of variables $\{x_1, \dots, x_d\}$ so that $\mathbb{R}[\boldsymbol{x}]$ denotes the polynomial ring $\mathbb{R}[x_1, \dots, x_d]$.

Given functions $f, g\colon \mathbb{N} \to \mathbb{N}$, the Landau symbol $f = O(g)$ means that there exists $c \geq 0$ such that $f(l) \leq c\, g(l)$ for all $l \in \mathbb{N}$. If we want to emphasize the dependence of the constant c on parameters, say d and k, we will write $f = O_{d,k}(g)$.

2.1. **Hilbert functions.** Given a homogeneous ideal $I \subset \mathbb{C}[z_0, \dots, z_d]$, the quotient $\mathbb{C}[z_0, \dots, z_d]/I$ is a graded $\mathbb{C}$-algebra. The *Hilbert function* of I is the function $\mathrm{H}_I\colon \mathbb{N} \to \mathbb{N}$ given, for $\ell \in \mathbb{N}$, by the dimension of the ℓ-th graded piece of this quotient, that is

$$\mathrm{H}_I(\ell) = \dim_{\mathbb{C}} \big(\mathbb{C}[z_0, \dots, z_d]/I\big)_\ell.$$

By Hilbert's theorem, there is a polynomial $\mathrm{P}_I \in \mathbb{Q}[t]$ and an integer $\ell_0 \in \mathbb{N}$ with

$$\mathrm{H}_I(\ell) = \mathrm{P}_I(\ell) \quad \text{for } \ell \geq \ell_0.$$

Let $\mathbb{P}^d(\mathbb{C})$ denote the d-dimensional projective space over the complex numbers. For an irreducible subvariety X of $\mathbb{P}^d(\mathbb{C})$, we denote by $I(X) \subset \mathbb{C}[z_0, \dots, z_d]$ its defining ideal, and by $\dim(X)$ and $\deg(X)$ its dimension and degree, respectively. Then $\mathrm{P}_{I(X)}$ is a polynomial of degree $\dim(X)$ and leading coefficient equal to the quotient $\deg(X)/\dim(X)!$.

In Theorem 2.3 below, we collect the upper and lower bounds for Hilbert functions that we will use later on. Because of our applications, we restrict to ideals coming from irreducible projective varieties, although these bounds are valid in greater generality.

Definition 2.1. Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety and $\delta \geq 1$. We say that X is *set-theoretically locally defined at degree* δ if there are homogeneous polynomials $g_1, \dots, g_t \in \mathbb{C}[z_0, \dots, z_d]$ of degree bounded by δ such that X is an irreducible component of the zero set of these polynomials in $\mathbb{P}^d(\mathbb{C})$. We denote by $\delta(X)$ the minimal integer $\delta \geq 1$ such that X is set-theoretically locally defined at degree δ.

With notation as in Definition 2.1, we have that

$$\delta(X) \leq \deg(X) \leq \delta(X)^{d-\dim(X)}. \tag{2.1}$$

The first inequality follows by considering the image of X under a generic linear map from $\mathbb{P}^d(\mathbb{C})$ onto $\mathbb{P}^{\dim(X)+1}(\mathbb{C})$, whereas the second one follows from Bézout theorem.

In the special case when X is of codimension 2, we have the following sharpening of the second inequality in (2.1). It is also an easy consequence of Bézout theorem.

Lemma 2.2. *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety of codimension 2. Let $\delta_1 \geq 1$ be the minimal degree of a hypersurface of $\mathbb{P}^d(\mathbb{C})$ containing X and set also $\delta_2 = \delta(X)$. Then*

$$\deg(X) \leq \delta_1 \delta_2.$$

Recall that binomial coefficients are defined, for $i, n \in \mathbb{Z}$, by

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety of dimension $e \geq 0$.*

(a) *For $\ell \geq 0$,*

$$\mathrm{H}_{I(X)}(\ell) \leq \deg(X) \binom{\ell + e}{e}.$$

(b) *For $\ell \geq (d-e)(\delta(X)-1)+1$,*

$$\mathrm{H}_{I(X)}(\ell) \geq \deg(X) \binom{\ell - (d-e)(\delta(X)-1) + e}{e}.$$

Proof. These are particular cases of [Cha89, Théorème] and [CP99, Corollaire 3], respectively. Similar, though weaker, bounds can be found in [Som14]. □

The following result is an easy consequence of Theorem 2.3(a), see [Cha89, Corollaire 3] for details.

Proposition 2.4. *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety of codimension 2. Then there are coprime polynomials $f_1, f_2 \in I(X)$ such that*

$$\deg(f_1)\deg(f_2) \leq d(d-1)\deg(X).$$

The next result gives a lower bound for the Hilbert function of the ideal of a variety X of codimension two. For $\ell \geq 0$, it exhibits three different behaviors, depending on the codimension of the zero set of the graded part $I(X)_\ell$.

Proposition 2.5. *There is a constant $c = c(d) > 0$ with the following property. Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible subvariety of codimension 2. Let $\delta_1 \geq 1$ be the minimal degree of a hypersurface of $\mathbb{P}^d(\mathbb{C})$ containing X and set $\delta_2 = \delta(X)$. Then*

$$\mathrm{H}_{I(X)}(\ell) \geq \begin{cases} c\,(\ell+1)^d + 1 & \text{if } 1 \leq \ell \leq \delta_1 - 1, \\ c\,\delta_1(\ell+1)^{d-1} + 1 & \text{if } \delta_1 \leq \ell \leq \delta_2 - 1, \\ c\,\delta_1\delta_2(\ell+1)^{d-2} + 1 & \text{if } \delta_2 \leq \ell. \end{cases}$$

Proof. For $1 \leq \ell \leq \delta_1 - 1$, we have that $(\mathbb{C}[\boldsymbol{z}]/I(X))_\ell = \mathbb{C}[\boldsymbol{z}]_\ell$. Hence

$$\mathrm{H}_{I(X)}(\ell) = \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}]_\ell = \binom{\ell+d}{d} \geq c_1(\ell+1)^d + 1 \tag{2.2}$$

for a suitable constant $c_1 > 0$ depending on d, which gives the first lower bound.

Now consider the case $\delta_1 \leq \ell \leq \delta_2 - 1$ and let f_1 be a nonzero polynomial in $I(X)$ of degree δ_1. Then $\mathrm{H}_{I(X)}(\ell) = \mathrm{H}_{(f_1)}(\ell)$. By the exact sequence

$$0 \longrightarrow \mathbb{C}[\boldsymbol{z}] \xrightarrow{\times f_1} \mathbb{C}[\boldsymbol{z}] \longrightarrow \mathbb{C}[\boldsymbol{z}](f_1) \longrightarrow 0,$$

we have that $\mathrm{H}_{(f_1)}(\ell)$ is equal to

$$\dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}]_\ell - \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}]_{\ell-\delta_1} = \binom{\ell+d}{d} - \binom{\ell-\delta_1+d}{d} = \sum_{j=0}^{\delta_1-1} \binom{\ell-j+d-1}{d-1}.$$

It follows that, for $\delta_1 \leq \ell \leq \delta_2 - 1$,

$$\mathrm{H}_{I(X)}(\ell) \geq \frac{\delta_1}{2(d-1)!}\Big(\ell - \frac{\delta_1}{2}\Big)^{d-1} + 1 \geq c_2\delta_1(\ell+1)^{d-1} + 1 \tag{2.3}$$

for another constant $c_2 = c_2(d) > 0$.

