New bounds for Betti numbers of semi-algebraic sets and algorithms for computing them

Saugata Basu
School of Mathematics
Georgia Institute of Technology.

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- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.

Complexity of Semi-algebraic Sets

Uniform bounds on the number of connected components, Betti numbers etc., in terms of the degrees and number of polynomials used to describe them.

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Number of polynomials : n (controls the *combinatorial complexity*)
Degree bound : d (controls the *algebraic complexity*)
Dimension of the ambient space : k

Classical Result on the Topology of Semi-algebraic Sets

Theorem 1. *(Oleinik and Petrovsky, Thom, Milnor) Let* $S \subset \mathbb{R}^k$ *be the set defined by the conjunction of* n *inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$deg(P_i) \leq d, 1 \leq i \leq n$. *Then,*

$$\sum_i \beta_i(S) = nd(2nd-1)^{k-1} = O(nd)^k.$$

Tightness

The above bound is actually quite tight. Example: Let

$$P_i = L_{i,1}^2 \cdots L_{i,\lfloor d/2 \rfloor}^2 - \epsilon,$$

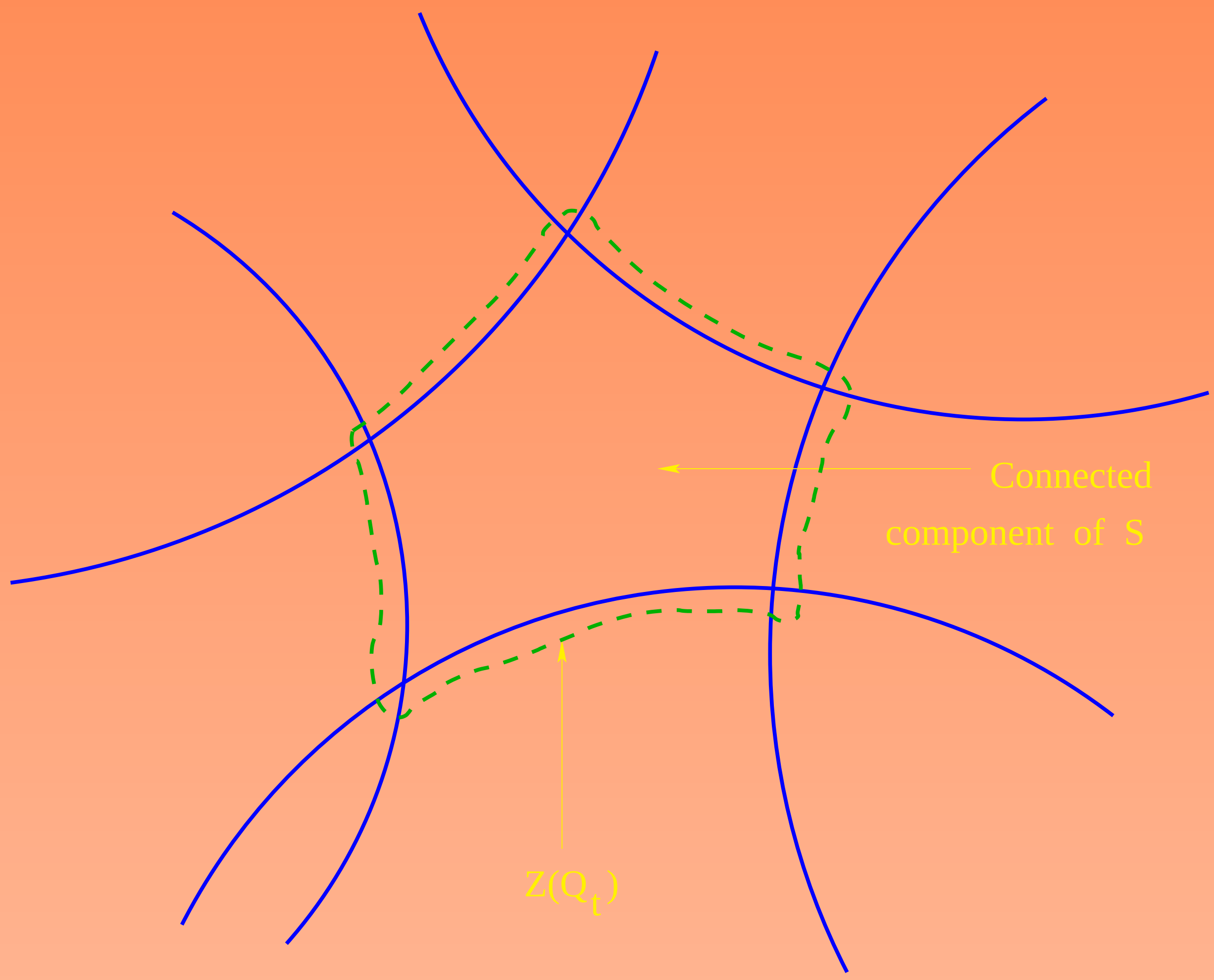
where the L_{ij}'s are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_n \geq 0$ has $\Omega(nd)^k$ connected components and hence $\beta_0(S) = \Omega(nd)^k$.

