Efficient Algorithms for Computing Betti Numbers of Semi-algebraic Sets

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Outline

Statement of the problem

- Let R be a real closed field and $S \subset \mathrm{R}^k$ a semi-algebraic set defined by a quantifier-free Boolean formula with atoms of the form $P > 0, P < 0, P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$. We call S a $\mathcal{P}$-semi-algebraic set. If, instead, the Boolean formula has atoms of the form $P = 0, P \geq 0, P \leq 0,\ P \in \mathcal{P}$, and additionally contains no negation, then we will call S a $\mathcal{P}$-closed semi-algebraic set.
- The sum of the Betti numbers of S is bounded by $O(sd)^k$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.
- Even though the Betti numbers of S are bounded singly exponentially in k, there is no singly exponential algorithm for computing the Betti numbers of S.

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Previous Work

- Doubly exponential algorithms (with complexity $(sd)^{2^{O(k)}}$) for computing all the Betti numbers are known, since it is possible to obtain a triangulation of S in doubly exponential time using cylindrical algebraic decomposition (Collins,Schwartz-Sharir).
- Algorithms with single exponential complexity are known only for the problems of testing emptiness, computing the zero-th Betti number (i.e. the number of semi-algebraically connected components of S) the Euler-Poincaré characteristic of S, as well as the dimension of S.

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The Critical Point Method

- Using infinitesimal perturbation to deform a given algebraic set to a basic, closed semi-algebraic set with smooth boundary, which has the same homotopy type.
- Compute the critical points (as well as the index of the Hessian at these points) efficiently, making use of the particular structure of the deformation.
- Use Morse theory to compute the Euler-Poincaré characteristic.

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Morse Lemma A

Lemma (Morse lemma A)

Let $[a, b]$ be an interval containing no critical value of π. Then $\mathrm{Z}(Q, \mathbb{R}^k)_{[a,b]}$ is homeomorphic to $\mathrm{Z}(Q, \mathbb{R}^k)_a \times [a, b]$ and $\mathrm{Z}(Q, \mathbb{R}^k)_{\leq a}$ is homotopy equivalent to $\mathrm{Z}(Q, \mathbb{R}^k)_{\leq b}$.

Morse Lemma B

Lemma (Morse lemma B)

Let $\mathrm{Z}(Q, \mathbb{R}^k)$ *be a non-singular bounded algebraic hypersurface such that the projection* π *to the* X_1*-axis is a Morse function. Let* p *be a non-degenerate critical point of* π *of index* λ *and such that* $\pi(p) = c$*.*
Then, for all sufficiently small $\epsilon > 0$*, the set* $\mathrm{Z}(Q, \mathbb{R}^k)_{\leq c+\epsilon}$ *has the homotopy type of the union of* $\mathrm{Z}(Q, \mathbb{R}^k)_{\leq c-\epsilon}$ *with a ball of dimension* $k - 1 - \lambda$*, attached along its boundary .*

Example of the torus

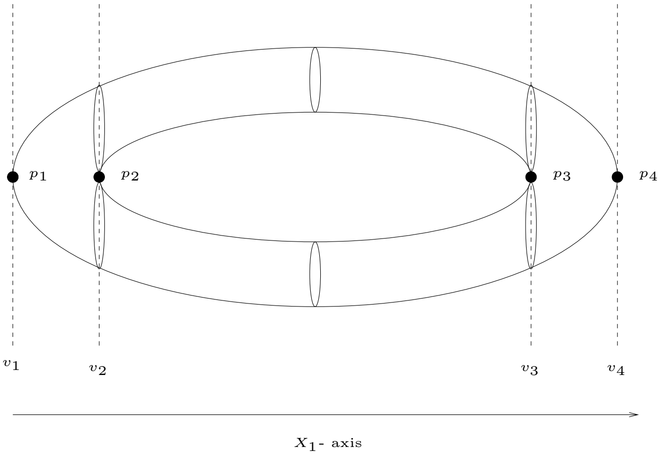

Figure: Critical values for the smooth torus in $\mathbb{R}^3$.

