Applications of algebraic geometry and model theory in incidence combinatorics

Workshop on Algebraic Methods in Discrete and Computational Geometry CG Week, Portland State University, Jun 21, 2019

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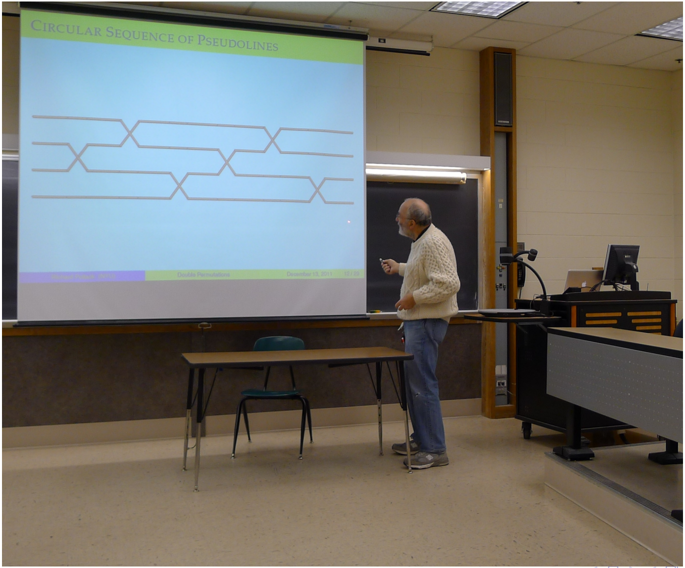
Ricky in Purdue, Dec 2011



Outline

1. Some history
2. Semi-algebraic case
   - Bounds on Betti numbers: methods
     - Method of effective triangulation
     - Critical point method
     - Method coming from complex algebraic geometry
3. More general structures

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On the number of "cells" ...

Theorem (B., Pollack, Roy (1996))

Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ *be a finite set with* $\deg(P) \leq d$, $P \in \mathcal{P}$ *and* $V \subset \mathrm{R}^k$ *be an algebraic set, and suppose that* V *is cut out by polynomials also of degree bounded by* d. *Then, the number of connected components of the realizations of all realizable sign conditions of* $\mathcal{P}$ *on* V *is bounded by*

$$s^{\dim_{\mathrm{R}}(V)}(O(d))^k,$$

where $\mathrm{card}(\mathcal{P}) = s$.

As a consequence ...

Corollary

The VC co-density of the family of real algebraic sets defined by a polynomials of degree d *on* V *is bounded by* $\dim_{\mathrm{R}}(V)$.

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Generalizations ...

1. Higher Betti numbers.
2. General (not just locally closed) semi-algebraic sets.
3. Better bounds for other families semi-algebraic sets – for example, symmetric ones, defined by randomly chosen polynomials etc.
4. More refined dependence on the degrees of the polynomials.
5. Dependence on the "geometric" degree of V rather than on the degrees of polynomials cutting it out.
6. More general structures rather than that of real semi-algebraic sets. For example, o-minimal structures, more generally arbitrary NIP structures where $\dim V$ makes sense.

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Fixing some notation

- Throughout, R will denote a real closed field and $\mathrm{C} = \mathrm{R}[i]$ the algebraic closure of R.
- Given $P \in \mathrm{R}[X_1, \ldots, X_k]$ we denote by $\mathrm{Zer}(P, \mathrm{R}^k)$ the set of zeros of P in R^k.
- Given a finite set $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, a subset $S \subset \mathrm{R}^k$ is $\mathcal{P}$-semi-algebraic if S is the realization of a Boolean formula with atoms $P = 0$, $P > 0$ or $P < 0$ with $P \in \mathcal{P}$ (we will call such a formula a quantifier-free $\mathcal{P}$-formula).
- We call a semi-algebraic set a $\mathcal{P}$-closed semi-algebraic set if it is defined by a Boolean formula with no negations with atoms $P = 0$, $P \geq 0$, or $P \leq 0$ with $P \in \mathcal{P}$.
- For any semi-algebraic set S, we will denote

$$b(S, \mathbb{F}) = \sum_i b_i(S, \mathbb{F}).$$

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- k the dimension of the ambient space.
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Upper bounds on Betti numbers: using effective triangulation

- Upper bounds on the Betti numbers of semi-algebraic sets follow from results on effective triangulation of semi-algebraic sets.
- Effective triangulation in turn uses cylindrical algebraic decomposition – Collins (1976), Wüthrich (1976).
- This yields bounds that are doubly exponential in k. That is,

$$b(S, \mathbb{F}) \leq (sd)^{2^{O(k)}}.$$

Open problems:

- Prove or disprove the existence of a semi-algebraic triangulation or stratification of semi-algebraic sets with single exponential complexity.
- Corresponding algorithmic question.

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Upper bounds on Betti numbers: using the critical point method/Morse theory

- Main idea was to use make a perturbation to reduce to the compact, non-singular, situation and then use Morse theory in order to bound the Betti numbers by the number of critical points of some affine function restricted to the hypersurface. The number of critical point is bounded by Bezout's theorem.
- In this way one obtains (Oleĭnik-Petrovskiĭ (1949), Thom, Milnor (1960s)) $b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{F}) \leq d(2d-1)^{k-1}$.
- Generalized to more general semi-algebraic sets – (to $\mathcal{P}$-closed s.a. sets by B.(1999), and then to arbitrary $\mathcal{P}$-s.a. sets Gabrielov-Vorobjov (2005)).
- Generalization uses additional techniques such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.

