ON A REAL ANALOGUE OF BEZOUT INEQUALITY AND THE NUMBER OF CONNECTED COMPONENTS OF SIGN CONDITIONS

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Abstract. Let R be a real closed field and $Q_1, \ldots, Q_\ell \in \mathrm{R}[X_1, \ldots, X_k]$ such that for each $i, 1 \leq i \leq \ell$, $\deg(Q_i) \leq d_i$. For $1 \leq i \leq \ell$, denote by $\mathcal{Q}_i = \{Q_1, \ldots, Q_i\}$, V_i the real variety defined by $\mathcal{Q}_i$, and k_i an upper bound on the real dimension of V_i (by convention $V_0 = \mathrm{R}^k$ and $k_0 = k$). Suppose also that

$$2 \leq d_1 \leq d_2 \leq \frac{1}{k+1} d_3 \leq \frac{1}{(k+1)^2} d_4 \leq \cdots \leq \frac{1}{(k+1)^{\ell-3}} d_{\ell-1} \leq \frac{1}{(k+1)^{\ell-2}} d_\ell,$$

and that $\ell \leq k$. We prove that the number of semi-algebraically connected components of V_ℓ is bounded by

$$O(k)^{2k} \left(\prod_{1 \leq j < \ell} d_j^{k_{j-1}-k_j} \right) d_\ell^{k_{\ell-1}}.$$

This bound can be seen as a weak extension of the classical Bezout inequality (which holds only over algebraically closed fields and is false over real closed fields) to varieties defined over real closed fields.

Additionally, if $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ is a finite family of polynomials with $\deg(P) \leq d$ for all $P \in \mathcal{P}$, $\operatorname{card} \mathcal{P} = s$, and $d_\ell \leq \frac{1}{k+1} d$, we prove that the number of semi-algebraically connected components of the realizations of all realizable sign conditions of the family $\mathcal{P}$ restricted to V_ℓ is bounded by

$$O(k)^{2k} (sd)^{k_\ell} \left(\prod_{1 \leq j \leq \ell} d_j^{k_{j-1}-k_j} \right).$$

These results have found applications in discrete geometry, for proving incidence bounds [11], as well as in effcient range-searching [20].

Contents

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1. Introduction

1.1. **History and motivation.** Let R be a fixed real closed field, and we denote by C the algebraic closure of R. Bounds on the number of semi-algebraically connected components, and in fact on all the Betti numbers of real algebraic varieties and of semi-algebraic subsets of R^k in terms of the number and the degrees of the polynomials used to define them is a well studied problem in quantitative real algebraic geometry. The classical bounds, going back to the work of Oleĭnik and Petrovskiĭ [23], Thom [27] and Milnor [21], bounded the sum of the Betti numbers of real algebraic varieties, as well as those of basic closed semi-algebraic sets. These and related bounds (see below) are extremely important in real algebraic geometry [10], but have also been used extensively in other areas such as combinatorics [3], discrete and computational geometry [15], and theoretical computer science [22] (the cited references are not by any means exhaustive but only given for illustrative purposes – we refer the reader to [9] for a more extensive survey).

An important application of the bounds mentioned above is in bounding the number of semi-algebraically connected components of the realizations of various sign conditions of a family of polynomials in R^k or more generally sign conditions restricted to a given real sub-variety of R^k. In order to state these results more precisely, we introduce some notation.

Notation 1.1. For $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ a finite family of polynomials, a *sign condition* σ on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization* $\mathrm{Reali}(\sigma, V)$ *of the sign condition* σ *on a semi-algebraic set* $V \subset \mathrm{R}^k$ is the semi-algebraic set defined by

$$\mathrm{Reali}(\sigma, V) \quad = \quad \{x \in V \mid \mathrm{sign}(P) = \sigma(P), P \in \mathcal{P}\}.$$

Notation 1.2. For any finite family of polynomials $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ we will denote by $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$ the set of real zeros of $\mathcal{Q}$ in R^k. If $\mathcal{Q} = \{Q\}$, then we will use the notation $\mathrm{Zer}(Q, \mathrm{R}^k)$ instead. We will denote by $\mathcal{Q}^h$ (respectively, Q^h) the *homogenizations* of the polynomials in $\mathcal{Q}$ (respectively, the polynomial Q), and denote by $\mathrm{Zer}(\mathcal{Q}^h, \mathbb{P}^k_{\mathrm{C}})$ (respectively, $\mathrm{Zer}(Q^h, \mathbb{P}^k_{\mathrm{C}})$) the common zeros of the family $\mathcal{Q}^h$ (respectively, the polynomial Q^h) in the projective space $\mathbb{P}^k_{\mathrm{C}}$.

Notation 1.3. For any $Q \in \mathrm{R}[X_1, \ldots, X_k]$ we will denote by $\deg(Q)$ the degree of Q. More, generally for a tuple of polynomials $\mathcal{Q} = (Q_1, \ldots, Q_\ell) \in \mathrm{R}[X_1, \ldots, X_k]^\ell$ we will denote $\deg(\mathcal{Q}) = (d_1, \ldots, d_\ell)$ where $d_i = \deg(Q_i)$, $1 \leq i \leq \ell$.

Notation 1.4. For any semi-algebraic set $S \subset \mathrm{R}^k$, we will denote by $b_i(S)$ the i-th Betti number of S. In particular, $b_0(S)$ is the number of semi-algebraically connected components of S.

Notation 1.5. For any semi-algebraic set $S \subset \mathrm{R}^k$, we will denote by $\dim S$ the *real dimension* of S. For any $x \in S$, we denote by $\dim_x S$ the local real dimension of S at x. Note that unlike complex varieties, an irreducible real variety can have have different local dimensions at different points.

Remark 1.6. We will at times slightly abuse notation and use the same letter to denote a tuple of polynomials as well as the ordered finite set whose elements are the elements of the tuple. This should not cause any confusion.

The following theorem gives a reasonably tight bound on the number of semi-algebraically connected components of the realizations of all realizable sign conditions of a finite family of polynomials restricted to a variety. It generalizes earlier results of Alon [3], Warren [29] and Pollack and Roy [24], and has found several applications in discrete geometry.

Theorem 1. [5] *Let $\mathcal{P}, \mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ be finite families of polynomials such that the degrees of the polynomials in $\mathcal{P}, \mathcal{Q}$ are bounded by d, $\operatorname{card} \mathcal{P} = s$, and $\dim_{\mathrm{R}}(V) = k'$, where $V = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$. Then,*

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma, V)) \leq O(1)^k s^{k'} d^k.$$

Notice that in the bound in Theorem 1, while the exponent of s depends on the dimension of the variety V, the exponent of d is that of the ambient space. Moreover, the bound depends only on the maximum degree of the polynomials in $\mathcal{P}$ and $\mathcal{Q}$. This is a consequence of the fact that the proof involves taking *sums of squares* of the polynomials in $\mathcal{P}$ and $\mathcal{Q}$, and thus only the maximum degree plays a role in the argument. This feature of taking the sum of squares is something that is common in the proofs of all the bounds mentioned above. As such they all depend on the *maximum* of the degrees of the polynomials used to define the given set or sign conditions.

More recently, a new application of the bounds described above in discrete and computational geometry, triggered by the work of Guth and Katz [16], raised the question whether even the part of the bound in Theorem 1 that depends only on the degree d could have a finer dependence on the degrees of the polynomials in $\mathcal{P}$ and $\mathcal{Q}$, in the case when the degrees of the polynomials in $\mathcal{Q}$ and those in $\mathcal{P}$ differ significantly (see [16, 26, 18, 17, 30, 20]). This is one of the primary motivations behind the results proved in the current paper (see Section 1.2 below for more detail). A second motivation is to prove a version of the Bezout inequality on bounding the number of isolated complex solutions (or more generally the number of connected components) of an affine polynomial system by the product of the degrees, over real closed fields where the original statement of the inequality does not hold (see Section 1.3, and in particular Example 1.8 and Remark 1.11 below).

A first step was taken in this direction in [4] where the authors of the current paper proved the following theorem (actually a more precise statement appears in [4] but the following simplified version is what is important for the present purpose).

Theorem 2. [4] *Let $\mathcal{P}, \mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ be finite subsets of polynomials such that $\deg(Q) \leq d_1$ for all $Q \in \mathcal{Q}$, $\deg(P) \leq d_2$ for all $P \in \mathcal{P}$. Suppose also that $d_1 \leq d_2$, and the real dimension of $V = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$ is $k_1 \leq k$, and that $\operatorname{card} \mathcal{P} = s$.*

Then,

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma, V)) \quad \leq \quad O(1)^k (sd_2)^{k_1} d_1^{k-k_1}.$$

Remark 1.7. One should compare Theorem 2 with Theorem 1. The new aspect of Theorem 2 is the more refined dependence on the two different degrees, taking into account the dimension of the variety V and the fact that $d_1 \leq d_2$.

Notice that Theorem 2 implies the following corollary about the number of semi-algebraically connected components of real varieties.

Corollary 3. *Let* $Q_1, Q_2 \in \mathrm{R}[X_1, \ldots, X_k]$ *such that* $\deg(Q_1) \leq d_1$,$\deg(Q_2) \leq d_2$, $d_1 \leq d_2$. *Let* $V_1 = \mathrm{Zer}(Q_1, \mathrm{R}^k)$ *and* $\dim V_1 \leq k_1$, *and let* $V_2 = \mathrm{Zer}(\{Q_1, Q_2\}, \mathrm{R}^k)$. *Then,*

$$b_0(V_2) \quad \leq \quad O(1)^k d_1^{k-k_1} d_2^{k_1}.$$

Proof. In Theorem 2, take $\mathcal{Q} = \{Q_1\}$, $\mathcal{P} = \{Q_2\}$ and $V = V_1$. □

1.2. **Applications in discrete geometry.** While Theorem 2 (in particular, also Corollary 3) has already proved useful in certain applications in discrete and computational geometry (see [2, 26, 20]), some even more recent developments seem to require a more detailed analysis.

The requirement of refined bounds from real algebraic geometry in the applications mentioned above originates in the so called "polynomial partitioning" method due to Guth and Katz [16], which provides a framework for proving bounds in several types of problems in discrete geometry involving finite sets of points (such as incidence problems [26], unit and distinct distance problems [16, 17, 30] etc.).

The original polynomial partitioning result states that given any set, S, of n points in R^k, and an auxiliary parameter r, $0 < r < n$, there exists a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$ of degree at most $O\left(r^{\frac{1}{k}}\right)$, having the property that each semi-algebraically connected component C of $\mathrm{R}^k \setminus \mathrm{Zer}(P, \mathrm{R}^k)$ contains at most $\frac{n}{r}$ of the points of S. The number of such semi-algebraically connected components C (using for instance Theorem 1) is bounded by $O(r)$, and it is at this point that a quantitative bound on the number of semi-algebraically connected components of a semi-algebraic set or sign conditions enters the proof. The polynomial partitioning theorem is a tool to decompose a given problem involving the set S into sub-problems of smaller size (corresponding to the point sets $C \cap S$ where C is a semi-algebraically connected component of $\mathrm{R}^k \setminus \mathrm{Zer}(P, \mathrm{R}^k)$). However, it might happen that most or even all the points of S are contained in $\mathrm{Zer}(P, \mathrm{R}^k)$ which is problematic for a "divide-and-conquer" type argument. In this case, an obvious idea is to try to extend the polynomial partitioning theorem to varieties of lower dimensions, and continue the partitioning recursively. However, in order to prove the strongest result possible using this approach, one requires tight bounds on the number of semi-algebraically connected components of real varieties defined by a sequence polynomials of strictly increasing degrees, which has a much more refined dependence on the sequence of degrees than what was provided in Theorem 2 mentioned above (where the length of the sequence is restricted to at most 2). Note however that in the applications related to the polynomial partitioning method, the length of the sequence of degrees could be as large as the dimension of the ambient space, and Theorem 2 is insufficient to deal with cases with degree sequences of lengths greater than 2. The main result of this paper (Theorem 4) is geared towards handling this situation, and has already proved useful in applications involving the polynomial partitioning technique. For example, Theorem 4 plays a crucial role in a recent application of multi-level polynomial partition technique for proving the tightest known bound for the point-hypersurface incidence problem in $\mathbb{R}^4$ [11, Theorem 1.5].

1.3. **Failure of the naive version of Bezout inequality over the reals.** Before stating our results let us consider what kind of refined bounds are plausible. In the case of a real variety V of R^k, which is a non-singular complete intersection (even at infinity) and defined by polynomials of degrees $d_1 \leq d_2 \leq \cdots \leq d_\ell$, the number of semi-algebraically connected components of V is bounded by (see Proposition 3.22 as well as Remark 3.23 below)

$$O(1)^k d_1 \ldots d_{\ell-1} d_\ell d_\ell^{k-\ell}.$$

Notice that $k-\ell = \dim V$. It is thus natural to hope that such a bound continues to hold even if the given variety is not a non-singular complete intersection – namely, one might hope that the number of semi-algebraically connected components of a real variety $V \subset \mathrm{R}^k$ defined by a sequence of ℓ polynomials having degrees $d_1 \leq d_2 \leq \cdots \leq d_\ell$ is bounded by $O(1)^k d_1 \ldots d_\ell d_\ell^{\dim V}$. However, the following well known (counter-)example (that appears in [14]) already shows that this is not the case.

Example 1.8. Let $k = 3$ and let

$$\begin{aligned} Q_1 &= X_{3,} \\ Q_2 &= X_{3,} \\ Q_3 &= \sum_{i=1}^{2} \left(\prod_{j=1}^{d} (X_i - j)^2 \right). \end{aligned}$$

The real variety defined by $\{Q_1, Q_2, Q_3\}$ is 0-dimensional, and has d^2 isolated (in R^3) points, whereas the degree sequence $(d_1, d_2, d_3) = (1, 1, 2d)$, and thus the conjectured bound is $d_1 \ldots d_\ell d_\ell^{\dim V} = O(d)$. In particular, this example shows that the (naive version of) Bezout inequality which states that the number of isolated complex zeros of a system of polynomial equations is bounded by the product of the degrees of the polynomials appearing in the system, is not true over if we replace the complex numbers by a real closed field. Notice however that the polynomials Q_1, Q_2, Q_3 do not define a non-singular complete intersection.

While this might seem discouraging at first glance, one way to repair the situation is to formulate a bound that depends not just on the degree sequence and the dimension of the last variety $V = V_3 = \mathrm{Zer}(\{Q_1, Q_2, Q_3\}, \mathrm{R}^k)$, but also takes into account the dimensions of the intermediate varieties $V_1 = \mathrm{Zer}(Q_1, \mathrm{R}^k)$, $V_2 = \mathrm{Zer}(\{Q_1, Q_2\}, \mathrm{R}^k)$ etc. Notice that in Example 1.8 the dimensions $k_1 = \dim V_1$, and $k_2 = \dim V_2$ are both equal to 2, whereas $k_3 = \dim V_3 = 0$. The number of semi-algebraically connected components in this case is bounded by $O(d_1^{k-k_1} d_2^{k_1-k_2} d_3^{k_2})$, where $d_i = \deg Q_i$. This is the starting point of the formulation of the new results proved in this paper.