Computing the Euler-Poincaré characteristic of a bounded algebraic set

- Let $Q \in \mathrm{D}[X_1, \ldots, X_k]$, $\mathrm{Z}(Q, \mathrm{R}^k) \subset B(0, 1/c)$ for some $0 < c \leq 1, c \in \mathrm{D}$.
-

$$G_k(d, c) = c^d(X_1^d + \cdots + X_k^d + X_2^2 + \cdots + X_k^2) - (2k-1),$$

$$\mathrm{Def}(Q^2, d, c, \zeta) = \zeta G_k(d, c) + (1-\zeta)Q^2,$$

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Computing the Euler-Poincaré characteristic of a bounded algebraic set (cont.)

$$\mathrm{Cr}(Q^2, d, c, \zeta) =$$
$$\left\{\mathrm{Def}(Q^2, d, c, \zeta), \frac{\partial \mathrm{Def}(Q^2, d, c, \zeta)}{\partial X_2}, \ldots, \frac{\partial \mathrm{Def}(Q^2, d, c, \zeta)}{\partial X_k}\right\},$$
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$\mathrm{Cr}(Q^2, d, c, \zeta)$ as well as $\mathrm{Cr}_+(Q^2, d, c, \zeta)$ are both (very nearly) Grobner bases.

Algorithm for a bounded algebraic set

Algorithm (Euler-Poincaré Characteristic of a Bounded Algebraic Set)

Input : $Q \in \mathrm{D}[X_1, \ldots, X_k]$ *with* $\mathrm{Z}(Q, \mathrm{R}^k) \subset B(0, 1/c)$.

Output : $\chi(\mathrm{Z}(Q, \mathrm{R}^k))$.

Procedure : *Compute the characteristic polynomial of the matrices*

$$H_1 = \left[\frac{\partial^2 \mathrm{Def}(Q^2, \zeta, d, c)}{\partial X_i \partial X_j}\right]_{2 \le i,j \le k}$$

$$H_2 = \left[\frac{\partial^2 \mathrm{Def}_+(Q^2, d, c, \zeta)}{\partial X_i \partial X_j}\right]_{2 \le i,j \le k+1}.$$

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Algorithm for a bounded algebraic set

Compute the signature $\sigma(H_1)$ (respectively $\sigma(H_2)$), of the matrix H_1 (respectively H_2) at the real roots of $\mathrm{Cr}(Q^2, d, c, \zeta)$ (respectively $\mathrm{Cr}_+(Q^2, d, c, \zeta)$).
For i from 0 to $k-1$ let,

$$\ell_i := \#\{x \in \mathrm{Z}(\mathrm{Cr}(Q^2, d, c, \zeta), \mathrm{C}\langle\zeta\rangle^k) \mid \frac{k-1+\sigma(H_1(x))}{2} = i\}.$$

For i from 0 to k, let

$$m_i := \#\{x \in \mathrm{Z}(\mathrm{Cr}_+(Q^2, d, c, \zeta), \mathrm{C}\langle\zeta\rangle^{k+1}) \mid \frac{k+\sigma(H_2(x))}{2} = i\}.$$

Output

$$\chi(\mathrm{Z}(Q, \mathrm{R}^k)) = \frac{1}{2}\left(\sum_{i=0}^{k-1}(-1)^{k-1-i}\ell_i + \sum_{i=0}^{k}(-1)^{k-i}m_i\right).$$

Algorithm for a general algebraic set

Algorithm (Euler-Poincaré Characteristic of an Algebraic Set)

Input : $Q \in \mathrm{D}[X_1, \ldots, X_k]$.

Output : $\chi(\mathrm{Z}(Q, \mathrm{R}^k))$.

Procedure : *Let*

$$Q_1 = Q^2 + (\varepsilon^2(X_1^2 + \ldots + X_k^2) - 1)^2.$$

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Algorithm (Euler-Poincaré Characteristic of an Algebraic Set)

Using previous Algorithm (Euler-Poincaré Characteristic of a Bounded Algebraic Set) compute $\chi(\mathrm{Z}(Q_1, \mathrm{R}\langle\varepsilon\rangle^k))$ *and* $\chi(\mathrm{Z}(Q_2, \mathrm{R}\langle\varepsilon\rangle^{k+1}))$.