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Upper bounds via critical points (cont).

For completeness ...

Theorem (B.(1999), B.,Pollack,Roy(2005))

Let S be a $\mathcal{P}$-closed semi-algebraic set $S \subset \mathrm{R}^k$, with $s = \mathrm{card}(\mathcal{P})$, and $d = \max_{P \in \mathcal{P}} \deg(P)$, and V a real algebraic set also defined by a polynomial of degree at most d. Then,

$$b(S \cap V, \mathbb{F}) \leq \sum_{i=0}^{\dim_{\mathrm{R}}(V)} \sum_{j=0}^{\dim_{\mathrm{R}}(V)-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1}$$
$$= s^{\dim_{R}(V)}(O(d))^k.$$

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Upper bounds on Betti numbers: using complex geometry

- Suppose that $V(\mathrm{C}) \subset \mathbb{P}^k(\mathrm{C})$ is a non-singular complete intersection defined by polynomials of degree $d_1 \le d_2 \le \cdots \le d_\ell$, $\ell \le k$.
- $V(\mathrm{C})$ is of complex dimension $k-\ell$, and real dimension $2(k-\ell)$. The cohomology of $V(\mathrm{C})$ is concentrated in dimension $k-\ell$, and there is a formula for $b(V(\mathrm{C}), \mathbb{Z})$ – namely:

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\begin{aligned}
b(V(\mathrm{C}),\mathbb{Z}) &= (1+(-1)^{k-\ell+1})\cdot(k-\ell+1) + \\
& \quad d_1\cdot d_2\cdots d_\ell\cdot\left(\sum_{i=0}^{k-\ell}(-1)^i\binom{k+1}{i}h_{k-\ell-i}(d_1,\ldots,d_\ell)\right) \\
&= (d_1\cdots d_\ell)d_\ell^{k-\ell} + \ O(1)^k \text{ lower order terms} \\
&\le \ O(1)^k \deg(V)\cdot d_\ell^{k-\ell} \\
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where $\delta(V)$ is the least degree such that V is cut out by polynomials of at most that degree and $h_j(\cdots)$ is the j-th complete homogeneous symmetric polynomial.

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- Since the cohomology of V is torsion free, $b(V(\mathrm{C}), \mathbb{Z}) = b(V(\mathrm{C}), \mathbb{Z}_2)$.
- If V is defined by real polynomials, then using the fact that $V(\mathrm{C})$ admits an involution (complex conjugation), the sum of the $\mathbb{Z}_2$-Betti numbers of the fixed points of this involution, i.e. $V(\mathrm{R})$, is bounded by $b(V(\mathrm{C}), \mathbb{Z}_2)$ (Smith inequality).
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Why is the complete intersection assumption necessary ?

- The Betti numbers of $V(\mathrm{C})$ can be deduced from the Euler characteristic of $V(\mathrm{C})$.
- The Euler characteristic of $V(\mathrm{C})$ equals the degree of the top Chern class of the tangent bundle of $V(\mathrm{C})$.
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Bounding the number of connected components on a variety

The situation of interest in "iterated" polynomial partitioning ...

Theorem (Simplified version of Barone-B. (2016))

Let $V(\mathrm{R}) \subset \mathrm{R}^k$ be a complete intersection variety and let $P \in \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P) \geq \delta(V)$. Then,

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"Heuristic but incorrect proof".

Perturb the polynomials defining V, as well as P, infinitesimally to go to the non-singular situation without losing any connected component. Then use formula for the Betti numbers + Smith inequality + Mayer-Vietoris + local conical structure ... But the perturbation has to be carefully done in order not to lose connected components ... □

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Theorem (Real analogue of Bezout bound, Barone-B. (2013))

Let $Q_1, \ldots, Q_\ell \in \mathrm{R}[X_1, \ldots, X_k]$ *with* $\deg(Q_i) = d_i$*;*
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- Model theory meets discrete geometry ...
- The following statement makes sense in any "structure" having a reasonable notion of "dimension".
- Let A, B be "definable" sets with $\dim(A), \dim(B) \leq 2$, and $V \subset A \times B$ a definable subset. Then, one of the following two alternatives must hold.
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  2. there exists $c = c(V) > 0$ such that for every finite subsets $P \subset A, Q \subset B$,

$$|V \cap P \times Q| \leq c(|P|^{2/3}|Q|^{2/3} + |P| + |Q|).$$

- The above statement is a Theorem for "o-minimal structures" (B.-Raz 2018), and more generally for all "distal structures" (Chernikov-Galvin-Starchenko (2018)).

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Theorem (B. 2010)

Let V, W and $X \subset V \times W$ be definable sets (in an o-minimal expansion of $(\mathrm{R}, +, \cdot, <)$). Then, there exists a constant $C = C(X) > 0$ such that for all s, $\bar{w} \in W^s$, and $i, 0 \leq i \leq \dim(V)$,

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"Proof".

Mayer-Vietoris inequalities + local contractibility of definable sets + finiteness of topological types amongst the fibers of any fixed definable map. □

The special case for $i = 0$ yields:

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The main fact used in the proof is just that all definable subsets of V have vanishing cohomology in dimensions greater than $\dim(V)$.

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