What about the higher Betti Numbers ?

- Cannot construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$.
- The technique used for proving the above result does not help:

  Replace the semi-algebraic set S by another set bounded by a smooth algebraic hypersurface of degree nd having the same homotopy type as S.

  Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.


Connected
component of S
Z(Q$_t$)

Graded Bounds

Theorem 2. *(B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

contained in a variety $Z(Q)$ of real dimension k', and $deg(Q) \leq d$. Then,

$$\beta_i(S) \leq \binom{n}{k' - i} (2d)^k.$$

The case of the union

Theorem 3. *(B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the disjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$deg(P_i) \leq d, 1 \leq i \leq n$. *Then,*

$$\beta_i(S) \leq \binom{n}{i+1}(2d)^k.$$

Sets defined by Quadratic Inequalities

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Theorem 4. *(B, 2001) Let* ℓ *be any fixed number and let* $S \subset \mathbb{R}^k$ *be defined by* $P_1 \geq 0, \ldots, P_n \geq 0$ *with* $\deg(P_i) \leq 2$. *Then,* $\beta_{k-\ell}(S) \leq \binom{n}{\ell} k^{O(\ell)}$.

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This bound is polynomial.
Notice that the lowest Betti numbers of S better not be polynomially bounded. Example:
S defined by $X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0$. Clearly, $\beta_0(S) = 2^k$.

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- Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathbb{R}[X_1, \ldots, X_k]$. A *sign condition* on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$.

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- Let $b_i(\sigma)$ denote the i-th Betti number of the realization of σ, and let $b_i(\mathcal{Q}, \mathcal{P}) = \sum_\sigma b_i(\sigma)$.

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- Let $b_i(d, k, k', n)$ be the maximum of $b_i(\mathcal{Q}, \mathcal{P})$ over all $\mathcal{Q}, \mathcal{P}$ where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of of $\mathbb{R}[X_1, \ldots, X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = n$ and the algebraic set $Z(\mathcal{Q})$ has dimension k'.

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- Previously known (B, Pollack, Roy (1995))

$$b_0(d,k,k',n) = \binom{4n}{k'} d(2d-1)^{k-1} = \binom{n}{k'} O(d)^k.$$

Betti Numbers of Sign Patterns III

Theorem 5. *(B, Pollack,Roy, 2002)*

$$b_i(d,k,k',n) \leq \sum_{0 \leq j \leq k'-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = \binom{n}{k'-i} O(d)^k.$$

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Applications ?

Generalized Mayer-Vietoris Exact Sequence

- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.
- We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
- The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0,\ldots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

Mayer-Vietoris Double Complex I

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

and ...

$$
\begin{array}{cccccccc}
 & & \vdots & & \vdots & & \vdots & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & \\
0 & \longrightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow \\
 & & \uparrow d & & \uparrow d & & \uparrow d & \\
0 & \longrightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow \\
 & & \uparrow d & & \uparrow d & & \uparrow d & \\
0 & \longrightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow \\
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 & & \uparrow d & & \uparrow d & & \uparrow d & \\
 & & 0 & & 0 & & 0 &
\end{array}
$$

