Efficient Algorithms for Computing Betti Numbers of Semi-algebraic Sets :3

Saugata Basu

School of Mathematics
Georgia Tech

Nov 24, 2005/ Minicourse (Advanced)/ IHP
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
There exists an algorithm that takes as input the description of a \( \mathcal{P} \)-closed semi-algebraic set \( S \subset \mathbb{R}^k \), and outputs a covering of \( S \) by a family of subsets of \( S \) which are closed and contractible. The complexity of the algorithm, as well as the complexity of the covering, is

\[(sd)^{kO(1)},\]

where \( s = \#(\mathcal{P}) \) and \( d = \max_{P \in \mathcal{P}} \deg(P) \).
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
Theorem

There exists an algorithm that takes as input the description of a $\mathcal{P}$-semi-algebraic set $S \subset \mathbb{R}^k$, and outputs $b_0(S), \ldots, b_\ell(S)$. The complexity of the algorithm is

$$(sd')^{kO(\ell)}$$

where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. 

Saugata Basu
Efficient Algorithms for Computing Betti Numbers
Main Ingredients

- The first ingredient is a result discussed in the previous lecture, which enables us to compute a singly exponential sized covering of the given closed and bounded semi-algebraic set, consisting of closed, contractible semi-algebraic sets, in single exponential time. The number and the degrees of the polynomials used to define the sets in this covering are also bounded singly exponentially.

- The second ingredient is an algorithm which uses the covering algorithm recursively and computes in singly exponential time a complex whose homology groups are isomorphic to the first $\ell$ homology groups of the input set. This complex is of singly exponential size for fixed $\ell$. 
Main Ingredients

- The first ingredient is a result discussed in the previous lecture, which enables us to compute a singly exponential sized covering of the given closed and bounded semi-algebraic set, consisting of closed, contractible semi-algebraic sets, in single exponential time. The number and the degrees of the polynomials used to define the sets in this covering are also bounded singly exponentially.

- The second ingredient is an algorithm which uses the covering algorithm recursively and computes in singly exponential time a complex whose homology groups are isomorphic to the first $\ell$ homology groups of the input set. This complex is of singly exponential size for fixed $\ell$. 
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex $A$ such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $R$-vector space of $i$ co-chains of $A$, and $C^*(A) = \bigoplus_i C^i(A)$.

We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

The following sequence of homomorphisms is exact.

\[
0 \longrightarrow C^*(A) \overset{\iota}{\longrightarrow} \prod_{\alpha_0} C^*(A_{\alpha_0}) \overset{\delta}{\longrightarrow} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1}) \rightarrow \cdots
\]

\[
\cdots \overset{\delta}{\longrightarrow} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \overset{\delta}{\longrightarrow} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0, \ldots, \alpha_{p+1}}) \rightarrow \cdots
\]
Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex $A$ such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $R$-vector space of $i$ co-chains of $A$, and $C^*(A) = \bigoplus_i C^i(A)$.

We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

The following sequence of homomorphisms is exact.

$$
\begin{align*}
0 & \longrightarrow C^*(A) \overset{\iota}{\longrightarrow} \prod_{\alpha_0} C^*(A_{\alpha_0}) \overset{\delta}{\longrightarrow} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1}) \\
& \quad \vdots \delta \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \cdots \delta \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0, \ldots, \alpha_{p+1}}) \cdots
\end{align*}
$$
Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex $A$ such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of $i$ co-chains of $A$, and $C^*(A) = \bigoplus_i C^i(A)$.

