Isotypic decomposition of cohomology modules of symmetric semi-algebraic sets: Polynomial bounds on the multiplicities

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Dagstuhl Seminar, Jun 9, 2015
(joint work with Cordian Riener, Aalto University)

Basic definitions

- Throughout, R will denote a real closed field.
- Given $P \in \mathrm{R}[X_1, \ldots, X_k]$ we denote by $\mathrm{Z}(P, \mathrm{R}^k)$ the set of zeros of P in R^k.
- Given any semi-algebraic subset $S \subset \mathrm{R}^k$ we will denote by $b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(\mathrm{H}^i(S, \mathbb{F})$ (i.e. the dimension of the i-th cohomology group of S with coefficients in $\mathbb{F}$ assumed to be of characterisic 0), and we will denote by $b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F})$.
- $b(S, \mathbb{F})$ is an important measure of the "complexity" of a semi-algebaric set S.
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in cetain models of computations.
- Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).

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Upper bounds on the Betti numbers

- Doubly exponential (in k) bounds on $b(S, \mathbb{F})$ follow from results on effective triangulation of semi-algebraic sets which in turn uses cylindrical algebraic decomposition.
- Singly exponential (in k) bounds: Long history – Oleĭnik and Petrovskiĭ (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.
- More precisely, if $P \in \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(P) \leq d$, then $b(\mathrm{Z}(P, \mathrm{R}^k), \mathbb{F}) \leq d(2d-1)^{k-1}$.
- Main idea was to use Morse theory and counting critical points.
- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).
- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.

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Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left(\prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = \mathrm{Z}(F_{k,d}, \mathrm{R}\langle\varepsilon\rangle^k)$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in k.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are "simple". For example, for every fixed degree d there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/\mathfrak{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$.

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Representations of finite groups

- A representation of G over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \to \mathrm{GL}(V)$ for some $\mathbb{F}$-vector space V. It is usual to refer to the representation ρ by V.
- A representation $\rho : G \to \mathrm{GL}(V)$is said to be *irreducible* iff the only G-invariant subspaces are 0 and V.
- The set, $\mathrm{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of G over $\mathbb{F}$, is finite.
- Every finite dimensional representation V of G admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \mathrm{Irred}(G,\mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components V_W are called the *isotypic components*, and m_W the *multiplicity* of the irreducible W in V.
- Clearly, $\dim_{\mathbb{F}}(V) = \sum_{W \in \mathrm{Irred}(G,\mathbb{F})} m_W \dim_{\mathbb{F}}(W)$.

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Partitions, Young diagrams and dominance ordering

- A partition λ of k (denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \cdots + \lambda_\ell = k$.
- We denote by $\mathrm{Par}(k)$ the set of partitions of k.
- We denote by $\mathrm{Young}(\lambda)$ the Young diagram associated with λ.
- For example, $\mathrm{Young}((4, 2, 1))$ is given by

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- For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathrm{Par}(k)$, we say that $\mu \trianglerighteq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order (called the *dominance order*).

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Partitions, Young diagrams and dominance ordering

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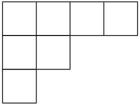

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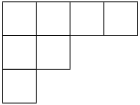

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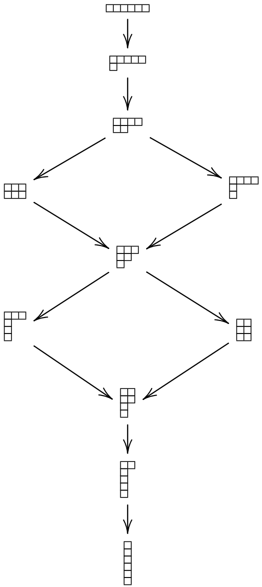
Dominance order on $\mathrm{Par}(6)$



Semi-standard tableau, Kostka numbers

- Given partitions $\mu, \lambda = (\lambda_1, \lambda_2, \ldots,) \vdash k$, a *semi-standard tableau* of shape μ and content λ is a Young diagram in $\mathrm{Young}(\mu)$ with entries in the boxes which are non-decreasing along rows and increasing along columns – and for each $i > 0$, the number of i's is equal to λ_i.
- For example,

1	1	1	2
2	2		
3			

  is a semi-standard of shape $(4, 2, 1)$ and content $(3, 3, 1)$.
- For $\lambda, \mu \vdash k$, the *Kostka number* $K(\mu, \lambda)$ is the number of semi-standard Young tableux of shape μ and content λ.
- Fact: for all $\mu, \lambda \vdash k$, $K(\mu, \mu) = K((k), \mu) = 1$, and $K(\mu, \lambda) \neq 0$ iff $\mu \trianglerighteq \lambda$.

