Computing the Betti Numbers of Arrangements

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ABSTRACT

In this paper, we consider the problem of computing the Betti numbers of an arrangement of n compact semi-algebraic sets, $S_1, \ldots, S_n \subset \mathbb{R}^k$, where each S_i is described using a constant number of polynomials with degrees bounded by a constant. Such arrangements are ubiquitous in computational geometry. We give an algorithm for computing ℓ-th Betti number, $\beta_\ell(\cup_i S_i), 0 \leq \ell \leq k-1$, using $O(n^{\ell+2})$ algebraic operations. Additionally, one has to perform linear algebra on matrices of size bounded by $O(n^{\ell+1})$. All previous algorithms for computing the Betti numbers of arrangements, triangulated the arrangement giving rise to a complex of size $O(n^{2^k})$ in the worst case. To our knowledge this is the first algorithm for computing $\beta_\ell(\cup_i S_i)$ that does not rely on such a global triangulation, and has a graded complexity which depends on ℓ.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

General Terms

Topology of Arrangements

Keywords

Semi-algebraic Sets, Betti Numbers, Spectral Sequence

1. INTRODUCTION

The combinatorial, algebraic and topological analysis of arrangements of real algebraic hypersurfaces in higher dimensions are active areas of research in computational geometry (see [1, 11, 21]). Arrangements of lines and hyperplanes have been studied quite extensively earlier. It was later realized that arrangements of curved surfaces are a significant generalization and have a wider range of applications. There has been substantial progress in analyzing the combinatorial complexity – that is the number of cells (appropriately defined) of various dimensions occurring in the boundary – of substructures in arrangements [21].

However, there is another source of geometric complexity in arrangements of hypersurfaces – namely topological complexity. Arrangements of hypersurfaces are distinguished from arrangements of hyperplanes by the fact that arrangements of hypersurfaces are topologically more complicated than arrangements of hyperplanes. For instance a single hypersurface or intersections of two or more hypersurfaces, can have non-vanishing higher homology groups and thus sets defined in terms of such hypersurfaces can be topologically more complicated in various non-intuitive ways. It is often necessary to estimate the topological complexity of arrangements [9] and sometimes these estimates even play a role in bounding the combinatorial complexity see [4].

An important measure of the topological complexity of a set S are the Betti numbers $\beta_i(S)$. Here and elsewhere in the paper the set S will always be semi-algebraic, (that is defined in terms of a finite number of real polynomial equalities and inequalities) and closed and $\beta_i(S)$ will denote the rank of the $H^i(S)$ (the i-th singular cohomology group with real coefficients). Intuitively, $\beta_i(S)$ measures the number of i-dimensional holes in S. The zero-th Betti number $\beta_0(S)$ is the number of connected components.

For example, if T is topologically a hollow torus, then $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1, \beta_i(T) = 0, i > 2$, confirming our intuition that the torus has two 1-dimensional holes and one 2-dimensional hole. Analogously, for the two dimensional sphere, S, $\beta_0(S) = 1, \beta_1(S) = 0, \beta_2(S) = 1, \beta_i(S) = 0, i > 2$.

1.1 Brief History

The basic result in bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovsky [15], Thom [22] and Milnor [14].

They proved:

THEOREM 1. *[15, 22, 14] Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

[*]The author was partially supported by NSF grant CCR-0049070 and an NSF Career Award CCR-0133597.

Then,

$$\sum_i \beta_i(S) = O(nd)^k.$$

However, the proof techniques used to prove theorem 1 does not allow us to prove bounds on the individual Betti numbers separately. Nor do they suggest an efficient algorithm for actually computing the Betti numbers.

The first bounds on the individual Betti numbers of semi-algebraic sets were proved in [3]. The following two theorems were proved bounding the Betti numbers of semi-algebraic sets obtained as intersections or as unions of sets, each defined by a single polynomial inequality.

THEOREM 2. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \dots, P_n \geq 0, P_i \in R[X_1, \dots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

contained in a variety $Z(Q)$ of real dimension k', and

$$deg(Q) \leq d.$$

Then,

$$\beta_i(S) \leq \binom{n}{k'-i} O(d)^k.$$

THEOREM 3. *Let $S \subset R^k$ be the set defined by the disjunction of n inequalities,*

$$P_1 \geq 0, \dots, P_n \geq 0, P_i \in R[X_1, \dots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq \binom{n}{i+1} O(d)^k.$$

Note that, a special case of the above theorem is the situation when S is the union of n sets each defined by a polynomial equation $P_i = 0$. We just replace the equalities by the inequalities, $-P_i^2 \geq 0$.

A crucial new ingredient in the proofs of these bounds is the use of a spectral sequence argument. In this paper, we extend this argument so as to be able to actually compute the Betti numbers of a union of semi-algebraic sets efficiently. We note that similar techniques would also work in the case of intersections.