Double Complex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \cdots \\
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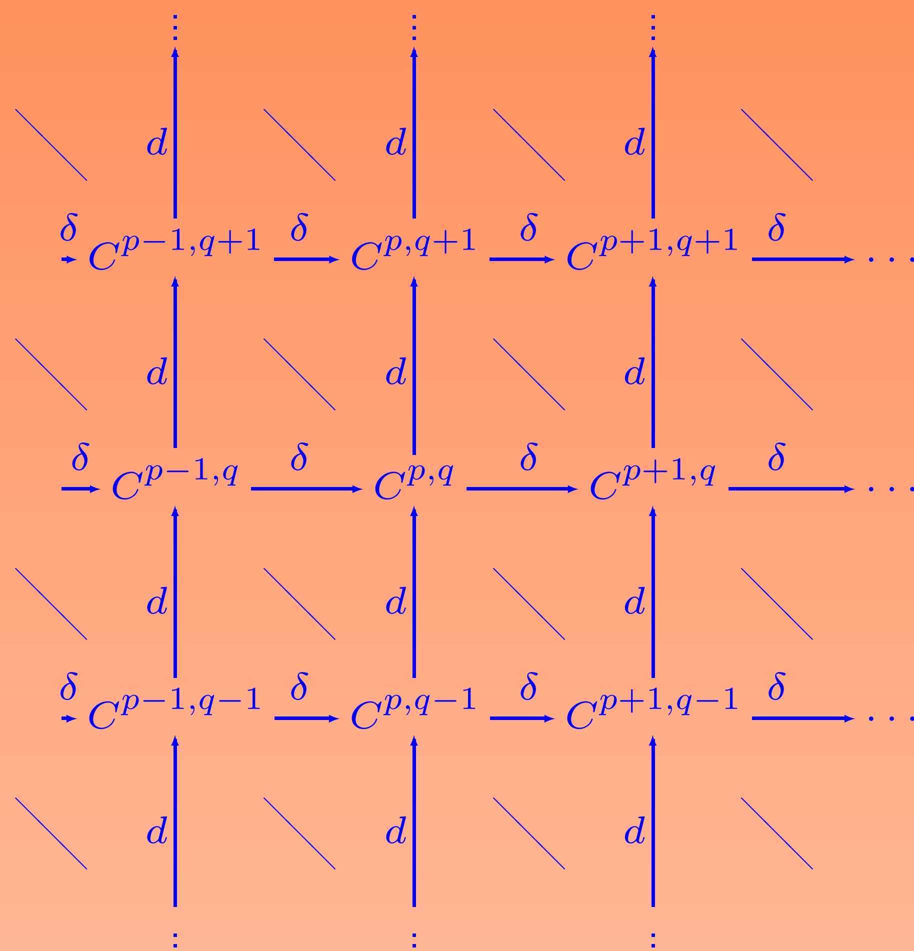
\end{array}
$$

The Associated Total Complex



Spectral Sequence

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- $E_{r+1} = H(E_r, d_r)$,
- $E_\infty = H^*(\text{Associated Total Complex})$.

Spectral Sequence

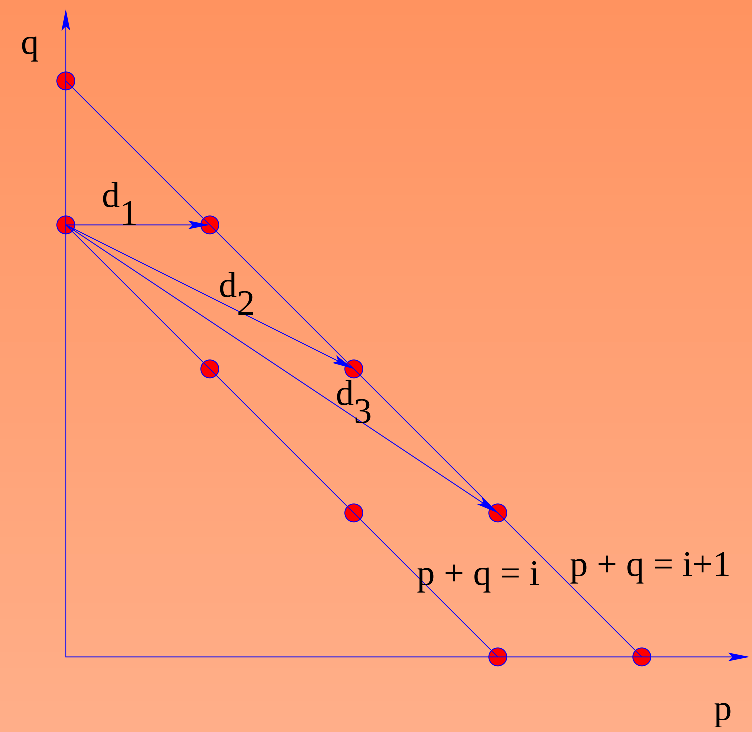

Figure 1: The differentials d_r in the spectral sequence (E_r, d_r)