Finally, we consider the case when $\ell \geq \delta_2$. When $\ell \leq 2\,d\,\delta_2 - 1$, we deduce from (2.3) that

$$\mathrm{H}_{I(X)}(\ell) \geq \mathrm{H}_{I(X)}(\delta_2 - 1) \geq c_2\delta_1\delta_2^{d-1} + 1 \geq c_3\delta_1\delta_2(\ell+1)^{d-2} + 1. \tag{2.4}$$

It follows immediately from Proposition 2.4 that $d(d-1)\deg(X) \geq \delta_1\delta_2$. Hence, for $\ell \geq 2\,d\,\delta_2$, by Theorem 2.3(b) we have that

$$\mathrm{H}_{I(X)}(\ell) \geq \deg(X)\binom{\ell - d\delta_2 + d - 2}{d-2} \geq c_4\delta_1\delta_2(\ell+1)^{d-2} + 1. \tag{2.5}$$

The result follows from (2.2), (2.3), (2.4) and (2.5) by taking $c = \min_i c_i$. □

2.2. **Connected components of semi-algebraic sets.** As explained in the introduction, the polynomial partitioning method has to be coupled with bounds for the number of connected components of semi-algebraic sets. When partitioning the Euclidean space $\mathbb{R}^d$, the appropriate bound follows from the Oleĭnik-Petrovskiĭ-Milnor-Thom's bounds for the Betti numbers of a semi-algebraic set [PO49, Mil64, Tho65]: with notation as in Theorem 1.1, the number of connected components of $\mathbb{R}^d \setminus V(g)$ is bounded by $\ell(2\ell-1)^{d-1} = O(\ell^d)$.

In our situation, we will need the Barone-Basu bound for the number of connected components, with a refined dependence on the degrees of the polynomials [BB12, BB13]. We recall a simplified version of this result in Theorem 2.6 below.

Given $f_1, \dots, f_e \in \mathbb{R}[x_1, \dots, x_d]$, we denote by $V(f_1, \dots, f_e)$ its set of common zeros in $\mathbb{C}^d$. For a variety $X \subset \mathbb{C}^d$, we denote by $X(\mathbb{R}) = X \cap \mathbb{R}^d$ its set of real points. For a semi-algebraic subset $S \subset \mathbb{R}^d$, we denote by $\mathrm{cc}(S)$ the set of connected components of S. The 0-th Betti number $\mathrm{b}_0(S)$ coincides with the cardinality of the set $\mathrm{cc}(S)$.

Theorem 2.6. *There is a constant $c = c(d)$ with the following property. Let $f_1, \dots, f_e, g \in \mathbb{R}[x_1, \dots, x_d]$ with $\deg(f_1) \leq \dots \leq \deg(f_e) \leq \deg(g)$ such that $\dim(V(f_1, \dots, f_i)) = d - i$ for $i = 1, \dots, e$. Then both*

$$\mathrm{b}_0(V(f_1, \dots, f_e)(\mathbb{R}) \setminus V(g)) \quad \text{and} \quad \mathrm{b}_0(V(f_1, \dots, f_e, g)(\mathbb{R}))$$

are bounded by $c\deg(f_1)\dots\deg(f_e)\deg(g)^{d-e}$.

Proof. The semi-algebraic set $V(f_1,\dots,f_e)(\mathbb{R})\setminus V(g)$ is the union of the realization of the sign conditions ±1 of g on $V(f_1,\dots,f_e)(\mathbb{R})$. Similarly, $V(f_1,\dots,f_e,g)(\mathbb{R})$ is the realization of the sign condition 0 of g on the same real algebraic variety.

The result follows from [BB13, Theorem 4] and the fact that $\dim(V(f_1,\dots,f_i))$ bounds from above the dimension of the semi-algebraic set $V(f_1,\dots,f_i)(\mathbb{R})$, see Remark 1.10 in *loc. cit.* □

We will also need the technical result below. Given $p\in\mathbb{R}^d$ and $r>0$, we denote by $\mathrm{B}(p,r)$ the open ball in $\mathbb{R}^d$ with center p and radius r. Given a variety $W\subset\mathbb{C}^d$ and a hypersurface $H\subset\mathbb{C}^d$, we denote by $B(W,H)$ the subset of $W(\mathbb{R})$ of points $p\in W(\mathbb{R})$ having an open neighborhood, in the Euclidean topology of $W(\mathbb{R})$, contained in H. We also set $G(W,H)=W(\mathbb{R})\setminus B(W,H)$.

Proposition 2.7. *Let $W\subset\mathbb{C}^4$ be a variety and $H,K\subset\mathbb{C}^4$ two hypersurfaces. Let $b\in\mathbb{R}[x_1,x_2,x_3,x_4]$ be a polynomial defining H. Then there exists $\varepsilon_0>0$ such that, for all $0<\varepsilon\le\varepsilon_0$ and any variety $\widetilde{W}\subset\mathbb{C}^4$ containing W, the number of connected components C of $\mathbb{R}^4\setminus K$ such that $C\cap G(W,H)\cap H\neq\emptyset$ is bounded by*

$$\mathrm{b}_0((\widetilde{W}(\mathbb{R})\cap V(b^2-\varepsilon))\setminus K).$$

Proof. Consider the set of connected components

$$\mathcal{C}=\{C\in\mathrm{cc}(\mathbb{R}^4\setminus K)\mid C\cap G(W,H)\cap H\neq\emptyset\}.$$

For each $C\in\mathcal{C}$ choose a point $p_C\in C\cap G(W,H)\cap H$. Since $\mathcal{C}$ is a finite set, the set of points $\{p_C\}_{C\in\mathcal{C}}$ is finite. For each $C\in\mathcal{C}$ and $r>0$, consider also the semi-algebraic set given by

$$U_r(p_C)=\mathrm{B}(p_C,r)\cap(W(\mathbb{R})\setminus K).$$

By the definition of $G(W,H)$, the set $U_r(p_c)$ is not contained in H. Semi-algebraic sets are locally contractible because of their local conical structure, see for instance [BPR06, Theorem 5.48]. Hence, there exists $r_C>0$ such that, for all $0<r\le r_C$, the set $U_r(p_C)$ is contractible and, in particular, connected. Set $r_0=\min_C r_C$.

Choose also $q_C\in U_{r_0}(p_C)\setminus H$ and a semi-algebraic path $\gamma_C:[0,1]\to U_r(p_C)$ with $\gamma(0)=p_C$ and $\gamma(1)=q_C$. We have that $b^2(q_C)>0$ because $q_C\notin H$. We set $\varepsilon_0=\min_C b^2(q_C)$.

By the intermediate value theorem, for all $C\in\mathcal{C}$ and $0<\varepsilon\le\varepsilon_0$, there exists $0<t_C\le1$ such that $b^2(z_C)=\varepsilon$ with $z_C=\gamma_C(t_C)$. By construction,

$$z_C\in(W(\mathbb{R})\cap V(b^2-\varepsilon))\setminus K\subset(\widetilde{W}(\mathbb{R})\cap V(b^2-\varepsilon))\setminus K.$$

Moreover, $z_C\in C$ because this point is connected by a path to p_C. For $C,C'\in\mathcal{C}$ with $C\neq C'$, the points z_C and $z_{C'}$ belong to distinct connected components of $\widetilde{W}(\mathbb{R})\cap V(b^2-\varepsilon)\setminus K$. Hence, the map $C\mapsto z_C$ induces an injection between the set of connected components $\mathcal{C}$ and $\mathrm{cc}(\widetilde{W}(\mathbb{R})\cap V(b^2-\varepsilon))\setminus K$, which proves the proposition. □

In connection with the application of the polynomial partitioning theorem to incidence problems in higher dimensions, we propose the following conjectural bound for the number of connected components of a semi-algebraic set in terms of the degree of the variety instead of the Bézout number of a set of defining equations.

Conjecture 2.8. *Let $X \subset \mathbb{C}^d$ be an irreducible variety and $g \in \mathbb{R}[x_1, \dots, x_d]$ a polynomial of degree $\ell \geq \delta(X)$. Then there exists a variety $Y \subset \mathbb{C}^d$ containing X as an irreducible component such that*

$$\mathrm{b}_0(Y(\mathbb{R}) \setminus V(g)) \quad \textit{and} \quad \mathrm{b}_0(Y(\mathbb{R}) \cap V(g))$$

are bounded by $O_d(\deg(X)\ell^{\dim(X)})$.

When X is an irreducible variety of codimension 2, this statement follows easily from Proposition 2.4 and Theorem 2.6. In this case, the variety Y is given, in the notation of Proposition 2.4, by the zero set of f_1 and f_2.

3. Partitioning finite sets on varieties

Given a set of points $\mathcal{P} \subset \mathbb{R}^d$ and a set of polynomials $\mathcal{G} \subset \mathbb{R}[x_1, \dots, x_d]$, for each the choice of signs $\gamma \in \{\pm 1\}^{\mathcal{G}}$ we put

$$\mathcal{P}(\gamma) = \{p \in \mathcal{P} \mid \gamma_q g(p) > 0 \text{ for all } g \in \mathcal{G}\}, \tag{3.1}$$

If the set of polynomials $\mathcal{G}$ is clear from the context, then we say that $\mathcal{P}(\gamma)$ is *realized* by γ.