In many applications in computational geometry one is often interested in understanding the topological complexity of the whole arrangement. For instance, unions of balls in R^3 has been studied by Edelsbrunner [9] from both combinatorial and topological view-point motivated by applications in molecular biology, and efficient algorithms for computing the various Betti numbers of such unions are currently being studied [10]. There is also a whole body of mathematical literature studying the topology of arrangements of hyperplanes in complex as well as real spaces (see [18]).

The standard technique of computing the Betti numbers of an arrangement is to associate a simplicial complex to such an arrangement, and compute the simplicial homology groups of such a complex (see [20]). Since, compact semi-algebraic sets are triangulable, there always exists such a simplicial complex (corresponding to that of the triangulation). Thus, in order to compute the Betti numbers of an arrangement of n real algebraic hypersurfaces in $\mathbb{R}^k$ it suffices to first triangulate the arrangement and then compute the Betti numbers of the corresponding simplicial complex. However, currently the most efficient way known to obtain such a triangulation is via the technique of cylindrical algebraic decomposition [7], and this produces $O(n^{2^k})$ simplices in the worst case.

More efficient ways of decomposing such an arrangement into topological balls have been proposed. In [6], the authors provide a decomposition into $O^*(n^{2k-3})$ cells (see [13] for a recent improvement of this result). However, this decomposition does not produce a cell complex and is therefore not useful in computing the Betti numbers of the arrangement.

1.2 The Nerve Lemma and the Leray Spectral Sequence

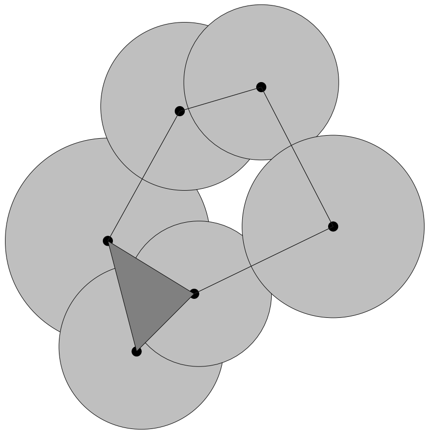

Figure 1: The nerve complex of a union of disks.

In certain simple situations, it is possible to compute the Betti numbers of an arrangement, without having to compute a triangulation. For instance, when the sets are compact and convex, a classical result of topology, the nerve lemma [19], gives us an efficient way of computing the Betti numbers of the union. The nerve lemma states that the homology groups of such a union is isomorphic to the homology groups of a combinatorially defined simplicial complex, the nerve complex. The nerve complex has n vertices, one vertex for each set in the union, and a simplex for each non-empty intersection among the sets. Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets. The ℓ-th Betti number of the union can then be computed from the $(\ell+1)$-skeleton using linear algebra. (Actually, the nerve lemma requires only that all finite intersections of the sets be topologically trivial and convex sets clearly satisfy this condition.) This technique would work, for instance, if one is interested in computing the Betti numbers of a union of balls in $\mathbb{R}^k$ (see [9]).

However, when the given sets are not necessarily convex, which would be the case in very many applications, the nerve lemma does not apply. However, even though one cannot directly associate a simplicial complex in general, it is possible to associate a more complicated combinatorial object – namely a *spectral sequence* of vector spaces – which converges to the cohomology groups of the union in finitely many steps. In fact, the nerve lemma is a particular case of this spectral sequence which degenerates in one step to yield the cohomology groups. Using the spectral sequence we will give an algorithm for computing the ℓ-th Betti number of an arrangement of n compact semi-algebraic sets, which needs to triangulate only $O(n^{\ell+2})$ sub-arrangements each of small (constant) size.

Spectral sequences were first introduced by Leray and are familiar objects in algebraic topology, and we refer the reader to [16] for a comprehensive survey. The spectral sequence method was used to prove the first graded bound on the individual Betti numbers of an arrangement of algebraic hypersurfaces [3]. In this paper, we show that using the spectral sequence, it is possible to compute the Betti numbers of an arrangement without having to compute a global triangulate of the whole arrangement. In order to compute the ℓ-th Betti number of the arrangement, it suffices to compute $O(n^{\ell+2})$ *independent* triangulations of all subarrangements consisting of at most $\ell + 2$ of the given sets. Each such triangulation will be of constant size and description complexity and the cost of computing such a triangulation is $O(1)$. Thus, in order to compute all the non-zero Betti numbers, $\beta_0, \ldots, \beta_{k-1}$, we will need to produce $O(n^{k+1})$ different constant sized triangulations.

By complexity of our algorithm we will mean the number of arithmetic operations including comparisons on elements of the ring generated by the coefficients of the input polynomials (those describing the input sets). Thus, we are only counting the cost of computing the different triangulations, and not the cost of performing the linear algebra in order to determine the Betti numbers.