We prove the following theorems where the shapes of the bounds should be seen in the light of Example 1.8.

1.4. **Main results.** For the rest of the paper we will use the following notation.

Notation 1.9. Let $Q_1, \ldots, Q_\ell \in \mathrm{R}[X_1, \ldots, X_k]$ such that for each $i, 1 \le i \le \ell$, $\deg(Q_i) \le d_i$. For $1 \le i \le \ell$, denote by $\mathcal{Q}_i = \{Q_1, \ldots, Q_i\}$, $V_i = \mathrm{Zer}(\mathcal{Q}_i, \mathrm{R}^k)$, and $\dim_{\mathrm{R}}(V_i) \le k_i$. We set $V_0 = \mathrm{R}^k$, and adopt the convention that $k_i = k$ for $i \le 0$. It is clear that $k = k_0 \ge k_1 \ge \cdots \ge k_\ell$.

With these assumptions we have the following generalization of Corollary 3.

Theorem 4. *Suppose that*

$$2 \le d_1 \le d_2 \le \frac{1}{k+1} d_3 \le \frac{1}{(k+1)^2} d_4 \le \cdots \le \frac{1}{(k+1)^{\ell-2}} d_\ell.$$

Then,

$$b_0(V_\ell) \quad \le \quad O(1)^k \sum_{\tau=(\tau_0,\tau_1,\ldots,\tau_{\ell-1})} F(k,\tau) \left(d_\ell^{\tau_{\ell-1}} \prod_{1 \le i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i} \right)$$

where the sum on the right hand side is taken over all $\tau \in \mathbb{N}^\ell$, with $k = \tau_0 \ge \tau_1 \ge \cdots \ge \tau_{\ell-1} \ge 0$, and $\tau_i \le k_i$, for each $i, 1 \le i < \ell$, and

$$F(k,\tau) = (k - \tau_{\ell-1} + 1) \binom{k - \tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \ldots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

This implies that

$$b_0(V_\ell) \quad \le \quad O(1)^\ell O(k)^{2k} \left(\prod_{1 \le j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}},$$

and in particular if $\ell \le k$,

$$b_0(V_\ell) \le O(k)^{2k} \left(\prod_{1 \le j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}}.$$

Remark 1.10. Note that since the real dimension of each variety V_i is at most the complex dimension of V_i, Theorem 4 remains true if we replace real dimension by complex dimension in the statement. This observation is important in the application of Theorem 4 to incidence problems (see [11]).

Remark 1.11. In view of Example 1.8 above, Theorem 4 can be viewed as a weak version of the Bezout inequality over real closed fields.

The following slight modification of Example 1.8 shows that the dependence on the degrees in the bound in Theorem 4 cannot be improved.

Example 1.12. Let $k = k_0 \ge k_1 \ge k_2 \ge \cdots \ge k_\ell = 0$, and $d_1, \ldots, d_\ell$ be even. For $1 \le i \le \ell$, let

$$Q_i \quad = \quad \sum_{j=k-k_{i-1}+1}^{k-k_i} \left(\prod_{h=1}^{d_i/2} (X_j - h) \right)^2.$$

Then, for $0 \le i \le \ell$, $\deg(Q_i) = d_i$, the real dimension of the variety $V_i = \mathrm{Zer}(\mathcal{Q}_i, \mathrm{R}^k)$ where $\mathcal{Q}_i = \{Q_1, \ldots, Q_i\}$, is clearly k_i, and

$$b_0(V_\ell) \quad = \quad \frac{1}{2^k} d_1^{k_0 - k_1} d_2^{k_1 - k_2} \cdots d_{\ell-1}^{k_{\ell-2} - k_{\ell-1}} d_\ell^{k_{\ell-1}}.$$

With the same assumptions as in Theorem 4, suppose additionally that $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_k]$ is a finite family of polynomials with $\deg(P) \leq d$ for all $P \in \mathcal{P}$, and $\mathrm{card}\,\mathcal{P} = s$, and suppose that $d_\ell \leq \frac{1}{k+1}d$.

Theorem 5.

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,V_\ell)) \quad\leq\quad \sum_{j=0}^{k_\ell} 4^j\binom{s}{j}O(1)^k\Delta \tag{1}$$

where Δ is defined by

$$\Delta = \sum_{\tau=(\tau_0,\tau_1,\ldots,\tau_\ell)} F(k,\tau)d^{\tau_\ell}\left(\prod_{1\leq i\leq\ell}((k-\tau_{i-1}+1)d_i)^{\tau_{i-1}-\tau_i}\right),$$

where the sum is taken over all $\tau \in \mathbb{N}^{\ell+1}$, with $k = \tau_0 \geq \tau_1 \geq \cdots \geq \tau_\ell \geq 0$, and $\tau_i \leq k_i$, for each $i, 1 \leq i \leq \ell$, and

$$F(k,\tau) = (k-\tau_\ell+1)\binom{k-\tau_\ell}{\tau_0-\tau_1,\tau_1-\tau_2,\ldots,\tau_{\ell-1}-\tau_\ell}.$$

This implies that

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,V_\ell)) \quad\leq\quad O(1)^\ell O(k)^{2k}(sd)^{k_\ell}\left(\prod_{1\leq j\leq\ell} d_j^{k_{j-1}-k_j}\right).$$

In particular, if $\ell \leq k$,

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,V_\ell)) \quad\leq\quad O(k)^{2k}(sd)^{k_\ell}\left(\prod_{1\leq j\leq\ell} d_j^{k_{j-1}-k_j}\right).$$

Remark 1.13. Notice that in the case $\ell = 1$, the bound (1) in Theorem 5 implies that of Theorem 2, and hence Theorem 5 is a strict generalization of Theorem 2.

With the same assumptions as in Theorem 5, let for $P \in \mathcal{P}$, $d_P = \deg(P)$, and for any subset $\mathcal{I} \subset \mathcal{P}$ let

$$d_{\mathcal{I}} \quad=\quad (k+1)^{\binom{\mathrm{card}\,\mathcal{I}}{2}+(k_\ell-\mathrm{card}\,\mathcal{I})(\mathrm{card}\,\mathcal{I}-1)}\left(\prod_{P\in\mathcal{I}} d_P\right)(\max_{P\in\mathcal{I}} d_P)^{k_\ell-\mathrm{card}\,\mathcal{I}}. \tag{2}$$

We have the following variant of Theorem 5 (the extra precision with respect to the degrees of polynomials in $\mathcal{P}$ might be useful in applications in incidence geometry).

Using Notation 1.9 and notation introduced in (2) above:

Theorem 6.

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,V_\ell)) \leq \sum_{\substack{\mathcal{I}\subset\mathcal{P}\\ j=\mathrm{card}\,\mathcal{I}\leq k_\ell}} 4^j O(1)^\ell O(k)^{2k} d_{\mathcal{I}}\left(\prod_{1\leq j\leq\ell} d_j^{k_{j-1}-k_j}\right).$$

Remark 1.14. The condition on the degrees in Theorems 4 and 5 might look unnatural at first glance but is forced on us by the method of the proof, which involves taking minors of matrices of size at most $(k+1)\times(k+1)$ with entries which are polynomials of degree d_i, $1 \leq i \leq \ell$. We want at each step, the degree d_i to majorize the degree of the polynomial obtained as a minor in the previous step whose entries have degree at most d_j, where $j < i$. Notice that in the case $\ell = 2$, the condition on the degree sequence is just $d_1 \leq d_2$, and this allows us to recover Theorem 2 from Theorem 5.

Remark 1.15. We also note that in [25] the authors define the "complexification" of a semi-algebraic set as the smallest complex variety containing it, and prove an effective bound on the geometric degree of this complexification which depend amongst other quantities on the real dimension of the given set. This degree could be thought of as the "real degree" of the semi-algebraic set. It is possible that Theorem 5 could serve as an alternative basis for a good definition of the "real degree" of a real variety – in the sense that the "real degree" of a real variety V should control the number of semi-algebraically connected components of the intersection of V with any real hypersurface of sufficiently large degree. We do not pursue this idea further in the current paper.

Finally, we conjecture that the bounds in Theorems 4 and 5 extend to the sum of all the Betti numbers (instead of just the zero-th one). The techniques developed in this paper are not sufficient to prove this conjecture.

1.5. **Outline of the proofs of the main theorems.** The main difficulty that one faces in order to prove bounds having the shapes of Theorems 4 and Theorem 5 is that in order to respect the degree sequence one has to be careful about taking "sums of squares" which spoil the dependence on the degrees. The crucial idea is to use the notion of "approximating" varieties. An approximating variety is a variety which is infinitesimally close to the given variety of the same dimension, but having good algebraic properties which allow one to give a precise bound on the number of its semi-algebraically connected components in terms of the sequence of degrees of polynomials defining it (rather than just the maximum degree). If the given variety can be covered (in a technical sense made precise later) by a small number of such approximating varieties, then the problem of bounding the number of semi-algebraically connected components of the given varieties reduces to the problem of bounding the total number of semi-algebraically connected components of these approximating varieties.

The idea of using approximating varieties originates in algorithmic semi-algebraic geometry and it was used in [7] to give efficient algorithms for computing sample points on varieties and in [8] to compute roadmaps of semi-algebraic sets. The combinatorial part of the complexities of these algorithms depends on the dimension of the given variety rather than that of the ambient space, and this is where the approximating varieties play an important role in those papers. In quantitative semi-algebraic geometry, the notion of approximating varieties was used in [4] in order to prove Theorem 2.

The approximation scheme that we use, which is a generalization of the one used in [4] is described in Section 3.1 below. One difficulty in generalizing the scheme in [4] is that the non-singularity of polar varieties of smooth hypersurfaces with respect to generic projections that is used in that paper no longer holds for smooth varieties of higher co-dimension. A second difficulty is that the sequence of local (real) dimensions at a point $x \in V_\ell$ of the varieties $V_1, \ldots, V_\ell$ is not globally constant, but is only a local invariant (see Example 3.3 and Figure 1). Thus, one cannot expect to have a single global approximating variety with good properties. We overcome the latter problem by taking into account all possible sequences of local dimensions whether they actually occur or not (indexed by the set A below), and construct approximating varieties with acceptable degree sequences to approximate each of them.

Consider the subset of points of U_i of V_ℓ having local dimension $i \leq k_\ell$. At each point $x \in U_i$ the dimension of $V_{\ell-1}$ is between i and $k_{_{\ell-1}}$. Suppose we have already constructed approximations of subsets of $V_{\ell-1}$ consisting points having some fixed local dimension at $V_{\ell-1}$. Using these approximations and adding appropriately many equations in each case we construct a set of approximations of U_i. Taking all these approximating varieties, for all $i, 0 \leq i \leq k_\ell$, and noticing that V_ℓ is the union of the U_i's we obtain a global approximation of V_ℓ (see Example 3.17 and Figures 3, 4, 5, 6 and 7 below).

More precisely, we construct a family of basic semi-algebraic sets each of the form,

$$\mathrm{Bas}(\mathcal{P}, \mathcal{Q}) \quad := \quad \{x \in \mathrm{R}'^k \mid P(x) = 0, Q(x) \leq 0, P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

where R' is some real closed extension of R depending on the particular approximating set. The family of pairs $\{(\mathcal{P}^\alpha_{\tau,\ell}, \mathcal{Q}^\alpha_{\tau,\ell})\}_{\tau \in A \subset \mathbb{N}^\ell, \alpha \in I(\tau)}$ defining these approximating varieties are indexed by a pair of indices τ, α coming from two finite set of indices $A_\ell \subset \mathbb{N}^\ell$, and $I_\ell(\tau)$. While the definition of the second, $I_\ell(\tau)$, is a bit technical and which we defer for later, the definition of the index set A_ℓ is the following.

$$A_\ell \quad := \quad \{\tau = (\tau_1, \ldots, \tau_\ell) \in \mathbb{N}^\ell \mid k \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_\ell, \tau_i \leq k_i, 1 \leq i < \ell\}.$$

For any given $\tau \in A_\ell$, let $V_\tau \subset V_\ell$ denote the closure of the set of points $x \in V_\ell$ such that the local real dimension of V_i at x is equal to τ_i, for each $i, 1 \leq i \leq \ell$. The union of the approximating sets $V^\alpha_{\sigma,\ell} = \mathrm{Bas}(\mathcal{P}^\alpha_\sigma, \mathcal{Q}^\alpha_\sigma)$ with $\sigma \leq \tau$, "approximates" V_τ in a certain precise sense (see Proposition 3.13 below), and since clearly $V_\ell = \bigcup_{\tau \in A} V_\tau$, the union of all the approximating sets $\{V^\alpha_{\tau,\ell}\}_{\tau \in A \subset \mathbb{N}^\ell, \alpha \in I(\tau)}$ approximate the whole variety V_ℓ. Because of the approximating property, in order to bound the number of semi-algebraically connected components of V_ℓ it suffices to bound the sum of the number of semi-connected components of each one of the approximating sets $V^\alpha_{\tau,\ell}$. The tuples $\mathcal{P}^\alpha_{\tau,\ell}, \mathcal{Q}^\alpha_{\tau,\ell}$ have the following properties that enable us to obtain good bounds on the number of semi-algebraically connected components of $V^\alpha_{\tau,\ell}$ (see Proposition 2.13 below).

a) The tuple of polynomials $\mathcal{P}^\alpha_{\tau,\ell}$ define a non-singular, bounded complete intersection of dimension $\tau_\ell \leq k_\ell$. In particular, this means that the cardinality of $\mathcal{P}^\alpha_{\tau,\ell}$ is equal to $k - \tau_\ell$. Suppose that $\mathcal{P}^\alpha_{\tau,\ell} = (P_1, \ldots, P_{k-\tau_\ell})$. Let for $1 \leq i \leq \ell$, $\ell_i = \tau_{i-1} - \tau_i$, with the convention that $\tau_0 = k$, and $L_i = \sum_{h=1}^i \ell_h$. Then for each $i, 1 \leq i \leq \ell$, the degrees of the polynomials $P_{L_{i-1}+1}, \ldots, P_{L_i}$ are bounded by $O(kd_i)$.
b) $\mathcal{Q}^\alpha_{\tau,\ell}$ is either empty or contains one polynomial, $Q^\alpha_{\tau,\ell}$, with $\deg(Q^\alpha_{\tau,\ell}) = O(d_\ell)$, and $\mathcal{P}', Q^\alpha_{\tau,\ell}$, where $\mathcal{P}'$ is any subset of $\mathcal{P}^\alpha_{\tau,\ell}$, defines a non-singular complete intersection.

It remains to bound the number of semi-algebraically connected components of each $V^\alpha_{\tau,\ell}$ and take the sum of these bounds, for which we use the same result as in [4] where a bound is derived using a classical formula for the Betti numbers of complex non-singular complete intersections and the Smith inequality (see Proposition 3.22 below). The number of approximating varieties (which is independent of the given degree sequence) and the bounds on the degree sequences of their defining polynomials as stated in Properties a) and b) above are good enough to give us the bound in Theorem 4.

Theorem 5 follows from Theorem 4 using standard techniques already used in [6] and no fundamentally new ingredients.