Two Spectral Sequences

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- There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H_D^*(\mathcal{M})$. The first terms of these are:
- $$E_1 = H_\delta(\mathcal{M}), E_2 = H_d H_\delta(\mathcal{M})$$
- $$E_1' = H_d(\mathcal{M}), E_2' = H_\delta H_d(\mathcal{M})$$

$$E_1 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ C^3(A) & 0 & 0 \\ C^2(A) & 0 & 0 \\ C^1(A) & 0 & 0 \\ C^0(A) & 0 & 0 \end{array}$$

$$E_2 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ H^3(A) & 0 & 0 \\ H^2(A) & 0 & 0 \\ H^1(A) & 0 & 0 \\ H^0(A) & 0 & 0 \end{array}$$

The degeneration of this sequence at E_2 shows that $H^*_D(\mathcal{M}) \cong H^*(A)$.

$$
E_1' = \begin{array}{ccc}
\vdots & \vdots & \vdots \\
\prod_{\alpha_0} H^3(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,} \\
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\end{array}
$$

Lemma 1

Lemma 6. *Let A be a finite simplicial complex and $A_1, \ldots, A_n$ subcomplexes of A such that $A = A_1 \cup \cdots \cup A_n$. Suppose that for every ℓ, $0 \leq \ell \leq i$, and for every $(\ell + 1)$ tuple $A_{\alpha_0}, \ldots, A_{\alpha_\ell}$, $\beta_{i-\ell}(A_{\alpha_0,\ldots,\alpha_\ell}) \leq M$. Then, $\beta_i(A) \leq \sum_{0\leq\ell\leq i} \binom{n}{\ell+1} M$.*

Lemma 2

Lemma 7. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the conjunction of the inequalities* $P_i \geq 0$. *Assume that* S *is bounded. Then,* $\sum_i \beta_i(S) = (2d)^k$.

Theorem 3 follows.
Theorem 2 follows by a dual argument.
Theorem 4 follows using a result of Barvinok (1995).

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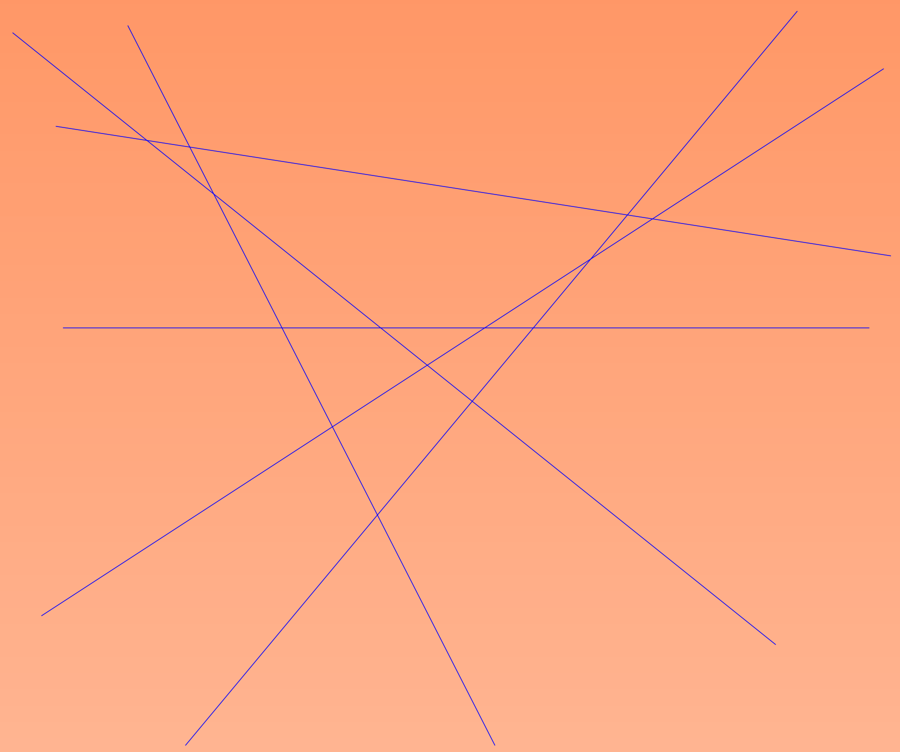
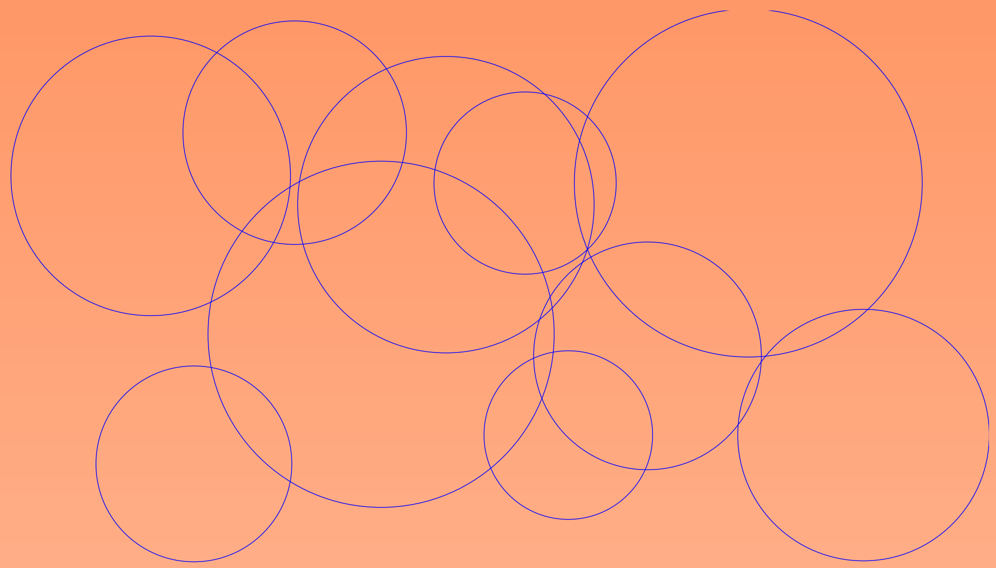
- Arrangements of lines in the plane, or more generally hyperplanes in $\mathbb{R}^k$.
- Arrangements of balls or simplices in $\mathbb{R}^k$.
- Arrangements of semi-algebraic objects in $\mathbb{R}^k$, each defined by a fixed number of polynomials of constant degree.