We prove the following theorem.

THEOREM 4. *Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity and let $S = \cup_{1 \le i \le n} S_i$, and $0 \le \ell \le k_1$. Then, there is an algorithm to compute $\beta_0(S), \ldots, \beta_\ell(S)$, whose complexity is $O(n^{\ell+2})$.*

The idea of using filtrations for computing Betti numbers has been used in [10] for incremental algorithms for computing the homology groups of certain complexes in low dimensions. However, our techniques in this paper are quite different.

Also note that, efficient decomposition of an arrangement of n algebraic surfaces of constant degree in $\mathbb{R}^k$, into simple cells remains one of the outstanding open problems in computational geometry. Here by simple we mean that the individual cells should be describable in terms of a fixed number of polynomials of fixed degree (independent of n). The dependence on the degrees of the input polynomials is allowed to be doubly exponential in k or even worse. The main conjecture is that there exists such a decomposition of size $O(n^k)$, which is also a bound on the Betti numbers of such an arrangement. Such a decomposition would lead to more efficient algorithms for a host of different problems in computational geometry. Even though in this paper, we do not produce a decomposition of the whole arrangement of size $O(n^k)$, we prove that $O(n^{k+1})$ independent decompositions are enough to compute important topological information about the arrangement (namely the Betti numbers).

Finally, computing the Betti numbers of semi-algebraic sets in single exponential time is a major open question in algorithmic semi-algebraic geometry. The algorithm described in this paper does not answer this question, as we use triangulations whose sizes are doubly exponential in the dimension. However, as is usual in computational geometry we will assume that the degree d of the defining polynomials, as well as the dimension k are fixed constants and hence these triangulations are of size $O(1)$ for the purposes of this paper.

The rest of the paper is organized as follows. In section 2, we state a result on triangulations of semi-algebraic sets which we will use heavily in the rest of the paper. In section 3, we give a brief introduction to two spectral sequences associated with a double complex. In section 4, we describe the generalized Mayer-Vietoris exact sequence and the double complex associated to it. Finally, in section 5 we describe our algorithm for computing the Betti numbers of a union of semi-algebraic sets, each of constant description complexity.

2. SEMI-ALGEBRAIC TRIANGULATIONS

Given a simplicial complex K we will denote by by $C^i(K)$ denote the $\mathbb{R}$-vector space of i co-chains of K (that is the dual vector space to the vector space of formal $\mathbb{R}$-linear combinations of the i-simplices in K), and denote by $C^*(K)$ the direct sum $\oplus_i C^i(K)$.

2.1 Triangulation of semi-algebraic sets

A triangulation of a compact semi-algebraic set S is a simplicial complex Δ together with a semi-algebraic homeomorphism from $|\Delta|$ to S. Given such a triangulation we will often identify the simplices in Δ with their images in S under the given homeomorphism, and will often refer to the triangulation by Δ.

Given a triangulation Δ, the cohomology groups $H^i(S)$ are isomorphic to the simplicial cohomology groups $H^i(\Delta)$ of the simplicial complex Δ and are in fact independent of the triangulation.

We call a triangulation $h_1 : |\Delta_1| \to S$ of a semi-algebraic set S, to be a *refinement* of a triangulation $h_2 : |\Delta_2| \to S$ if for every simplex $\sigma_1 \in \Delta_1$, there exists a simplex $\sigma_2 \in \Delta_2$ such that $h_1(\sigma_1) \subset h_2(\sigma_2)$.

If Δ_1, Δ_2 are two triangulations of a compact semi-algebraic set S, and Δ_1 is a refinement of Δ_2, then there there exists a natural homomorphism $\lambda : C^*(\Delta_1) \to C^*(\Delta_2)$, such that the induced map $\lambda^* : H^*(\Delta_1) \to H^*(\Delta_2)$ is an isomorphism [17].

Let $S_1 \subset S_2$ be two compact semi-algebraic subsets of $\mathbb{R}^k$. We say that a semi-algebraic triangulation $h : |\Delta| \to S_2$ of S_2, *respects* S_1 if for every simplex $\sigma \in \Delta$, $h(\sigma) \cap S_1 = h(\sigma)$ or $\emptyset$. In this case, $h^{-1}(S_1)$ is naturally identified with a sub-complex of Δ and $h|_{h^{-1}(S_1)} : h^{-1}(S_1) \to S_1$ is a semi-algebraic triangulation of S_1. We will refer to this sub-complex by $\Delta|_{S_1}$.

We will need the following theorem which can be deduced from section 9.2 in [5].

