Different bounds on the different Betti numbers of semi-algebraic sets *

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ABSTRACT

A classic result in real algebraic geometry due to Oleinik-Petrovsky, Thom and Milnor, bounds the *topological complexity* (the sum of the Betti numbers) of basic semi-algebraic sets. This bound is tight as one can construct examples having that many connected components. However, till now no significantly better bounds were known on the individual higher Betti numbers.

In this paper we prove separate bounds on the different Betti numbers of basic semi-algebraic sets, as well as arrangements of algebraic hypersurfaces. These are the first results in this direction.

1. INTRODUCTION

The combinatorial, algebraic and topological analysis of arrangements of real algebraic hypersurfaces in higher dimensions are active areas of research in computational geometry (see [1, 10, 16]). Arrangements of lines and hyperplanes have been studied quite extensively earlier. It was later realized that arrangements of curved surfaces are a significant generalization and have a wider range of applications. Many geometric problems reduces to problems involving such arrangements, such as motion planning for a robot with several degrees of freedom, generalized Voronoi diagrams where the Voronoi cells need not be polyhedral, geometric problems involving transversals, and many problems in geometric optimization. All such problems can be rephrased in terms of certain substructures (eg. single cell, lower/upper envelopes etc.) of an arrangement of surfaces in higher dimensions. There has been a lot of progress in analyzing the combinatorial complexity – that is the number of cells (appropriately defined) of various dimensions occurring in the boundary – of substructures in arrangements [16].

However, there is another source of geometric complexity in arrangements of hypersurfaces – namely topological complexity. Arrangements of hypersurfaces are distinguished from arrangements of hyperplanes by the fact that arrangements of hypersurfaces are topologically more complicated than arrangements of hyperplanes. For instance a single hypersurface or intersections of two or more hypersurfaces, can have non-vanishing higher homologies and thus sets defined in terms of such hypersurfaces can be topologically more complicated in various non-intuitive ways. It is often necessary to estimate the topological complexity of arrangements [7] and sometimes these estimates even play a role in bounding the combinatorial complexity see [4].

An important measure of the topological complexity of a set S are the Betti numbers $\beta_i(S)$. Here and elsewhere in the paper the set S will always be semi-algebraic, that is defined in terms of a finite number of real polynomial equalities and inequalities) and closed and $\beta_i(S)$ will denote the rank of the $H_i(S)$ (the i-th singular homology group with integer coefficients). Intuitively, $\beta_i(S)$ measures the number of i-dimensional holes in S. The zero-th Betti number $\beta_0(S)$ is the number of connected components.

For example, if T is topologically a hollow torus, then $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1, \beta_i(T) = 0, i > 2$, confirming our intuition that the torus has two 1-dimensional holes and one 2-dimensional hole. Analogously, for the two dimensional sphere, S, $\beta_0(S) = 1, \beta_1(S) = 0, \beta_2(S) = 1, \beta_i(S) = 0, i > 2$.

The basic result in bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovsky [13], Thom [17] and Milnor [12].

They proved:

THEOREM 1. *[13, 17, 12] Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \dots, P_n \geq 0, P_i \in R[X_1, \dots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\sum_i \beta_i(S) = O(nd)^k.$$

The above bound is actually quite tight. One can see this by considering the following example: Let

$$P_i = L_{i,1}^2 \cdots L_{i,\lfloor d/2 \rfloor}^2 - \epsilon,$$

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where the L_{ij}'s are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_n \geq 0$ has $(nd)^k$ connected components and hence $\beta_0(S) = (nd)^k$.

However, one fails to construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$. Thus, one is led to believe that tighter bounds can be proved for the higher Betti numbers. The proof technique used for proving the above result (see [12]) first replaces the semi-algebraic set S by another set bounded by a smooth algebraic hypersurface of degree nd having the same homotopy type as S. It then proceeds to bound the Betti numbers of this hypersurface using Morse theory and Bezout's bound on the number of solutions of a system of polynomial equations. This technique does not allow one to prove separate bounds on the individual Betti numbers smaller than the general bound.