Given $g \in \mathbb{R}[x_1, \dots, x_d] \setminus \{0\}$, we denote by $\mathrm{irr}(g) \subset \mathbb{R}[x_1, \dots, x_d]$ a complete and irredundant set of irreducible factors of g. These irreducible factors are unique up to scalars in $\mathbb{R}^\times$. To fix their indeterminacy, we choose them to be monic with respect to some fixed monomial order on $\mathbb{R}[x_1, \dots, x_d]$. With this convention, the set $\mathrm{irr}(g)$ is uniquely defined and

$$g = \lambda \prod_{q \in \mathrm{irr}(g)} q^{e_q}$$

with $\lambda \in \mathbb{R}^\times$ and $e_q \in \mathbb{N}$.

For $\ell \geq 0$, we denote by $\mathbb{R}[x_1, \dots, x_d]_{\leq \ell}$ the linear subspace of $\mathbb{R}[x_1, \dots, x_d]$ of polynomials of degree bounded by ℓ. Recall that, for a variety $X \subset \mathbb{C}^d$, we denote by $X(\mathbb{R}) = X \cap \mathbb{R}^d$ its set of real points.

We state and prove our polynomial partitioning theorem in terms of sign conditions. For convenience, we state it for varieties of codimension *at most* two, even though we prove it only when the codimension is two. The cases when the codimension is smaller are simpler and can be proven as in [GK10, Zah13, KMSS12].

Theorem 3.1. *Let $X \subset \mathbb{C}^d$ be an irreducible variety of codimension at most two, $\mathcal{P} \subset X(\mathbb{R})$ a finite subset and $\ell \geq 6\, d\, \delta(X)$. Then there exists $g \in \mathbb{R}[x_1, \dots, x_d]_{\leq \ell}$ with $\dim(X \cap V(g)) = \dim(X) - 1$ such that, for each $\gamma \in \{\pm 1\}^{\mathrm{irr}(g)}$,*

$$\mathrm{card}(\mathcal{P}(\gamma)) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^{\dim(X)}}\Big).$$

Remark 3.2. Let $S \subset \mathbb{R}^d$ be an arbitrary subset. For each connected component C of $S \setminus V(g)$, the set $\mathcal{P} \cap C$ is contained in a set of the form $\mathcal{P}(\gamma)$ with $\gamma \in \{\pm 1\}^{\mathrm{irr}(g)}$. Hence, Theorem 1.2 in the introduction follows from Theorem 3.1 above by choosing $S = \mathbb{R}^d$.

Given $\ell \geq 0$, we denote by v_ℓ the Veronese embedding $\mathbb{C}^d \hookrightarrow \mathbb{C}^{\binom{\ell+d}{d}-1}$ given, for a point $\boldsymbol{p} = (p_1, \dots, p_d) \in \mathbb{C}^d$, by

$$v_\ell(\boldsymbol{p}) = (\boldsymbol{p}^{\boldsymbol{a}})_{\boldsymbol{a}} \tag{3.2}$$

where $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$ runs over all nonzero vectors of length $|\boldsymbol{a}| = \sum_i a_i$ bounded by ℓ, and where $\boldsymbol{p}^{\boldsymbol{a}}$ denotes the monomial $p_1^{a_1} \dots p_d^{a_d}$. We also denote by ι the standard inclusion $\mathbb{C}^d \to \mathbb{P}^d(\mathbb{C})$ given by

$$\iota(\boldsymbol{p}) = (1 : p_1 : \dots : p_d).$$

For a subset $E \subset \mathbb{R}^d$, we denote by $\mathrm{aff}(E)$ the smallest affine subspace of $\mathbb{R}^d$ containing E. We also denote by $I(\iota(E)) \subset \mathbb{C}[z_0, \dots, z_d]$ the homogeneous ideal of polynomials vanishing identically on the subset $\iota(E) \subset \mathbb{P}^d(\mathbb{C})$.

Lemma 3.3. *Let $E \subset \mathbb{R}^d$ be a subset and $\ell \geq 0$. Then*

$$\dim_{\mathbb{R}}(\mathrm{aff}(v_\ell(E))) = \mathrm{H}_{I(\iota(E))}(\ell) - 1.$$

Proof. The ideal $I(\iota(E))$ is generated over $\mathbb{R}[\boldsymbol{z}]$, because it is defined by the vanishing of a set of real points. Setting $I = I(\iota(E)) \cap \mathbb{R}[\boldsymbol{z}]$, then

$$\mathrm{H}_{I(\iota(E))}(\ell) = \dim_{\mathbb{C}} \left(\mathbb{C}[\boldsymbol{z}]/I(\iota(E))\right)_\ell = \dim_{\mathbb{R}}(\mathbb{R}[\boldsymbol{z}]_\ell) - \dim_{\mathbb{R}}(I_\ell), \tag{3.3}$$

where I_ℓ denotes the ℓ-th graded part of I.

Let $\mathbb{R}^{\binom{\ell+d}{d}}$ with coordinates indexed by the vectors of $\mathbb{N}^{d+1}$ of length equal to ℓ and consider the pairing

$$\mathbb{R}[\boldsymbol{z}]_\ell \times \mathbb{R}^{\binom{\ell+d}{d}} \longrightarrow \mathbb{R}, \quad \left(\sum_{|\boldsymbol{b}|=\ell} \alpha_{\boldsymbol{b}} \boldsymbol{z}^{\boldsymbol{b}}, \boldsymbol{w}\right) \longmapsto \sum_{\boldsymbol{b}} \alpha_{\boldsymbol{b}} \boldsymbol{w}_{\boldsymbol{b}},$$

where $\boldsymbol{b}$ runs over all vectors of $\mathbb{N}^{d+1}$ of length ℓ.

The graded part I_ℓ coincides with the annihilator of $\{1\} \times v_\ell(E)$ with respect to this pairing. Since I_ℓ is a linear subspace, it also coincides with the annihilator of the linear span in $\mathbb{R}^{\binom{\ell+d}{d}}$ of this subset. Denote by $\mathrm{lin}(\{1\} \times v_\ell(E))$ this linear span, which is a linear space containing $\{1\} \times \mathrm{aff}(v_\ell(E))$ as an affine hyperplane. Hence

$$\dim_{\mathbb{R}}(\mathbb{R}[\boldsymbol{z}]_\ell) - \dim_{\mathbb{R}}(I_\ell) = \dim_{\mathbb{R}}(\mathrm{lin}(\{1\} \times v_\ell(E))) = \dim_{\mathbb{R}}(\mathrm{aff}(v_\ell(E))) + 1. \tag{3.4}$$

The result then follows from (3.3) and (3.4). □

Proof of Theorem 3.1. We assume that X is of codimension 2. Let $\delta_1 \geq 1$ be the minimal degree of a hypersurface of $\mathbb{P}^d(\mathbb{C})$ containing X and set also $\delta_2 = \delta(X)$. Let $\eta \geq \delta_2$ be an integer to be fixed later on.