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Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called *Specht modules*) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\mathrm{Par}(k)$, of partitions of k.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathrm{Par}(\lambda)$, we denote by $\mathbb{S}^\lambda$ the corresponding Specht module.
- In particular, $\mathbb{S}^{(k)} = \mathbf{1}_{\mathfrak{S}_k}, \mathbb{S}^{(1^k)} = \mathrm{sign}_{\mathfrak{S}_k}$.
- The dimension of $\mathbb{S}^\lambda$ equals the number of standard of Young tableau of shape λ. Its also give by the *hook length formula* below.
- For a box b in the Young diagram, $\mathrm{Young}(\lambda)$, of a partition λ, let h_b denote the length of the *the hook of b* i.e. h_b is the number of boxes in $\mathrm{Young}(\lambda)$ strictly to the right and below b plus 1.
- Hook length formula:

$$\dim_{\mathbb{F}} \mathbb{S}^\lambda = \frac{k!}{\prod_{b \in \mathrm{Young}(\lambda)} h_b}$$

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Young modules and Specht modules

- For $\lambda \vdash k$, we will denote

$$M^\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\mathbf{1}_{\mathfrak{S}_\lambda})$$

 (the Young module of λ). It is isomorphic to the permutation representation of $\mathfrak{S}_k$ on the set of cosets in $\mathfrak{S}_k$ of the subgroup $\mathfrak{S}_\lambda$.
- Clearly, $\dim_{\mathbb{F}} M^\lambda = \binom{k}{\lambda}$.
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$$M^\lambda \cong_{\mathfrak{S}_k} \bigoplus_{\mu \trianglerighteq \lambda} K(\mu, \lambda) \mathbb{S}^\mu.$$

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Action of a finite group on a space X

- Let a finite group G act on a topological space X.
- The action of G on X induces an action of G on the cohomology group $\mathrm{H}^*(X, \mathbb{F})$, making $\mathrm{H}^*(X, \mathbb{F})$ into a G-module.
- If $\mathrm{card}(G)$ is invertible in $\mathbb{F}$ (and so in particular, if $\mathbb{F}$ is a field of characteristic 0) we have the isomorphisms

$$\mathrm{H}^*(X/G, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*_G(X, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{F})^G.$$

- In particular, if $S \subset \mathrm{R}^k$, is a symmetric semi-algebraic set, $\mathrm{H}^*(S, \mathbb{F})$ is a finite dimensional $\mathfrak{S}_k$-module, and

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Key example

- Let

$$F_k = \sum_{i=1}^{k} (X_i(X_i - 1))^2 - \varepsilon,$$

$$V_k = \mathrm{Z}(F_k, \mathrm{R}^k).$$

- $$\mathrm{H}^0(V_k, \mathbb{F}) \cong \bigoplus_{0 \le i \le k} \mathrm{H}^0(V_{k,i}, \mathbb{F}),$$

  where for $0 \le i \le k$, $V_{k,i}$ is the $\mathfrak{S}_k$-orbit of the connected component of V_k infinitesimally close (as a function of ε) to the point $\mathbf{x}^i = (\underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{k-i})$, and $\mathrm{H}^0(V_{k,i}, \mathbb{F})$ is an invariant subspace of $\mathrm{H}^0(V_k, \mathbb{F})$.

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Key example (cont).