THEOREM 5. *Let $S_1 \subset S_2 \subset \mathbb{R}^k$ be closed and bounded semi-algebraic sets, and let $h_i : \Delta_i \to S_i, i = 1, 2$ be semi-algebraic triangulations of S_1, S_2. Then, there exists a semi-algebraic triangulation $h : \Delta \to S_2$ of S_2, such that Δ re-*

spects S_1, Δ is a refinement of Δ_2, and $\Delta|_{S_1}$ is a refinement of Δ_1.

Moreover, there exists an algorithm which computes such a triangulation and if the sets S_1, S_2, as well as the triangulations Δ_1, Δ_2 are of constant description complexity, then the triangulation Δ produced by the algorithm is also of constant description complexity.

Also, all the homomorphisms in the following diagram,

$$\begin{tikzcd}
C^*(\Delta_2) & C^*(\Delta_1) \\
C^*(\Delta) \arrow[u, "\lambda_2"] \arrow[r, "r"] & C^*(\Delta|_{S_1}) \arrow[u, "\lambda_1"]
\end{tikzcd}$$

are all computable in constant time. (Here the vertical homomorphisms λ_2, λ_1 are as described above and r is induced by restriction.)

3. DOUBLE COMPLEXES

In this section, we introduce the basic notions of a double complex of vector spaces and associated spectral sequences.

A *double complex* is a bigraded vector space,

$$C = \oplus C^{p,q},$$

with co-boundary operators $d : C^{p,q} \rightarrow C^{p,q+1}$ and $\delta : C^{p,q} \rightarrow C^{p+1,q}$ and such that $d\delta + \delta d = 0$. In our case, the double complex would be a single quadrant double complex, which means that we can assume that $C^{p,q} = 0$ if either $p < 0$ or $q < 0$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{0,2} \arrow[u, "d"] \arrow[r, "\delta"] & C^{1,2} \arrow[u, "d"] \arrow[r, "\delta"] & C^{2,2} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
C^{0,1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{1,1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{2,1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
C^{0,0} \arrow[u, "d"] \arrow[r, "\delta"] & C^{1,0} \arrow[u, "d"] \arrow[r, "\delta"] & C^{2,0} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

Out of a double complex we can form an ordinary complex of vector spaces, namely the *associated total complex*, which is a graded vector space, defined by $C^n = \oplus_{p+q=n} C^{p,q}$, with co-boundary operator $\mathrm{D} = d + \delta : C^n \rightarrow C^{n+1}$.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\arrow[r, "\delta"] & C^{p-1,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
\arrow[r, "\delta"] & C^{p-1,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
\arrow[r, "\delta"] & C^{p-1,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{p+1,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}$$

There is a natural decreasing filtration that we can define on the associated total complex, by restricting p to be greater or equal k, with $k \geq 0$. In other words, let $C_k^n = \oplus_{p+q=n, p \geq k} C^{p,q}$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{k,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+1,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+2,q+1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
C^{k,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+1,q} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+2,q} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
C^{k,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+1,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & C^{k+2,q-1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}$$

Also, let $Z_k^n = \{z \in C_k^n | \mathrm{D}z = 0\}$ and $B^n = \mathrm{D}C_{n-1}$.

Let $H_k^n = Z_k^n / Z_k^n \cap B^n$. We thus have a decreasing filtration, $\cdots \supset H_{k-1}^n \supset H_k^n \supset H_{k+1}^n \cdots$ of the cohomology group $H^n(C, \mathrm{D})$. We denote the successive quotients H_k^n / H_{k+1}^n by $H^{k,n-k}$.

We will now construct a sequence of complexes (E_r, d_r) such that $E_{r+1} = H(E_r, d_r)$. Now, any element in $C^n = \oplus_{i+j=n} C^{i,j}$ will have a leading term at a position (p, q), where p denotes the smallest i such that the component at position $(i, n-i)$ does not vanish.

Let $Z^{p,q}$ denote the set of the (p, q) components of cocycles whose leading term is at position (p, q). In other words, $Z^{p,q}$ denotes the set of all $a \in C^{p,q}$ such that the following system of equations has a solution.

$$
\begin{aligned}
da &= 0 \qquad (1)\\
\delta a &= -da^{(1)}\\
\delta a^{(1)} &= -da^{(2)}\\
\delta a^{(2)} &= -da^{(3)}\\
&\vdots
\end{aligned}
$$

Here, $a^{(i)} \in C^{p+i,q-i}$. Hence, the element $a \oplus a^{(1)} \oplus a^{(2)} \cdots$ lies in Z_p^{p+q} and has leading term a.

Now, let $B^{p,q} \subset C^{p,q}$ consist of all b with the property that the following system of equations admits a solution.

$$
\begin{aligned}
db^{(0)} + \delta b^{(-1)} &= b \qquad (2)\\
db^{(-1)} + \delta b^{(-2)} &= 0\\
db^{(-2)} + \delta b^{(-3)} &= 0\\
&\vdots
\end{aligned}
$$

Here, $b^{(-i)} \in C^{p-i,q+i-1}$.