The first attempt in trying to prove better bounds for higher Betti numbers was made in [4]. It was motivated by the long-standing problem in computational geometry of bounding the combinatorial complexity of a single cell in an arrangement of surface patches. The following result was proved in [4] which bounds the higher Betti numbers of a single connected component of a basic semi-algebraic set.

THEOREM 2. *[4] Let $C \subset R^k$ be a connected component of the set S defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\sum_i \beta_i(C) = n^{k-1} O(d)^k.$$

The proof of the above result involved Morse theory of stratified spaces in an essential way. The technique used to prove this result was also crucial in extending the method developed by Halperin and Sharir [11] for bounding the combinatorial complexity of a single cell in an arrangement of surface patches to higher dimensions (see [4] for more details).

The main contribution of this paper is a generalization of theorem 2. We show that it is possible to obtain a "graded" bound on the Betti numbers of a basic semi-algebraic set. In particular we prove the following theorem.

THEOREM 3. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq n^{k-i} O(d)^k.$$

Our method for proving the above theorem actually gives a stronger bound in the case when the given set $S \subset R^k$ is contained in a real variety of dimension k', defined by polynomials of degree at most d. More precisely we have,

THEOREM 4. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

contained in a variety $Z(Q)$ of real dimension k', and

$$deg(Q) \leq d.$$

Then,

$$\beta_i(S) \leq n^{k'-i} O(d)^k.$$

One should compare the above theorem to the main results in [3, 6] where similar bounds are proved on the number of connected components and the sum of the Betti numbers of more general (not just basic) semi-algebraic sets contained in a variety.

The combinatorial part (the part depending on n) in the above bound is tight, as one can easily construct examples using only hyperplanes such that the bound on β_i is actually realized. To my knowledge this is also the first bound on the Betti numbers of semi-algebraic sets that bounds the individual Betti numbers rather than their sum.

In many computational geometry applications one is often interested in understanding the topological complexity of the whole arrangement. For instance, unions of balls in R^3 has been studied by Edelsbrunner [7] from both combinatorial and topological view-point motivated by applications in molecular biology, and efficient algorithms for computing the various Betti numbers of such unions are currently being studied [8]. There is also a whole body of mathematical literature studying the topology of arrangements of hyperplanes in complex as well as real spaces (see [14]).

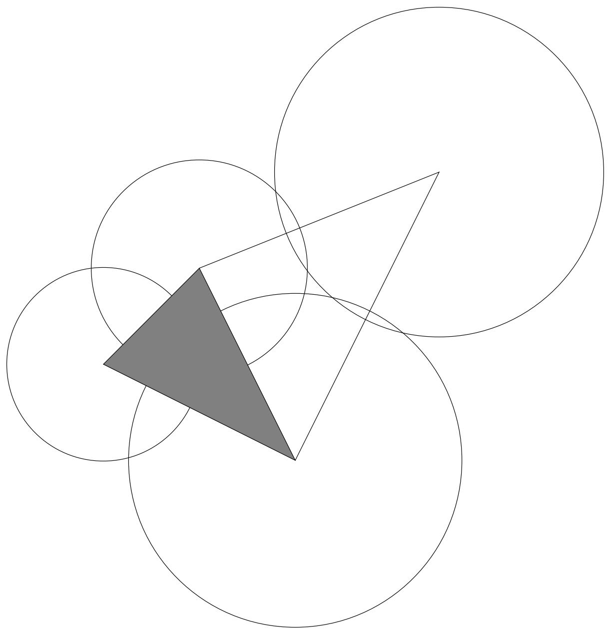

Figure 1: The nerve complex of a union of disks.

In the situation where the sets are convex, a classical result of topology, the nerve lemma [15] gives us a bound on the individual Betti numbers of the union. The nerve lemma states that the homology groups of the union is isomorphic to the homology groups of a combinatorially defined simplicial complex, the nerve complex. The nerve complex has n vertices and thus the i-th Betti number is bounded by $\binom{n}{i+1}$.