- The isotropy subgroup of the point $\mathbf{x}^i$ under the action of $\mathfrak{S}_k$ is $\mathfrak{S}_i \times \mathfrak{S}_{k-i}$, and $\mathrm{orbit}(\mathbf{x}^i)$ is thus in 1-1 correspondence with the cosets of the subgroup $\mathfrak{S}_i \times \mathfrak{S}_{k-i}$.
- It now follows from the definition of Young's module:

$$\begin{aligned} \mathrm{H}^0(V_{k,i}, \mathbb{F}) &\cong_{\mathfrak{S}_k} M^{(i,k-i)} \text{ if } i \geq k-i, \\ &\cong_{\mathfrak{S}_k} M^{(k-i,i)} \text{ otherwise.} \end{aligned}$$

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Key example (cont).

- It follows that for k odd,

$$\begin{aligned}
\mathrm{H}^0(V_k,\mathbb{F}) &\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\le 2}} (M^\lambda \oplus M^\lambda)\\
&\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\le 2}} \bigoplus_{\mu \trianglerighteq \lambda} 2K(\mu,\lambda)\mathbb{S}^\mu\\
&\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\le 2}} \bigoplus_{\mu \trianglerighteq \lambda} 2\mathbb{S}^\mu\\
&\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\mu\vdash k\\ \ell(\mu)\le 2}} m_{0,\mu}\mathbb{S}^\mu,
\end{aligned}$$

where for each $\mu = (\mu_1,\mu_2) \vdash k$,

$$\begin{aligned}
m_{0,\mu} &= 2(\mu_1 - \lfloor k/2 \rfloor)\\
&= 2\mu_1 - k + 1\\
&= \mu_1 - \mu_2 + 1.
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Key example (cont).

- For k even:

$$\mathrm{H}^0(V_k, \mathbb{F}) \cong_{\mathfrak{S}_k} M^{(k/2,k/2)} \oplus \Big(\bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq 2 \\ \lambda \neq (k/2,k/2)}} (M^\lambda \oplus M^\lambda) \Big)$$

$$\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\mu \vdash k \\ \ell(\mu) \leq 2}} m_{0,\mu} \mathbb{S}^\mu,$$

where for each $\mu = (\mu_1, \mu_2) \vdash k$,

$$\begin{aligned} m_{0,\mu} &= 2(\mu_1 - k/2) + 1 \\ &= \mu_1 - \mu_2 + 1. \end{aligned}$$

- We deduce for all k,

$$\begin{aligned} m_{0,\mu} &= \mu_1 - \mu_2 + 1 \\ &\leq k + 1. \end{aligned}$$

Key example (cont).

- For k even:

$$\mathrm{H}^0(V_k,\mathbb{F}) \cong_{\mathfrak{S}_k} M^{(k/2,k/2)} \oplus \Big(\bigoplus_{\substack{\lambda\vdash k \\ \ell(\lambda)\leq 2 \\ \lambda\neq(k/2,k/2)}} (M^\lambda \oplus M^\lambda)\Big)$$

$$\cong_{\mathfrak{S}_k} \bigoplus_{\substack{\mu\vdash k \\ \ell(\mu)\leq 2}} m_{0,\mu}\mathbb{S}^\mu,$$

where for each $\mu = (\mu_1,\mu_2) \vdash k$,

$$\begin{aligned} m_{0,\mu} &= 2(\mu_1 - k/2) + 1 \\ &= \mu_1 - \mu_2 + 1. \end{aligned}$$

- We deduce for all k,

$$\begin{aligned} m_{0,\mu} &= \mu_1 - \mu_2 + 1 \\ &\leq k + 1. \end{aligned}$$

$\mathfrak{S}_k$-equivariant Poincaré duality

What about $\mathrm{H}^{k-1}(V_k, \mathbb{F})$?

Theorem

Let $V \subset \mathbb{R}^k$ be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of $\mathfrak{S}_k$ on $\mathbb{R}^k$. Then, for each $p, 0 \le p \le k-1$, there is a $\mathfrak{S}_k$-module isomorphism

$$\mathrm{H}^p(V, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^{k-p-1}(V, \mathbb{F}) \otimes \mathbf{sign}_k.$$

This implies in our example that

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Key example (cont).