We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

The following sequence of homomorphisms is exact.

$$0 \rightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0, \ldots, \alpha_{p+1}}) \cdots$$
Mayer-Vietoris Double Complex

\[ \begin{array}{ccccccccc}
0 & \rightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1} C^3(A_{\alpha_0, \alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1 < \alpha_2} C^3(A_{\alpha_0, \alpha_1, \alpha_2}) & \xrightarrow{\delta} & \cdots \\
& & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \rightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1} C^2(A_{\alpha_0, \alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1 < \alpha_2} C^2(A_{\alpha_0, \alpha_1, \alpha_2}) & \xrightarrow{\delta} & \cdots \\
& & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \rightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1} C^1(A_{\alpha_0, \alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1 < \alpha_2} C^1(A_{\alpha_0, \alpha_1, \alpha_2}) & \xrightarrow{\delta} & \cdots \\
& & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \rightarrow & \prod_{\alpha_0} C^0(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1} C^0(A_{\alpha_0, \alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0 < \alpha_1 < \alpha_2} C^0(A_{\alpha_0, \alpha_1, \alpha_2}) & \xrightarrow{\delta} & \cdots \\
& & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \rightarrow & & & & & & \\
\end{array} \]
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
Double Complex

\[ C^{0,2} \xrightarrow{\delta} C^{1,2} \xrightarrow{\delta} C^{2,2} \xrightarrow{\delta} \cdots \]

\[ d \quad \delta \quad d \quad \delta \quad d \quad \delta \quad \cdots \]

\[ C^{0,1} \xrightarrow{\delta} C^{1,1} \xrightarrow{\delta} C^{2,1} \xrightarrow{\delta} \cdots \]

\[ d \quad \delta \quad d \quad \delta \quad d \quad \delta \quad \cdots \]

\[ C^{0,0} \xrightarrow{\delta} C^{1,0} \xrightarrow{\delta} C^{2,0} \xrightarrow{\delta} \cdots \]
The Associated Total Complex

\[ \begin{array}{c}
\vdots \\
\delta C^{p-1,q+1} \to C^p,q+1 \\
\delta \\
\vdots \\
\delta C^{p,q} \to C^{p+1,q} \\
\delta \\
\vdots \\
\delta C^{p-1,q-1} \to C^p,q-1 \\
\delta \\
\vdots \\
\delta C^{p-1,q-1} \to C^p,q-1 \\
\delta \\
\vdots \\
\delta C^{p,q-1} \to C^{p+1,q-1} \\
\delta \\
\vdots \\
\delta C^{p,q-1} \to C^{p+1,q-1} \\
\delta \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]
Spectral Sequences of a Double Complex

- A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
- a sequence of bi-graded vector spaces and differentials $(E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})$,
- $E_{r+1} = H(E_r, d_r)$,
- $E_\infty = H^*(\text{Associated Total Complex})$. 
Spectral Sequences of a Double Complex

- A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
- a sequence of bi-graded vector spaces and differentials \((E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})\),
- \(E_{r+1} = H(E_r, d_r)\),
- \(E_\infty = H^*(\text{Associated Total Complex})\).
Spectral Sequences of a Double Complex

- A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
- a sequence of bi-graded vector spaces and differentials 
  \((E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})\),
- \(E_{r+1} = H(E_r, d_r)\),
- \(E_\infty = H^*(\text{Associated Total Complex})\).
A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,

- a sequence of bi-graded vector spaces and differentials 
  \((E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})\),
- \(E_{r+1} = H(E_r, d_r)\),
- \(E_\infty = H^*(\text{Associated Total Complex})\).
Introduction
Algebraic Topological Preliminaries
Double complexes associated to certain coverings

Generalized Mayer-Vietoris Sequence
Double Complexes and Spectral Sequences
Mayer-Vietoris Spectral Sequence

Spectral Sequence

\[ E_1^{p,q}, d_1, d_2, d_3, \ldots \]

Figure: The differentials \( d_r \) in the spectral sequence \( (E_r^{p,q}, d_r) \)

Saugata Basu
Efficient Algorithms for Computing Betti Numbers
There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H^*_D(\mathcal{M})$. The first terms of these are:

$\prime E_1 = H_\delta(\mathcal{M}), \quad E_2 = H_d H_\delta(\mathcal{M})$

$\prime\prime E_1 = H_d(\mathcal{M}), \quad E_2 = H_\delta H_d(\mathcal{M})$
There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H^*_D(\mathcal{M})$. The first terms of these are:

\['E_1 = H_\delta(\mathcal{M}),' \ E_2 = H_dH_\delta(\mathcal{M})\]

\"E_1 = H_d(\mathcal{M})," \ E_2 = H_\delta H_d(\mathcal{M})\
There are two spectral sequences associated with $M^{p,q}$ both converging to $H^*_D(M)$. The first terms of these are:

$'E_1 = H_\delta(M),'$ $E_2 = H_dH_\delta(M)$

$"E_1 = H_d(M),"$ $E_2 = H_\delta H_d(M)$
Homomorphisms of Double Complexes

Given two (first quadrant) double complexes, $C^{\bullet,\bullet}$ and $\overline{C}^{\bullet,\bullet}$, a homomorphism of double complexes,

$$\phi : C^{\bullet,\bullet} \rightarrow \overline{C}^{\bullet,\bullet},$$

is a collection of homomorphisms, $\phi^{p,q} : C^{p,q} \rightarrow \overline{C}^{p,q}$, such that the following diagrams commute.

\[
\begin{array}{cccc}
C^{p,q} & \overset{\delta}{\rightarrow} & C^{p+1,q} \\
\downarrow{\phi^{p,q}} & & \downarrow{\phi^{p+1,q}} \\
\overline{C}^{p,q} & \overset{\delta}{\rightarrow} & \overline{C}^{p+1,q} \\
\end{array}
\]

\[
\begin{array}{cccc}
C^{p,q} & \overset{d}{\rightarrow} & C^{p,q+1} \\
\downarrow{\phi^{p,q}} & & \downarrow{\phi^{p,q+1}} \\
\overline{C}^{p,q} & \overset{d}{\rightarrow} & \overline{C}^{p,q+1} \\
\end{array}
\]
A homomorphism of double complexes,
\[ \phi : C^{\bullet,\bullet} \rightarrow \bar{C}^{\bullet,\bullet}, \]
induces homomorphisms, \( '\phi_s : 'E_s \rightarrow '\bar{E}_s \) (respectively, \( ''\phi_s : ''E_s \rightarrow ''\bar{E}_s \)) between the associated spectral sequences. If \( '\phi_s \) (respectively, \( ''\phi_s) \) is an isomorphism for some \( s \geq 1 \), then \( 'E_r^{p,q} \) and \( '\bar{E}_r^{p,q} \) (respectively, \( ''E_r^{p,q} \) and \( ''\bar{E}_r^{p,q} \)) are isomorphic for all \( r \geq s \). In particular, the induced homomorphism,
\[ \phi : \text{Tot}^\bullet(C^{\bullet,\bullet}) \rightarrow \text{Tot}^\bullet(\bar{C}^{\bullet,\bullet}) \]
is a quasi-isomorphism.
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
<table>
<thead>
<tr>
<th>Degree</th>
<th>$C^3(A)$</th>
<th>$C^2(A)$</th>
<th>$C^1(A)$</th>
<th>$C^0(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ E_2^{p,q} = \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  H^3(A) & 0 & 0 \\
  H^2(A) & 0 & 0 \\
  H^1(A) & 0 & 0 \\
  H^0(A) & 0 & 0 \\
\end{array} \]
<table>
<thead>
<tr>
<th>[E_1]</th>
<th>[\ldots]</th>
<th>[\ldots]</th>
<th>[\ldots]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\prod_{\alpha_0} H^3(A_{\alpha_0}))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1} H^3(A_{\alpha_0}, \alpha_1))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1 &lt; \alpha_2} H^3(A_{\alpha_0}, \alpha_1, \alpha_2))</td>
<td></td>
</tr>
<tr>
<td>(\prod_{\alpha_0} H^2(A_{\alpha_0}))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1} H^2(A_{\alpha_0}, \alpha_1))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1 &lt; \alpha_2} H^2(A_{\alpha_0}, \alpha_1, \alpha_2))</td>
<td></td>
</tr>
<tr>
<td>(\prod_{\alpha_0} H^1(A_{\alpha_0}))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1} H^1(A_{\alpha_0}, \alpha_1))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1 &lt; \alpha_2} H^1(A_{\alpha_0}, \alpha_1, \alpha_2))</td>
<td></td>
</tr>
<tr>
<td>(\prod_{\alpha_0} H^0(A_{\alpha_0}))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1} H^0(A_{\alpha_0}, \alpha_1))</td>
<td>(\prod_{\alpha_0 &lt; \alpha_1 &lt; \alpha_2} H^0(A_{\alpha_0}, \alpha_1, \alpha_2))</td>
<td></td>
</tr>
</tbody>
</table>
"$E_1$ in this case