(Actually, the nerve lemma requires only that all finite intersections of the sets be topologically trivial and convex sets clearly satisfy this condition.) However, when the given sets are not necessarily convex, which would be the case in very many applications, the nerve lemma does not apply, and there is no a priori bound on the Betti numbers of the union. We prove the following bound dual to theorem 3:

THEOREM 5. *Let $S \subset R^k$ be the set defined by the disjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq n^{i+1} O(d)^k.$$

Note that, a special case of the above theorem is the situation when S is the union of n sets each defined by a polynomial equation $P_i = 0$. We just replace the equalities by the inequalities, $-P_i^2 \geq 0$.

Even though we state the result in terms of sets defined as the union of sets of a special kind, namely each defined by a single polynomial inequality, the combinatorial part (depending on n) of the bound remains true, even if one considers unions of closed sets, each of constant description complexity (that is defined by some fixed number of polynomials of constant degree). This is the situation which is of most interest in computational geometry.

2. PROOFS OF THE MAIN RESULTS:

Our basic tool is the Mayer-Vietoris sequence for homologies. Let $A, B \subset R^k$ be compact semi-algebraic sets. Then, there is the following exact sequence of homology groups known as the Mayer-Vietoris sequence [15]:

$$0 \rightarrow H_{k-1}(A \cap B) \rightarrow H_{k-1}(A) \oplus H_{k-1}(B) \rightarrow H_{k-1}(A \cup B) \rightarrow$$

$$H_{k-2}(A \cap B) \rightarrow \cdots \rightarrow H_{i+1}(A \cup B) \rightarrow H_i(A \cap B)$$

$$\rightarrow H_i(A) \oplus H(B) \rightarrow H_i(A \cup B) \rightarrow \cdots$$

We start with some preliminary results, all consequences of the Mayer Vietoris sequence.

PROPOSITION 1. *Let $S_1, \ldots, S_n \subset R^k$ be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cup \cdots \cup S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq k - i$ (that is the sum of the Betti numbers of the union of any ℓ of the sets for all $\ell \leq k - i$ is bounded by M).

Let $S = \cap_{1 \leq j \leq n} S_j$. Then,

$$\beta_i(S) \leq n^{k-i} M.$$

PROOF. Let $T_j = \cap_{1 \leq i \leq j} S_i$. Hence, $T_n = S$. Recall the Mayer-Vietoris exact sequence of homologies:

$$0 \rightarrow H_{k-1}(T_{n-1} \cap S_n) \rightarrow H_{k-1}(T_{n-1}) \oplus H_{k-1}(S_n) \rightarrow$$

$$H_{k-1}(T_{n-1} \cup S_n) \rightarrow H_{k-2}(T_{n-1} \cap S_n) \rightarrow \cdots$$

$$\rightarrow H_{i+1}(T_{n-1} \cup S_n) \rightarrow H_i(T_{n-1} \cap S_n) \rightarrow$$

$$H_i(T_{n-1}) \oplus H_i(S_n) \rightarrow H_i(T_{n-1} \cup S_n) \rightarrow \cdots$$

From the first three terms of the exact sequence we get that,

$\beta_{k-1}(T_n) = \beta_{k-1}(T_{n-1} \cap S_n) \leq \beta_{k-1}(T_{n-1}) + \beta_{k-1}(S_n)$. Unwinding the first term of right hand side we obtain that $\beta_{k-1}(S) \leq nM$.

The claim for $i < k - 1$ is proved by double induction on i and n. Assume that the induction hypothesis holds for all $i' > i$ and for $i' = i$ for all $n' < n$.

Again from the Mayer-Vietoris sequence we get that,

$$\beta_i(S) \leq \beta_{i+1}(T_{n-1} \cup S_n) + \beta_i(T_{n-1}) + \beta_i(S_n).$$

Now,

$T_{n-1} \cup S_n = \cap_{1 \leq i \leq n-1}(S_i \cup S_n)$.

The $n-1$ sets $S_i \cup S_n$ satisfies the assumption on at most $(k-i-1)$-ary unions and we can apply the induction hypothesis, and assume that $\beta_{i+1}(T_{n-1} \cup S_n) \leq (n-1)^{k-i-1} M$ and $\beta_i(T_{n-1}) \leq (n-1)^{k-i} M$.