Clearly, $H^{p,q} \cong Z^{p,q}/B^{p,q}$.

Let

$$Z_r^{p,q} = \{a \in C^{p,q} | \exists (a^{(1)}, \dots, a^{(r-1)} | (a, a^{(1)}, \dots, a^{(r-1)})$$

$$\text{satisfy equations 1}\}.$$

Similarly, let

$$B_r^{p,q} = \{b \in C^{p,q} | \exists (b^{(0)}, b^{(-1)}, \dots, b^{(-r+1)}) | (b, b^{(0)}, \dots, b^{(-r+1)})$$

$$\text{satisfy equations 2}\}.$$

Let $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$.

Let $[a] \in E_r^{p,q}$ for some $a \in Z_r^{p,q}$. Then, there exists $a^{(1)}, \dots, a^{(r-1)}$ satisfying equations 1. We let

$$d_r[a] = [\delta a^{(r-1)}] \in E_r^{p+r,q-r+1}. \qquad (3)$$

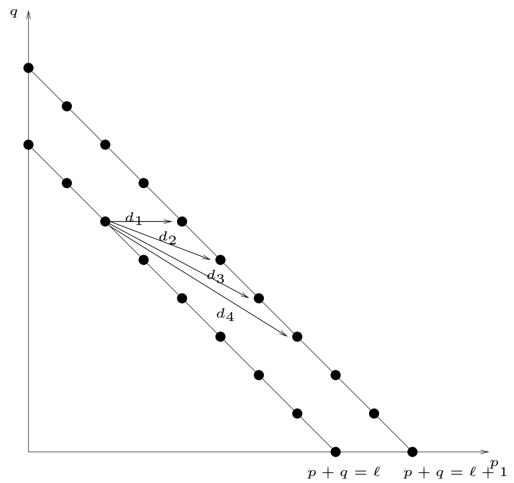

Figure 2: $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$

The sequence of graded complexes, (E_r, d_r), where the complex E_{r+1} is obtained from E_r by taking its homology with respect to d_r (that is $E_{r+1} = H(E_r, d_r)$) is called the *spectral sequence* associated to the double complex $C^{p,q}$.

The new differential d_{r+1} in the complex E_{r+1} is defined as above.

If the double complex $C^{p,q}$ is non-zero in only the first quadrant, it is can be shown (see [16]) that the spectral sequence $E_r^{p,q}$ stabilizes after a finitely many steps to $E_\infty^{p,q} = H^{p,q}$. Also, observe that the map d_r takes E_r^{pq} to $E_r^{p+r,q-r+1}$, and hence if the double complex C^{pq} is non-zero in only the first quadrant, then $d_r = 0$ for all $r > q+1$. Thus, $E_{q+2}^{pq} = E_\infty^{pq}$.

4. THE MAYER-VIETORIS DOUBLE COMPLEX

Let $A_1, \dots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \dots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.

We will denote by $A_{\alpha_0,\dots,\alpha_p}$ the sub-complex $A_{\alpha_0} \cap \dots \cap A_{\alpha_p}$.

Consider the following sequence of homomorphisms.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0<\dots<\alpha_p} C^*(A_{\alpha_0,\dots,\alpha_p}) \cdots \xrightarrow{\delta}$$

$$\prod_{\alpha_0<\dots<\alpha_{p+1}} C^*(A_{\alpha_0,\dots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

where r is induced by restriction and the connecting homomorphisms δ are defined as follows:

For $\omega \in \prod_{\alpha_0<\dots<\alpha_p} C^*(A_{\alpha_0,\dots,\alpha_p})$, and a simplex $s \in A_{\alpha_0,\dots,\alpha_p}$, we define

$$(\delta\omega)_{\alpha_0,\dots,\alpha_{p+1}}(s) =$$

$$\sum_{0 \le i \le p+1} (-1)^i \omega_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_{p+1}}(s)$$

(here and everywhere else in the paper ˆ denotes omission).

Recall that a sequence of vector space homomorphisms

$$\cdots \xrightarrow{d_{i-1}} V_i \xrightarrow{d_i} V_{i+1} \xrightarrow{d_{i+1}} \cdots$$

is said to be exact if $\ker(d_i) = \mathrm{Im}(d_{i-1})$ for each i.