Let $c = c(d)$ be the constant in Proposition 2.5 and set $c_1 = \min\{c, 2^{-d}\}$. Put

$$s_0 = \log(c_1 \delta_1^d), \quad s_1 = \log(c_1 \delta_1 \delta_2^{d-1}), \quad t = \lfloor \log(c_1 \delta_1 \delta_2 \eta^{d-2}) \rfloor$$

and

$$\ell_i = \begin{cases} \lfloor (c_1^{-1} 2^i)^{\frac{1}{d}} \rfloor & \text{for } 0 \leq i < s_0, \\ \lfloor (c_1^{-1} \delta_1^{-1} 2^i)^{\frac{1}{d-1}} \rfloor & \text{for } s_0 \leq i < s_1, \\ \lfloor (c_1^{-1} \delta_1^{-1} \delta_2^{-1} 2^i)^{\frac{1}{d-2}} \rfloor & \text{for } s_1 \leq i \leq t, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. We verify that the following conditions hold:

$$\begin{aligned} &\text{if } 0 \leq i < s_0 \text{ then } 1 \leq \ell_i \leq \delta_1 - 1, \\ &\text{if } s_0 \leq i < s_1 \text{ then } \delta_1 \leq \ell_i \leq \delta_2 - 1, \\ &\text{if } s_1 \leq i \leq t \text{ then } \delta_2 \leq \ell_i \leq \eta. \end{aligned} \tag{3.5}$$

Let v_{ℓ_i} be the Veronese map of degree ℓ_i as in (3.2) and set $A_i \subset \mathbb{R}^{\binom{\ell_i+e}{e}} - 1$ for the affine hull of the image of $X(\mathbb{R})$ under v_{ℓ_i}. Let $I(\iota(X(\mathbb{R})))$ be the ideal of polynomials vanishing on the image under ι of the set of real points of X. By Lemma 3.3,

$$\dim_{\mathbb{R}}(A_i) = \mathrm{H}_{I(\iota(X(\mathbb{R})))}(\ell_i) - 1. \tag{3.6}$$

Since $\iota(X(\mathbb{R})) \subset \iota(X)$, we have that $I(\iota(X(\mathbb{R}))) \supset I(\iota(X))$ and so

$$\mathrm{H}_{I(\iota(X(\mathbb{R})))}(\ell_i) \leq \mathrm{H}_{I(\iota(X))}(\ell_i). \tag{3.7}$$

We consider first the case when (3.7) is an equality for all i. Since the affine variety X is irreducible and locally set-theoretically defined at degree $\delta(X)$, the same holds for $\overline{\iota(X)}$, the Zariski closure of $\iota(X)$ in projective space. It follows from Proposition 2.5 and the conditions in (3.5) that

$$\mathrm{H}_{I(\iota(X(\mathbb{R})))}(\ell_i) \geq 2^i + 1, \quad i = 0, \dots, t. \tag{3.8}$$

As in the Guth-Katz polynomial partitioning, we will inductively subdivide the set of points $\mathcal{P}$. We start with $\mathcal{C}_0 = \{\mathcal{P}\}$. Having constructed $\mathcal{C}_i$ with at most 2^i sets, we apply the ham sandwich theorem to the image of these sets under the map v_{ℓ_i}. These images lie in A_i and, by (3.6) and (3.8), this is an affine space of dimension $\geq 2^i$. Hence, there is a nonzero linear form on A_i that bisects each of these images or, equivalently, there is a polynomial $g_i \in \mathbb{R}[x_1, \dots, x_d]_{\leq \ell_i}$ bisecting each of the sets in $\mathcal{C}_i$.

For each $\mathcal{Q} \in \mathcal{C}_i$, we put $\mathcal{Q}^+$ and $\mathcal{Q}^-$ for the sets of points of $\mathcal{Q}$ at which $g_i > 0$ and $g_i < 0$, respectively. We then put

$$\mathcal{C}_{i+1} = \bigcup_{\mathcal{Q} \in \mathcal{C}_i} \{\mathcal{Q}^+, \mathcal{Q}^-\}.$$

Hence, each of the sets in $\mathcal{C}_t$ has cardinality bounded by $2^{-t}\mathrm{card}(\mathcal{P})$.

Set $g = \prod_{i=0}^{t} g_i$. To bound the degree of g, we write $\deg(g) = S_0 + S_1 + S_2$ with

$$S_0 = \sum_{0 \leq i < s_0} \ell_i, \quad S_1 = \sum_{s_0 \leq i < s_1} \ell_i, \quad S_0 = \sum_{s_1 \leq i \leq t} \ell_i.$$

We have that

$$S_0 \leq \sum_{i=0}^{s_0 - 1} (c_1^{-1} 2^i)^{\frac{1}{d}} \leq c_1^{-\frac{1}{d}} \frac{2^{\frac{s_0+1}{d}} - 1}{2^{\frac{1}{d}} - 1} \leq \frac{2^{\frac{1}{d}}}{2^{\frac{1}{d}} - 1} \delta_1.$$

Similarly, one can verify that

$$S_1 \leq \frac{2^{\frac{1}{d-1}}}{2^{\frac{1}{d-1}} - 1} \delta_2 \quad \text{and} \quad S_2 \leq \frac{2^{\frac{1}{d-2}}}{2^{\frac{1}{d-2}} - 1} \eta.$$

Using that $d \geq 3$ and $\delta_1 \leq \delta_2$, we deduce that $\deg(g) \leq 4\, d\, \delta_2 + 2\, d\, \eta$. Finally, set

$$\eta = \frac{\ell}{2d} - 2\delta_2.$$

Since $\ell \geq 6\, d\, \delta_2$, we have that $\eta \geq \delta_2$ as required and $\deg(g) \leq \ell$, as stated.

On the other hand, the sets in $\mathcal{C}_t$ are realized by sign conditions given in terms of the g_i's. The sets realized by sign conditions on $\mathrm{irr}(g)$ have cardinality bounded by those in $\mathcal{C}_t$. Since $\eta \geq \frac{\ell}{6d}$, it follows that for each $\gamma \in \{\pm 1\}^{\mathrm{irr}(g)}$,

$$\mathrm{card}(\mathcal{P}(\gamma)) \leq \frac{\mathrm{card}(\mathcal{P})}{2^t} \leq c_2 \frac{\mathrm{card}(\mathcal{P})}{\deg(X) \eta^{d-2}},$$

where the last inequality follows from Lemma 2.2, and c_2 denotes a suitable constant. This proves the statement in the case when X is of codimension 2 and the inequality (3.7) are equalities for all i.

If the inequality (3.7) is strict for some i, then there is a polynomial $g_i \in I(X(\mathbb{R})) \setminus I(X)$ of degree bounded by $\ell_i \le \ell$. Hence, the hypersurface $V(g_i)$ cuts X properly and contains its set of real points. In particular, $\mathcal{P} \subset V(g_i)$. It follows that $g = g_i$ has the appropriate degree and $\mathcal{P}(\gamma) = \emptyset$ for all $\gamma \in \{\pm 1\}^{\mathrm{irr}(g)}$, which completes the proof for the case when X is of codimension two.

The cases when the codimension of X is either zero or one are simpler and can be proven as in [GK10, Zah13, KMSS12]. □

A previous version of this paper contained a polynomial partitioning theorem on varieties of arbitrary dimension. Whereas the proof of this result contained a gap, we still think that its statement is correct, and we propose it as a conjecture.

Conjecture 3.4. *There is a constant $c = c(d) > 0$ with the following property. Let $X \subset \mathbb{C}^d$ be an irreducible variety of dimension e, $\mathcal{P} \subset X(\mathbb{R})$ a finite subset and $\ell \ge c\,\delta(X)$. Then there exists $g \in \mathbb{R}[x_1, \dots, x_d]_{\le \ell}$ with $\dim(X \cap V(g)) = \dim(X) - 1$ such that, for each $\gamma \in \{\pm 1\}^{\mathrm{irr}(g)}$,*

$$\mathrm{card}(\mathcal{P}(\gamma)) \le c\,\frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^e}.$$

4. Point-hypersurface incidences

In this section we prove Theorem 1.5. To this end, we use three levels of polynomial partitioning. This leads to a partition of the Euclidean space $\mathbb{R}^4$ into semi-algebraic pieces of various dimensions. We bound separately the number of incidences contributed by the points of the set $\mathcal{P}$ in each piece. The contribution from each level of the partitioning is the essentially same, up to constant factors, as the claimed bound.

Proof of Theorem 1.5. The procedure performed at each level is similar. For clarity and ease of exposition, we prefer to describe each of these level separately, even at the expense of repeating some of the arguments.

