Combinatorial and Topological Complexity in Computational Geometry

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- A *cell* is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches.
- The combinatorial complexity of an ℓ-dimensional cell C is the number of cells in the relative boundary of C.

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2. For $k = 3$:
   •: Complexity of the whole arrangement : $O(n^3)$.
   •: Complexity of a single cell : $O(n^{2+\epsilon})$

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Topological Complexity of Semi-Algebraic Sets

Oleinik and Petrovsky (1949) Thom (1964) and Milnor (1965) proved that the sum of the Betti numbers of a semi-algebraic set $S \subset R^k$, defined by

$$P_1 \geq 0, \ldots, P_n \geq 0, deg(P_i) \leq d, 1 \leq i \leq s,$$

is bounded by $(O(nd))^k$.

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- In analogy to the single cell results computational geometry, one might conjecture that the sum of the Betti numbers of a single connected component of a basic semi-algebraic set is bounded by $n^{k-1}O(d)^k$.

- It is easy to construct a basic semi-algebraic set such that it has one connected component whose other Betti numbers sum to $\Omega(nd)^{k-1}$.

- Let

$$P_i = (X_k^2 + L_{i,1}^2) \cdots (X_k^2 + L_{i,\lfloor d/2 \rfloor}^2) - \epsilon,$$

where the $L_{ij} \in R[X_1, \ldots, X_{k-1}]$ are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_s \geq 0$ has one connected component with $\sum_i \beta_i(S) = \Omega(nd)^{k-1}$.

New Results

Theorem 1. *(B98) Let* $C_1, \ldots, C_m \subset R^k$ *be* m *different connected components of a basic semi-algebraic set defined by* $P_1 \geq 0, \ldots, P_n \geq 0$, *with the degrees of the polynomials* P_i *bounded by* d. *Then* $\sum_{i,j} \beta_i(C_j)$ *is bounded by* $m + \binom{n}{k-1} O(d)^k$.

New Results

Theorem 2. *(B98) Let* C *be a* k*-dimensional cell in an arrangement of* n *surface patches* $S_1, \ldots, S_n$ *in* R^k*. Then the combinatorial complexity of* C *is bounded by* $O(n^{k-1+\epsilon})$ *for every* $\epsilon > 0$*.*

Ideas behind the proofs:

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- There are $\sum_{1\leq i\leq k}\binom{n}{i}$ strata of dimension > 0. Thus, there are $\binom{n}{k-1}(O(d))^k$ critical values coming from strata of dimension > 0.

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- The bound on the number of good vertices plays an essential role in the proof of Theorem 7.

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- What if the intersections are not acyclic but have bounded topology ?

Betti numbers for union

Theorem 3. *Let* $S \subset R^k$ *be the set defined by the disjunction of* n *inequalities,* $P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k], deg(P_i) \leq d, 1 \leq i \leq n$. *Then,*

$$\beta_i(S) \leq n^{i+1} O(d)^k.$$

Note that, a special case of the above theorem is the situation when S is the union of n sets each defined by $-P_i^2 \geq 0$.

Betti numbers for intersections

Theorem 4. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq n^{k-i} O(d)^k.$$

Ideas behind the proofs:

Let $A, B \subset R^k$ be compact semi-algebraic sets.
Mayer-Vietoris exact sequence:

$$0 \rightarrow H_{k-1}(A \cap B) \rightarrow H_{k-1}(A) \oplus H_{k-1}(B) \rightarrow H_{k-1}(A \cup B) \rightarrow$$

$$H_{k-2}(A \cap B) \rightarrow \cdots \rightarrow H_{i+1}(A \cup B) \rightarrow H_i(A \cap B) \rightarrow$$
$$H_i(A) \oplus H(B) \rightarrow H_i(A \cup B) \rightarrow \cdots$$

A preliminary lemma

Lemma 5. *Let* $S_1, \ldots, S_n \subset R^k$ *be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cup \cdots \cup S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq k-i$ *(that is the sum of the Betti numbers of the union of any* ℓ *of the sets for all* $\ell \leq k-i$ *is bounded by* M*). Let* $S = \cap_{1 \leq j \leq n} S_j$*. Then,*

$$\beta_i(S) \leq n^{k-i} M.$$

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$$0 \to H_{k-1}(T_{n-1} \cap S_n) \to H_{k-1}(T_{n-1}) \oplus H_{k-1}(S_n)$$

$$\to H_{k-1}(T_{n-1} \cup S_n) \to H_{k-2}(T_{n-1} \cap S_n) \to \cdots$$
$$\to H_{i+1}(T_{n-1} \cup S_n) \to H_i(T_{n-1} \cap S_n) \to$$
$$H_i(T_{n-1}) \oplus H_i(S_n) \to H_i(T_{n-1} \cup S_n) \to \cdots$$

Proof (cont):

- $\beta_{k-1}(T_n) = \beta_{k-1}(T_{n-1} \cap S_n) \leq \beta_{k-1}(T_{n-1}) + \beta_{k-1}(S_n)$. Unwinding the first term of right hand side we obtain that $\beta_{k-1}(S) \leq nM$.

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- Again from the Mayer-Vietoris sequence we get that,

$$\beta_i(S) \leq \beta_{i+1}(T_{n-1} \cup S_n) + \beta_i(T_{n-1}) + \beta_i(S_n).$$

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- $T_{n-1} \cup S_n = \cap_{1 \leq i \leq n-1} (S_i \cup S_n)$. The $n-1$ sets $S_i \cup S_n$ satisfies the assumption on at most $(k-i-1)$-ary unions and we can apply the induction hypothesis.

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- Thus, we have that $\beta_i(S) \leq (n-1)^{k-i-1}M + (n-1)^{k-i}M + M \leq n^{k-i}M$.

Dual lemma

Lemma 6. *Let* $S_1, \ldots, S_n \subset R^k$ *be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cap \cdots \cap S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq i+1$. *Let* $S = \cup_{1 \leq j \leq n} S_j$. *Then,*

$$\beta_i(S) \leq n^{i+1} M.$$

Sets defined by few inequalities:

Lemma 7. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the conjunction of the inequalities* $P_i \geq 0$. *Let* S *be bounded. Then,* $\sum_i \beta_i(S) = O(d)^k$.

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Lemma 8. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the disjunction of the inequalities* $P_i \geq 0$. *Then,* $\sum_i \beta_i(S) = O(d)^k$.