Lemma 1. *The sequence defined above is exact.*

Proof. We first prove that $\delta^2 = 0$. For,

$$\omega \in \prod_{\alpha_0<\dots<\alpha_p} C^*(A_{\alpha_0,\dots,\alpha_p}),$$

$$\delta^2(\omega)_{\alpha_0,\dots,\alpha_{p+2}} = \sum_{0 \le i \le p+2} (-1)^i (\delta\omega)_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_{p+2}} =$$

$$\sum_{i,j<i} (-1)^i (-1)^j \omega_{\alpha_0,\dots,\hat{\alpha_j},\dots,\hat{\alpha_i}\dots\alpha_{p+2}} +$$

$$\sum_{i,j>i} (-1)^i (-1)^j \omega_{\alpha_0,\dots,\hat{\alpha_i},\dots,\hat{\alpha_j}\dots\alpha_{p+2}} = 0.$$

Now let, $\omega \in \prod_{\alpha_0<\cdots<\alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p})$, such that $\delta\omega = 0$. For a simplex $s \in A$ let $n(s) = |\{A_\alpha | s \in A_\alpha\}|$.

Define, $\tau \in \prod_{\alpha_0<\cdots<\alpha_{p-1}} C^*(A_{\alpha_0,\ldots,\alpha_{p-1}})$ by,

$$\tau_{\alpha_0,\ldots,\alpha_{p-1}}(s) = \frac{1}{n(s)} \sum_{\alpha | s \in A_\alpha} \omega_{\alpha,\alpha_0,\ldots,\alpha_{p-1}}(s).$$

Now,

$$(\delta\tau)_{\alpha_0,\ldots,\alpha_p}(s) = \sum_{0\le i\le p} (-1)^i \tau_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_p}(s) =$$

$$\sum_{0\le i\le p} (-1)^i \left(\frac{1}{n(s)} \sum_{\alpha|s\in A_\alpha} \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_p}(s)\right).$$

Now, $\delta\omega = 0$. Hence,

$$(\delta\omega)_{\alpha,\alpha_0,\ldots,\alpha_p}(s) =$$

$$\omega_{\alpha_0,\ldots,\alpha_p}(s) - \sum_{0\le i\le p} (-1)^{i+1} \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_p}(s) = 0.$$

Hence,

$$\omega_{\alpha_0,\ldots,\alpha_p}(s) = \sum_{0\le i\le p} (-1)^i \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_p}(s).$$

Substituting the above in the expression for $\delta\tau$ above, we have,

$$(\delta\tau)_{\alpha_0,\ldots,\alpha_p}(s) =$$

$$\frac{1}{n(s)} \sum_{\alpha|s\in A_\alpha} \left(\sum_{0\le i\le p} (-1)^i \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_p}(s)\right) =$$

$$\frac{1}{n(s)} \sum_{\alpha|s\in A_\alpha} \omega_{\alpha_0,\ldots,\alpha_p}(s) = \omega_{\alpha_0,\ldots,\alpha_p}(s).$$

This proves that the sequence is exact. $\square$

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & \prod_{\alpha_0} C^3(A_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \prod_{\alpha_0} C^2(A_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \prod_{\alpha_0} C^1(A_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \prod_{\alpha_0} C^0(A_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0<\alpha_1} C^0(A_{\alpha_0,\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] &
\end{tikzcd}$$

Interchanging the roles of the horizontal and the vertical differentials, we obtain two spectral sequences, $(E'_r, d'_r), (E_r, d_r)$, associated with $\mathcal{M}^{p,q}$ both converging to $H^*_D(\mathcal{M})$. The first terms of these are $E'_1 = H_\delta\mathcal{M}, E'_2 = H_d H_\delta \mathcal{M}$, and $E_1 = H_d\mathcal{M}, E_2 = H_\delta H_d \mathcal{M}$. Because of the exactness of the generalized Mayer-Vietoris sequence, we have that, E'_1 is:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^3(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & C^2(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & C^1(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & C^0(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
 & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] &
\end{tikzcd}$$

and E'_2 is:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & H^3(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & H^2(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & H^1(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & H^0(A) \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r, "\delta"] \arrow[u, "d"] & 0 \arrow[r] \arrow[u, "d"] & \cdots \\
 & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] &
\end{tikzcd}$$

The degeneration of this sequence at E'_2 shows that $H^*_D(\mathcal{M}) \cong H^*(A)$.

The initial term E_1 of the second spectral sequence is given by:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\prod_{\alpha_0} H^3(A_{\alpha_0}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,\alpha_1,\alpha_2}) \arrow[u, "d"] \arrow[r] & \cdots \\
\prod_{\alpha_0} H^2(A_{\alpha_0}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0,\alpha_1,\alpha_2}) \arrow[u, "d"] \arrow[r] & \cdots \\
\prod_{\alpha_0} H^1(A_{\alpha_0}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0,\alpha_1,\alpha_2}) \arrow[u, "d"] \arrow[r] & \cdots \\
\prod_{\alpha_0} H^0(A_{\alpha_0}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0,\alpha_1,\alpha_2}) \arrow[u, "d"] \arrow[r] & \cdots \\
0 \arrow[u, "d"] & 0 \arrow[u, "d"] & 0 \arrow[u, "d"] &
\end{tikzcd}
$$

5. COMPUTING THE BETTI NUMBERS

Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity. We now describe the algorithm for computing $\beta_\ell(S)$.