The *set of incidences* between $\mathcal{P}$ and $\mathcal{H}$ is the subset of $\mathcal{P} \times \mathcal{H}$ defined by

$$\mathcal{I}(\mathcal{P}, \mathcal{H}) = \{(p, H) \in \mathcal{P} \times \mathcal{H} \mid p \in H\}.$$

Hence $I(\mathcal{P}, \mathcal{H}) = \mathrm{card}(\mathcal{I}(\mathcal{P}, \mathcal{H}))$. For a subset $\mathcal{Q} \subset \mathcal{P}$, we denote by

$$\mathcal{I}_{<k}(\mathcal{Q}, \mathcal{H}) = \{(p, H) \in \mathcal{I}(\mathcal{Q}, \mathcal{H}) \mid \mathrm{card}(H \cap \mathcal{Q}) < k\},$$
$$\mathcal{I}_{\ge k}(\mathcal{Q}, \mathcal{H}) = \{(p, H) \in \mathcal{I}(\mathcal{Q}, \mathcal{H}) \mid \mathrm{card}(H \cap \mathcal{Q}) \ge k\}$$

the set of incidences between $\mathcal{Q}$ and hypersurfaces of $\mathcal{H}$ containing at most $k-1$ points of $\mathcal{Q}$ and at least k points of $\mathcal{Q}$, respectively. We also set $I_{<k}(\mathcal{Q}, \mathcal{H}) = \mathrm{card}(\mathcal{I}_{<k}(\mathcal{Q}, \mathcal{H}))$ and $I_{\ge k}(\mathcal{Q}, \mathcal{H}) = \mathrm{card}(\mathcal{I}_{\ge k}(\mathcal{Q}, \mathcal{H}))$. Clearly,

$$I(\mathcal{Q}, \mathcal{H}) = I_{<k}(\mathcal{Q}, \mathcal{H}) + I_{\ge k}(\mathcal{Q}, \mathcal{H}). \tag{4.1}$$

In the sequel, the dimension d of the ambient space is fixed to 4. Hence, all implicit constants in the O-notation depend only on the parameters k and c in the statement of the theorem.

First level partitioning. Let $D \geq 24$ to be fixed later on. By Theorem 3.1, there exists $f \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq D} \setminus \{0\}$ such that, for each $\gamma \in \{\pm 1\}^{\mathrm{irr}(f)}$,

$$\mathrm{card}(\mathcal{P}(\gamma)) = O\Big(\frac{m}{D^4}\Big), \tag{4.2}$$

where $\mathcal{P}(\gamma)$ denotes the subset of $\mathcal{P}$ realized by the signs γ as in (3.1). Choose a minimal subset $\Sigma_1 \subset \{\pm 1\}^{\mathrm{irr}(f)}$ realizing all nonempty subsets of this form.

We partition $\mathcal{P}$ into the disjoint subsets $\mathcal{P}_0 = \mathcal{P} \cap V(f)$ and $\mathcal{P}(\gamma)$, $\gamma \in \Sigma_1$. Set $m_0 = \mathrm{card}(\mathcal{P}_0)$ and $m_\gamma = \mathrm{card}(\mathcal{P}(\gamma))$ for each γ. Clearly,

$$m_0 + \sum_{\gamma \in \Sigma_1} m_\gamma = m.$$

We first bound the number of incidences with hypersurfaces that contain at least k points in one of the subsets $\mathcal{P}(\gamma)$. By the hypothesis (c), for each $\gamma \in \Sigma_1$ and each subset of k points of $\mathcal{P}(\gamma)$, there are at most c hypersurfaces in $\mathcal{H}$ containing these points. Hence,

$$I_{\geq k}(\mathcal{P}(\gamma), \mathcal{H}) \leq ck \binom{m_\gamma}{k} = O(m_\gamma^k). \tag{4.3}$$

The cardinality of Σ_1 or equivalently, the number of nonempty subsets of the form $\mathcal{P}(\gamma)$, is bounded by the number of connected components of $\mathbb{R}^4 \setminus V(f)$. By Theorem 2.6, this number is bounded by $O(D^4)$. With (4.2) and (4.3), this implies that

$$\sum_\gamma I_{\geq k}(\mathcal{P}(\gamma), \mathcal{H}) = O\Big(\sum_\gamma \Big(\frac{m}{D^4}\Big)^k\Big) = O(m^k D^{4-4k}). \tag{4.4}$$

We now bound the number of incidences with hypersurfaces that contain at most $k-1$ points in every $\mathcal{P}(\gamma)$. For each $H \in \mathcal{H}$, the number of subsets $\mathcal{P}(\gamma)$ having nonempty intersection with H is bounded by $\mathrm{b}_0(H(\mathbb{R}) \setminus V(f))$. By Theorem 2.6, this number of connected components is bounded by $O(D^3)$, because the degree of H is bounded by a constant. Hence $\sum_\gamma I_{<k}(\mathcal{P}(\gamma), \{H\}) \leq (k-1)\, \mathrm{b}_0(H \setminus V(f)) = O(D^3)$. It follows that

$$\sum_\gamma I_{<k}(\mathcal{P}(\gamma), \mathcal{H}) = O(nD^3). \tag{4.5}$$

From (4.1), (4.4) and (4.5) we deduce that

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = \sum_\gamma I(\mathcal{P}(\gamma), \mathcal{H}) = O(nD^3 + m^k D^{4-4k}). \tag{4.6}$$

We then set

$$D = \max\Big(24, \frac{m^{\alpha_1}}{n^{\beta_1}}\Big) \quad \text{with } \alpha_1 = \frac{k}{4k-1} \text{ and } \beta_1 = \frac{1}{4k-1}. \tag{4.7}$$

If $D = 24$, then $m^{\alpha_1} n^{-\beta_1} \leq 24$ and so $m^k = O(n)$. In this case, it follows from (4.6) that $I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(n + m^k) = O(n)$. Otherwise,

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(m^{3\alpha_1} n^{1-3\beta_1}) = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}).$$

In either case,

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n). \tag{4.8}$$

Second level partitioning. Let $V(f) = \bigcup_{i\in I} V_i$ be the decomposition of the hypersurface $V(f)$ into irreducible components. Set $D_i = \deg(V_i)$ for each $i \in I$. Then

$$\sum_{i\in I} D_i = \deg(V(f)) \leq D. \tag{4.9}$$

We choose a partition of the finite set $\mathcal{P}_0 = \mathcal{P} \cap V(f)$ into disjoint subsets $\mathcal{Q}_i$, $i \in I$, by assigning each point in $\mathcal{P}_0$ to one of the subsets $\mathcal{Q}_i$ corresponding to an irreducible component V_i it belongs to. Set $l_i = \mathrm{card}(\mathcal{Q}_i)$ for each $i \in I$. Then

$$\sum_i l_i = m_0. \tag{4.10}$$

Fix $i \in I$ and let $E_i \geq 24D_i$. By Theorem 3.1, there exists $g_i \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq E_i}$ such that $\dim(V_i \cap V(g_i)) = 2$ and, for each $\delta \in \{\pm 1\}^{\mathrm{irr}(g_i)}$,

$$\mathrm{card}(\mathcal{Q}_i(\delta)) = O\Big(\frac{l_i}{D_i E_i^3}\Big). \tag{4.11}$$

Choose a minimal subset $\Sigma_{2,i} \subset \{\pm 1\}^{\mathrm{irr}(g_i)}$ realizing all nonempty subsets of the form $\mathcal{Q}_i(\delta)$.

Consider the surface $W_i = V_i \cap V(g_i) = V(f_i, g_i)$ and partition $\mathcal{Q}_i$ into the disjoint subsets $\mathcal{Q}_{i,0} = \mathcal{Q}_i \cap W_i$ and $\mathcal{Q}_i(\delta)$, $\delta \in \Sigma_{2,i}$. We set $l_{i,0} = \mathrm{card}(\mathcal{Q}_{i,0})$ and $l_{i,\delta} = \mathrm{card}(\mathcal{Q}_i(\delta))$ for each δ. Clearly,

$$l_{i,0} + \sum_{\delta\in\Sigma_{2,i}} l_{i,\delta} = l_i \quad \text{and} \quad \sum_i l_{i,0} = \mathrm{card}\Big(\mathcal{P} \cap \bigcup_i W_i\Big).$$

We follow the same approach as in the previous case, and we first bound the number of incidences with hypersurfaces that contain at least k points in some $\mathcal{Q}_i(\delta)$. Similarly as in (4.3), the hypothesis (c) implies that, for each δ,

$$I_{\geq k}(\mathcal{Q}_i(\delta), \mathcal{H}) \leq ck\binom{l_{i,\delta}}{k} = O(l_{i,\delta}^k). \tag{4.12}$$

The cardinality of $\Sigma_{2,i}$ is bounded by $\mathrm{b}_0(V_i(\mathbb{R}) \setminus V(g_i))$ which, by Theorem 2.6, is bounded by $O(D_i E_i^3)$. With (4.11) and (4.12), this implies that

$$\sum_\delta I_{\geq k}(\mathcal{Q}_i(\delta), \mathcal{H}) = O\Big(\sum_\delta \Big(\frac{l_i}{D_i E_i^3}\Big)^k\Big) = O(l_i^k D_i^{1-k} E_i^{3-3k}). \tag{4.13}$$

We now bound the number of incidences with hypersurfaces that contain at most $k-1$ points in every $\mathcal{Q}_i(\delta)$. Let $H \in \mathcal{H}$ and, for the moment, suppose that $V_i \not\subset H$. The number of subsets of the form $\mathcal{Q}_i(\delta)$ with nonempty intersection with H is bounded by $\mathrm{b}_0((H \cap V_i)(\mathbb{R}) \setminus V(g_i))$. By Theorem 2.6, this number is bounded by $O(D_i E_i^2)$, since $\dim(V_i) = 3$ and either $H \cap V_i$ is empty or of dimension 2, and the degree of H is bounded by a constant.