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,\alpha_1,\alpha_2}) \\
0 & \prod_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0,\alpha_1,\alpha_2}) \\
0 & \prod_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0,\alpha_1,\alpha_2}) \\
\prod_{\alpha_0} H^0(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0,\alpha_1,\alpha_2}) \\
\end{array}
\]
The following proposition is classical and follows from the exactness of the generalized Mayer-Vietoris sequence.

**Proposition**

The spectral sequences, $E_r$, $E_r$, associated to $\mathcal{N}^{\bullet,\bullet}(A)$ converge to $H^\ast(A, \mathbb{Q})$ and thus,

$$H^\ast(\text{Tot}^{\bullet}(\mathcal{N}^{\bullet,\bullet}(A))) \cong H^\ast(A, \mathbb{Q}).$$

Moreover, the homomorphism $\psi : C^{\bullet}(A) \to \text{Tot}^{\bullet}(\mathcal{N}^{\bullet,\bullet}(A))$ induced by the homomorphism $r$ (in the generalized Mayer-Vietoris sequence) is a quasi-isomorphism.
Consider a fixed family of polynomials, $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, as well as a fixed $\mathcal{P}$-closed and bounded semi-algebraic set, $S \subset \mathbb{R}^k$. We also fix a number, $\ell$, $0 \leq \ell \leq k$.

We identify certain closed and bounded semi-algebraic subsets of $S$ (which we call the admissible subsets of $S$). We associate to each admissible subset $X \subset S$, its level denoted $\text{level}(X)$, with $\text{level}(S) = 0$.

For each such admissible subset, $X \subset S$, we define a double complex, $\mathcal{M}^{\bullet, \bullet}(X)$, such that

$$H^i(\text{Tot}^{\bullet}(\mathcal{M}^{\bullet, \bullet}(X))) \cong H^i(X, \mathbb{Q}), \ 0 \leq i \leq \ell - \text{level}(X).$$
Admissible Subsets

Consider a fixed family of polynomials, \( \mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k] \), as well as a fixed \( \mathcal{P} \)-closed and bounded semi-algebraic set, \( S \subset \mathbb{R}^k \). We also fix a number, \( \ell, 0 \leq \ell \leq k \).

We identify certain closed and bounded semi-algebraic subsets of \( S \) (which we call the admissible subsets of \( S \)). We associate to each admissible subset \( X \subset S \), its level denoted \( \text{level}(X) \), with \( \text{level}(S) = 0 \).

For each such admissible subset, \( X \subset S \), we define a double complex, \( \mathcal{M}^{\bullet, \bullet}(X) \), such that

\[
H^i(\text{Tot}^\bullet(\mathcal{M}^{\bullet, \bullet}(X))) \cong H^i(X, \mathbb{Q}), \quad 0 \leq i \leq \ell - \text{level}(X).
\]
Admissible Subsets

Consider a fixed family of polynomials, \( \mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k] \), as well as a fixed \( \mathcal{P} \)-closed and bounded semi-algebraic set, \( S \subset \mathbb{R}^k \). We also fix a number, \( \ell, 0 \leq \ell \leq k \).