The rest of the paper is organized as follows. In Section 2, we recall some basic facts about real closed fields of Puiseux series that we need for making deformation arguments. We also recall some results proved in [4] on the choice of generic coordinates. Finally, in Section 3 we prove the main theorems.

2. Preliminary results

2.1. **Deformation of several equations to general position.** In this section we describe how to deform a system of equations using infinitesimals so that the set of common zeros of the deformed equations (in certain real closed non-archimedean extensions of the ground field) has good properties. For this we first need to recall some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [10] for further detail.

We begin with some notation.

Notation 2.1. For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\cdots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$, $0<\varepsilon_m\ll\varepsilon_{m-1}\ll\cdots\ll\varepsilon_1\ll 1$. Also, note that both fields $\mathrm{R}\langle\varepsilon\rangle,\mathrm{R}\langle\delta\rangle$ are sub-fields in a natural way of $\mathrm{R}\langle\varepsilon,\delta\rangle$.

Notation 2.2. If R' is a real closed extension of a real closed field R, and $S\subset\mathrm{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by $\mathrm{Ext}(S,\mathrm{R}')\subset\mathrm{R}'^k$ the semi-algebraic subset of R'^k defined by the same formula. It is well-known that $\mathrm{Ext}(S,\mathrm{R}')$ does not depend on the choice of the formula defining S [10].

Notation 2.3. For $x\in\mathrm{R}^k$ and $r\in\mathrm{R}$, $r>0$, we will denote by $B_k(x,r)$ the open Euclidean ball centered at x of radius r. If R' is a real closed extension of the real closed field R and when the context is clear, we will continue to denote by $B_k(x,r)$ the extension $\mathrm{Ext}(B_k(x,r),\mathrm{R}')$. This should not cause any confusion.

Notation 2.4. For elements $x\in\mathrm{R}\langle\varepsilon\rangle$ which are bounded over R we denote by $\lim_\varepsilon x$ to be the image in R under the usual map that sets ε to 0 in the Puiseux series x.

Notation 2.5. Let $Q\in\mathrm{R}[X_1,\ldots,X_k]$, $0\le q\le k$, and $H\in\mathrm{R}[X_{q+1},\ldots,X_k]$. Let ζ be a new variable. We denote

$$\mathrm{Def}(Q,\zeta,q,H) \quad = \quad (1-\zeta)Q-\zeta H.$$

Notation 2.6. For $\mathcal{P}=(P_1,\ldots,P_m)$, with each $P_i\in\mathrm{R}[X_1,\ldots,X_k]$, $1\le q\le k$, and $\mathcal{G}=(G_1,\ldots,G_m)$ with each $G_i\in\mathrm{R}[X_{q+1},\ldots,X_k]$, and ζ a new variable, we denote by $\mathrm{Def}(\mathcal{P},\zeta,q,\mathcal{G})$ the tuple

$$(\mathrm{Def}(P_1,\zeta,q,G_1),\ldots,\mathrm{Def}(P_m,\zeta,q,G_m)),$$

and by $\mathrm{Def}(\mathcal{P},\zeta,q,\mathcal{G})^h$ the corresponding tuple of homogenized polynomials

$$(\mathrm{Def}(P_1,\zeta,q,G_1)^h,\ldots,\mathrm{Def}(P_m,\zeta,q,G_m)^h).$$

Notation 2.7. For $\mathcal{F} = (F_1, \ldots, F_{k-p}), q \le p \le k$, we denote the jacobian matrix

$$\mathrm{Jac}(\mathcal{F}, p, q) \quad := \quad \begin{pmatrix} \frac{\partial F_1}{\partial X_{q+1}} & \cdots & \frac{\partial F_{k-p}}{\partial X_{q+1}} \\ \vdots & & \vdots \\ \frac{\partial F_1}{\partial X_k} & \cdots & \frac{\partial F_{k-p}}{\partial X_k} \end{pmatrix}$$

whose rows are indexed by $[q+1, k]$ and columns by $[1, k-p]$.

For $J \subset [q+1, k]$, $\mathrm{card}\, J = k - p$ and $k \in J$, let Jac_J denote the $(k-p) \times (k-p)$ matrix extracted from the matrix $\mathrm{Jac}(\mathcal{F}, p, q)$ by extracting the rows whose index are in J, and let

$$\mathrm{jac}_J = \det(\mathrm{Jac}_J).$$

Let

$$\mathcal{F}_J \quad := \quad \mathcal{F} \cup \bigcup_{i \in [q+1,k] \setminus J} \{\mathrm{jac}_{J \cup \{i\} \setminus \{k\}}\}, \tag{3}$$

and the finite constructible set

$$C_J(\mathcal{F}) \quad := \quad \{x \in \mathrm{Zer}(\mathcal{F}_J, \mathrm{R}^k) \mid \mathrm{jac}_J(x) \neq 0\}. \tag{4}$$

Proposition 2.8. *Let $\mathcal{F} = (F_1, \ldots, F_{k-p})$, each $F_i \in \mathrm{R}[X_1, \ldots, X_k]$, and such that the variety $\mathrm{Zer}(\mathcal{F}^h, \mathbb{P}^k_{\mathrm{C}})$ is a non-singular complete intersection. Let $x \in \mathrm{R}^k$ be a non-generate critical point of the projection map to the X_k-coordinate restricted to the variety $V = \mathrm{Zer}(\mathcal{F}, \mathrm{R}^k)$. Then, there exists a subset $J \subset [1, k]$, $\mathrm{card}\, J = k - p$, $k \in J$, satisfying the following two conditions.*

1. *The $(k-p) \times (k-p)$ matrix, Jac_J, extracted from the matrix $\mathrm{Jac}(\mathcal{F}, p, 0)$ by extracting the rows whose index are in J, evaluated at x is non-singular.*
2. *The point x is a simple zero of the system $\mathcal{F}_J$ (see (3) for definition).*

Proof. First note that using the Jacobian criteria for non-singularity of real algebraic varieties (see for example [12, Definition 3.3.4]), we have that the variety V is of dimension equal to p and non-singular. Moreover, x is a critical point of the projection map to the X_k coordinate restricted to V, by the inverse function theorem we can choose p coordinates (not including X_k) such that the remaining $k-p$ co-ordinates of points of V in a small enough neighborhood U of x are smooth functions of these chosen p co-ordinates. Without loss of generality let these p coordinates be $X_1, \ldots, X_p$. We will denote the remaining co-ordinate functions on U by $X_{p+1}(X_1, \ldots, X_p), \ldots, X_k(X_1, \ldots, X_p)$ noting that they are smooth semi-algebraic functions of $X_1, \ldots, X_p$.

We use that

(1) $\mathrm{Jac}(\mathcal{F}, p, 0)(x)$ has full rank since x is a non-singular point of V, and
(2) $\mathrm{Hess}(X_k(X_1, \ldots, X_p))(x)$ is non-singular since x is a non-degenerate critical point with respect to X_k.

Let $J = [p+1, k]$, and consider the Jacobian matrix $\mathrm{Jac}(\mathcal{F}_J, 0, 0)$.

$$\mathrm{Jac}(\mathcal{F}_J, 0, 0) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_{k-p}}{\partial X_1} & \frac{\partial\, \mathrm{jac}_{J \cup \{1\} \setminus \{k\}}}{\partial X_1} & \cdots & \frac{\partial\, \mathrm{jac}_{J \cup \{p\} \setminus \{k\}}}{\partial X_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_1}{\partial X_k} & \cdots & \frac{\partial F_{k-p}}{\partial X_k} & \frac{\partial\, \mathrm{jac}_{J \cup \{1\} \setminus \{k\}}}{\partial X_k} & \cdots & \frac{\partial\, \mathrm{jac}_{J \cup \{p\} \setminus \{k\}}}{\partial X_k} \end{pmatrix}.$$

Since, by definition of the functions $X_{p+1}(X_1, \ldots, X_p), \ldots, X_k(X_1, \ldots, X_p)$

$$F_i(X_1, \ldots, X_p, X_{p+1}(X_1, \ldots, X_p), \ldots, X_k(X_1, \ldots, X_p)) \equiv 0,$$

for $1 \leq i \leq k-p$, by the chain rule for $1 \leq j \leq p$,

$$
\begin{aligned}
0 &= \frac{\partial F_1}{\partial X_j} + \frac{\partial F_1}{\partial X_{p+1}}\frac{\partial X_{p+1}}{\partial X_j} + \ldots + \frac{\partial F_1}{\partial X_k}\frac{\partial X_k}{\partial X_j}, \\
&\vdots \\
0 &= \frac{\partial F_{k-p}}{\partial X_j} + \frac{\partial F_{k-p}}{\partial X_{p+1}}\frac{\partial X_{p+1}}{\partial X_j} + \ldots + \frac{\partial F_{k-p}}{\partial X_k}\frac{\partial X_k}{\partial X_j}.
\end{aligned} \tag{5}
$$

Let $\Delta = \det \mathrm{Jac}(\mathcal{F}, p, p)$. Notice that in the sub-matrix $\mathrm{Jac}(\mathcal{F}, p, 0)$ of $\mathrm{Jac}(\mathcal{F}_J, 0, 0)$, for each $1 \leq j \leq p$, adding

$$\sum_{i=p+1}^{k} \frac{\partial X_i}{\partial X_j} \cdot \mathrm{row}_i(\mathrm{Jac}(\mathcal{F}, p, 0))$$

to the j-th row and using (5) we can clear out the first p rows. Since, $\mathrm{rank}(\mathrm{Jac}(\mathcal{F}, p, 0)(x)) = k-p$, this implies that $\Delta(x) \neq 0$.

From Cramer's Rule, we have

$$
\begin{aligned}
\frac{\partial X_k}{\partial X_1} &= \frac{-\mathrm{jac}_{J\cup\{1\}\setminus\{k\}}}{\Delta}, \\
&\vdots \\
\frac{\partial X_k}{\partial X_p} &= \frac{-\mathrm{jac}_{J\cup\{p\}\setminus\{k\}}}{\Delta}.
\end{aligned}
$$

Let for $1 \leq i \leq p$,

$G_i(X_1, \ldots, X_p) = -\mathrm{jac}_{J\cup\{i\}\setminus\{k\}}(X_1, \ldots, X_p, X_{p+1}(X_1, \ldots, X_p), \ldots, X_k(X_1, \ldots, X_p))$.

Substituting above we get that

$$
\begin{aligned}
\frac{\partial X_k}{\partial X_1} &= \frac{G_1(X_1, \ldots, X_p)}{\Delta}, \\
&\vdots \\
\frac{\partial X_k}{\partial X_p} &= \frac{G_p(X_1, \ldots, X_p)}{\Delta}.
\end{aligned}
$$

From the quotient rule,

$$\frac{\partial^2 X_k}{\partial X_i \partial X_j} = \frac{\frac{\partial G_1}{\partial X_i}\Delta - G_i \frac{\partial}{\partial X_j}}{\Delta^2},$$

and in particular

$$\mathrm{Hess}(X_k)(x) = \left(\ \frac{\partial^2 X_k}{\partial X_i \partial X_j}(x) \ \right)_{1 \leqslant i,j \leqslant p} = \left(\ \frac{\frac{\partial G_i}{\partial X_j}(x)}{\Delta(x)} \ \right)_{1 \leqslant i,j, \leqslant p}$$

noticing that since x is a critical point of the function X_k restricted to V, $G_1(x) = \cdots = G_p(x) = 0$.

Applying the chain rule again we have that for $1 \leq i, j \leq p$,

$$\frac{\partial G_i}{\partial X_j} = -\frac{\partial\,\mathrm{jac}_{J\cup\{i\}\setminus\{k\}}}{\partial X_j} - \frac{\partial\,\mathrm{jac}_{J\cup\{i\}\setminus\{k\}}}{\partial X_{p+1}}\frac{\partial X_{p+1}}{\partial X_j} - \cdots - \frac{\partial\,\mathrm{jac}_{J\cup\{i\}\setminus\{k\}}}{\partial X_k}\frac{\partial X_k}{\partial X_j}. \tag{6}$$

Finally, for each $1 \leq j \leq p$, adding

$$\sum_{i=p+1}^{k} \frac{\partial X_i}{\partial X_j} \cdot \mathrm{row}_i(\mathrm{Jac}(\mathcal{F}_J, 0, 0))$$

to the j-th row, and using (5) and (6), we see that $\mathrm{Jac}(\mathcal{F}_J, 0, 0)(x)$ is row equivalent to the matrix

$$\begin{pmatrix} \mathbf{0} & -\frac{\mathrm{Hess}(X_k)}{\Delta}(x) \\ \mathbf{I}_{k-p} & * \end{pmatrix}$$

which is clearly non-singular, since x is a non-degenerate critical point of X_k, which implies that the $\mathrm{Hess}(X_k)(x)$ is non-singular, and we have already observed that $\Delta(x) \neq 0$. $\square$

Definition 2.9. Let $X \subset \mathbb{P}^k_{\mathrm{C}}$ be a non-singular variety, and $(H_\mu)_{\mu=(\mu_0:\mu_1)\in\mathbb{P}^1_{\mathrm{C}}}$ a pencil of hyperplanes. We call the pencil of varieties $(X_\mu = X \cap H_\mu)_\mu$ a *Lefschetz pencil* if it satisfies the two following conditions.

1. The base locus B is smooth of co-dimension two in X.
2. Each member X_μ of the pencil has at most one ordinary double point as a singularity.

The main result about Lefschetz pencil we will require is the following well known result from complex algebraic geometry (see for example [28, Corollary 2.10]).

Proposition 2.10. *If $X \subset \mathbb{P}^k_{\mathrm{C}}$ is a non-singular variety, then any generic pencil of hyperplane sections of X is Lefschetz.*

Remark 2.11. Observe that a generic tuple of polynomials $\mathcal{G} = (G_1, \ldots, G_{k-p})$ where each $G_i \in \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(H_i) = d_i$ and is chosen generically, will have the property that the variety $W = \mathrm{Zer}(\mathcal{G}, \mathbb{P}^k_{\mathrm{C}})$ is non-singular and the pencil of hyperplane sections $(W_\mu = W \cap H_\mu)_\mu$ indexed by $\mu = (\mu_0 : \mu_1)$, where $H_\mu \subset \mathbb{P}^k_{\mathrm{C}}$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety W by Proposition 2.10 above.

Let $0 \leq p \leq k$, $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$ be a tuple of variables, and $\mathcal{P} = (P_1, \ldots, P_{k-p})$, $P_i \in \mathrm{R}\langle\bar{\varepsilon}\rangle[X_1, \ldots, X_k]$ with $\deg P_i \leq d_i$, and $P \in \mathrm{R}[X_1, \ldots, X_k]$, $\deg P \leq d$. Let $0 \leq q < p \leq k$ and $\mathcal{G} = (G_1, \ldots, G_{k-p})$ be a tuple of polynomials with $G_i \in \mathrm{R}[X_{q+1}, \ldots, X_k]$ with $\deg(G_i) = d_i$, and $G \in \mathrm{R}[X_{q+1}, \ldots, X_k]$ be another polynomial with $\deg(G) = d$, such that

1. The variety $W = \mathrm{Zer}(\mathcal{G} \cup \{G\}, \mathbb{P}^{k-q}_{\mathrm{C}})$ is a non-singular complete intersection.
2. The pencil of hyperplane sections $(W_\mu = W \cap H_\mu)_\mu$ indexed by $\mu = (\mu_0 : \mu_1) \in \mathbb{P}^1_{\mathrm{C}}$, where $H_\mu \subset \mathbb{P}^{k-q}_{\mathrm{C}}$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety W.

