Betti numbers of semi-algebraic sets defined by partly quadratic polynomials

Saugata Basu
(joint work with Dima Pasechnik and Marie-Francoise Roy)

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   • Semi-algebraic sets

2 Quantitative Bounds
   • Quantitative Bounds on Betti Numbers – Old and New

3 Proof of the main theorem

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Let $\mathbb{R}$ be a real closed field, for example the field $\mathbb{R}$ of the real numbers.

A semi-algebraic set, $S \subset \mathbb{R}^k$, is a subset of $\mathbb{R}^k$ defined by a Boolean formula whose atoms are polynomial equalities and inequalities.

If all the polynomials involved belong to $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call $S$ a $\mathcal{P}$-semi-algebraic set.

If the atoms of the Boolean formula are of the form $P \geq 0, P \leq 0, P \in \mathcal{P}$, and there are no negations, then we call $S$ a $\mathcal{P}$-closed semi-algebraic set.
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Quantitative Questions

1. Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\#\mathcal{P} = s$ and $\max_{P \in \mathcal{P}} \deg(P) = d$.
2. If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then how large can the Betti numbers of $S$ be?
3. How many of the possible $3^s$ sign patterns in $\{0, +, -\}^\mathcal{P}$ can be possibly realized by points in $\mathbb{R}^k$?
4. Into how many regions do the sign patterns decompose $\mathbb{R}^k$? How large can be the sum of the Betti numbers of all the sets in this decomposition?
5. If $f : X \to Y$ is a semi-algebraic map, defined in terms of $\mathcal{P}$, then how many topological types can occur amongst the semi-algebraic sets, $f^{-1}(y), y \in Y$. 

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Classical result (Oleinik, Petrovsky, Thom, Milnor): If $S$ is defined by $P_1 \geq 0, \ldots, P_s \geq 0$, then,

$$\sum_{0 \leq i \leq k} b_i(S) \leq (O(sd))^k.$$ 

The same bound extends (with a different constant) if $S$ is $\mathcal{P}$-closed semi-algebraic set (B 99).
Quantitative bounds – Singly exponential

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Extension to arbitrary $\mathcal{P}$-semi-algebraic set

- Extension to arbitrary $\mathcal{P}$-semi-algebraic sets is more technical and achieved only quite recently by Gabrielov and Vorobjov (2005, 2007) (with a slight worsening of the bound).

  - If $S$ is a $\mathcal{P}$-semi-algebraic set then
    
    $$
    \sum_{0 \leq i \leq k} b_i(S) \leq \min((O(s^2d'))^k, (O(skd'))^k).
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- All the above bounds are **singly exponential** in the number of variables $k$ and **polynomial** in $s$ and $d$.

- This single exponential dependence is unavoidable.
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Let $S \subset \mathbb{R}^\ell$ be a semi-algebraic set defined by $Q_1 \geq 0, \ldots, Q_m \geq 0$, with $\deg(Q_i) \leq 2, 1 \leq i \leq m$.

As in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large.

**Example**

The set $S \subset \mathbb{R}^\ell$ defined by

$$Y_1(Y_1 - 1) \geq 0, \ldots, Y_\ell(Y_\ell - 1) \geq 0$$

satisfies $b_0(S) = 2^\ell$. 

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Bounds on Betti Numbers of Sets Defined by Quadratic Inequalities

Theorem (Barvinok (1997))

Let \( S \subset \mathbb{R}^\ell \) be defined by

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Then

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Unlike the previous bound this bound is polynomial in \( \ell \) and exponential in \( m \).
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Unlike the previous bound this bound is polynomial in $\ell$ and exponential in $m$. 
The bound depends crucially on the assumption that the degrees of the polynomials $Q_1, \ldots, Q_m$ are at most two.

For instance, the semi-algebraic set defined by a single polynomial of degree 4 can have Betti numbers exponentially large in $\ell$. For instance the semi-algebraic set $S \subset \mathbb{R}^\ell$ defined by

$$\sum_{i=0}^{\ell} Y_i^2(Y_i - 1)^2 \leq 0.$$ 

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Other classes with polynomial bounds?

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Bounds for partly quadratic systems

Theorem (B., Pasechnik, Roy, 2007)

Let
- $Q \subset \mathbb{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$ with $\deg_Y(Q) \leq 2, \deg_X(Q) \leq d, Q \in Q, \#(Q) = m;$
- $P \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\deg_X(P) \leq d, P \in P, \#(P) = s;$
- $S \subset \mathbb{R}^{\ell+k}$ a $(P \cup Q)$-closed semi-algebraic set.

Then

$$b(S) \leq \ell^2 (O(s + \ell + m)\ell d)^{k+2m}.$$ 

In particular, for $m \leq \ell$, we have $b(S) \leq \ell^2 (O(s + \ell)\ell d)^{k+2m}$. 

