Betti numbers of semi-algebraic sets defined by partly quadratic polynomials

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(joint work with Dima Pasechnik and Marie-Francoise Roy)

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Outline

1. Introduction
   - Semi-algebraic sets
2. Quantitative Bounds
   - Quantitative Bounds on Betti Numbers – Old and New
3. Proof of the main theorem
4. Algorithmic Implications

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Semi-algebraic sets

- Let R be a real closed field, for example the field $\mathbb{R}$ of the real numbers.
- A semi-algebraic set, $S \subset \mathrm{R}^k$, is a subset of R^k defined by a Boolean formula whose atoms are polynomial equalities and inequalities.
- If all the polynomials involved belong to $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call S a $\mathcal{P}$-semi-algebraic set.
- If the atoms of the Boolean formula are of the form $P \geq 0, P \leq 0, P \in \mathcal{P}$, and there are no negations, then we call S a $\mathcal{P}$-closed semi-algebraic set.

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Quantitative Questions

- Let $\mathcal{P} \subset \mathrm{R}[\mathbf{X_1}, \ldots, \mathbf{X_k}]$ with $\#\mathcal{P} = \mathbf{s}$ and $\max_{\mathbf{P} \in \mathcal{P}} \deg(\mathbf{P}) = \mathbf{d}$.
- If $S \subset \mathrm{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then how large can the Betti numbers of S be ?
- How many of the possible 3^s sign patterns in $\{0, +, -\}^{\mathcal{P}}$ can be possibly realized by points in R^k ?
- Into how many regions do the sign patterns decompose R^k? How large can be the sum of the Betti numbers of all the sets in this decomposition ?
- If $f : X \to Y$ is a semi-algebraic map, defined in terms of $\mathcal{P}$, then how many topological types can occur amongst the semi-algebraic sets, $f^{-1}(\mathbf{y}), \mathbf{y} \in Y$.

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Quantitative bounds – Singly exponential

- Classical result (Oleinik, Petrovsky, Thom, Milnor): If S is defined by $P_1 \geq 0, \ldots, P_s \geq 0,$ then,

$$\sum_{0 \leq i \leq k} b_i(S) \leq (O(sd))^k.$$

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Extension to arbitrary $\mathcal{P}$-semi-algebraic set

- Extension to arbitrary $\mathcal{P}$- semi-algebraic sets is more technical and achieved only quite recently by Gabrielov and Vorobjov (2005, 2007) (with a slight worsening of the bound).
- If S is a $\mathcal{P}$-semi-algebraic set then

$$\sum_{0 \leq i \leq k} b_i(S) \leq \min((O(s^2 d))^k, (O(skd))^k).$$

- All the above bounds are **singly exponential** in the number of variables k and **polynomial** in s and d.
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- Let $S \subset \mathrm{R}^\ell$ be a semi-algebraic set defined by $Q_1 \geq 0, \ldots, Q_m \geq 0,$ with $\deg(Q_i) \leq 2, 1 \leq i \leq m.$
- As in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large.

Example

The set $S \subset \mathrm{R}^\ell$ defined by

$$Y_1(Y_1 - 1) \geq 0, \ldots, Y_\ell(Y_\ell - 1) \geq 0$$

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Other classes with polynomial bounds ?

- The bound depends crucially on the assumption that the degrees of the polynomials $Q_1, \ldots, Q_m$ are at most two.
- For instance, the semi-algebraic set defined by a *single* polynomial of degree 4 can have Betti numbers exponentially large in ℓ. For instance the semi-algebraic set $S \subset \mathbb{R}^\ell$ defined by

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Bounds for partly quadratic systems

Theorem (B.,Pasechnik, Roy, 2007)

Let

- $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$ *with* $\deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in \mathcal{Q}, \#(\mathcal{Q}) = m$;
- $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ *with* $\deg_X(P) \leq d, P \in \mathcal{P}, \#(\mathcal{P}) = s$;
- $S \subset \mathrm{R}^{\ell+k}$ *a* $(\mathcal{P} \cup \mathcal{Q})$*-closed semi-algebraic set.*

Then

$$b(S) \leq \ell^2 (O(s + \ell + m)\ell d)^{k+2m}.$$

In particular, for $m \leq \ell$, *we have* $b(S) \leq \ell^2 (O(s + \ell)\ell d)^{k+2m}$.