In particular for $k = 2, 3$ we have:

$$\begin{aligned}
\mathrm{H}^0(V_2, \mathbb{F}) &\cong_{\mathfrak{S}_2} 3\mathbb{S}^{(2)} \oplus \mathbb{S}^{(1,1)}, \\
\mathrm{H}^0(V_3, \mathbb{F}) &\cong_{\mathfrak{S}_3} 4\mathbb{S}^{(3)} \oplus 2\mathbb{S}^{(2,1)}, \\
\mathrm{H}^1(V_2, \mathbb{F}) &\cong_{\mathfrak{S}_2} 3\mathbb{S}^{(1,1)} \oplus \mathbb{S}^{(2)}, \\
\mathrm{H}^2(V_3, \mathbb{F}) &\cong_{\mathfrak{S}_3} 4\mathbb{S}^{(1,1,1)} \oplus 2\mathbb{S}^{(2,1)}.
\end{aligned}$$

Key example (cont).

- For $\mu = (\mu_1, \mu_2) \vdash k$, by the hook-length formula we have,

$$\dim \mathbb{S}^\mu = \frac{k!\,(\mu_1 - \mu_2 + 1)}{(\mu_1 + 1)!\mu_2!}.$$

- Since $\mathrm{H}^0(V_k, \mathbb{F}) \cong_{\mathfrak{S}_k} \bigoplus_{\mu=(\mu_1,\mu_2)\vdash k} m_{0,\mu}\mathbb{S}^\mu$, and hence $\dim_{\mathbb{F}}(\mathrm{H}^0(V_k, \mathbb{F}) = \sum_{\mu=(\mu_1,\mu_2)\vdash k} m_{0,\mu} \dim_{\mathbb{F}}(\mathbb{S}^\mu) = 2^k$, we obtain as a consequence the identity

$$k! \left(\sum_{\substack{\mu_1 \geq \mu_2 \geq 0 \\ \mu_1 + \mu_2 = k}} \frac{(\mu_1 - \mu_2 + 1)^2}{(\mu_1 + 1)!\mu_2!} \right) = 2^k.$$

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Previous Results

Theorem (B., Riener (2013))

Let $P \in \mathrm{R}[X_1, \ldots, X_k]$, be non-negative polynomial of degree bounded by d, and and such that $V = \mathrm{Z}(P, \mathrm{R}^k)$ is invariant under the action of $\mathfrak{S}_k$. Then,

$$b(V/\mathfrak{S}_k, \mathbb{F}) \quad \leq \quad (k)^{2d}(O(d))^{2d+1}.$$

Note that $\mathrm{H}^*(V/\mathfrak{S}_k, \mathbb{F})$ is isomorphic to the isotypic component of $\mathrm{H}^*(V, \mathbb{F})$ belonging to the trivial representation $\mathbf{1}_{\mathfrak{S}_k}$, and $b(V/\mathfrak{S}_k, \mathbb{F})$ is its multiplicity.

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More notation

- For any $\mathfrak{S}_k$-symmetric semi-algebraic subset $S \subset \mathrm{R}^k$, and $\lambda \vdash k$, we denote

$$\begin{aligned} m_{i,\lambda}(S,\mathbb{F}) &= \mathrm{mult}(\mathbb{S}^\lambda, \mathrm{H}^i(S,\mathbb{F})), \\ m_\lambda(S,\mathbb{F}) &= \sum_{i \geq 0} m_{i,\lambda}(S,\mathbb{F}). \end{aligned}$$

New Results

Theorem (B., Riener (2014))

Let $P \in \mathrm{R}[X_1, \ldots, X_k]$ *be a* $\mathfrak{S}_k$*-symmetric polynomial, with* $\deg(P) \leq d$. *Let* $V = \mathrm{Z}(P, \mathrm{R}^K)$. *Then, for all* $\mu = (\mu_1, \mu_2, \ldots) \vdash k$, $m_\mu(V, \mathbb{F}) > 0$ *implies that*

$$\mathrm{card}(\{i \mid \mu_i \geq 2d\}) \leq 2d, \mathrm{card}(\{j \mid \tilde{\mu}_j \geq 2d\}) \leq 2d.$$