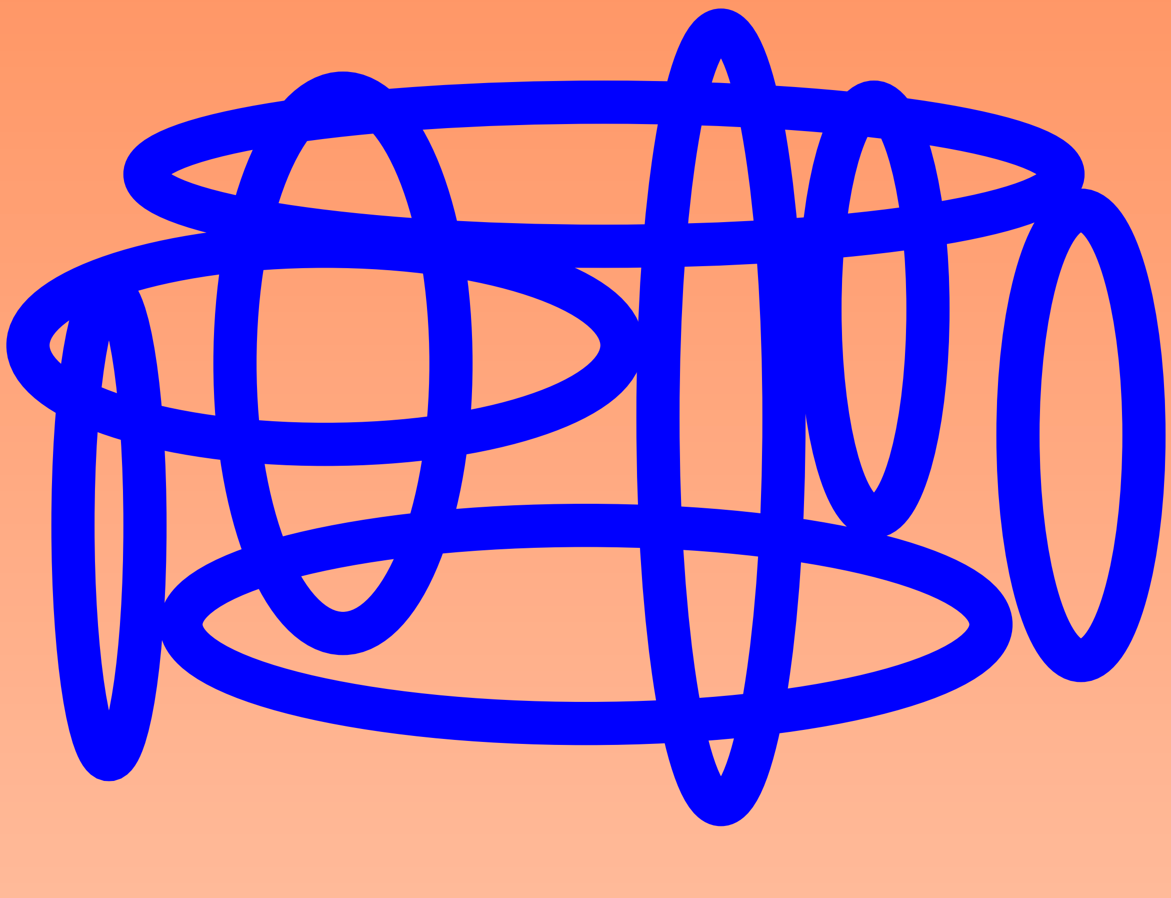
Arrangements of lines in the $\mathbb{R}^2$