Let for every $z \in \mathrm{R}^q$,

$$\mathcal{F}_z \quad = \quad \mathrm{Def}(\mathcal{P}, \zeta, q, \mathcal{G})(z, \cdot), \mathrm{Def}(P, \delta, q, G)(z, \cdot). \tag{7}$$

We also need the following notation.

Notation 2.12. For $1 \leq p \leq q \leq k$, we denote by $\pi_{[p,q]}$ the projection map on the coordinates $X_p, \ldots, X_q$, and also denote by $\mathrm{R}^{[p,q]}$ the subspace spanned by these coordinates. For any set $S \subset \mathrm{R}^k$, and $z \in \mathrm{R}^{[1,p]}$ we will denote by S_z the fiber $S \cap \pi^{-1}_{[1,p]}(z)$.

Proposition 2.13. *For every $z \in \mathrm{R}^q$, the following holds.*

1. $\mathrm{Def}(\mathcal{P},\eta,q,\mathcal{G})(z,\cdot)^h, \mathrm{Def}(P,\delta,q,G)(z,\cdot)^h$ *defines a non-singular complete intersection $V_z \subset \mathbb{P}^{k-q}_{\mathrm{C}\langle\delta,\bar{\varepsilon},\eta\rangle}$ of dimension $p-q-1$.*
2. *The pencil of hyperplane sections $(V_{z,\mu} = V_z \cap H_\mu)_\mu$ indexed by $\mu = (\mu_0 : \mu_1)$, where $H_\mu \subset \mathbb{P}^{k-q}_{\mathrm{C}\langle\delta,\bar{\varepsilon},\eta\rangle}$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety V_z.*
3. *For each singular point $x \in \mathrm{C}^k$ of the pencil $(V_{z,\mu})_\mu$, there exists $J \subset [k-q+1,k]$,* $\mathrm{card}\, J = k-p$ *and $k \in J$, such that $x \in C_J(\mathcal{F}_z)$ (see (4) and (7) for definitions), and x is a simple zero of the system $(\mathcal{F}_z)_J$ (see (3) for definition).*

Proof. Replacing η and δ by new variables s and t (respectively), and setting $s = t = 1$ we have that $(\mathrm{Def}(\mathcal{P},1,q,\mathcal{G})(z,\cdot)^h, \mathrm{Def}(P,1,q,G)(z,\cdot)^h) = (\mathcal{G}^h, G^h)$ define a non-singular complete intersection in $W \subset \mathbb{P}^{k-q}_{\mathrm{C}\langle\bar{\varepsilon}\rangle}$ (by hypothesis). Moreover, the pencil of hyperplane sections $(W_\mu = W \cap H_\mu)_\mu$ is Lefschetz by hypothesis. Since the property of being a non-singular complete intersection as well as a fixed pencil of hyperplane section being Lefschetz is stable, it also holds for an open neighborhood of the point $(s,t) = (1,1)$. The set of pairs (s,t) for which any of these two properties is violated is Zariski closed, defined over $\mathrm{C}\langle\bar{\varepsilon}\rangle$, and is not the whole of $\mathbb{P}^1_{\mathrm{C}\langle\bar{\varepsilon}\rangle} \times \mathbb{P}^1_{\mathrm{C}\langle\bar{\varepsilon}\rangle}$. In particular the complement contains the point (η,δ) (since η,δ are algebraically independent over $\mathrm{C}\langle\bar{\varepsilon}\rangle$). This proves parts 1. and 2. of the proposition. Part 3. follows from Proposition 2.8. $\square$

We also need the following proposition.

Proposition 2.14. *Let C be a bounded s.a. connected component of* $\mathrm{Bas}(\mathcal{P},\mathcal{Q})$. *Then, there exists a subset subset $\mathcal{Q}' \subset \mathcal{Q}$, and a semi-algebraically connected component D of* $\mathrm{Zer}(\mathcal{P} \cup \mathcal{Q}', \mathrm{R}^k)$ *such that $D \subset C$.*

Proof. See Proposition 13.1 in [10]. $\square$

Proposition 2.15. *Let $\mathcal{F} = (F_1,\ldots,F_k)$ be a tuple of polynomials with $F_i \in$* $\mathrm{R}[X_1,\ldots,X_k]$ *and let $\mathcal{G} = (G_1,\ldots,G_k)$, with $G_i \in \mathrm{R}[X_1,\ldots,X_k]$, be a tuple of polynomials with* $\deg(\mathcal{G}) \le \deg(\mathcal{F})$. *Let $x \in \mathrm{R}^k$ a simple zero of $\mathcal{F}$. Then, there exists a simple zero $\tilde{x} \in \mathrm{Zer}(\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G}),\mathrm{R}\langle\zeta\rangle^k)$, such that $\lim_\zeta \tilde{x} = x$.*

Proof. It follows from the fact that x is a simple zero of the family $\mathcal{F}$ that any infinitesimal perturbation of the family $\mathcal{F}$ will have a simple zero, $\tilde{x} \in \mathrm{C}\langle\zeta\rangle^k$, in an infinitesimal neighborhood of x. To see this observe that since $x \in \mathrm{R}^k$ (and is thus in particular bounded over R), it belongs to the image under the $\lim_\zeta$ map (extended to elements of $\mathrm{C}\langle\zeta\rangle^k$ which are bounded over R) of $\mathrm{Zer}(\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G}),\mathrm{C}\langle\zeta\rangle^k)$. Thus, there exists $\tilde{x} \in \mathrm{Zer}(\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G}),\mathrm{C}\langle\zeta\rangle^k)$, such that $\lim_\zeta \tilde{x} = x$. Moreover, since x is a simple zero of $\mathrm{Zer}(\mathcal{F},\mathrm{R}^k)$, we have that $\det(\mathrm{Jac}(\mathcal{F},0,0))(x) \neq 0$ (see Notation 2.7). Since,

$$\lim_\zeta(\det(\mathrm{Jac}(\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G}),0,0))(\tilde{x})) = \det(\mathrm{Jac}(\mathcal{F},0,0))(x),$$

this implies that

$$\det(\mathrm{Jac}((\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G}),0,0)))(\tilde{x}) \neq 0$$

as well, since $\det(\mathrm{Jac}(\mathcal{F},0,0))(x) \in \mathrm{R} \setminus \{0\}$, and hence $\tilde{x}$ is a simple zero of $\mathrm{Def}(\mathcal{F},\zeta,0,\mathcal{G})$.

Moreover, $\tilde{x}$ must belong to $\mathrm{R}\langle\zeta\rangle^k$ as long as the perturbed polynomials also have real coefficients. Otherwise, since complex zeros must occur in conjugate pairs, if $\tilde{x} \notin \mathrm{R}\langle\zeta\rangle^k$, then $\tilde{x} \neq \overline{\tilde{x}}$, while $\lim_\zeta \tilde{x} = \lim_\zeta \overline{\tilde{x}} = x$, and this implies that x is not a simple zero of $\mathcal{F}$. □

2.2. **Generic coordinates.** We recall in this section a result proved in [4] that we will require.

Notation 2.16. For a real algebraic set $V = \mathrm{Zer}(Q, \mathrm{R}^k)$we let $\mathrm{reg}(V)$ denote the non-singular points in dimension $\dim V$ of V (Definition 3.3.9 in [12]).

Definition 2.17. Let $V = \mathrm{Zer}(Q, \mathrm{R}^k)$ be a real algebraic set. Define $V^k = V$, and for $0 \leq i \leq k-1$ define

$$V^{(i)} = V^{(i+1)} \setminus \mathrm{reg}(V^{(i+1)}).$$

Let $d_V(i)$ denote the dimension of $V^{(i)}$.

Definition 2.18. Let $V = \mathrm{Zer}(Q, \mathrm{R}^k)$ be a real algebraic set, $1 \leq j \leq k$, and $\ell \in \mathrm{Gr}(k, k-j)$. We say that the linear space ℓ is j-good with respect to V if either:

- $j \notin d_V([0,k])$,
- or $d_V(i) = j$, and the set

$$\{x \in \mathrm{reg}(V^{(i)}) \mid \dim \mathrm{T}_x V^{(i)} \cap \ell = 0\}$$

is a non-empty dense Zariski open subset of $V^{(i)}$.

Definition 2.19. Let $V = \mathrm{Zer}(Q, \mathrm{R}^k)$ and $B = \{v_1, \ldots, v_k\}$ be a basis of R^k. We say that the basis B is *generic* with respect to V if for each $j, 1 \leq j \leq k$, the linear space $\mathrm{span}(v_1, \ldots, v_{k-j})$ is j-good with respect to V.

The following proposition appears in [4].

Proposition 2.20. *Let $V = \mathrm{Zer}(Q, \mathrm{R}^k)$ and $\{v_1, \ldots, v_k\}$ be a basis of R^k. Then, there exists a non-empty open semi-algebraic subset U of linear transformations $\mathrm{GL}(k, \mathrm{R})$ such that for every $T \in U$ the basis $\{T(v_1), \ldots, T(v_k)\}$ is generic with respect to V.*

3. Proofs of the main theorems

We now fix polynomials $Q_1, \ldots, Q_\ell$ and and the varieties $V_1, \ldots, V_\ell$ as in Theorem 4. We will assume if necessary by initially squaring each polynomial that each Q_i is non-negative over R^k. Since this increases each degree by a multiplicative factor of 2, this does not affect the asymptotics of the bound.

The section is organized as follows. In Subsection 3.1 we define certain approximating semi-algebraic sets and prove their important properties. In Subsection 3.2 we recall and then apply in the current context certain well-known bounds on the Betti numbers of non-singular cmplete intersections. Finally, we prove the main theorems of this paper in Subsections 3.3 and 3.4.

3.1. **Definitions and main properties of approximating semi-algebraic sets.** We first introduce in 3.1.1 some necessary notation, and then in Subsection 3.1.2 below we describe the construction of certain semi-algebraic sets approximating the varieties V_j. The main properties of these sets is then proved in Subsection 3.1.3. The approximating properties of these sets are proved in Proposition 3.13, and the quantitative estimates on the degrees of the polynomials appearing in the description of these approximating sets is proved in Proposition 3.21.

3.1.1. *Notation.*

Notation 3.1. For any semi-algebraic set S and $x \in S$, we denote by $\dim_x S$ the local dimension of S at x. For $0 \leq j \leq \ell$ and $x \in V_j$, we denote

$$\dim^{(j)}(x) = (\dim_x V_1, \ldots, \dim_x V_j).$$

Notation 3.2. We will use the natural partial order on the sets $\mathbb{N}^j$, and denote for $\sigma = (\sigma_1, \ldots, \sigma_j), \tau = (\tau_1, \ldots, \tau_j) \in \mathbb{N}^j$, $\sigma \leq \tau$, if $\sigma_i \leq \tau_i$ for all $1 \leq i \leq j$.

Before proceeding further we illustrate the notation introduced above by considering some examples.

We first consider again Example 1.8 from Section 1.

In this example, (following Notation 1.9) we have

$$\begin{aligned} V_1 &= \mathrm{Zer}(X_3, \mathrm{R}^k), \\ V_2 &= \mathrm{Zer}(X_3, \mathrm{R}^k), \\ V_3 &= \{1, \ldots, d\}^3. \end{aligned}$$

The various functions $\dim^{(j)} : V_j \to \mathbb{N}^j, j = 1, 2, 3$ (cf. Notation 3.1) are as follows.

$$\begin{aligned} \dim^{(1)}(x) &= (2) \text{ if } x \in V_1, \\ \dim^{(2)}(x) &= (2, 2) \text{ if } x \in V_2 = V_1, \\ \dim^{(3)}(x) &= (2, 0) \text{ if } x \in V_3. \end{aligned}$$

The next example is slightly more involved but is helpful in understanding the proof of Proposition 3.13 below.

Example 3.3. Let $k = 4$, $\ell = 3$, and

$$\begin{aligned} Q_1 &= (X_1 + X_2 + X_3 + X_4)(X_3^2 + X_4^2), \\ Q_2 &= X_3^2 + X_4^2, \\ Q_3 &= X_3^2 + X_4^2 + (X_1^2 - X_2^3)^2. \end{aligned}$$

We denote by $(e_i)_{1 \leq i \leq 4}$ the elementary basis vectors in R^4. Denote by L_1 the hyperplane defined by $X_1 + X_2 + X_3 + X_4 = 0$, by $L_2 = \mathrm{span}(e_1, e_2)$, the linear subspace defined by $X_3 = X_4 = 0$, and by C_3 the cubic curve contained in L_2, defined by the equation $X_1^2 - X_2^3 = 0$. Notice that $L_1 \cap L_2$ is the line in $\mathrm{span}(e_1, e_2)$ defined by $X_1 + X_2 = 0$, and it meets C_3 at the points $x^0 = \mathbf{0}$ (which is a singular point of C_3), and $x^1 = (-1, 1, 0, 0, 0)$ (which is a regular point of C_3). These sets are depicted in Figure 1.

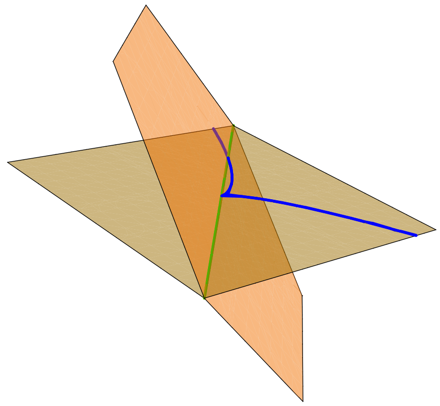

FIGURE 1. The sets L_1, L_2 and C_3 in Example 3.3 restricted to the hyperplane $X_4 = 0$

We have using Notation 1.9,

$$\begin{aligned} V_1 &= L_1 \cup L_2, \\ V_2 &= L_2, \\ V_3 &= C_3. \end{aligned}$$

The various functions $\dim^{(j)} : V_j \to \mathbb{N}^j, j = 1,2,3$ can now be described as follows.

$$\begin{aligned} \dim^{(1)}(x) &= (3) \text{ if } x \in L_1, \\ \dim^{(1)}(x) &= (2) \text{ if } x \in L_2 \setminus L_1, \\ \dim^{(2)}(x) &= (3,2) \text{ if } x \in L_2 \cap L_1, \\ \dim^{(2)}(x) &= (2,2) \text{ if } x \in L_2 \setminus L_1, \\ \dim^{(3)}(x) &= (3,2,1) \text{ if } x = x^0, x^1, \\ \dim^{(3)}(x) &= (2,2,1) \text{ if } x \in C_3 \setminus \{x^0, x^1\}. \end{aligned}$$

Remark 3.4. Observe that in Example 3.3 above, the point $x^0 = \mathbf{0}$ is a singular point of V_3, and $x^0 \in V_{(3,2,1)}$. However, for any open neighborhood $U \subset V_3$ of x_0 in V_3, and $x' \in U \setminus \{x^0\}$, we have that x' is a regular point of V_3, and if moreover $x' \neq x^1$, $\dim^{(3)}(x') = (2,2,1) < (3,2,1)$ (cf. Notation 3.2). Notice also that $(2,2,1)$ is a minimal element of the set $\{\dim^{(3)}(x) \mid x \in V_3\}$, and the semi-algebraic subset of V_3 defined by

$$\{x \in V_3 \mid \dim^{(3)}(x) = (2,2,1)\} = V_3 \setminus \{x^0, x^1\}$$

is open in V_3 (cf. Proposition 3.5 below).