If we note by $\mathcal{H}_i$ the set of hypersurfaces of $\mathcal{H}$ not containing V_i, then

$$\sum_\delta I_{<k}(\mathcal{Q}_i(\delta), \mathcal{H}_i) = O(nD_i E_i^2).$$

On the other hand, by the hypothesis (b), there are at most 3 hypersurfaces $H \in \mathcal{H}$ containing V_i, and each of them contains the l_i points of $\mathcal{Q}_i$. Hence

$$I_{<k}(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H} \setminus \mathcal{H}_i) \leq I(\mathcal{Q}_i, \mathcal{H} \setminus \mathcal{H}_i) \leq 3l_i. \tag{4.14}$$

By (4.13) and (4.14),

$$I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) = \sum_{\delta} I(\mathcal{Q}_i(\delta), \mathcal{H}) = O(nD_iE_i^2 + l_i^k D_i^{1-k} E_i^{3-3k} + l_i). \tag{4.15}$$

We set

$$E_i = \max\Big(24D_i, \Big(\frac{l_i}{D_i}\Big)^{\alpha_2} \frac{1}{n^{\beta_2}}\Big) \quad \text{with } \alpha_2 = \frac{k}{3k-1} \text{ and } \beta_2 = \frac{1}{3k-1}. \tag{4.16}$$

If $E_i = 24D_i$, then $(\frac{l_i}{D_i})^{\alpha_2} n^{-\beta_2} \le 24D_i$. In this case, the first term in the right-hand side of (4.15) controls the second one. Otherwise, both terms are equal up to a constant factor. We deduce from (4.15) that

$$I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) = \begin{cases} O(nD_i^3 + l_i) & \text{if } E_i = 24D_i, \\ O(l_i^{2\alpha_2} D_i^{1-2\alpha_2} n^{1-2\beta_2} + l_i) & \text{otherwise.} \end{cases} \tag{4.17}$$

By (4.9),

$$\sum_i nD_i^3 \le nD^3 = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n), \tag{4.18}$$

as the term nD^3 appears in (4.6) and is accounted for in (4.8). Using the Hölder inequality as well as (4.9) and (4.10), we get

$$\sum_i l_i^{2\alpha_2} D_i^{1-2\alpha_2} n^{1-2\beta_2} \le n^{1-2\beta_2} \Big(\sum_i l_i\Big)^{2\alpha_2} \Big(\sum_i D_i\Big)^{1-2\alpha_2}$$
$$\le n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2}. \tag{4.19}$$

We now substitute the value of D from (4.7) and those of $\alpha_1, \alpha_2, \beta_1$ and β_2 in the above expression. If $D = 24$, then $m^k = O(n)$ and so $n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2} = n^{1-2\beta_2} m_0^{2\alpha_2} = O(n)$. Otherwise,

$$n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2} \le n^{1-2\beta_2} m^{2\alpha_2} (m^{\alpha_1} n^{-\beta_1})^{1-2\alpha_2} = m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}. \tag{4.20}$$

It follows from (4.17), (4.18), (4.19), (4.20) and (4.10) that

$$\begin{aligned} I\Big(\mathcal{P}_0 \setminus \bigcup_i \mathcal{Q}_{i,0}, \mathcal{H}\Big) &= \sum_i I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) \\ &= O\Big(\sum_i nD_i^3 + \sum_i n^{1-2\beta_2} l_i^{2\alpha_2} D_i^{1-2\alpha_2} + \sum_i l_i\Big) \\ &= O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n + m_0). \end{aligned} \tag{4.21}$$

Third partitioning polynomials. For each $i \in I$, let $W_i = \bigcup_{j \in J_i} W_{i,j}$ be the decomposition of the surface $W_i = V(f_i, g_i)$ into irreducible components. Set $\Delta_{i,j} = \deg(W_{i,j})$ for each j. By Bézout theorem,

$$\sum_{j \in J_i} \Delta_{i,j} = \deg(W_i) \le D_i E_i.$$

We denote by $W_i(\mathbb{R})_0$ and $W_{i,j}(\mathbb{R})_0$ the set of isolated points of the semi-algebraic sets $W_i(\mathbb{R})$ and $W_{i,j}(\mathbb{R})$, respectively. We then choose an arbitrary partition of the set $\mathcal{Q}_{i,0} = \mathcal{Q}_i \cap W_i$ into disjoints subsets $\mathcal{R}_{i,j}$, $j \in J_i$, such that

$$\mathcal{R}_{i,j} \subset W_{i,j}(\mathbb{R}) \quad \text{and} \quad \mathcal{R}_{i,j} \cap W_{i,j}(\mathbb{R})_0 \subset W_i(\mathbb{R})_0.$$

Set $e_{i,j} = \mathrm{card}(\mathcal{R}_{i,j})$ for each j. Then

$$\sum_j e_{i,j} = l_{i,0}. \tag{4.22}$$

Let $j \in J_i$. Being an irreducible component of $W_i = V(f_i, g_i)$, the variety $W_{i,j}$ is locally set-theoretically defined at degree E_i. Let $F_{i,j} \geq 24\max(D_i, E_i)$, to be fixed later on. By Theorem 3.1, there exists $h_{i,j} \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq F_{i,j}}$ such that $\dim(W_{i,j} \cap V(h_{i,j})) = 1$ and, for each $\eta \in \{\pm 1\}^{\mathrm{irr}(h_{i,j})}$,

$$\mathrm{card}(\mathcal{R}_{i,j}(\eta)) = O\Big(\frac{e_{i,j}}{\Delta_{i,j} F_{i,j}^2}\Big). \tag{4.23}$$

Similarly as before, choose a minimal subset $\Sigma_{3,i,j} \subset \{\pm 1\}^{\mathrm{irr}(h_{i,j})}$ realizing all nonempty subsets of the form $\mathcal{R}_{i,j}(\eta)$.

Consider the curve $Y_{i,j} = W_{i,j} \cap V(h_{i,j})$ and partition of $\mathcal{R}_{i,j}$ into the disjoint subsets $\mathcal{R}_{i,j,0} = \mathcal{R}_{i,j} \cap Y_{i,j}$ and $\mathcal{R}_{i,j,\eta} = \mathcal{R}_{i,j}(\eta)$, $\eta \in \Sigma_{3,i,j}$. Set also $e_{i,j,0} = \mathrm{card}(\mathcal{R}_{i,j,0})$ and $e_{i,j,\eta} = \mathrm{card}(\mathcal{R}_{i,j}(\eta))$ for each η. Hence,

$$e_{i,j,0} + \sum_{\eta \in \Sigma_{3,i,j}} e_{i,j,\eta} = e_{i,j}.$$

We first bound the number of incidences of $\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}$ with hypersurfaces that contain at least k points in some $\mathcal{R}_{i,j}(\eta)$. Similarly as for (4.3) and (4.12), the hypothesis (c) implies that, for each η,

$$I_{\geq k}(\mathcal{R}_{i,j}(\eta), \mathcal{H}) \leq ck\binom{e_{i,j,\eta}}{k} = O(e_{i,j,\eta}^k). \tag{4.24}$$

By Proposition 2.4, there are coprime polynomials $\widetilde{f}_{i,j}, \widetilde{g}_{i,j} \in \mathbb{R}[x_1, x_2, x_3, x_4]$ such that $W_{i,j}$ is an irreducible component of the variety $\widetilde{W}_{i,j} = V(\widetilde{f}_{i,j}, \widetilde{g}_{i,j})$ and

$$\deg(\widetilde{f}_{i,j})\deg(\widetilde{g}_{i,j}) = O(\Delta_{i,j}). \tag{4.25}$$

Since $W_{i,j}$ is locally set-theoretically defined at degree E_i, we can furthermore that $\deg(\widetilde{f}_i), \deg(\widetilde{g}_i) \leq E_i$.