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Generalization of the previous bounds

Notice that the previous Theorem is a common generalization of the previous theorems in the sense that we recover similar bounds (that is bounds having the same shape) by setting ℓ and m (respectively, s, d and k) to $O(1)$.

Bound for semi-algebraic sets defined over a quadratic map

Corollary

Let $Q = (Q_1, \ldots, Q_k) : \mathrm{R}^\ell \to \mathrm{R}^k$ *be a quadratic map. and* $V \subset \mathrm{R}^k$ *be a* $\mathcal{P}$*-closed semi-algebraic with* $\#(\mathcal{P}) = s$ *and* $\deg(P) \leq d, P \in \mathcal{P}$.
Let $S = Q^{-1}(V)$. *Then,*

$$b(S) \leq \ell^2 (O(s + \ell + k)\ell d)^{3k}.$$

Homogeneous Case

We denote by:

- $\mathcal{Q}^h$ the family of polynomials obtained by homogenizing $\mathcal{Q}$ with respect to the variables Y, i.e.
- Φ a formula defining a $\mathcal{P}$-closed semi-algebraic set V,
- $$A^h = \bigcup_{Q \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\},$$
- $$W^h = \bigcap_{Q \in \mathcal{Q}^h} \{(y, x) \mid |y| = 1 \wedge Q(y, x) \leq 0 \wedge \Phi(x)\}.$$

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Result in a very special case

Proposition

$$b(A^h), b(W^h) \leq \ell^2(O((s+\ell+m)\ell d))^{m+k}.$$

Auxillary construction

- Let $\Omega = \{\omega \in \mathrm{R}^m \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq m\}$.
- For $\omega \in \Omega$ let $\langle \omega, \mathcal{Q}^h \rangle \in \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ be defined by

$$\langle \omega, \mathcal{Q}^h \rangle = \sum_{i=1}^{m} \omega_i Q_i^h.$$

- For $(\omega, x) \in \Omega \times V$ let $\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)$ be the quadratic form in $Y_0, \ldots, Y_\ell$ obtained from $\langle \omega, \mathcal{Q}^h \rangle$ by specializing $X_i = x_i, 1 \leq i \leq k$.

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Auxillary construction (cont).

Let $B \subset \Omega \times \mathbf{S}^\ell \times V$ be the semi-algebraic set defined by

$$B = \{(\omega, y, x) \mid \omega \in \Omega, y \in \mathbf{S}^\ell, x \in V, \ \langle \omega, \mathcal{Q}^h \rangle (y, x) \geq 0\}.$$

We have the following diagram.

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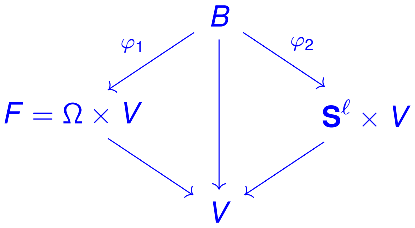

$B \sim A^h$

Proposition

The semi-algebraic set B is homotopy equivalent to $\varphi_2(B) = A^h$.

Filtration by index

- For a quadratic form Q let $\lambda_i(Q), 0 \leq i \leq \ell$ be the eigenvalues of Q in non-decreasing order, i.e.

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_\ell(Q).$$

- For $F = \Omega \times V$ let

$$F_j = \{(\omega, x) \in F \mid \mathrm{index}(\langle \omega, \mathcal{Q}^h \rangle(\cdot, x)) \leq j\}.$$

- It is clear that each F_j is a closed semi-algebraic subset of F and

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The fibre of the map φ_1 over a point $(\omega, x) \in F_j \setminus F_{j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.

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Example

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In this example $m = 2, \ell = 3, k = 0$, and $\mathcal{Q}^h = \{Q_1^h, Q_2^h\}$ with

$$Q_1^h = - Y_0^2 - Y_1^2 - Y_2^2,$$
$$Q_2^h = Y_0^2 + 2Y_1^2 + 3Y_2^2.$$

The set Ω is the part of the unit circle in the third quadrant of the plane, and $F = \Omega$ in this case. We display the fibers of the map $\varphi_1^{-1}(\omega) \subset B$ for a sequence of values of ω starting from $(-1, 0)$ and ending at $(0, -1)$. We also show the spheres, $C \cap \varphi_1^{-1}(\omega)$, of dimensions $0, 1$, and 2, that these fibers retract to.