For $J \subset \{1, \ldots, n\}$, we denote by T_J the set $\cup_{j \in J} S_j$.

For each j, $0 \leq j \leq \ell + 1$, and for each $(j+1)$-tuple $(i_0, \ldots, i_j)$ with $1 \leq i_0 < i_1 < \cdots < i_j \leq n$, we compute a semi-algebraic triangulation $\Delta^j_{i_0,\ldots,i_j}$ of the set $T_{i_0,\ldots,i_j} = S_{i_0} \cup \ldots \cup S_{i_j}$, such that for each $J \subset \{i_0, \ldots, i_j\}$, $\Delta^j_{i_0,\ldots,i_j}$ respects T_J and $\Delta^j_{i_0,\ldots,i_j}|_{T_J}$ is a refinement of the triangulation $\Delta^{|J|}_J$.

We denote by λ the homomorphisms,

$$\lambda : C^*(\Delta^j_{i_0,\ldots,i_j}|T_J) \to$$

$$C^*(\Delta^{|J|}_J)$$

as in theorem 5 and by λ^* the induced homomorphism in the co-homology.

Let Δ be a semi-algebraic triangulation (which we do not compute) of $S = \cup_{1 \leq i \leq n} S_i$, which respects the sets $T_{i_0,\ldots,i_j}$ and such that $\Delta|_{T_{i_0,\ldots,i_j}}$ is a refinement of $\Delta^j_{i_0,\ldots,i_j}$.

Let A be the simplicial complex corresponding to the triangulation Δ, and $A_1, \ldots, A_n$ the sub-complexes corresponding to the sets $S_1, \ldots, S_n$. Since the triangulation Δ respects the sets S_i, it is clear that each set S_i give rise to a sub-complex of A. Now consider the Mayer-Vietoris double complex $\mathcal{M}$ corresponding to $A_1, \ldots, A_n$ and the spectral sequence E associated with it.

The first term of the spectral sequence is $E_1 = H_d(\mathcal{M})$ after taking the vertical cohomology. We will compute the successive terms of this spectral sequence starting from E_1.

The term E_1^{pq} in the spectral sequence is the vector space

$$\prod_{0 \leq i_0 < \ldots < i_p \leq n} H^q(A_{i_0,\ldots,i_p}).$$

Now, the sub-complex $A_{i_0} \cap \cdots \cap A_{i_p}$ corresponds to the set $S_{i_0,\ldots,i_p}$ and we have a basis for $H^q(S_{i_0,\ldots,i_p})$ in terms of p-cochains in the co-chain complex $C^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$.

Note that, even though we do not have in hand the triangulation Δ of the whole arrangement, we are still able to compute the term of $E_1^{p,q}$ using the triangulations $\Delta^p_{i_0,\ldots,i_p}$ in a basis consisting of q-cochains in $C^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$.

We next show how to compute the different terms in $(E_r^{p,q}, d_r)$ starting from $r = 1$.

We first compute $E_1^{p,q} = \prod_{0 \leq i_0 < \cdots < i_p \leq n} H^q(A_{i_0,\ldots,i_p})$, using the different triangulations $\Delta^p_{i_0,\ldots,i_p}$.

We have a basis for $H^q(A_{i_0,\ldots,i_p})$ in terms of the q co-chains of $C^q(\Delta_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$, and we compute a basis for $H^q(A_{i_0,\ldots,i_p})$ in terms of these q co-chains.

Next, we compute the homomorphisms $d_1 : E_1^{p,q} \to E_1^{p+1,q}$. We first compute the homomorphisms,

$$\delta : \prod_{1 \leq i_0 < \cdots < i_p \leq n} C^q(\Delta^{p+1}_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}}) \to$$

$$\prod_{1 \leq i_0 < \cdots < i_{p+1} \leq n} C^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}}),$$

and the corresponding induced homomorphism

$$\delta^* : \prod_{1 \leq i_0 < \cdots < i_p \leq n} H^q(\Delta^{p+1}_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}}) \to$$

$$\prod_{1 \leq i_0 < \cdots < i_{p+1} \leq n} H^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}}),$$

noting that this is possible since each triangulation

$$\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_p}}$$

refines the triangulations $\Delta^p_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}|_{S_{i_0,\ldots,\hat{i_j},\ldots,i_p}}$.

More precisely, for every

$$[a] \in \prod_{1 \leq i_0 < \cdots < i_p \leq n} H^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$$

(where

$$a \in \prod_{1 \leq i_0 < \cdots < i_p \leq n} C^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$$

is a representative of the cohomology class $[a]$), let $a_{i_0,\ldots,i_p}$ be the component of a in $C^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$ and let

$$a'_{i_0,\ldots,i_p;i_{p+1}} \in C^q(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_p}}),$$

such that $\lambda(a'_{i_0,\ldots,i_p;i_{p+1}}) = a_{i_0,\ldots,i_p}$.