The following property of the function $\dim^{(j)} : V_j \to \mathbb{N}^j$ will be important later. Following Notation 1.9 as before we have the following proposition.

Proposition 3.5. *Let $\sigma \in \mathbb{N}^j, 1 \le j \le \ell$. Then, the semi-algebraic subset $V_j^{\le\sigma} \subset V_j$ defined by*

$$V_j^{\le\sigma} = \{x \in V_j \ \mid \ \dim^{(j)}(x) \le \sigma\}$$

is open in V_j. In particular, if $U \subset V_j$ is an open semi-algebraic subset of V_j, and $\sigma \in \mathbb{N}^j$ is such that σ is a minimal element of the set $\{\dim^{(j)}(x) \mid x \in U\}$, then the semi-algebraic subset $\{x \in U \mid \dim^{(j)}(x) = \sigma\}$ is open in V_j.

Proof. The proof is by induction on j. If $j = 1$, then the proposition follows immediately from the upper semi-continuity property of the dimension function. Now suppose that the proposition is true for all smaller values of j. Let $\sigma = (\sigma_1, \ldots, \sigma_j)$ and let $\sigma' = (\sigma_1, \ldots, \sigma_{j-1})$. Using the induction hypothesis we have that $V_{j-1}^{\le\sigma'}$ is open in V_{j-1}. This implies that there exists an open semi-algebraic subset $U \subset \mathrm{R}^k$, such that $V_{j-1}^{\le\sigma'} = V_{j-1} \cap U$. Also, the semi-algebraic set $V_j^{\le\sigma_j} = \{x \in V_j \ \mid \ \dim_x(V_j) \le \sigma_j\} \subset V_j$ is open in V_j. Thus, there exists an open semi-algebraic set $U' \subset \mathrm{R}^k$, such that $V^{\le\sigma_j} = V_j \cap U'$. Now,

$$\begin{aligned} V_j^{\le\sigma} &= V_j^{\le\sigma_j} \cap V_{j-1}^{\le\sigma'} \\ &= (V_j \cap U') \cap (V_{j-1} \cap U) \\ &= V_j \cap U \cap U' \quad (\text{since } V_j \subset V_{j-1}). \end{aligned}$$

Hence, $V_j^{\le\sigma}$ is open in V_j. □

Notation 3.6. For $0 \le j \le \ell$ we call $\tau = (\tau_1, \ldots, \tau_j) \in \mathbb{N}^j$ *admissible* if it satisfies the following two conditions.

1. $\tau_1 \ge \cdots \ge \tau_j$,
2. for $1 \le i < j$, $\tau_i \le k_i$.

We denote the subset of admissible tuples of $\mathbb{N}^j$ by A_j, and denote by A the set A_ℓ. For $\sigma = (\sigma_1, \ldots, \sigma_j), \tau = (\tau_1, \ldots, \tau_j) \in A_j$, we say $\sigma \le \tau$, if $\sigma_i \le \tau_i$ for each $i, 1 \le i \le j$.

Notation 3.7. For each $j, 1 \le j \le \ell$, we denote by R_j the real closed field

$$\mathrm{R}\langle \delta_j, \ldots, \delta_1, \eta_1, \zeta_1, \ldots, \eta_j, \zeta_j \rangle.$$

Notice that R_j is a real closed extension of the field R_{j-1}. For any semi-algebraic subset $S \subset \mathrm{R}_j$, we will denote by S_b the union of semi-algebraically connected components of S which are bounded over R.

Remark 3.8. For readers familiar with arguments in real algebraic geometry involving multiple infinitesimals, this ordering of the infinitesimals in Notation 3.7 might seem somewhat counter-intuitive, since we will consider the varieties V_i's in the order V_1, V_2, etc., and the infinitesimal δ_i will be used to perturb the variety V_i, one would expect that the infinitesimals δ_i's to be ordered the other way round. The reason behind this ordering of the infinitesimals will become clear in the proof of Proposition 3.13 below.

3.1.2. *Definition of sequences of approximating semi-algebraic sets.* We now describe the construction of certain semi-algebraic sets approximating the varieties V_j. We assume that V_1, and hence each V_j, are bounded over R.

We will also use the following notation.

Notation 3.9. For any set X and $j \geq 0$ we will denote by $\binom{X}{j}$ the set of all subsets of X of cardinality j.

Definition 3.10. For any $\tau \in A_j$ we define an index set $I_j(\tau)$, and a family $(V^\alpha_{\tau,j} \subset \mathrm{R}^k_j)_{\alpha \in I_j(\tau)}$ as follows. Each $V^\alpha_{\tau,j} = \left(\mathrm{Bas}(\mathcal{P}^\alpha_{\tau,j}, \{Q^\alpha_{\tau,j}\})\right)_b$, where $\mathcal{P}^\alpha_{\tau,j}$, is an ordered tuple of polynomials, and $Q^\alpha_{\tau,j} \in \mathrm{R}_j[X_1,\ldots,X_k]$ defined inductively as follows.

1. If $j = 0$, then for $\tau = ()$, define $I_0(\tau) = \{-1\}$, and $\mathcal{P}^{(-1)}_{\tau,0} = (0), Q^{(-1)}_{\tau,0} = 0$.
2. Otherwise, we denote by $\tau' = (\tau_1,\ldots,\tau_{j-1})$ and let $p = \tau_{j-1}$, $q = \tau_j$. Let G be a generic polynomial in $\mathrm{R}[X_{q+1},\ldots,X_k]$ strictly positive over R^{k-q} with $\deg(G) = \deg(\bar{P}_j)$,
$$\begin{aligned} \bar{P}_j &= \sum_{1 \leq i \leq j} Q_i \in \mathrm{R}[X_1,\ldots,X_k], \\ \tilde{P}_j &= \mathrm{Def}(\bar{P}_j, \delta_j, q, H) \in \mathrm{R}\langle\delta_j\rangle[X_1,\ldots,X_k]. \end{aligned}$$
3. $$\begin{aligned} I_j(\tau) &= I_{j-1}(\tau') \times \{-1\}, \text{ if } \tau_{j-1} = \tau_j, \\ &= I_{j-1}(\tau') \times \binom{[\tau_j+1,k]}{k-\tau_{j-1}+1}, \text{ else} \end{aligned}$$
(where $\times$ denotes the usual Cartesian product).
4. For each triple $(\alpha \in I_{j-1}(\tau'), \mathcal{P}^\alpha_{\tau',j-1}, Q^\alpha_{\tau',j-1})$
   - if $\tau_{j-1} = \tau_j$, then denoting $\beta = (\alpha, -1)$ let
   $$\begin{aligned} \mathcal{P}^\beta_{\tau,j} &= \mathcal{P}^\alpha_{\tau',j-1}, \\ Q^\beta_{\tau,j} &= \tilde{P}_j. \end{aligned}$$
   - otherwise, suppose that
   $$\mathcal{P} = \mathcal{P}^\alpha_{\tau',j-1} = (P_1,\ldots,P_{k-p}) \subset \mathrm{R}_{j-1}[X_1,\ldots,X_k]^{k-p},$$
   with $\deg(P_i) = d'_i$, for $1 \leq i \leq k-p$, and $\overline{d'} = (d'_1,\ldots,d'_{k-p})$. Let
   $$\mathcal{G} = (G_1,\ldots,G_{k-p})$$
   be generic polynomials in $\mathrm{R}[X_{q+1},\ldots,X_k]$ with $\deg(G_i) = d'_i$ and strictly positive over R^{k-q}, $1 \leq i \leq k-p$.
   We define (using Notation 2.6)
   $$\begin{aligned} \tilde{\mathcal{P}} &= \mathrm{Def}(\mathcal{P}, \eta_j, q, \mathcal{G}) \\ \mathcal{F} &= (\tilde{\mathcal{P}}, \tilde{P}_j). \end{aligned} \tag{8}$$
   Finally, for each $J \in \binom{[\tau_j+1,k]}{k-\tau_{j-1}+1}$, denoting $\beta = (\alpha, J)$, and following the notation introduced above (and using Notation 2.7)
   $$\begin{aligned} \mathcal{P}^\beta_{\tau,j} &= \mathrm{Def}(\mathcal{F}_J, \zeta_j, k, \mathcal{G}'), \\ Q^\beta_{\tau,j} &= Q^\alpha_{\tau',j-1}, \end{aligned} \tag{9}$$
   where $\mathcal{G}' = (G'_1,\ldots,G'_{k-q})$ is another tuple of generic polynomials strictly positive over R^k with $\deg(\mathcal{G}') = (\bar{d}_\alpha, d_j, d',\ldots,d')$, where $d' = (k-p+1)d_j$ and $\bar{d}_\alpha = \deg(\mathcal{P})$.

Notation 3.11. For each $j, 0 \leq j \leq \ell$, $\tau \in A_j$, let (cf. Notation 3.1)

$$\begin{aligned} \tilde{V}_\tau &= \bigcup_{\alpha \in I_j(\tau)} V^\alpha_{\tau,j}, \\ V_\tau &= \overline{\{x \in V_j \mid \dim^{(j)}(x) = \tau\}}. \end{aligned}$$

Using Notation 3.11:

Proposition 3.12. *For each j, $1 \leq j \leq \ell$,*

$$V_j = \bigcup_{\tau=(\tau_1,\ldots,\tau_j)\in A_j, \tau_j \leq k_j} V_\tau.$$

Proof. This is immediate from the definition of A_j and the various V_τ,

$$\tau = (\tau_1, \ldots, \tau_j) \in A_j, \tau_j \leq k_j,$$

and the fact that $\dim V_i \leq k_i$ for $0 \leq i \leq j$. □

3.1.3. *Properties of the approximating sets.* The following proposition and its corollary guarantees the approximating properties of the sets $V^\alpha_{\tau,j}$ defined above and is the main technical proposition of the paper.

Assume that the given system of coordinates is generic with respect to the finite number of varieties V_τ (cf. Proposition 2.20).

Proposition 3.13. *For all $\tau = (\tau_1, \ldots, \tau_j) \in A_j$, with $\tau_j \leq k_j$,*

$$V_\tau \subset W_\tau \subset V_j,$$

where

$$W_\tau = \bigcup_\sigma \lim_{\delta_j} \tilde{V}_\sigma,$$

and the union is taken over all $\sigma \in A_j$ with $\sigma_j = \tau_j$, and $\sigma_i \leq \tau_i$ for all $1 \leq i < j$.

In the proof of Proposition 3.13 we need the following technical lemma that we prove first. We draw the attention of the reader to the ordering of the infinitesimals in this lemma, which is particularly delicate and plays a very important role in the proof of the lemma. In particular, notice that if $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m), \delta$ are variables, then we have the following diagram of real closed subfields of the real closed field $\mathrm{R}\langle \delta, \bar{\varepsilon} \rangle$.

$$\begin{tikzcd} & \mathrm{R}\langle \delta, \bar{\varepsilon} \rangle & \\ \mathrm{R}\langle \delta \rangle \arrow[ur, hook] & & \mathrm{R}\langle \bar{\varepsilon} \rangle \arrow[ul, hook'] \\ & \mathrm{R} \arrow[ul, hook'] \arrow[uu, hook] \arrow[ur, hook] & \end{tikzcd}$$

In particular, if V (respectively, Z) is a semi-algebraic subset of $\mathrm{R}\langle \bar{\varepsilon} \rangle^k$ (respectively, $\mathrm{R}\langle \delta \rangle^k$), then $\mathrm{Ext}(V, \mathrm{R}\langle \delta, \bar{\varepsilon} \rangle), \mathrm{Ext}(Z, \mathrm{R}\langle \delta, \bar{\varepsilon} \rangle)$ (recall Notation 2.2) are semi-algebraic subsets of $\mathrm{R}\langle \delta, \bar{\varepsilon} \rangle^k$.

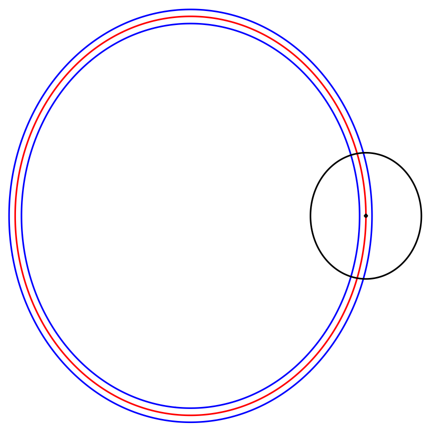

FIGURE 2. Example illustrating Lemma 3.14

Lemma 3.14. *Let* $P, H \in \mathrm{R}[X_1, \ldots, X_k]$*,* P *non-negative, and* H *strictly positive at all points of* R^k*. Let* $V \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^k$ *be a semi-algebraic set bounded over* R*, where* $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$*. Let* $\tilde{P} = (1-\delta)P - \delta H$*, and* Z *a semi-algebraically connected component of* $\mathrm{Zer}(\tilde{P}, \mathrm{R}\langle\delta\rangle^k)$*, such that* $Z = \mathrm{Zer}(\tilde{P}, \mathrm{R}\langle\delta\rangle^k) \cap B_k(x, r)$*, for some* $x \in \mathrm{R}^k$ *and* $r > 0$*,* $r \in \mathrm{R}$ *. Suppose that* $\lim_{\varepsilon_1}(\mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle)) \cap Z \neq \emptyset$*. Then,* $\mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle) \cap \mathrm{Ext}(Z, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle) \neq \emptyset$*.*

Before proving Lemma 3.14 we illustrate it with a simple example.