The number of nonempty subsets of the form $\mathcal{R}_{i,j}(\eta)$ is bounded by the number of connected components of $\widetilde{W}_{i,j}(\mathbb{R}) \setminus V(h_{i,j})$, as explained in Remark 3.2. By Theorem 2.6 and (4.25), this number of connected components is bounded by

$$\mathrm{b}_0(\widetilde{W}_{i,j}(\mathbb{R}) \setminus V(h_{i,j})) = O(\deg(\widetilde{f}_{i,j})\deg(\widetilde{g}_{i,j}) F_{i,j}^2) = O(\Delta_{i,j} F_{i,j}^2). \tag{4.26}$$

By (4.23), (4.24) and (4.26),

$$\sum_\eta I_{\geq k}(\mathcal{R}_{i,j}(\eta), \mathcal{H}) = O\Big(\sum_\eta \Big(\frac{e_{i,j}}{\Delta_{i,j} F_{i,j}^2}\Big)^k\Big) = O(e_{i,j}^k \Delta_{i,j}^{1-k} F_{i,j}^{2-2k}). \tag{4.27}$$

We now bound the number of incidences of $\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}$ with hypersurfaces that contain at most $k-1$ points in every $\mathcal{R}_{i,j}(\eta)$. Given $H \in \mathcal{H}$, we denote by $B_{i,j}(H) \subset W_{i,j}(\mathbb{R})$ the semi-algebraic subset of points $p \in W_{i,j}(\mathbb{R})$ having an open neighborhood, in the Euclidean topology of $W_{i,j}(\mathbb{R})$, contained in H. We also set $G_{i,j}(H) = W_{i,j}(\mathbb{R}) \setminus B_{i,j}(H)$. Notice that $W_{i,j}(\mathbb{R})_0 \cap H \subset B_{i,j}(H)$.

For any finite subset $\mathcal{R} \subset W_{i,j}(\mathbb{R})$ we set

$$\mathcal{I}^{\mathrm{B}}(\mathcal{R}, \mathcal{H}) = \mathcal{I}^{\mathrm{B}}(\mathcal{R} \cap B_{i,j}(H), \mathcal{H}) \quad \text{and} \quad \mathcal{I}^{\mathrm{G}}(\mathcal{R}, \mathcal{H}) = \mathcal{I}^{\mathrm{G}}(\mathcal{R} \cap G_{i,j}(H), \mathcal{H}).$$

We also set $I^{\mathrm{B}}(\mathcal{R},\mathcal{H}) = \mathrm{card}(\mathcal{I}^{\mathrm{B}}(\mathcal{R},\mathcal{H}))$ and $I^{\mathrm{G}}(\mathcal{R},\mathcal{H}) = \mathrm{card}(\mathcal{I}^{\mathrm{G}}(\mathcal{R},\mathcal{H}))$. Clearly,

$$I(\mathcal{R},\mathcal{H}) = I^{\mathrm{B}}(\mathcal{R},\mathcal{H}) + I^{\mathrm{G}}(\mathcal{R},\mathcal{H}).$$

We first treat the incidences in $G_{i,j}(H)$. Write $H = V(b)$ with $b \in \mathbb{R}[x_1,x_2,x_3,x_4]$. The number of nonempty subsets of the form $\mathcal{R}_{i,j}(\eta) \cap G_{i,j}(H)$ is bounded by the number of connected components of $\mathbb{R}^4 \setminus V(h_{i,j})$ having nonempty intersection with $H \cap G_{i,j}(H)$. By Proposition 2.7, this number of connected components is bounded by the number of connected components of the semi-algebraic set

$$(\widetilde{W}_{i,j} \cap V(b^2-\varepsilon))(\mathbb{R}) \setminus V(h_{i,j}) = V(b^2-\varepsilon, \widetilde{f}_{i,j}, \widetilde{g}_{i,j})(\mathbb{R}) \setminus V(h_{i,j}) \tag{4.28}$$

for any $\varepsilon > 0$ sufficiently small. Choosing a possibly smaller $\varepsilon > 0$, we also have that $\dim(\widetilde{W}_{i,j} \cap V(b^2-\varepsilon)) = 1$. For any such a choice of ε, by Theorem 2.6 and (4.25), the number of connected components of the semi-algebraic set in (4.28) is bounded by $O(\deg(b^2-\varepsilon)\deg(\widetilde{f}_{i,j})\deg(\widetilde{g}_{i,j})\deg(h_{i,j})) = O(\Delta_{i,j}F_{i,j})$. Thus

$$\sum_{\eta} I^{\mathrm{G}}_{<k}(\mathcal{R}_{i,j}(\eta),\mathcal{H}) = O(n\Delta_{i,j}F_{i,j}). \tag{4.29}$$

Gathering together (4.27) and (4.29), we obtain that

$$\begin{aligned} I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0},\mathcal{H}) &= \sum_{\eta} I^{\mathrm{G}}(\mathcal{R}_{i,j}(\eta),\mathcal{H}) \\ &= O\big(n\Delta_{i,j}F_{i,j} + e_{i,j}^{k}\Delta_{i,j}^{1-k}F_{i,j}^{2-2k}\big). \end{aligned} \tag{4.30}$$

We set

$$F_{i,j} = \max\Big(24E_i, \Big(\frac{e_{i,j}}{\Delta_{i,j}}\Big)^{\alpha_3}\frac{1}{n^{\beta_3}}\Big) \quad \text{with } \alpha_3 = \frac{k}{2k-1} \text{ and } \beta_3 = \frac{1}{2k-1}.$$

If $F_{i,j} = 24E_i$, then $(\frac{e_{i,j}}{\Delta_{i,j}})^{\alpha_3} n^{-\beta_3} = O(E_i)$. In this case, the first term in the right-hand side of (4.30) controls the second one and, otherwise, both terms are equal up to a constant factor. Hence,

$$I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0},\mathcal{H}) = \begin{cases} O(n\Delta_{i,j}E_i) & \text{if } F_{i,j} = 24E_i, \\ O(n^{1-\beta_3}e_{i,j}^{\alpha_3}\Delta_{i,j}^{1-\alpha_3}) & \text{otherwise.} \end{cases} \tag{4.31}$$

By (4.9) and Bézout theorem,

$$\sum_{i,j} n\Delta_{i,j}E_i \le \sum_i nD_iE_i^2 = O(m^{1-\frac{k-1}{4k-1}}n^{1-\frac{3}{4k-1}} + n), \tag{4.32}$$

as shown when passing from (4.15) to (4.21). Else, applying the Hölder inequality together with (4.22) and (4.9),

$$\begin{aligned} \sum_{i,j} n^{1-\beta_3}e_{i,j}^{\alpha_3}\Delta_{i,j}^{1-\alpha_3} &\le n^{1-\beta_3}\Big(\sum_{i,j} e_{i,j}\Big)^{\alpha_3}\Big(\sum_{i,j}\Delta_{i,j}\Big)^{1-\alpha_3} \\ &\le n^{1-\beta_3}m^{\alpha_3}\Big(\sum_i D_iE_i\Big)^{1-\alpha_3}. \end{aligned} \tag{4.33}$$

Recall that $E_i = \max\left(24D_i, \left(\frac{l_i}{D_i}\right)^{\alpha_2} n^{-\beta_2}\right)$ as in (4.16). Hence

$$\begin{aligned}\sum_i D_i E_i &= O\Big(\sum_i D_i^2 + n^{-\beta_2}\sum_i {l_i}^{\alpha_2} {D_i}^{1-\alpha_2}\Big)\\ &= O\Big(\sum_i D_i^2 + n^{-\beta_2}\Big(\sum_i l_i\Big)^{\alpha_2}\Big(\sum_i D_i\Big)^{1-\alpha_2}\Big)\\ &= O\big(D^2 + n^{-\beta_2} m^{\alpha_2} D^{1-\alpha_2}\big). \end{aligned} \tag{4.34}$$