Let,

$$r(a'_{i_0,\ldots,i_p;i_{p+1}}) \in C^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})$$

be the image of $a'_{i_0,\ldots,i_p;i_{p+1}}$ under the restriction homomorphism.

Let

$$\eta_{i_0,\ldots,i_p;i_{p+1}} : H^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}}) \to$$

$$H^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})$$

denote the homomorphism which takes

$$[a_{i_0,\ldots,i_p}] \mapsto [r(a'_{i_0,\ldots,i_p;i_{p+1}})].$$

Then,

$$\delta^*([a])_{i_0,\ldots,i_{p+1}} =$$

$$\sum_{0\leq j\leq p+1} (-1)^j \eta_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1};i_j}([a_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}])$$

We compute the matrix of the homomorphism δ^* that takes an element of $H^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$ (in the basis of $C^q(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}})$) to its image in $H^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})$ (in the basis of $C^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})$).

We also have isomorphisms (which we compute),

$$\lambda^* : H^q(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,\hat{i_j},\ldots,i_p}}) \rightarrow$$

$$H^q(\Delta^p_{i_0,\ldots,\hat{i_j},\ldots,i_p}|_{S_{i_0,\ldots,\hat{i_j},\ldots,i_p}}).$$

We let $d_1 : \delta^* \circ (\lambda^*)^{-1}$.

We have the following commutative diagram:

$$\begin{tikzcd}
C^*(\Delta|_{S_{i_0,\ldots,i_p}}) \arrow[r, "r"] \arrow[d, "\lambda"] & C^*(\Delta|_{S_{i_0,\ldots,i_{p+1}}}) \arrow[d, "\lambda"] \\
C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_p}}) \arrow[r, "r"] \arrow[d, "\lambda"] & C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}}) \arrow[d, "Id"] \\
C^*(\Delta^p_{i_0,\ldots,i_p}|_{S_{i_0,\ldots,i_p}}) & C^*(\Delta^{p+1}_{i_0,\ldots,i_p,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})
\end{tikzcd}$$

The vertical homomorphisms, λ are as in theorem 5 and induce isomorphisms in the cohomology groups. The horizontal homomorphisms, r, are induced by restriction.

We next compute the vector space $E_2 = H(E_1, d_1)$ by taking the homology with respect to the differential d_1. In order, to compute d_2 we follow the method outlined in section 3.

If $[a] \in E_2^{p,q}$ then there exists

$$a^{(1)} \in \prod_{1\leq i_0<\cdots<i_{p+1}\leq n} C^{q-1}(\Delta^{p+1}_{i_0,\ldots,i_{p+1}}|_{S_{i_0,\ldots,i_{p+1}}})$$

such that

$$(da^{(1)})_{i_0,\ldots,i_{p+1}} = -\sum_{0\leq j\leq p+1} (-1)^j r(a'_{i_0,\ldots,i_p;i_{p+1}}),$$

$1 \leq i_0 < \cdots < i_{p+1} \leq n$.

We let $d_2([a]) = [\delta(a_1)]$.

We iterate this process as in section 3 and compute (E_r, d_r) for $1 \leq r \leq \ell+2$, by computing the sequence $a, a^{(1)}, a^{(2)}, \ldots$.

The crucial observation here is that the maps, $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$, can be computed just from the data of the various local triangulations, $\Delta^{p+r}_{i_0,\ldots,i_{p+r}}$.

The rank of ℓ-th cohomology group of S is equal to

$$\sum_{p+q=\ell} \mathrm{rank}(E^{pq}_{\ell+2}).$$

In order to compute $\beta_\ell(S)$ it suffices to compute the ranks of $E^{p,\ell-p}_{\ell+2}, 0 \leq p \leq \ell$. For this we compute all the homomorphisms $d_r : E_r^{p,\ell-p} \rightarrow E_r^{p+r,\ell-p-r+1}$ with $0 \leq p \leq \ell$ and $1 \leq r \leq \ell - p + 1$. Thus, $p + r \leq \ell + 1$ and hence we do not have to consider intersections of more than $\ell + 2$ sets at a time.

5.1 Complexity

We only count the number of arithmetic operations in the ring of the coefficients. Since each triangulation takes $O(1)$ time, it suffices to count the number of distinct triangulations we need to compute to compute the maps d_r.

The triangulations needed are computed at the beginning, and there are $\sum_{1\leq j\leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ of these. Thus, the total number of triangulations needed is $O(n^{\ell+2})$.

The linear algebra needed in order to compute the bases for the different E_r^{pq} involve Gaussian elimination on matrices of size at most $O(n^{\ell+1})$.

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