Thus, we have that

$$\beta_i(S) \leq (n-1)^{k-i-1} M + (n-1)^{k-i} M + M$$

$$\leq n^{k-i} M.$$

□

In case, the given semi-algebraic sets have real dimension $k' \leq k$, then an easy modification of the above proof gives us:

PROPOSITION 2. *Let $S_1, \ldots, S_n \subset R^k$ be compact semi-algebraic sets, each having real dimension k' and such that,*

$$\sum_i \beta_i(S_{i_1} \cup \cdots \cup S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq k' - i$ (that is the sum of the Betti numbers of the union of any ℓ of the sets for all $\ell \leq k' - i$ is bounded by M).

Let $S = \cap_{1 \leq j \leq n} S_j$. Then,

$$\beta_i(S) \leq n^{k'-i} M,$$

for $0 \leq i < k'$.

PROOF. We use the same argument as above, starting the induction at $i = k' - 1$, rather than an $i = k - 1$, noting that $\beta_i(S) = 0$ for $i \geq k'$. □

PROPOSITION 3. *Let $S_1, \ldots, S_n \subset R^k$ be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cap \cdots \cap S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq i + 1$ (that is the sum of the Betti numbers of the intersection of any ℓ of the sets for all $\ell \leq i + 1$ is bounded by M).

Let $S = \cup_{1 \leq j \leq n} S_j$. Then,

$$\beta_i(S) \leq n^{i+1} M.$$

PROOF. Let $T_j = \cup_{1 \le i \le j} S_i$. Hence, $T_n = S$. The Mayer-Vietoris exact sequence of homologies gives us:

$$\cdots \to H_{i+1}(T_{n-1} \cup S_n) \to H_i(T_{n-1} \cap S_n) \to$$

$$H_i(T_{n-1}) \oplus H_i(S_n) \to H_i(T_{n-1} \cup S_n) \to$$

$$\cdots \to H_0(T_{n-1} \cap S_n) \to H_0(T_{n-1}) \oplus H_0(S_n)$$

$$\to H_0(T_{n-1} \cup S_n) \to 0.$$

From the last three terms of the exact sequence we get that,

$\beta_0(T_n) = \beta_0(T_{n-1}) + \beta_0(S_n)$. Unwinding the first term of right hand side we obtain that $\beta_0(S) \le nM$.

The claim for $i > 0$ is proved by double induction on i and n. Assume that the induction hypothesis holds for all $i' < i$ and for $i' = i$ for all $n' < n$.

Again from the Mayer-Vietoris sequence we get that,

$$\beta_i(S) \le \beta_i(T_{n-1}) + \beta_i(S_n) + \beta_{i-1}(T_{n-1} \cap S_n).$$

Now,

$T_{n-1} \cap S_n = \cup_{1 \le i \le n-1}(S_i \cap S_n)$.

The $n-1$ sets $S_i \cap S_n$ satisfies the assumption on at most i-ary intersections and we can apply the induction hypothesis, and assume that $\beta_{i-1}(T_{n-1} \cup S_n) \le (n-1)^i M$ and $\beta_i(T_{n-1}) \le (n-1)^{i+1} M$.

Thus, we have that

$$\beta_i(S) \le (n-1)^i M + (n-1)^{i+1} M + M$$

$$\le n^{i+1} M.$$

$\square$

LEMMA 1. *Let $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \le d$, and $l \le k$. Let S be the set defined by the conjunction of the inequalities $P_i \ge 0$. Let S be bounded. Then, $\sum_i \beta_i(S) = O(d)^k$.*

Before proving this lemma we introduce a new definition and recall few facts from real algebraic geometry.