We identify certain closed and bounded semi-algebraic subsets of \( S \) (which we call the admissible subsets of \( S \)). We associate to each admissible subset \( X \subset S \), its level denoted \( \text{level}(X) \), with \( \text{level}(S) = 0 \).

For each such admissible subset, \( X \subset S \), we define a double complex, \( \mathcal{M}^{\bullet, \bullet}(X) \), such that

\[
H^i(\operatorname{Tot}^{\bullet}(\mathcal{M}^{\bullet, \bullet}(X))) \cong H^i(X, \mathbb{Q}), \quad 0 \leq i \leq \ell - \text{level}(X).
\]
Main Idea

- If the sets occurring in the covering of $X$ are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row.
- In order to compute $b_0(X), \ldots, b_{\ell_{\text{level}}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex.
- However, we do not know how to efficiently compute (even the truncated) Mayer-Vietoris double complex.
- However, making use of the covering construction recursively, we are able to compute another double complex, $M^{\bullet, \bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex.
Main Idea

If the sets occurring in the covering of $X$ are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row.

In order to compute $b_0(X), \ldots, b_{\ell-\text{level}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex.

However, we do not know how to efficiently compute (even the truncated) Mayer-Vietoris double complex.

However, making use of the covering construction recursively, we are able to compute another double complex, $\mathcal{M}^{\bullet, \bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex.
Main Idea

- If the sets occurring in the covering of $X$ are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row.

- In order to compute $b_0(X), \ldots, b_{\ell-\text{level}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex.

- However, we do not know how to efficiently compute (even the truncated) Mayer-Vietoris double complex.

- However, making use of the covering construction recursively, we are able to compute another double complex, $M^{\bullet, \bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex.
Main Idea

- If the sets occurring in the covering of $X$ are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row.

- In order to compute $b_0(X), \ldots, b_{\ell-\text{level}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex.

- However, we do not know how to efficiently compute (even the truncated) Mayer-Vietoris double complex.

- However, making use of the covering construction recursively, we are able to compute another double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex.
Admissible Sets

Given any closed and bounded semi-algebraic set $X \subset \mathbb{R}^k$, we will denote by $\mathcal{C}'(X)$, a fixed covering of $X$ by a finite family of closed, bounded and acyclic semi-algebraic sets.

We have that, $V \subset X$ for each $V \in \mathcal{C}'(X)$ and $X = \bigcup_{V \in \mathcal{C}'(X)} V$. We will index the sets in $\mathcal{C}'(X)$ as $V_1, \ldots, V_{n_X}$ where $n_X = \# \mathcal{C}'(X)$, and for $1 \leq \alpha_0 < \cdots < \alpha_p \leq n_X$, we will denote $V_{\alpha_0, \ldots, \alpha_p} = \bigcap_{0 \leq i \leq p} V_{\alpha_i}$. For $I \subset J \subset \{1, \ldots, n_X\}$ we will call $V_I$ an ancestor of $V_J$ and $X$ an ancestor of all the $V_I$'s. We will transitively close the ancestor relation.
Admissible Sets

- Given any closed and bounded semi-algebraic set $X \subset \mathbb{R}^k$, we will denote by $C'(X)$, a fixed covering of $X$ by a finite family of closed, bounded and acyclic semi-algebraic sets.

- We have that, $V \subset X$ for each $V \in C'(X)$ and $X = \bigcup_{V \in C'(X)} V$. We will index the sets in $C'(X)$ as $V_1, \ldots, V_{n_X}$ where $n_X = \#C'(X)$, and for $1 \leq \alpha_0 < \cdots < \alpha_p \leq n_X$, we will denote $V_{\alpha_0, \ldots, \alpha_p} = \bigcap_{0 \leq i \leq p} V_{\alpha_i}$. For $I \subset J \subset \{1, \ldots, n_X\}$ we will call $V_I$ an ancestor of $V_J$ and $X$ an ancestor of all the $V_I$’s. We will transitively close the ancestor relation.
We now associate to certain closed semi-algebraic subsets $X$ of $S$ (which we call the admissible subsets of $S$), a covering, $\mathcal{C}(X)$, of $X$ by closed, bounded, acyclic semi-algebraic sets, obtained by enlarging the covering $\mathcal{C}'(X)$.