Recall also that $D = \max\left(24, m^{\alpha_1} n^{-\beta_1}\right)$ as in (4.7). If $D = 24$, then $m^k = O(n)$. In this case, $\sum_i D_i E_i = O(1)$. Otherwise, substituting $D = m^{\alpha_1} n^{-\beta_1}$ in (4.34) and the sum $\sum_i D_i E_i$ into (4.33),

$$\begin{aligned} n^{1-\beta_3} m^{\alpha_3}\Big(\sum_i D_i E_i\Big)^{1-\alpha_3} &= O\big(n^{1-\beta_3} m^{\alpha_3} (n^{-\beta_2} m^{\alpha_2} (m^{\alpha_1} n^{-\beta_1})^{1-\alpha_2})^{1-\alpha_3}\big)\\ &= O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}\big). \end{aligned} \tag{4.35}$$

It follows from (4.31), (4.32), (4.33), (4.34) and(4.35), that

$$\begin{aligned} I^{\mathrm{G}}\Big(\bigcup_i \mathcal{Q}_{i,0} \setminus \bigcup_{i,j} \mathcal{R}_{i,j,0}, \mathcal{H}\Big) &= \sum_{i,j} I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}, \mathcal{H})\\ &= O\Big(\sum_{i,j}\Big(n\Delta_{i,j} F_{i,j} + e_{i,j}^k \Delta_{i,j}^{1-k} F_{i,j}^{2-2k}\Big)\Big) \qquad (4.36)\\ &= O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n\big). \qquad (4.37) \end{aligned}$$

Finally, we treat the incidences in $B_{i,j}(H)$. We claim that for each $p \in \mathcal{R}_{i,j} \setminus W_i(\mathbb{R})_0$ there are at most 3 hypersurfaces in $\mathcal{H}$ such that $p \in B_{i,j}(H)$. To see this, observe that $p \in B_{i,j}(H)$ implies that H contains an open neighborhood $U \subset W_{i,j}(\mathbb{R})$ of p. Since p is not an isolated point of $W_{i,j}(\mathbb{R})$, if follows that U is of real dimension at least 1. The claim then follows from the hypothesis (b).

Hence,

$$I^{\mathrm{B}}(\mathcal{R}_{i,j} \setminus W_i(\mathbb{R})_0, \mathcal{H}) \le 3e_{i,j}. \tag{4.38}$$

The incidences of $\mathcal{Q}_i$ with hypersurfaces $H \in \mathcal{H}$ containing V_i are already accounted for in (4.14). Hence, we can suppose that V_i is not contained in H. In this case, by Theorem 2.6, $\mathrm{card}(H \cap W_i(\mathbb{R})_0) \le \mathrm{b}_0(V(b, f_i, g_i)) = O(D_i E_i^2)$, where b is the polynomial defining H. Together with (4.14), this implies that

$$\sum_j I^{\mathrm{B}}(\mathcal{R}_{i,j} \cap W_i(\mathbb{R})_0, \mathcal{H}) = O(nD_i E_i^2 + l_i) \tag{4.39}$$

It follows from (4.38), (4.22) and (4.39) that

$$\sum_j I^{\mathrm{B}}(\mathcal{R}_{i,j}, \mathcal{H}) = O(nD_i E_i^2 + l_i).$$

This bound already appears in (4.15). The contribution of the sum of these terms over $i \in I$ is accounted for in (4.21) and can be absorbed into the bound (4.37), after adding the term m. We conclude that

$$I\Big(\bigcup_i \mathcal{Q}_{i,0} \setminus \bigcup_{i,j} \mathcal{R}_{i,j,0}, \mathcal{H}\Big) = O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + m + n\big). \tag{4.40}$$

The case of curves and conclusion of the proof. Finally, we bound the number of incidences that occur on the curves $Y_{i,j} = W_{i,j} \cap V(h_{i,j})$.

For each i, j, set $\mathcal{R}_{i,j,0} = \mathcal{R}_{i,j} \cap Y_{i,j}$. Let $Y_{i,j} = \bigcup_{l \in L_{i,j}} Y_{i,j,l}$ be the decomposition of $Y_{i,j}$ into irreducible components and consider an arbitrary partition of $\mathcal{R}_{i,j,0}$ into disjoint subsets $\mathcal{S}_{i,j,l}$, $l \in L_{i,j}$, with $\mathcal{S}_{i,j,l} \subset Y_{i,j,l}$ for all l.

Let $l \in L_{i,j}$ and $H \in \mathcal{H}$. If $Y_{i,j,l}$ is not contained in H, then the number of incidences between $\mathcal{S}_{i,j,l}$ and this hypersurface is bounded by $\mathrm{card}(Y_{i,j,l} \cap H)$. From Bézout theorem, we deduce that

$$I(\mathcal{S}_{i,j,l}, \{H\}) \le \begin{cases} \deg(H)\deg(Y_{i,j,l}) & \text{if } Y_{i,j,l} \not\subset H, \\ \mathrm{card}(\mathcal{S}_{i,j,l}) & \text{if } Y_{i,j,l} \subset H. \end{cases} \tag{4.41}$$

The hypothesis (b) implies that, for each l, there are at most 3 hypersurfaces in $\mathcal{H}$ containing $Y_{i,j,l}$. It follows from (4.41) that

$$\begin{aligned} I(\mathcal{R}_{i,j,0}, \mathcal{H}) &= \sum_{l \in L_{i,j}} \sum_{H \in \mathcal{H}} I(\mathcal{S}_{i,j,l}, \{H\}) \\ &= O\Big(\sum_{l,H} \deg(Y_{i,j,l})\Big) + 3 \sum_{l} \mathrm{card}(\mathcal{S}_{i,j,l}) \\ &= O(n \deg(Y_{i,j}) + \mathrm{card}(\mathcal{R}_{i,j,0})). \end{aligned} \tag{4.42}$$

By Bézout theorem, $\deg(Y_{i,j}) \le \Delta_{i,j} F_{i,j}$. Using (4.42),

$$I\Big(\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}, \mathcal{H}\Big) = \sum_{i,j} I(\mathcal{R}_{i,j,0}, \mathcal{H}) = O\Big(\sum_{i,j} n \Delta_{i,j} F_{i,j} + \sum_{i,j} \mathrm{card}(\mathcal{R}_{i,j,0})\Big). \tag{4.43}$$

The first sum in the right-hand side of (4.43) appears in (4.36) and is already accounted for in (4.37). By construction, the family of sets $\{\mathcal{R}_{i,j,0}\}_{i,j}$ is a partition of $\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}$. Therefore, the sum of their cardinalities is bounded by m. Hence,

$$I\Big(\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}, \mathcal{H}\Big) = O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + m + n\big). \tag{4.44}$$

The statement now follows by summing up the contributions from (4.8), (4.21), (4.40) and (4.44). □

We close this paper by proposing the next conjecture on the number of point-hypersurfaces incidences in higher dimension.

Conjecture 4.1. *Let $d, h, k \ge 1$, and let $\mathcal{P}$ be a finite set of points of $\mathbb{R}^d$ and $\mathcal{H}$ a finite set of hypersurfaces of $\mathbb{C}^d$ satisfying the following conditions:*

(a) *the degrees of the hypersurfaces in $\mathcal{H}$ are bounded by h;*
(b) *the intersection of any family of d distinct hypersurfaces in $\mathcal{H}$ is finite;*
(c) *for any subset of k distinct points in $\mathcal{P}$, the number of hypersurfaces in $\mathcal{H}$ containing them is bounded by h.*

Set $m = \mathrm{card}(\mathcal{P})$ and $n = \mathrm{card}(\mathcal{H})$. Then

$$I(\mathcal{P}, \mathcal{H}) = O_{d,h,k}(m^{1-\frac{k-1}{dk-1}} n^{1-\frac{d-1}{dk-1}} + m + n).$$

This conjecture is suggested by the bound that follows from the first level of the polynomial partitioning method applied to this problem. It contains the statements of the Szemerédi-Trotter theorem 1.4, the results of Zahl and Kaplan, Matoušek, Sharir and Safernová in three dimensions [Zah13, KMSS12], and Theorem 1.5.

Concurrently with this paper, a proof of a weaker version of this conjecture, with an extra factor of m^{ε} in the bound, has appeared in [FPS+14].

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