In the following we will perturb polynomials by various *infinitesimals* so that our geometric objects live over the field of Puiseux series in these infinitesimals. We write $R\langle\zeta\rangle$ for the real closed field of Puiseux series in ζ with coefficients in R [2]. The sign of such a Puiseux series agrees with the sign of the coefficient of the lowest degree term in ζ. This order makes ζ positive and smaller than any positive element of R. In particular ζ is infinitesimal over R. If S is a semi-algebraic subset of R^k we denote by $S_{R\langle\zeta\rangle}$ the subset of $R\langle\zeta\rangle^k$ defined by the same equalities and inequalities that define S. Note that in the following, in all arguments involving infinitesimals, the infinitesimals can be replaced by sufficiently small positive real numbers. We refer the reader to [5] for more details on the use of infinitesimals.

We denote by $Z(P)$ the set of real zeros of the polynomial P.

PROOF. (Proof of lemma 1) Using the Oleinik-Petrovsky-Thom-Milnor bound from theorem 1 will give us a bound of $(O(kd))^k$. We make use of certain techniques introduced in [6] to obtain the more refined bound claimed above.

Consider the set S^* defined by the inequalities $P_i + \delta_i \ge 0$, with $0 \ll \delta_1 \ll \cdots \ll \delta_l$ infinitesimals.

Then, $H_*(S^*) \cong H_*(S)$. Let T be the set defined by $P_1 + \delta_1 \ge 0, \ldots, P_{l-1} + \delta_{l-1} \ge 0$.

Let $0 \ll \delta \ll \delta_1$ be a new infinitesimal and let U be the intersection of the set defined by $(P_l + \delta_l)^2 \ge \delta$ with T and V the intersection of the set defined by $(P_l + \delta_l)^2 \le \delta$ with T.

For any set X let $r(X)$ denote the sum of its Betti numbers.

We have,

$$r(S) \le r(U).$$

Also, $U \cup V = T$, and $U \cap V$ has the same homology groups as the intersection of T with $Z(P_l + \delta_l)$. Using Mayer-Vietoris we get,

$r(U) + r(V) \le r(U \cup V) + r(U \cap V)$.

Let $r(l, d, k)$ be maximum of $r(X)$ for all basic semi-algebraic sets $X \subset R^k$ contained in a real variety defined by polynomials of degree at most d, and defined by a conjunction of l weak inequalities of degree at most d.

Then we have that, $r(l, d, k) \le 2r(l-1, d, k)$.

Moreover, using the Oleinik-Petrovsky-Thom-Milnor bound from theorem 1, we have that $r(0, d, k) = O(d)^k$.

For $l \le k$ we get $r(l, d, k) = 2^l O(d)^k = O(d)^k$. $\square$

LEMMA 2. *Let $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \le d$, and $l \le k$. Let S be the set defined by the disjunction of the inequalities $P_i \ge 0$. Then, $\sum_i \beta_i(S) = O(d)^k$.*

PROOF. Let S_i be the set defined by the inequality $P_i \ge 0$. We use the Mayer-Vietoris sequence again. Define $T_j = \cup_{1 \le i \le j} S_i$.

Then, $S = T_l$.

Also,

$\beta_i(S) \le \beta_i(T_{l-1}) + \beta_i(S_l) + \beta_{i-1}(T_{l-1} \cap S_l)$.

Now,

$T_{l-1} \cap S_l = \cup_{1 \le j \le l-1}(S_j \cap S_i)$. Applying Mayer-Vietoris again on the first and the last term and doing it repeatedly we will finally end up with terms of the form, $\beta_j(S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_j})$, $j \le l$.

There are at most 2^{l+1} such terms and each can contribute at most $O(d)^k$ by lemma 1. This proves the lemma. $\square$

PROOF. (Proof of theorem 3) Let $1/\Omega$ be an infinitesimal, and let the set S_j be defined by $P_j \ge 0 \wedge (\sum_i X_i^2 \le \Omega^2)$. Each S_j is compact and $\cap_j S_j$ has the same homology as S. Now apply proposition 1 with the sets S_j defined above, and noting using lemma 2 that we can take $M = O(d)^k$. $\square$

PROOF. (Proof of theorem 4) Same as above using proposition 2 instead of proposition 1. $\square$

We now prove theorem 5.

PROOF. (Proof of theorem 5) Apply proposition 3 with the sets S_j defined as in the proof of theorem 3, and noting using lemma 1 that we can take $M = O(d)^k$. $\square$

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