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In particular, for \( m \leq \ell \), we have \( b(S) \leq \ell^2 (O(s + \ell)ld)^{k+2m}. \)
Notice that the previous Theorem is a common generalization of the previous theorems in the sense that we recover similar bounds (that is bounds having the same shape) by setting $\ell$ and $m$ (respectively, $s$, $d$ and $k$) to $O(1)$. 
Bound for semi-algebraic sets defined over a quadratic map

Corollary

Let $Q = (Q_1, \ldots, Q_k) : \mathbb{R}^\ell \to \mathbb{R}^k$ be a quadratic map and $V \subset \mathbb{R}^k$ be a $\mathcal{P}$-closed semi-algebraic with $\#(\mathcal{P}) = s$ and $\deg(P) \leq d$, $P \in \mathcal{P}$. Let $S = Q^{-1}(V)$. Then,

$$b(S) \leq \ell^2(O(s + \ell + k)\ell d)^{3k}.$$
Homogeneous Case

We denote by:

- $Q^h$ the family of polynomials obtained by homogenizing $Q$ with respect to the variables $Y$, i.e.
- $\Phi$ a formula defining a $P$-closed semi-algebraic set $V$,

$A^h = \bigcup_{Q \in Q^h} \{ (y, x) \mid |y| = 1 \land Q(y, x) \leq 0 \land \Phi(x) \}$,

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Result in a very special case

Proposition

\[ b(A^h), b(W^h) \leq \ell^2 (O((s + \ell + m)\ell d))^{m+k}. \]
Let $\Omega = \{ \omega \in \mathbb{R}^m \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq m \}$.

For $\omega \in \Omega$ let $\langle \omega, Q^h \rangle \in \mathbb{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ be defined by

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\langle \omega, Q^h \rangle = \sum_{i=1}^{m} \omega_i Q^h_i.
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For $(\omega, x) \in \Omega \times V$ let $\langle \omega, Q^h \rangle(\cdot, x)$ be the quadratic form in $Y_0, \ldots, Y_\ell$ obtained from $\langle \omega, Q^h \rangle$ by specializing $X_i = x_i, 1 \leq i \leq k$. 
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For $(\omega, x) \in \Omega \times V$ let $\langle \omega, Q^h \rangle(\cdot, x)$ be the quadratic form in $Y_0, \ldots, Y_\ell$ obtained from $\langle \omega, Q^h \rangle$ by specializing $X_i = x_i, 1 \leq i \leq k$. 
Let $B \subset \Omega \times S^l \times V$ be the semi-algebraic set defined by

$$B = \{(\omega, y, x) \mid \omega \in \Omega, y \in S^l, x \in V, \langle \omega, Q^h \rangle(y, x) \geq 0\}.$$
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We have the following diagram.
Proposition

The semi-algebraic set $B$ is homotopy equivalent to $\varphi_2(B) = A^h$. 
Filtration by index

For a quadratic form $Q$ let $\lambda_i(Q), 0 \leq i \leq \ell$ be the eigenvalues of $Q$ in non-decreasing order, i.e.

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_\ell(Q).$$

For $F = \Omega \times V$ let

$$F_j = \{(\omega, x) \in F \mid \text{index}(\langle \omega, Q^h \rangle(\cdot, x)) \leq j \}.$$ 

It is clear that each $F_j$ is a closed semi-algebraic subset of $F$ and

$$F_0 \subset F_1 \subset \cdots \subset F_{\ell+1} = F.$$
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The fibre of the map $\varphi_1$ over a point $(\omega, x) \in F_j \setminus F_{j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.

In fact by simultaneous retraction of the fibers to the positive eigenspace we actually obtain a $S^{\ell-j}$ bundle $C^j$ over $F_j \setminus F_{j-1}$. 
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In fact by simultaneous retraction of the fibers to the positive eigenspace we actually obtain a $S^{\ell-j}$ bundle $C^j$ over $F_j \setminus F_{j-1}$.
In this example $m = 2$, $\ell = 3$, $k = 0$, and $Q^h = \{Q_1^h, Q_2^h\}$ with

$$Q_1^h = -Y_0^2 - Y_1^2 - Y_2^2,$$

$$Q_2^h = Y_0^2 + 2Y_1^2 + 3Y_2^2.$$ 

The set $\Omega$ is the part of the unit circle in the third quadrant of the plane, and $F = \Omega$ in this case. We display the fibers of the map $\varphi_1^{-1}(\omega) \subset B$ for a sequence of values of $\omega$ starting from $(-1, 0)$ and ending at $(0, -1)$. We also show the spheres, $C \cap \varphi_1^{-1}(\omega)$, of dimensions 0, 1, and 2, that these fibers retract to.
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Figure: Type change: $\emptyset \rightarrow S^0 \rightarrow S^1 \rightarrow S^2$. $\emptyset$ is not shown.
Outline of the remaining argument

Each $C_j$ is a $S^{\ell-j}$-bundle over $F_j \setminus F_{j-1}$ under the map $\varphi_1$, and $C = \bigcup_{0 \leq j \leq \ell} C_j$.