Output,

$$\chi(\mathrm{Z}(Q, \mathrm{R}^k)) = \frac{1}{2}(\chi(\mathrm{Z}(Q_2, \mathrm{R}\langle\varepsilon\rangle^{k+1})) - \chi(\mathrm{Z}(Q_1, \mathrm{R}\langle\varepsilon\rangle^k))).$$

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Outline

Euler-Poincaré Queries

Given S a locally closed semi-algebraic set contained in Z, we denote by $\chi(S)$ the Euler-Poincaré characteristic of S. Given $P \in \mathrm{R}[X_1, \ldots, X_k]$, we denote

$$\mathcal{R}(P = 0, S) = \{x \in S \mid P(x) = 0\},$$
$$\mathcal{R}(P > 0, S) = \{x \in S \mid P(x) > 0\},$$
$$\mathcal{R}(P > 0, S) = \{x \in S \mid P(x) < 0\},$$

and $\chi(P = 0, S), \chi(P > 0, S), \chi(P < 0, S)$ the Euler-Poincaré characteristics of the corresponding sets The Euler-Poincaré-query of P for S is

$$\mathrm{EQ}(P, S) = \chi(P > 0, S) - \chi(P < 0, S).$$

Euler-Poincaré of sign conditions

Let $\mathcal{P} = P_1, \ldots, P_s$ be a finite list of polynomials in $\mathrm{R}[X_1, \ldots, X_k]$.
Let $Q \in \mathrm{R}[X_1, \ldots, X_k]$, $Z = \mathrm{Z}(Q, \mathrm{R}^k)$. We denote as usual by $\mathrm{Sign}(\mathcal{P}, Z)$ the list of $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ such that $\mathcal{R}(\sigma, Z)$ is non-empty. We denote by $\chi(\mathcal{P}, Z)$ the list of Euler-Poincaré characteristics $\chi(\sigma, Z) = \chi(\mathcal{R}(\sigma, Z))$ for $\sigma \in \mathrm{Sign}(\mathcal{P}, Z)$.

Algorithm for Computing Euler-Poincaré of sign conditions

- Same as that of B-K-R sign determination algorithm, except we we compute Euler-Poincaré characteristics of realizations of sign conditions rather than cardinalities of sign conditions on a finite set, using the notion of Euler-Poincaré-query rather than that of Sturm-query.
- Complexity:

$$s^{k'+1}O(d)^k + s^{k'}((k' \log_2(s) + k \log_2(d))d)^{O(k)}.$$

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What is a roadmap ?

Let S be a semi-algebraic set. We denote by π the projection on the X_1-axis and set

$$S_x = \{y \in \mathrm{R}^{k-1} \mid (x, y) \in S\}.$$

A **roadmap** for S is a semi-algebraic set M of dimension at most one contained in S which satisfies the following roadmap conditions:

RM_1 For every semi-algebraically connected component D of S, $D \cap M$ is semi-algebraically connected.

RM_2 For every $x \in \mathrm{R}$ and for every semi-algebraically connected component D' of S_x, $D' \cap M \neq \emptyset$.

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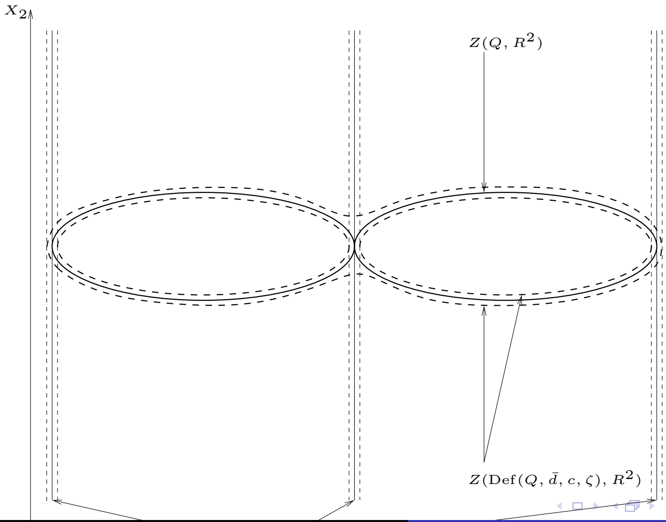
Pseudo-critical values