Example 3.15. In this example, $k = 2$, and

$$\begin{aligned} V &= \mathrm{Zer}((X_1^2 + X_2^2 - 1)^2 - \varepsilon, \mathrm{R}\langle\varepsilon\rangle^2) \text{ (shown in blue in Figure 2)}, \\ P &= (X_1 - 1)^2 + X_2^2, \\ H &= 1. \end{aligned}$$

We display the various sets occurring in this example in Figure 2 after choosing ε, δ, to be certain sufficiently small positive real numbers, with $\varepsilon \ll \delta$. It is clear from definition that P is non-negative, $\mathrm{Zer}(P, \mathrm{R}^k)$ consists of the single point $(1, 0)$ (shown in black in Figure 2), and the variety $\mathrm{Zer}(\tilde{P}, \mathrm{R}\langle\delta\rangle^2)$, where $\tilde{P} = (1-\delta)P - \delta H$, has one semi-algebraically connected component, Z, which is also depicted in black. The semi-algebraic set $\lim_\varepsilon V = \mathrm{Zer}((X_1^2 + X_2^2 - 1)^2, \mathrm{R}^2)$ is the unit circle centered at the origin (shown in red), and $\mathrm{Ext}(V, \mathrm{R}\langle\delta, \varepsilon\rangle)$ meets $\mathrm{Ext}(Z, \mathrm{R}\langle\delta, \varepsilon\rangle)$ in two points, and $\mathrm{Ext}(V, \mathrm{R}\langle\delta, \varepsilon\rangle) \cap \mathrm{Ext}(Z, \mathrm{R}\langle\delta, \varepsilon\rangle)$ is not empty, and consists of four points as can be seen in Figure 2.

Proof of Lemma 3.14. Let $G \in \mathrm{R}(X_1, \ldots, X_k)$ denote the rational function $\frac{P}{P+H}$ which is continuous, and takes non-negative values at all points of R^k by hypothesis. Let $y \in \mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle)$ be such that $z = \lim_{\varepsilon_1} y \in Z$. Since, Z is contained in $\overline{B_k(x, r)}$, and $y \in \mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle)$ is ε_1-infinitesimally close to $z \in Z$, it is clear that $\mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle) \cap \overline{B_k(x, r)}$ contains y and in particular is not empty. Let C be the semi-algebraically connected component of $\mathrm{Ext}(V, \mathrm{R}\langle\delta, \bar{\varepsilon}\rangle) \cap \overline{B_k(x, r)}$ which contains y.

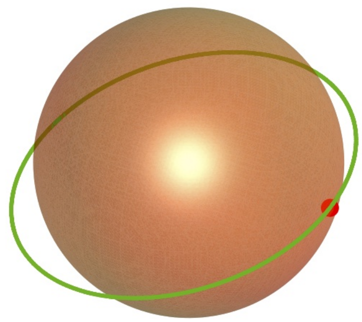

FIGURE 3. The variety V_1.

We prove that $\operatorname{Ext}(C, \mathrm{R}\langle\delta,\bar{\varepsilon}\rangle) \cap \operatorname{Ext}(Z, \mathrm{R}\langle\delta,\bar{\varepsilon}\rangle) \neq \emptyset$. Suppose otherwise. Then, $G(y) \neq \delta$. Suppose without loss of generality that $G(y) - \delta > 0$. Since, $z = \lim_{\varepsilon_1} y \in \operatorname{Zer}(\tilde{P}, \mathrm{R}\langle\delta\rangle^k)$, it is clear that $\lim_{\varepsilon_1}(G(y) - \delta) = 0$. Let $h = \inf_{x \in C} G(x)$. Since, C is a semi-algebraic set defined over $\mathrm{R}\langle\bar{\varepsilon}\rangle$, and G is a continuous rational function defined over R, it follows that $h \in \mathrm{R}\langle\bar{\varepsilon}\rangle$. Moreover, since $\operatorname{Ext}(C, \mathrm{R}\langle\delta,\bar{\varepsilon}\rangle) \cap \operatorname{Zer}(\tilde{P}, \mathrm{R}\langle\delta,\bar{\varepsilon}\rangle^k) = \emptyset$, $G(y) - \delta > 0$, and C is closed and bounded, the infimum of G over C is achieved at a point, and hence $h > \delta$. On the other hand, from the fact that $\lim_{\varepsilon_1}(G(y) - \delta) = 0$, it follows that $\lim_{\varepsilon_1} h = \delta$. This is impossible, since $\lim_{\varepsilon_1} h \in \mathrm{R}$. $\square$

Lemma 3.16. *Suppose that* $\sigma = (\sigma_1, \ldots, \sigma_j), \tau = (\tau_1, \ldots, \tau_j) \in A_j$ *with* $\sigma_i \leq \tau_i, 1 \leq i < j$, *and* $\sigma_j = \tau_j$. *Then,* $W_\sigma \subset W_\tau$.

Proof. Obvious from the definitions of W_σ and W_τ. $\square$

Before giving the proof of Proposition 3.13 we consider the following three-dimensional example which illustrates some of the finer points.

Example 3.17. Let $k = 3$, $\ell = 3$, and

$$\begin{aligned}
Q_1 &= (X_1^2 + X_2^2 + X_3^2 - 1)(X_3^2 + (X_1^2 + \frac{1}{2}X_2^2 - 1)^2), \\
Q_2 &= (X_3^2 + (X_1^2 + \frac{1}{2}X_2^2 - 1)^2), \\
Q_3 &= X_3^2 + X_2^2 + (X_1 - 1)^2.
\end{aligned}$$

The variety V_1 (shown in Figure 3) is bounded, and equal to the union of the unit sphere $S \subset \mathrm{R}^3$ (shown in ornage), and an ellipse, C (shown in green), contained in the plane $\operatorname{span}(e_1, e_2)$, with $S \cap C = \{(\pm 1, 0, 0)\}$ (shown in red). The variety $V_2 = C$, and $V_3 = \{(1, 0, 0)\}$.

The various functions $\dim^{(j)} : V_j \to \mathbb{N}^j$ are as follows.

$$\begin{aligned}
\dim^{(1)}(x) &= (2) \text{ if } x \in S,\\
\dim^{(1)}(x) &= (1) \text{ if } x \in C \setminus S,\\
\dim^{(2)}(x) &= (2,1) \text{ if } x \in S \cap C,\\
\dim^{(2)}(x) &= (1,1) \text{ if } x \in C \setminus S,\\
\dim^{(3)}(x) &= (2,1,0) \text{ if } x = (1,0,0).
\end{aligned}$$

It follows that

$$\begin{aligned}
V_{(2)} &= S,\\
V_{(1)} &= C,\\
V_{(1,1)} &= C,\\
V_{(2,1)} &= S \cap C.
\end{aligned}$$

In the Figures 4, 5 and 6 below we depict the approximating semi-algebraic sets $\tilde{V}_\tau$, for $\tau = (2), (1)$ and $(1,1)$, respectively. Note that in order to be able to draw these pictures we used (small) finite values of the infinitesimals, and so the pictures are for illustrative purposes only.

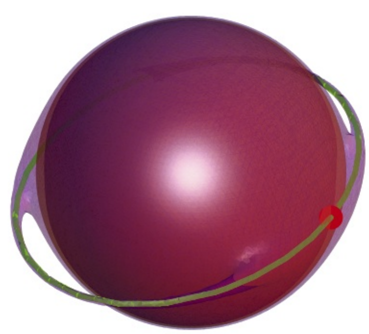

FIGURE 4. The approximating set $\tilde{V}_{(2)}$.

In Figure 4 we depict the approximating set $\tilde{V}_{(2)}$. Notice that in this example $V_{(2)} \subset \lim_{\delta_1} \tilde{V}_{(2)} \subset V_1$. The first inclusion is proper, while the second one is an equality.

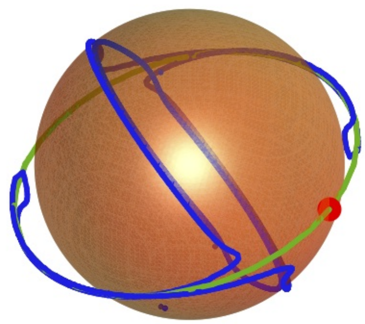

FIGURE 5. The approximating set $\tilde{V}_{(1)}$ in blue.

In Figure 5 we depict the approximating set $\tilde{V}_{(1)}$ (in blue). Note that $V_{(1)} \subset \lim_{\delta_1} \tilde{V}_{(1)} \subset V_1$. Observe that both inclusions are proper in this case. The image of the curve $\tilde{V}_{(1)}$ (shown in blue in Figure 5) under the $\lim_{\delta_1}$ map contains the curve C (shown in green), as well as an additional part contained in the sphere S.

The set $\tilde{V}_{(1,1)}$ is the intersection of $\tilde{V}_{(1)}$ with the set defined by the inequality $\tilde{P}_2 \leq 0$, which is a tube containing the set C, and $V_{(1,1)} = C \subset \lim_{\delta_2} \tilde{V}_{(1,1)} \subset V_2$. Both inclusions are equalities in this case. This is depicted in Figure 6.

In Figure 7 we depict the approximating set $\tilde{V}_{(2,1)}$ (in blue). Note that the set $V_{(2,1)} = \{(\pm 1, 0, 0)\}$, and we have the inclusions $V_{(21)} \subset \lim_{\delta_2} \tilde{V}_{(21)} \subset V_2$. Observe that both inclusions are proper in this case.

We now prove Proposition 3.13. While reading the proof particular attention should be paid to the ordering of the infinitesimals, $\delta_i, \eta_i, \zeta_i, 1 \leq i \leq j$ which plays a crucial role.

Proof of Proposition 3.13. We first prove the inclusion $V_\tau \subset W_\tau$.

Let $x \in V_\tau$ with $\dim^{(j)}(x) = \tau$. We will prove that $x \in W_\tau$ which suffices to prove the inclusion $V_\tau \subset W_\tau$, since W_τ is closed and V_τ is the closure of the set of points y with $\dim^{(j)}(y) = \tau$. The proof of the claim that $x \in W_\tau$ is by induction on j. Suppose the claim holds for $j-1$. There are two cases to consider.

(1) $\tau_j = \tau_{j-1}$: The induction hypothesis implies that $x \in \lim_{\delta_{j-1}} \tilde{V}_{\sigma'}$, where $\sigma' \in A_{j-1}$ with $\sigma'_{j-1} = \tau_{j-1} = \tau_j$, and $\sigma'_i \leq \tau_i$ for $1 \leq i < j-1$. Let $\alpha \in I_{j-1}(\sigma')$ be such that $x \in \lim_{\delta_{j-1}}(V^{\alpha}_{\sigma',j-1})$. Hence, there exists $x' \in$

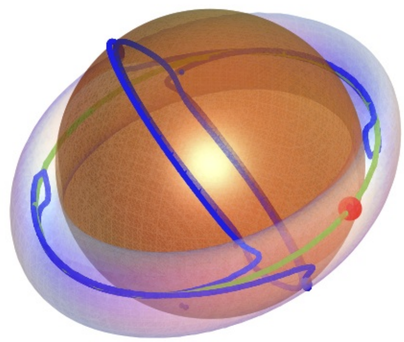

FIGURE 6. The approximating set $\tilde{V}_{(11)}$.

$V^{\alpha}_{\sigma',j-1}$ such that $\lim_{\delta_{j-1}} x' = x$. Moreover, since, $\bar{P}_j(x) = 0$, we have that $\lim_{\delta_{j-1}} \bar{P}_j(x') = 0$. From the definition of $\tilde{P}_j$ and the fact that $\delta_j \gg \delta_{j-1} > 0$, we obtain that $\tilde{P}_j(x') \leq 0$, and hence $x' \in V^{\beta}_{\sigma,j}$, and $x \in \lim_{\delta_j} V^{\beta}_{\sigma,j}$ where $\sigma = (\sigma', \tau_j)$, and $\beta = (\alpha, -1)$.

(2) $q = \tau_j < \tau_{j-1}$: We prove that every neighborhood, U, of x in V_j contains a point of W_τ. Let U be a small enough neighborhood of x in V_j. Then there exists a non-empty open subset $U' \subset U$ such that each $x' \in U'$ is a regular point of V_j of dimension q.

For each $x' \in U'$, clearly $q \leq \dim_{x'} V_{j-1} \leq \dim_x V_{j-1} = \tau_{j-1}$, the second inequality coming from upper semi-continuity property of the dimension function. There are two subcases.

Case (a) If there exists $x' \in U'$, with $\dim_{x'} V_{j-1} = q = \dim_{x'} V_j$, we are reduced to Case (1) as follows. Let $\sigma = (\tau_1, \ldots, \tau_{j-2}, q, q)$. Then, $\sigma \leq \tau$, and using Case (1), $x' \in W_\sigma$, and $W_\sigma \subset W_\tau$ (using Lemma 3.16).

Case (b) We assume that $q < \dim_{x'} V_{j-1} \leq \tau_{j-1}$ for each $x' \in U'$. Using the genericity of the given co-ordinates and shrinking U' if necessary by subtracting a Zariski closed set of co-dimension at least one we can assume that the tangent space $T_{x'}V_j$ is transversal to $\pi^{-1}_{[1,q]}(z')$ (recall Notation 2.12), where $z' = \pi_{[1.q]}(x')$, and hence in particular that x' is an isolated point of $(V_j)_{z'}$ for all $x' \in U'$.

Shrinking U' further if necessary we can also assume that x' is not an isolated point of $(V_{j-1})_{z'}$ where $z' = \pi_{[1.q]}(x')$ for all $x' \in U'$. To see this

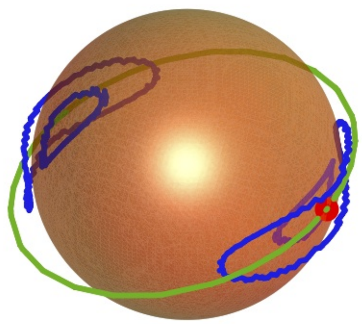

FIGURE 7. The approximating set $\tilde{V}_{(21)}$ in blue.

suppose that there exists a non-empty open subset U'' of U' such that for all $x'' \in U''$, x'' is an isolated point of $(V_{j-1})_{z''}$ where $z'' = \pi_{[1.q]}(x'')$. Then, there exists for any $x'' \in U''$ an open neighborhood W of x'' in V_{j-1} contained in $(V_{j-1})_{\pi_{[1,q](U'')}}$ such that the dimension of W is $\leq q$, which is contrary to our assumption.

Now for each $x' \in U'$, since x' is an isolated point of $(V_j)_{z'}$, and $\lim_{\delta_j}((\mathrm{Bas}(\emptyset, \{\tilde{P}_j\}))_{z'})_b = (V_j)_{z'}$, there exists a unique semi-algebraically connected component of $((\mathrm{Bas}(\emptyset, \{\tilde{P}_j\}))_{z'})_b$, and hence of $\mathrm{Zer}(\tilde{P}_j, \mathrm{R}\langle\delta_j\rangle^k)_{z'}$ (which we will denote by $Z(x')$) such that $\lim_{\delta_j} Z(x') = x'$. Since x' is not an isolated point of $(V_{j-1})_{z'}$, $\mathrm{Ext}((V_{j-1})_{z'}, \mathrm{R}\langle\delta_j\rangle) \cap Z(x') \neq \emptyset$.

We claim that there exist, $\sigma' = (\sigma'_1, \ldots, \sigma'_{j-1}) \in A_{j-1}$, $\sigma' \leq \tau' := (\tau_1, \ldots, \tau_{j-1})$, $\alpha \in I_{j-1}(\sigma')$, $x' \in U'$, $z' = \pi_{[1.q]}(x')$, such that

$$\lim_{\delta_{j-1}}((V^{\alpha}_{\sigma',j-1})_{z'})_b \cap \mathrm{Ext}((V_{\tau'})_{z'}, \mathrm{R}\langle\delta_j\rangle) \cap Z(x') \neq \emptyset,$$

where $Z(x')$ is the unique semi-algebraically connected component of

$$\mathrm{Zer}(\tilde{P}_j, \mathrm{R}\langle\delta_j\rangle^k)_{z'}$$

such that $\lim_{\delta_j} Z(x') = x'$ (see previous paragraph).