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Picture

Figure: Type change: $\emptyset \to \mathbf{S}^0 \to \mathbf{S}^1 \to \mathbf{S}^2$. $\emptyset$ is not shown.

Outline of the remaining argument

- Each C_j is a $\mathbf{S}^{\ell-j}$-bundle over $F_j \setminus F_{j-1}$ under the map φ_1, and $C = \cup_{0 \leq j \leq \ell} C_j$.
- Since we have good bounds on the number as well as the degrees of polynomials used to define the bases, $F_j \setminus F_{j-1}$, we are able to bound the Betti numbers of each C_j by the following proposition:

Proposition

Let $B \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set and let $\pi : E \to B$ be a semi-algebraic sphere bundle with base B. Then

$$b(E, \mathbb{Z}_2) \leq 2 \cdot b(B, \mathbb{Z}_2).$$

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Outline (cont.)

- However, the C_j's could be possibly glued to each other in complicated ways, and thus knowing upper bounds on the Betti numbers of each C_j does not immediately produce a bound on Betti numbers of C.
- In order to get around this difficulty, we consider certain closed subsets, $F'_j \subset F$, where each F'_j is an infinitesimal deformation of $F_j \setminus F_{j-1}$, and form the base of a $\mathbf{S}^{\ell-j}$-bundle C'_j.
- Additionally, the C'_j are glued to each other along sphere bundles over $F'_j \cap F'_{j-1}$, and their union, C', is homotopy equivalent to C.
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Complexity of the bases

- Let $\Lambda \in \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k, T]$ be the polynomial defined by

$$\begin{aligned} \Lambda &= \det(T \cdot \mathrm{Id}_{\ell+1} - M_{Z \cdot \mathcal{Q}^h}), \\ &= T^{\ell+1} + D_\ell T^\ell + \cdots + D_0, \end{aligned}$$

  where each $D_i \in \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

- It then follows from Descartes' rule of signs that for each $(\omega, x) \in \Omega \times \mathrm{R}^k$, $\mathrm{index}(\langle \omega, \mathcal{Q}^h \rangle (\cdot, x))$ is determined by the sign vector

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$$\mathcal{D} = \{D_0, \ldots, D_\ell\} \subset \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$$

we have

Lemma

F_j is the intersection of F with a $\mathcal{D}$-closed semi-algebraic set for each $0 \leq j \leq \ell + 1$.

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$$\#\mathcal{D} = \ell + 1,$$

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Finishing the argument

- Now use the O-P-T-M type bounds to bound the Betti numbers of the various F_j' and hence the C_j'.
- Notice that only the adjacent C_j''s intersect and then use Mayer-Vietoris inequalities to bound the Betti numbers of C.
- Hence, obtain a bound on $b(A^h)$.
- Again Mayer-Vietoris inequalities give a bound on $b(W^h)$.
- Reduce the general case to the basic case using standard arguments.

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Algorithms for computing the Euler-Poincaré characteristic and Betti numbers

Theorem

There exists an algorithm that takes as input the description of a $(\mathcal{P} \cup \mathcal{Q})$-closed semi-algebraic set S and outputs its the Euler-Poincaré characteristic $\chi(S)$. The complexity of this algorithm is bounded by $(\ell s m d)^{O(m(m+k))}$. There exists an algorithm for computing all the Betti numbers whose complexity is $(\ell s m d)^{2^{O(m+k)}}$.

The complexity of both the algorithms is *polynomial* for fixed m and k.

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Computational hardness

- The problem of computing the Betti numbers of semi-algebraic sets in general is a PSPACE-hard problem. The same is true for semi-algebraic sets defined by many quadratic inequalities.
- On the other hand it was knwn before that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of quadratic inequalities is solvable in polynomial time.
- We have shown that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of polynomial inequalities is solvable in polynomial time, even if we allow a small (constant sized) subset of the variables to occur with degrees larger than two in the polynomials defining the given set.

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