Arrangement of circles in $\mathbb{R}^2$



Arrangement of tori in $\mathbb{R}^3$



Computing the Betti Numbers: Previous Work

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- Computing the Betti numbers of triangulated manifolds (Edelsbrunner, Dey, Guha et al).

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- In computational geometry it is customary to study the *combinatorial complexity* of algorithms. The *algebraic complexity* (dependence on the degree) is considered to be a constant.
- We only count the number of algebraic operations and ignore the cost of doing linear algebra.

Two Approaches

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Global

vs

Local

First Approach (Global): Using Triangulations

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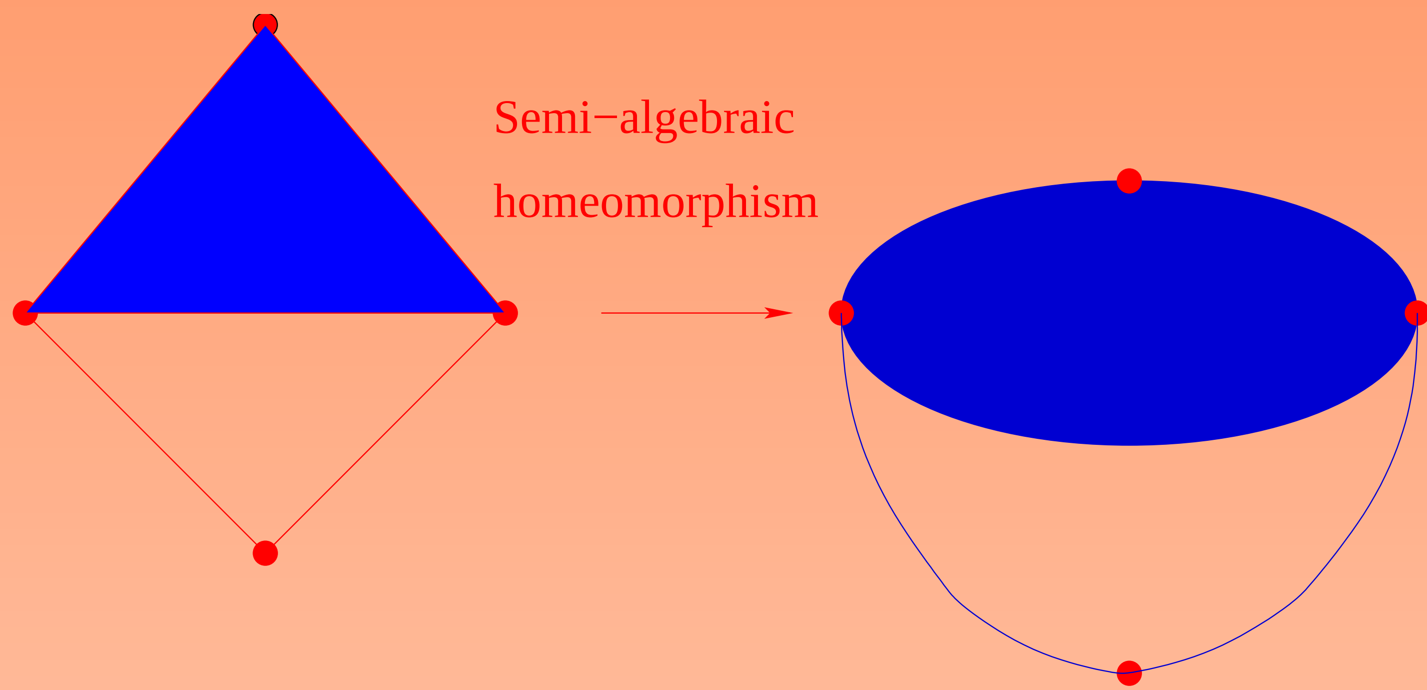
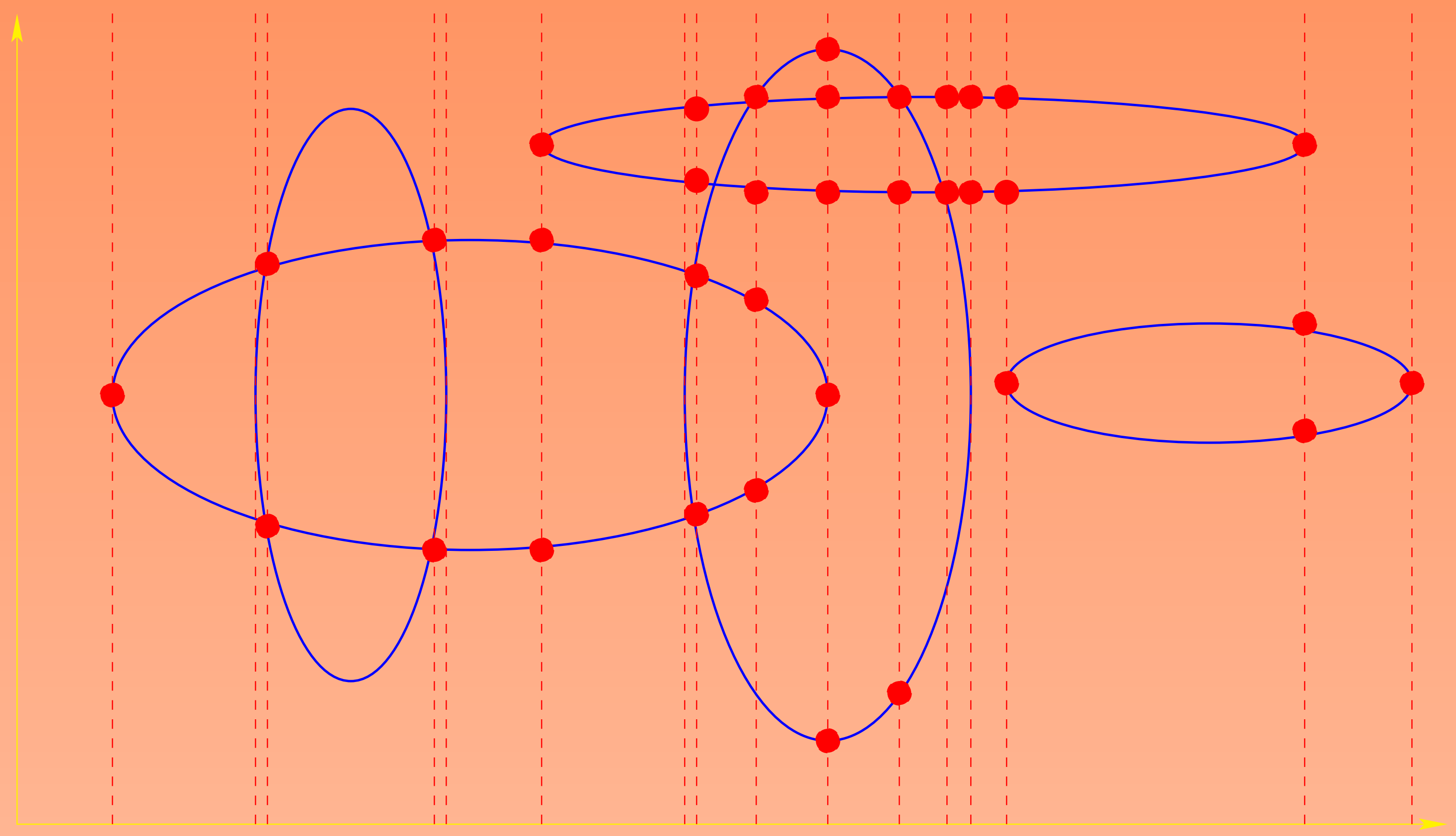


Triangulation via Cylindrical Algebraic Decomposition



Computing Betti Numbers using Global Triangulations

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- But ...