To see this let σ' be a minimal element in A_{j-1} such that $U'' := V^{\leq\sigma'}_{j-1} \cap U' \neq \emptyset$. By Proposition 3.5, (see also Example 3.3 and Remark 3.4 following it) U'' is a non-empty open subset of U'. Now suppose that

$$\lim_{\delta_{j-1}}((V^{\alpha}_{\sigma',j-1})_{z'})_b \cap \mathrm{Ext}((V_{\sigma'})_{z'}, \mathrm{R}\langle\delta_j\rangle) \cap Z(x') = \emptyset,$$

for every $x' \in U''$ and $z' = \pi_{[1,q]}(x')$. Now, $\pi_{[1,q]}(U')$, and hence $\pi_{[1,q]}(U'')$, is a non-empty open subset of $\mathrm{R}^{[1,q]}$, since the map $\pi_{[1,q]}|U'$ is a semi-algebraic diffeomorphism by the semi-algebraic implicit function theorem, and the fact that $T_{x'}V_j$ is transversal to $\pi^{-1}_{[1,q]}(\pi_{[1,q]}(x'))$ for every $x' \in U'$ (see above). This contradicts the inductive hypothesis, which implies that $V_{\sigma'} \subset W_{\sigma'}$.

Now fix x', z', σ', α as above. Notice that since $\lim_{\delta_j} Z(x') = x'$, there exists $r > 0$, such that

$$Z(x') = \mathrm{Zer}(\tilde{P}_j, \mathrm{R}\langle\delta_j\rangle^k)_{z'} \cap \overline{B_k(x', r)}_{z'}.$$

It now follows from Lemma 3.14 (applied after taking $\delta = \delta_j$ and $\bar{\varepsilon} = (\delta_{j-1}, \ldots, \delta_1, \eta_1, \zeta_1, \ldots, \eta_{j-1}, \zeta_{j-1})$) that

$$\lim_{\delta_{j-1}} ((V^{\alpha}_{\sigma',j-1})_{z'})_b \cap Z(x') \neq \emptyset$$

implies that

$$\mathrm{Ext}((V^{\alpha}_{\sigma',j-1})_{z'}, \mathrm{R}') \cap \mathrm{Ext}(Z(x'), \mathrm{R}') \neq \emptyset, \tag{10}$$

where $\mathrm{R}' = \mathrm{R}\langle\delta_j, \ldots, \delta_1, \eta_1, \zeta_1, \ldots, \eta_{j-1}, \zeta_{j-1}\rangle$. Moreover, it is clear that (10) implies that

$$\mathrm{Ext}((V^{\alpha}_{\sigma',j-1})_{z'}, \mathrm{R}_j) \cap \mathrm{Ext}(Z(x'), \mathrm{R}_j) \neq \emptyset.$$

Note that the order $\delta_j \gg \delta_{j-1}$ is important here (cf. Remark 3.8).

It follows that there exists a semi-algebraically connected component C of $\mathrm{Zer}((\mathcal{P}^{\alpha}_{\sigma',j-1}, \tilde{P}_j), \mathrm{R}^k_j)_{z'}$, such that $x' \in \lim_{\delta_j} C$, and $C \subset (V^{\alpha}_{\sigma',j-1})_{z'}$. Moreover, using the fact that $z' \in \mathrm{R}^{[1,q]}$, and applying Proposition 2.13 with $\eta = \eta_j, \delta = \delta_j$, and $\bar{\varepsilon} = (\delta_{j-1}, \ldots, \delta_1, \eta_1, \zeta_1, \ldots, \eta_{j-1}, \zeta_{j-1})$, we deduce that the polynomials in $(\mathcal{P}^{\alpha}_{\sigma',j-1}(z', \cdot), \tilde{P}_j(z', \cdot))$ define a non-singular complete intersection of dimension $p - q - 1$ in $\mathrm{R}^{[q+1,k]}_j$, where $p = \sigma'_{j-1}$. Let $\tilde{\mathcal{P}} = \mathcal{P}^{\alpha}_{\sigma',j-1}$, and let $\mathcal{F}$ be the tuple of polynomials defined by (8). Then, there exists a semi-algebraically connected component $\tilde{C}$ of $\mathrm{Zer}(\mathcal{F}, \mathrm{R}^k_j)_{z'}$ such that $x' \in \lim_{\delta_j} \tilde{C}$. There are a finite number of X_k-critical points (all of which are simple) on $\tilde{C}_z$ by Remark 2.11 and Proposition 2.13. If (z', w'), $w' \in \mathrm{R}^{[q+1+,k]}_j$, is one such critical point, then (z', w') is contained in the finite constructible set $C_J(\mathcal{F})$ (cf. (4) and part 3. of Proposition 2.13) for some $J \in \binom{[q+1,k]}{k-p+1}$, and such that w' is a simple zero of the system $\mathcal{F}_J(z', \cdot)$. Hence, applying Proposition 2.15 (with the field of coefficients R in the Proposition 2.15 taken to be the real closed field $\mathrm{R}\langle\delta_j, \ldots, \delta_1, \eta_1, \zeta_1, \eta_{j-1}, \zeta_{j-1}, \eta_j\rangle$ and $\zeta = \zeta_j$) we have that there exists a simple zero, w'', of the system $\mathcal{P}^{\beta}_{\sigma,j}(z, \cdot)$ (cf. (9)) where $\sigma = (\sigma', \tau_j)$ and $\beta = (\alpha, J)$, such that $\lim_{\zeta_j} w'' = w'$. Clearly, then $x'' = (z', w'') \in V^{\beta}_{\sigma,j}$, , and $x' = \lim_{\delta_j} x''$ and thus $x' \in \lim_{\delta_j} V^{\beta}_{\sigma,j}$. Notice that $\sigma_j = \tau_j$ and $\sigma \leq \tau$.

The inclusion $\lim_{\delta_j} \tilde{V}_\tau \subset V_j$, from which the second inclusion $W_\tau \subset V_j$ follows immediately, is due to the fact that for each $\beta \in I_j(\tau)$, $V^{\beta}_{\tau,j}$ is either contained in the part of the semi-algebraic set defined by $\tilde{P}_j \leq 0$ which is bounded over R, or in the algebraic variety $\mathrm{Zer}(\tilde{P}_j, \mathrm{R}^k_j)_b$ depending on whether $\tau_{j-1} = \tau_j$ or $\tau_{j-1} > \tau_j$

respectively. It is clear from definition of $\tilde{P}_j$, that the images under $\lim_{\delta_j}$ of the last two sets are contained in V_j. $\square$

The following slight refinement of Proposition 3.13 is required to ensure that the degree of the last polynomial does not enter the bound with a factor of $(k-\tau_{i-1}-1)$ as is the case of the other degrees d_i, with $i < \ell$, but rather just as d_ℓ. This slight improvement is possible since we do not need to ensure that the dimension of the approximating varieties drops appropriately (to k_ℓ) when we approximate the last variety V_ℓ. If we were not interested in obtaining the tightest possible dependence on k in the multiplicative factor in the bound (the factor that is independent of the degrees), then this refinement would not have been necessary. However, in order to ensure that the results in the current paper properly generalize the results in [4] we need to take this extra care.

Notation 3.18. For all $\sigma = (\sigma_1, \ldots, \sigma_j) \in A_j, 2 \leq j \leq \ell$, denote by

$$\hat{\sigma} = (\sigma_1, \ldots, \sigma_{j-1}, \sigma_{j-1}).$$

Corollary 3.19. *For all $\tau \in A_j$,*

$$V_\tau \subset W'_\tau \subset V_j,$$

where

$$W'_\tau = \bigcup_{\substack{\sigma=(\sigma_1,\ldots,\sigma_j)\in A_j \\ \sigma_i \leq \tau_i, 1\leq i<j, \sigma_j=\tau_j}} \lim_{\delta_j} \tilde{V}_{\hat{\sigma}}.$$

Proof. It is clear from the definition that for all $\sigma \in A_j$

$$\tilde{V}_\sigma \subset \tilde{V}_{\hat{\sigma}},$$

and that $\lim_{\delta_j} \tilde{V}_{\hat{\sigma}} \subset V_j$. The corollary now follows from Proposition 3.13. $\square$

Corollary 3.20.

$$b_0(V_\ell) \leq \sum_{\tau \in A_\ell} \sum_{\beta \in I_\ell(\hat{\tau})} b_0(V^{\beta}_{\hat{\tau},\ell}).$$

Proof. Follows immediately from Corollary 3.19 after noting that (using Proposition 3.12)

$$V_\ell = \bigcup_{\tau \in A} V_\tau.$$

$\square$

Following notation introduced above we have the following proposition.

Proposition 3.21. *Let $\tau \in A_j$, $\tau_{j-1} = p$, and $\alpha \in I_j(\hat{\tau})$.*

1. *Then* $\mathrm{card}\, \mathcal{P}^{\alpha}_{\hat{\tau},j} = k - p$.
2. *Suppose that $\mathcal{P}^{\alpha}_{\hat{\tau},j} = (P_1, \ldots, P_{k-p})$. Let for $1 \leq i \leq j-1$, $\ell_i = \tau_{i-1} - \tau_i$, with the convention that $\tau_0 = k$, and $L_i = \sum_{h=1}^{i} \ell_h$. Then for each $i, 1 \leq i < j$, the degrees of the polynomials $P_{L_{i-1}+1}, \ldots, P_{L_i}$ are bounded by $(k - \tau_{i-1} + 1)d_i \leq (k+1)d_i$.*
3. $\deg(Q^{\alpha}_{\hat{\tau},j}) \leq d_\ell$.
4. $b_0(V^{\alpha}_{\hat{\tau},j}) \leq b_0(\mathrm{Zer}(\mathcal{P}^{\alpha}_{\hat{\tau},j} \cup \{Q^{\alpha}_{\tau,j}\}, \mathrm{R}^k_j)_b) + b_0(\mathrm{Zer}(\mathcal{P}^{\alpha}_{\hat{\tau},j}, \mathrm{R}^k_j)_b)$.

Proof. Follows from the definitions of the tuples $\mathcal{P}^{\alpha}_{\hat{\tau},j}$, and the polynomials $Q^{\alpha}_{\hat{\tau},j}$ (see Definition 3.10), as well as Proposition 2.14. $\square$

3.2. **Bounds on the Betti numbers of non-singular complete intersections.** The following proposition appears in [4], and is a consequence of the classical formula for the Euler-Poincaré characteristic of non-singular complex projective intersections and the Smith inequality.

Proposition 3.22. *Let $\mathcal{F} = \{F_1, \ldots, F_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(F_i) = d_i$, $d_1 \leq d_2 \leq \cdots \leq d_m$. Moreover, assume that $\mathcal{F}^h = \{F_1^h, \ldots, F_m^h\}$ defines a non-singular complete intersection in $\mathbb{P}^k_{\mathrm{C}}$. Then,*

$$b_0(\mathrm{Zer}(\mathcal{F}, \mathrm{R}^k)_b) \leq \binom{k+1}{m+1} d_1 \cdots d_{m-1} d_m^{k-m+1} + 2(k-m+1).$$

Remark 3.23. We note that in Proposition 3.22 if the polynomials in $\mathcal{F}$ do not define a non-singular complete intersection, it is still possible to bound the sum of the Betti numbers of the corresponding complex variety by $O(1)^m O(md_m)^k$ using a result of Katz [19], which in turn uses previous results of Bombieri [13], and Adolphson and Sperber [1]. These results use the theory of exponential sums over finite fields, and are of a much deeper nature than the classical formula giving the Betti numbers in terms of the degree sequence in the non-singular complete intersection case which is used to prove Proposition 3.22. However, the results of Katz [19] which do not assume non-singularity and are very general, do not have the finer dependence on the degree sequence (see the bound given above), and this finer dependence on the degree sequence is the key point in Proposition 3.22 above.

Corollary 3.24. *For each $\tau = (\tau_1, \ldots, \tau_\ell) \in A_\ell$ and $\alpha \in I_\ell(\hat{\tau})$ and $\mathcal{Q} \subset \{Q^\alpha_{\hat{\tau},\ell}\}$,*

$$b_0(\mathrm{Zer}((\mathcal{P}^\alpha_{\hat{\tau},\ell} \cup \mathcal{Q})_b, \mathrm{R}^k_\ell)) \leq O(1)^k d_\ell^{\tau_{\ell-1}} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}.$$

Proof. Follows from parts 1., 2. and 3. of Proposition 3.21, and Proposition 3.22. $\square$

It now follows from Corollary 3.24 and part 4. of Proposition 3.21 that

Corollary 3.25. *For each $\tau = (\tau_1, \ldots, \tau_\ell) \in A_\ell$ and $\alpha \in I_\ell(\hat{\tau})$*

$$b_0(V^\alpha_{\hat{\tau},\ell}) \quad \leq \quad O(1)^k d_\ell^{\tau_{\ell-1}} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}.$$

Let $\tau \in A_\ell$ and $d_1, \ldots, d_\ell$ satisfy the hypothesis of Theorem 4.

Lemma 3.26. *Then,*

$$\frac{d_\ell^{\tau_{\ell-1}} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}}{d_\ell^{k_{\ell-1}} \prod_{1 \leq i < \ell} ((k - k_{i-1} + 1) d_i)^{k_{i-1} - k_i}} \quad \leq \quad O(k)^k.$$

Proof. Using the inequality that for $2 \leq i \leq \ell$,

$$\frac{d_{i-1}}{d_i} \quad \leq \tfrac{1}{k+1} \leq \tfrac{1}{k - k_{i-2} + 1}$$

we get that the expression on the left hand side of the proposition is bounded by

$$\frac{\prod_{1 \leq i < \ell} (k - \tau_{i-1} + 1)^{\tau_{i-1} - \tau_i}}{\prod_{1 \leq i < \ell} (k - k_{i-1} + 1)^{k_{i-1} - \tau_i}}.$$

The sum of the various exponents of the numerator is

$$\sum_{i=1}^{\ell-1}(\tau_{i-1}-\tau_i) \quad = \quad \tau_0-\tau_{\ell-1}\leq k,$$

and for each $i, 1\leq i<\ell$, $(k-\tau_{i-1}+1)\leq(k+1)$. The denominator is a non-zero integer. $\square$

We next bound the cardinality of the index set A_ℓ.