Definition

*An X_1-**pseudo-critical point** on $\mathrm{Z}(Q, \mathrm{R}^k)$ is the $\lim_\zeta$ of an X_1-critical point on $\mathrm{Z}(\mathrm{Def}(Q, \bar{d}, c, \zeta), \mathrm{R}\langle\zeta\rangle^k)$.*
*An X_1-**pseudo-critical value** on $\mathrm{Z}(Q, \mathrm{R}^k)$ is the projection to the X_1-axis of an X_1-pseudo-critical point on $\mathrm{Z}(Q, \mathrm{R}^k)$.*

Pseudo-critical values



Proposition

Let $\mathrm{Z}(Q,\mathrm{R}^k)$ *be a bounded algebraic set and* S *a semi-algebraically connected component of* $\mathrm{Z}(Q,\mathrm{R}^k)_{[a,b]}$*. If* $v \in (a,b)$ *and* $[a,b] \setminus \{v\}$ *contains no* X_1*-pseudo-critical value on* $\mathrm{Z}(Q,\mathrm{R}^k)$*, then* S_v *is semi-algebraically connected.*

Proposition

Let $\mathrm{Z}(Q, \mathrm{R}^k)$ be a bounded algebraic set and let S be a semi-algebraically connected component of $\mathrm{Z}(Q, \mathrm{R}^k)_{[a,b]}$. If $S_{[a,b)}$ is not semi-algebraically connected, then b is an X_1-pseudo-critical value of $\mathrm{Z}(Q, \mathrm{R}^k)$.

Outline

Roadmap of a bounded $\mathrm{Z}(Q, \mathrm{R}^k)$ containing finite $\mathcal{N} \subset \mathrm{Z}(Q, \mathrm{R}^k)$.

- We first construct X_2-pseudo-critical points on $\mathrm{Z}(Q, \mathrm{R}^k)$ in a parametric way along the X_1-axis. This results in curve segments and their endpoints on $\mathrm{Z}(Q, \mathrm{R}^k)$.
- Since these curves and their endpoints include, for every $x \in \mathrm{R}$, the X_2-pseudo-critical points of $\mathrm{Z}(Q, \mathrm{R}^k)_x$, they meet every connected component of $\mathrm{Z}(Q, \mathrm{R}^k)_x$. Thus the set of curve segments and their endpoints already satisfy RM_2.
- We add additional curve segments to ensure that M is connected by recursing in certain distinguished hyperplanes defined by $X_1 = z$ for distinguished values z.

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Roadmap of a bounded $\mathrm{Z}(Q, \mathrm{R}^k)$

- The set of **distinguished values** is the union of the X_1-pseudo-critical values, the first coordinates of the input points $\mathcal{N}$ and the first coordinates of the endpoints of the curve segments. The input points, the endpoints of the curve segments and the intersections of the curve segments with the distinguished hyperplanes define the set of **distinguished points** .
- We then repeat this construction in each each distinguished hyperplane H_i defined by $X_1 = v_i$ with input $Q(v_i, X_2, \ldots, X_k)$ and the distinguished points in $\mathcal{N}_i$. The process is iterated until for

$$I = (i_1, \ldots, i_{k-2}), 1 \leq i_1 \leq \ell, \ldots, 1 \leq i_{k-2} \leq \ell(i_1, \ldots, i_{k-3}),$$

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Complexity Analysis and open problem

- $$T(d,k) = d^{O(k)} + d^{O(k)} T(d,k-1)$$
  $$= d^{O(k^2)}.$$
- Question: Can this be improved to $d^{O(k)}$?
- Remark: Using Crofton's formula from integral geometry this also shows that any two points in a connected component of a real algebraic variety of defined by polynomials of degree d contained in the unit ball can be connected by a semi-algebraic path of length $d^{O(k^2)}$. Can this be improved to $d^{O(k)}$ also ?

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Complexity Analysis and open problem

- $$\begin{aligned} T(d,k) &= d^{O(k)} + d^{O(k)} T(d,k-1) \\ &= d^{O(k^2)}. \end{aligned}$$
- Question: Can this be improved to $d^{O(k)}$?
- Remark: Using Crofton's formula from integral geometry this also shows that any two points in a connected component of a real algebraic variety of defined by polynomials of degree d contained in the unit ball can be connected by a semi-algebraic path of length $d^{O(k^2)}$. Can this be improved to $d^{O(k)}$ also ?