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- Compact semi-algebraic sets are finitely triangulable.
- First triangulate the arrangement using *Cylindrical algebraic decomposition* and then compute the Betti numbers of the corresponding simplicial complex.
- But ... CAD produces $O(n^{2^k})$ simplices in the worst case.

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- If the sets have the special property that all their non-empty intersections are contractible we can use the *nerve lemma* (Leray, Folkman).

- The homology groups of the union are then isomorphic to the homology groups of a combinatorially defined complex called the *nerve complex*.

The Nerve Complex

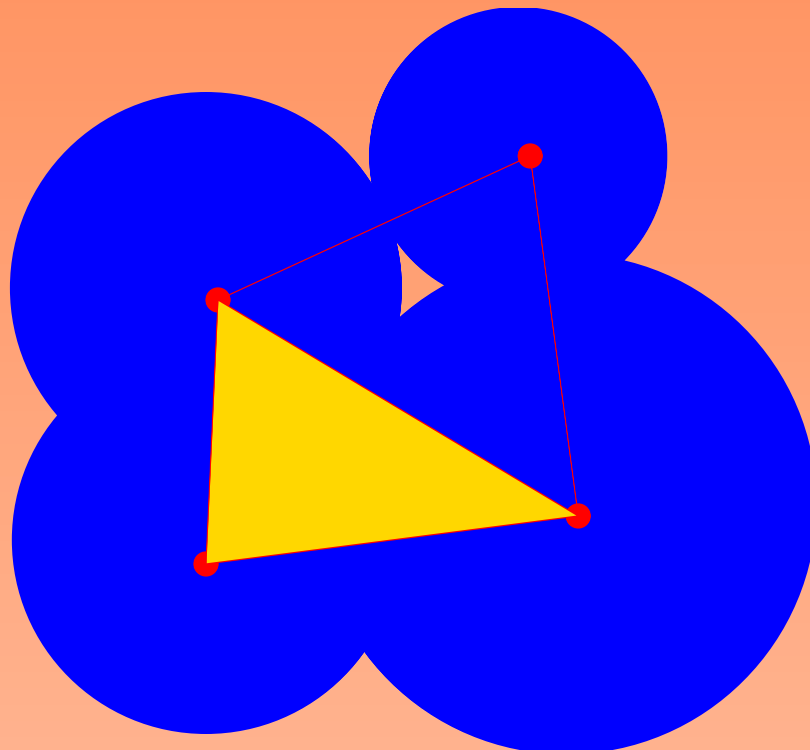

Figure 2: The nerve complex of a union of disks

Computing the Betti Numbers via the Nerve Complex (local algorithm)

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- The nerve complex has n vertices, one vertex for each set in the union, and a simplex for each *non-empty* intersection among the sets.

- Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1\leq j\leq \ell+2}\binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets.

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- If the sets are such that the topology of the "small" intersections are controlled, then
- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
- The algorithmic version gives the first efficient algorithm for computing the Betti numbers, without the double-exponential complexity entailed in CAD.

Main Result

Theorem 8. *Let* $S_1, \ldots, S_n \subset \mathbb{R}^k$ *be compact semi-algebraic sets of constant description complexity and let* $S = \cup_{1 \leq i \leq n} S_i$, *and* $0 \leq \ell \leq k-1$. *Then, there is an algorithm to compute* $\beta_0(S), \ldots, \beta_\ell(S)$, *whose complexity is* $O(n^{\ell+2})$.

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- In order to compute β_ℓ, we only need to compute upto $E'_{\ell+2}$.*But the punchline is that:*

- In order to compute the differentials $d_r, 1 \leq r \leq \ell+1$, it suffices to have *independent triangulations of the different unions taken upto* $\ell+2$ *at a time.*

- For instance, it should be intuitively clear that in order to compute $\beta_0(\cup_i S_i)$ it suffices to triangulate pairs.

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- To what extent does topological simplicity aid algorithms in computational geometry ?