Lemma 3.27. *The cardinality of A_ℓ is bounded by*

$$O(1)^{k+\ell}.$$

Proof. The number of tuples $\tau=(\tau_1,\ldots,\tau_\ell)$ in which $k\geq\tau_1>\tau_2>\cdots>\tau_\ell\geq 0$ is bounded by the volume of the corresponding ℓ-dimensional simplex in $\mathbb{R}^\ell$ which is equal to $\frac{(k+1)^\ell}{\ell!}$. Allowing some of the τ_i's to be equal, the number of tuples is bounded by

$$\sum_{0\leq i\leq\ell}\binom{\ell}{i}\frac{(k+1)^{\ell-i}}{(\ell-i)!}\leq 2^\ell\sum_{0\leq i\leq\ell}\frac{(k+1)^{\ell-i}}{(\ell-i)!}=O(1)^{k+\ell}.$$

$\square$

Lemma 3.28. *For each $\tau=(\tau_1,\ldots,\tau_\ell)$ the cardinality of the index set $I_\ell(\tau)$ is bounded by*

$$(k-\tau_\ell+1)\binom{k-\tau_\ell}{\tau_1-\tau_2,\ldots,\tau_{\ell-1}-\tau_\ell}.$$

Proof. It is clear from the definition that the cardinality of the index set $I_\ell(\tau)$ is bounded by

$$\begin{aligned}\prod_{1\leq j\leq\ell}\binom{k-\tau_j+1}{k-\tau_{j-1}+1} \quad &= \quad \frac{(k-\tau_\ell+1)!}{(\tau_0-\tau_1)!(\tau_1-\tau_2)!\cdots(\tau_{\ell-1-}\tau_\ell)!}\\ &= \quad (k-\tau_\ell+1)\binom{k-\tau_\ell}{\tau_0-\tau_1,\tau_1-\tau_2,\ldots,\tau_{\ell-1}-\tau_\ell}.\end{aligned}$$

$\square$

3.3. **Proof of Theorem 4.** We now prove Theorem 4.

Proof of Theorem 4. We first prove the theorem in case V_0 is bounded. It follows from Corollary 3.20 and Corollary 3.25 that

$$b_0(V_\ell) \quad \leq \quad \sum_{\tau\in A_\ell}\sum_{\alpha\in I_\ell(\hat{\tau})}\left(O(1)^k d_\ell^{\tau_{\ell-1}}\prod_{1\leq i<\ell}((k-\tau_{i-1}+1)d_i)^{\tau_{i-1}-\tau_i}\right).$$

Using Lemma 3.28 to bound the cardinality of the index set $I_\ell(\hat{\tau})$, we get that the right hand side of the above inequality is bounded by

$$O(1)^k\sum_{\tau\in A_\ell}F(k,\tau)\left(d_\ell^{\tau_{\ell-1}}\prod_{1\leq i<\ell}((k-\tau_{i-1}+1)d_i)^{\tau_{i-1}-\tau_i}\right),$$

where

$$F(k,\tau) = (k - \tau_{\ell-1} + 1)\binom{k-\tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \ldots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

The theorem in the bounded case now follows from Lemma 3.26 and Lemma 3.27.

In the general case, we first replace the given sequence of polynomials $Q_1, \ldots, Q_\ell$, by a new sequence, $Q_0, Q_1, \ldots, Q_\ell$, where

$$Q_0 = \sum_{i=1}^{k+1} X_i^2 - \Omega,$$

where Ω is infinitely large and positive over R. For each $i, 0 \leq i \leq \ell$, defining $\hat{\mathcal{Q}}_i = \{Q_0, \ldots, Q_i\}$, and $\hat{V}_i = \mathrm{Zer}(\hat{\mathcal{Q}}_i, \mathrm{R}\langle 1/\Omega\rangle^{k+1})$ we have that each $\widehat{V_i}$ is bounded over $\mathrm{R}\langle 1/\Omega\rangle$, and also that $b_0(V_\ell) \leq b_0(\hat{V}_\ell)$. Applying the same arguments as in the bounded case we obtain that

$$b_0(\hat{V}_\ell) \leq O(1)^k \sum_{\tau=(\tau_{-1},\tau_0,\ldots,\tau_{\ell-1})} F(k,\tau)\left(d_\ell^{\tau_{\ell-1}} \prod_{1\leq i<\ell} ((k-\tau_{i-1}+1)d_i)^{\tau_{i-1}-\tau_i}\right),$$

where the sum is taken over all $\tau \in \mathbb{N}^\ell$, with $k+1 = \tau_{-1} > k = \tau_0 \geq \tau_1 \cdots \geq \tau_{\ell-1} \geq 0$, and $\tau_i \leq k_i$, for each $i, 1 \leq i < \ell$, and

$$F(k,\tau) = (k - \tau_{\ell-1} + 1)\binom{k-\tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \ldots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

Notice that since the local dimension of the variety $\hat{V}_0$ is constant, it suffices to fix $\tau_0 = k$ in the sum above, and the contribution of the degree of the polynomial Q_0 (note that $\deg(Q_0) = 2$) gets absorbed into the $O(1)^k$ term.

□

3.4. **Proof of Theorem 5.** We now prove Theorem 5.

We introduce a new family of polynomials defined as follows:

$$\tilde{\mathcal{P}} = \bigcup_{1\leq i\leq s} \{P_i \pm \varepsilon\gamma_i, P_i \pm \delta\gamma_i\},$$

where $\varepsilon, \delta, \gamma_1, \ldots, \gamma_s$ new variables.

For any subset $I = \{(\epsilon_1, \sigma_1, i_1), \ldots, (\epsilon_m, \sigma_m, i_m)\} \subset \{+1,-1\} \times \{\varepsilon, \delta\} \times \{1, \ldots s\}$, we denote by $\tilde{\mathcal{P}}_I$ the subset of $\tilde{\mathcal{P}}$ defined by

$$\tilde{\mathcal{P}}_I \quad = \quad \bigcup_{1\leq j\leq m} \{P_{i_j} + \epsilon_j\sigma_j\gamma_{i_j}\}.$$

Let R' denote the real closed field $\mathrm{R}\langle\varepsilon, \delta, \gamma_1, \ldots \gamma_s\rangle$.

Proposition 3.29. *For each $I \subset \{+1,-1\}\times\{\varepsilon,\delta\}\times\{1,\ldots,s\}$, the dimension of the variety $\mathrm{Zer}(\tilde{\mathcal{P}}, \mathrm{R}'^k) \cap \mathrm{Ext}(V_\ell, \mathrm{R}')$ is at most $k_\ell - \mathrm{card}\, I$. In particular, $\mathrm{Zer}(\tilde{\mathcal{P}}, \mathrm{R}'^k) \cap \mathrm{Ext}(V_\ell, \mathrm{R}')$ is empty if $\mathrm{card}\, I > k_\ell$.*

Proof. It follows immediately from the fact that the various γ_i's are algebraically independent over R. □

Notation 3.30. For any finite family $\mathcal{F} \subset \mathrm{R}[X_1, \ldots, X_k]$ we call a formula

$$\bigwedge_{F\in\mathcal{F}} F\sigma_F 0,$$

where each $\sigma_F \in \{\geq, \leq\}$, a *weak sign condition* on $\mathcal{F}$.

Proposition 3.31. *Let V_1 be bounded, and let $\sigma \in \{-1,0,1\}^{\mathcal{P}}$ and C a semi-algebraically connected component of* $\mathrm{Reali}(\sigma, V_\ell) \subset \mathrm{R}^k$*. Then, there exists a weak sign condition $\tilde{\sigma}$ on $\tilde{\mathcal{P}}$, and a semi-algebraically connected component $\tilde{C}$ of*

$$\mathrm{Reali}(\tilde{\sigma}, \mathrm{Ext}(V_\ell, \mathrm{R}'))$$

such that

$$\lim_\delta \tilde{C} \subset \mathrm{Ext}(C, \mathrm{R}').$$

Proof. The proof is similar to the proof of Proposition 4 in [6] and omitted. □

The following proposition occurs in [10] (Proposition 13.1).

Proposition 3.32. *Let V_1 be bounded and let $\mathcal{F} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials and $\tilde{\sigma}$ a weak sign condition on $\mathcal{F}$. Let C be a semi-algebraically connected component of* $\mathrm{Reali}(\tilde{\sigma}, \mathrm{Ext}(V_\ell, \mathrm{R}'))$*. Then there exists a subset $\mathcal{F}' \subset \mathcal{F}$, and a semi-algebraically connected component D of* $\mathrm{Zer}(\mathcal{F}', \mathrm{Ext}(V_\ell, R'))$*, such that $D \subset C$.*

Proof of Theorem 5. In the case V_1 is bounded, using successively Propositions 3.31 and 3.32 it suffices to bound the total number of semi-algebraically connected components of the real algebraic sets

$$\mathrm{Zer}(\mathcal{Q}_\ell \cup \tilde{\mathcal{P}}_I, \mathrm{R}'^k)$$

for subsets $I \subset \{+1,-1\} \times \{\varepsilon, \delta\} \times \{1, \ldots s\}$. Moreover, using Proposition 3.29, the set of different subsets I that we need to consider is bounded by

$$\sum_{j=0}^{k_\ell} 4^j \binom{s}{j} = (O(s))^{k_\ell}.$$

Notice that each $\mathrm{Zer}(\mathcal{Q}_\ell \cup \tilde{\mathcal{P}}_I, \mathrm{R}'^k) = \mathrm{Zer}((Q_1, \ldots, Q_\ell, P_I), \mathrm{R}'^k)$, where $P_I = \sum_{P \in \tilde{\mathcal{P}}_I} P^2$. Also, notice that

$$(\deg(Q_1), \ldots, \deg(Q_\ell), \deg(P_I)) = (d_1, \ldots, d_\ell, 2d).$$

Now apply Theorem 4 to finish the proof. In the general case, use the same technique as in the proof of Theorem 4 to reduce to the bounded case. □

Proof of Theorem 6. In the proof of Theorem 5 instead of bounding the number of semi-algebraically connected components of the various algebraic sets

$$\mathrm{Zer}(\{Q_1, \ldots, Q_\ell, P_I\}, \mathrm{R}'^k)$$

using Theorem 4, apply Theorem 4 directly to the sequence $\mathcal{Q}, \tilde{\mathcal{P}}_I$, noting that its real zeros are the same as $\mathrm{Zer}(\{Q_1, \ldots, Q_\ell, P_I\}, \mathrm{R}'^k)$, and also that the degree sequence associated to $\tilde{\mathcal{P}}_I$ can be made to satisfy the requirement of Theorem 4 by multiplying, for each i, the i-th largest degree in the sequence by $(k+1)^{i-1}$. □

References

[1] A. Adolphson and S. Sperber. On the degree of the L-function associated with an exponential sum. *Compositio Math.*, 68(2):125–159, 1988. 29

[2] P. Agarwal, J. Matouek, and M. Sharir. On range searching with semialgebraic sets ii. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 420 –429, oct. 2012. 4

[3] N. Alon. Tools from higher algebra. In *Handbook of combinatorics, Vol. 1, 2*, pages 1749–1783. Elsevier, Amsterdam, 1995. 2, 3

[4] S. Barone and S. Basu. Refined bounds on the number of connected components of sign conditions on a variety. *Discrete Comput. Geom.*, 47(3):577–597, 2012. 3, 8, 9, 10, 15, 28, 29
[5] S. Basu, R. Pollack, and M.-F. M.-F. Roy. On the Betti numbers of sign conditions. *Proc. Amer. Math. Soc.*, 133(4):965–974 (electronic), 2005. 3
[6] S. Basu, R. Pollack, and M.-F. Roy. On the number of cells defined by a family of polynomials on a variety. *Mathematika*, 43(1):120–126, 1996. 10, 32
[7] S. Basu, R. Pollack, and M.-F. Roy. On computing a set of points meeting every cell defined by a family of polynomials on a variety. *J. Complexity*, 13(1):28–37, 1997. 8
[8] S. Basu, R. Pollack, and M.-F. Roy. Computing roadmaps of semi-algebraic sets on a variety. *J. Amer. Math. Soc.*, 13(1):55–82, 2000. 8
[9] S. Basu, R. Pollack, and M.-F. Roy. Betti number bounds, applications and algorithms. In *Current Trends in Combinatorial and Computational Geometry: Papers from the Special Program at MSRI*, volume 52 of *MSRI Publications*, pages 87–97. Cambridge University Press, 2005. 2
[10] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2006 (second edition). Revised version of the second edition online at http://perso.univ-rennes1.fr/marie-francoise.roy/. 2, 10, 14, 32
[11] S. Basu and M. Sombra. Polynomial partitioning on varieties and point-hypersurface incidences in four dimensions. *ArXiv e-prints*, June 2014. 1, 5, 6
[12] J. Bochnak, M. Coste, and M.-F. Roy. *Géométrie algébrique réelle (Second edition in english: Real Algebraic Geometry)*, volume 12 (36) of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1987 (1998). 11, 15
[13] E. Bombieri. On exponential sums in finite fields. II. *Invent. Math.*, 47(1):29–39, 1978. 29
[14] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998. 5
[15] J. E. Goodman, R. Pollack, and R. Wenger. Bounding the number of geometric permutations induced by k-transversals. *J. Combin. Theory Ser. A*, 75(2):187–197, 1996. 2
[16] L. Guth and N. H. Katz. On the Erdös distinct distance problem in the plane. *arXiv:1011.415v1 [math.CO]*. 3, 4
[17] H. Kaplan, J. Matoušek, Z. Safernová, and M. Sharir. Unit distances in three dimensions. *Combin. Probab. Comput.*, 21(4):597–610, 2012. 3, 4
[18] H. Kaplan, J. Matoušek, and M. Sharir. Simple proofs of classical theorems in discrete geometry via the Guth-Katz polynomial partitioning technique. *Discrete Comput. Geom.*, 48(3):499–517, 2012. 3
[19] N. M. Katz. Sums of Betti numbers in arbitrary characteristic. *Finite Fields Appl.*, 7(1):29–44, 2001. Dedicated to Professor Chao Ko on the occasion of his 90th birthday. 29
[20] J. Matousek and Z. Safernova. Multilevel polynomial partitions and simplified range searching. *ArXiv e-prints*, June 2014. 1, 3, 4
[21] J. Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, 15:275–280, 1964. 2
[22] J. L. Montaña and L. M. Pardo. Lower bounds for arithmetic networks. *Applicable Algebra in Engineering, Communication and Computing*, 4(1):1–24, 1993. 2
[23] I. G. Petrovskiĭ and O. A. Oleĭnik. On the topology of real algebraic surfaces. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 13:389–402, 1949. 2
[24] R. Pollack and M.-F. Roy. On the number of cells defined by a set of polynomials. *C. R. Acad. Sci. Paris*, 316:573–577, 1993. 3
[25] M.-F. Roy and N. Vorobjov. Computing the complexification of a semi-algebraic set. *Math. Zeitschrift*, 239:131–142, 2002. 8
[26] J. Solymosi and T. Tao. An incidence theorem in higher dimensions. *Discrete Comput. Geom.*, 48(2):255–280, 2012. 3, 4
[27] R. Thom. Sur l'homologie des variétés algébriques réelles. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 255–265. Princeton Univ. Press, Princeton, N.J., 1965. 2

[28] C. Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps. 13

[29] H. E. Warren. Lower bounds for approximation by nonlinear manifolds. *Trans. Amer. Math. Soc.*, 133:167–178, 1968. 3

[30] J. Zahl. An improved bound on the number of point-surface incidences in three dimensions. *arXiv preprint arXiv:1104.4987*, 2011. 3, 4

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