Moreover, for

$$m_\mu(V, \mathbb{F}) \leq k^{O(d^2)} d^d.$$

Pictorially

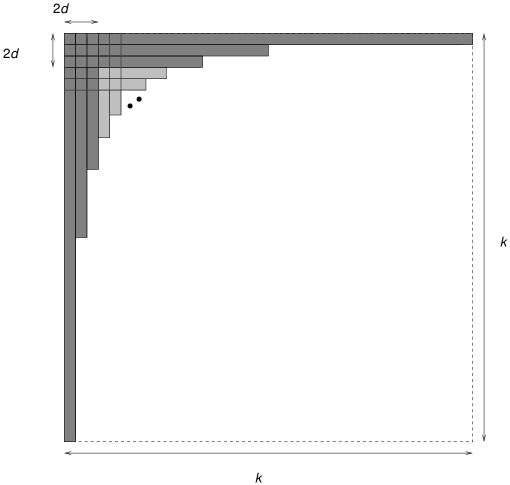

Figure : The shaded area contains all Young diagrams of partitions in $\mathrm{Par}(k)$, while the darker area contains the Young diagrams of the partitions which can possibly appear in the $\mathrm{H}^*(V, \mathbb{F})$ for fixed d and large k.

Asymptotics

- Note that by a famous result of Hardy and Ramanujan (1918)

$$\mathrm{card}(\mathrm{Par}(k)) \sim \frac{1}{4\sqrt{3}k} e^{\pi\sqrt{\frac{2k}{3}}}, k \to \infty$$

which is exponential in k;

- whereas it follows from the last theorem that

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- Degree principle.
- Equivariant Morse theory, equivariant Mayer-Vietoris sequence.
- Some tableau combinatorics. Pieri's rule.

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More results

Similar results bounding multiplicities in th eisotypic decomposition of the cohomology modules of:

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- Symmetric semi-algebraic sets.
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Algorithmic conjecture

Conjecture

For any fixed $d > 0$, there is an algorithm that takes as input the description of a symmetric semi-algebraic set $S \subset \mathrm{R}^k$, defined by a $\mathcal{P}$-closed formula, where $\mathcal{P}$ is a set symmetric polynomials of degrees bounded by d, and computes $m_{i,\lambda}(S, \mathbb{Q})$, for each $\lambda \vdash k$ with $m_{i,\lambda}(S, \mathbb{Q}) > 0$, as well as all the Betti numbers $b_i(S, \mathbb{Q})$, with complexity which is polynomial in $\mathrm{card}(\mathcal{P})$ and k.

Representational stability question

- Let $F \in \mathrm{R}[X_1, \ldots, X_d]^{\mathfrak{S}_d}_{\leq d}$ be a symmetric polynomial of degree at most d, and let for $k \geq d$ $F_k = \phi_{d,k}(F) \in \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}$ where $\phi_{d,k} : \mathrm{R}[X_1, \ldots, X_d]^{\mathfrak{S}_d}_{\leq d} \to \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}$ is the canonical injection.
- Let $(V_k = \mathrm{Z}(F_k, \mathrm{R}^k))_{k \geq d}$ be the corresponding sequence of symmetrc real varieties.
- Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let $\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k$.
- It is a consequence of the hook-length formula that

$$\dim_{\mathbb{F}}(\mathbb{S}^{\{\mu\}_k}) = \frac{\dim_{\mathbb{F}}(\mathbb{S}_\mu)}{|\mu|!} P_\mu(k),$$

where $P_\mu(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_\mu) = |\mu|$.

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Question

For any fixed number $p \geq 0$ we pose the following question.

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Does there exist a polynomial $P_{F,p,\mu}(k)$ such that for all sufficiently large k, $m_{p,\{\mu\}_k}(V_k, \mathbb{F}) = P_{F,p,\mu}(k)$? Note that a positive answer would imply that

$$\dim_{\mathbb{F}}(\mathrm{H}^p(V_k, \mathbb{F}))_{\{\mu\}_k} = \frac{\dim_{\mathbb{F}}(\mathbb{S}_\mu)}{|\mu|!} P_{F,p,\mu}(k) P_\mu(k)$$

is also given by a polynomial for all large enough k.
A stronger question is to ask for a bound on the degree of $P_{F,p,\mu}(k)$ as a function of d, μ and p.

The conjecture holds in the “key example”.

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Reference

S. Basu, C. Riener. On the isotypic decomposition of the cohomology modules of symmetric semi-algebraic sets: polynomial bounds on multiplicities. arXiv:1503.00138.