Since we have good bounds on the number as well as the degrees of polynomials used to define the bases, $F_j \setminus F_{j-1}$, we are able to bound the Betti numbers of each $C_j$ by the following proposition:

**Proposition**

Let $B \subset \mathbb{R}^k$ be a closed and bounded semi-algebraic set and let $\pi : E \to B$ be a semi-algebraic sphere bundle with base $B$. Then

$$b(E, \mathbb{Z}_2) \leq 2 \cdot b(B, \mathbb{Z}_2).$$
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However, the $C_j$’s could be possibly glued to each other in complicated ways, and thus knowing upper bounds on the Betti numbers of each $C_j$ does not immediately produce a bound on Betti numbers of $C$.

In order to get around this difficulty, we consider certain closed subsets, $F'_j \subset F$, where each $F'_j$ is an infinitesimal deformation of $F_j \setminus F_{j-1}$, and form the base of a $S^{\ell-j}$-bundle $C'_j$.

Additionally, the $C'_j$ are glued to each other along sphere bundles over $F'_j \cap F'_{j-1}$, and their union, $C'$, is homotopy equivalent to $C$.

Now we can use Mayer-Vietoris inequalities to bound the Betti numbers of $C'$, which in turn are equal to the Betti numbers of $C$. 

Saugata Basu (joint work with Dima Pasechnik and Marie-Françoise Roy)
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Saugata Basu (joint work with Dima Pasechnik and Marie-Francoise Roy) Betti numbers of semi-algebraic sets defined by partly quadratic polynomials
Let $\Lambda \in \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k, T]$ be the polynomial defined by

$$
\Lambda = \det(T \cdot \text{Id}_{\ell+1} - M_Z \cdot Q^h), \\
= T^{\ell+1} + D_\ell T^\ell + \cdots + D_0,
$$

where each $D_i \in \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

It then follows from Descartes’ rule of signs that for each $(\omega, x) \in \Omega \times \mathbb{R}^k$, index($\langle \omega, Q^h \rangle(\cdot, x)$) is determined by the sign vector $(\text{sign}(D_\ell(\omega, x)), \ldots, \text{sign}(D_0(\omega, x)))$. 
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$$(\text{sign}(D_\ell(\omega, x)), \ldots, \text{sign}(D_0(\omega, x))).$$
Denoting

\[ \mathcal{D} = \{D_0, \ldots, D_\ell\} \subset \mathbb{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k] \]

we have

**Lemma**

\( F_j \) is the intersection of \( F \) with a \( \mathcal{D} \)-closed semi-algebraic set for each \( 0 \leq j \leq \ell + 1 \).

Note that

\[ \#\mathcal{D} = \ell + 1, \]
\[ \deg(D_j) \leq (\ell + 1)d. \]
Complexity of the bases (cont.)

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**Lemma**

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Note that
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\[
\deg(D_j) \leq (\ell + 1)d.
\]
Now use the O-P-T-M type bounds to bound the Betti numbers of the various $F'_j$ and hence the $C'_j$.

Notice that only the adjacent $C'_j$’s intersect and then use Mayer-Vietoris inequalities to bound the Betti numbers of $C$.

Hence, obtain a bound on $b(A^h)$.

Again Mayer-Vietoris inequalities give a bound on $b(W^h)$.

Reduce the general case to the basic case using standard arguments.
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Again Mayer-Vietoris inequalities give a bound on $b(W^h)$.

Reduce the general case to the basic case using standard arguments.
Theorem

There exists an algorithm that takes as input the description of a \((\mathcal{P} \cup \mathcal{Q})\)-closed semi-algebraic set \(S\) and outputs its the Euler-Poincaré characteristic \(\chi(S)\). The complexity of this algorithm is bounded by \((\ell \text{smd})^{O(m(m+k))}\). There exists an algorithm for computing all the Betti numbers whose complexity is \((\ell \text{smd})^{2^{O(m+k)}}\).

The complexity of both the algorithms is polynomial for fixed \(m\) and \(k\).
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The complexity of both the algorithms is polynomial for fixed \(m\) and \(k\).
Computational hardness

- The problem of computing the Betti numbers of semi-algebraic sets in general is a PSPACE-hard problem. The same is true for semi-algebraic sets defined by many quadratic inequalities.
- On the other hand it was known before that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of quadratic inequalities is solvable in polynomial time.
- We have shown that the problem of computing the Betti numbers of semi-algebraic sets defined by a constant number of polynomial inequalities is solvable in polynomial time, even if we allow a small (constant sized) subset of the variables to occur with degrees larger than two in the polynomials defining the given set.
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