The set $S$ itself is admissible of level 0 and $\mathcal{C}(S) = \mathcal{C}'(S)$. All intersections of the sets in $\mathcal{C}(S)$ taken upto $\ell + 2$ at a time are admissible and have level 1.
We now associate to certain closed semi-algebraic subsets $X$ of $S$ (which we call the admissible subsets of $S$), a covering, $\mathcal{C}(X)$, of $X$ by closed, bounded, acyclic semi-algebraic sets, obtained by enlarging the covering $\mathcal{C}'(X)$.

The set $S$ itself is admissible of level 0 and $\mathcal{C}(S) = \mathcal{C}'(S)$. All intersections of the sets in $\mathcal{C}(S)$ taken up to $\ell + 2$ at a time are admissible and have level 1.
Admissible Sets (cont.)

The admissible subsets of $S$ are the smallest family of subsets of $S$ containing the above sets and satisfying the following. For any admissible subset $X \subset S$ at level $i$, we define $C(X)$ as follows. Let $\{Y_1, \ldots, Y_N\}$ be the set of admissible sets which are ancestors of $X$. Then,

$$C(X) = \bigcup_{U_i \in C(Y_i), 1 \leq i \leq N} C'(U_1 \cap \cdots \cap U_N \cap X).$$

All intersections of the sets in $C(X)$ taken at most $\ell - i + 2$ at a time are admissible, have level $i + 1$, and have $X$ as an ancestor. For $I \subset J \subset \{1, \ldots, n_X\}$, $V_I$ is an ancestor of $V_J$ and $X$ is an ancestor of all the $V_I$'s. Moreover, for $V \in C'(U_1 \cap \cdots \cap U_N \cap X)$, each $U_i$ is an ancestor of $V$. This clearly implies that each $V \in C(X)$ has a unique ancestor in each $C(Y_i)$ (namely, $U_i$).
We have a procedure (recall last lecture) for computing $C'(X)$, for any given $\mathcal{P}'$-closed and bounded semi-algebraic set, $X$, such that the number and the degrees of the polynomials appearing in the output of this procedure is bounded by \((mD)^{k^{O(1)}}\). where \(#\mathcal{P}' = m\) and \(\text{deg}(P) \leq D\), for \(P \in \mathcal{P}'\).
Proposition

Let $S \subset \mathbb{R}^k$ be a $\mathcal{P}$-closed semi-algebraic set, where $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$ is a family of $s$ polynomials of degree at most $d$. Then the number of admissible sets, the number of polynomials used to define them, the degrees of these polynomials, are all bounded by $(sd)^{kO(\varepsilon)}$.

Proof.

By induction on level($X$).
Proposition

Let $S \subset \mathbb{R}^k$ be a $P$-closed semi-algebraic set, where $P \subset \mathbb{R}[X_1, \ldots, X_k]$ is a family of $s$ polynomials of degree at most $d$. Then the number of admissible sets, the number of polynomials used to define them, the degrees of these polynomials, are all bounded by $(sd)^{kO(\ell)}$.

Proof.

By induction on $\text{level}(X)$. 
Outline

1. Introduction
   - Recall from last lecture
   - Main Result

2. Algebraic Topological Preliminaries
   - Generalized Mayer-Vietoris Sequence
   - Double Complexes and Spectral Sequences
   - Mayer-Vietoris Spectral Sequence

3. Double complexes associated to certain coverings
   - Inductive Construction of a Double Complex
Double complex Associated to an Admissible Set

Given the different coverings described above, we now associate to each admissible set $X \subseteq S$ a double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, satisfying the following:

1. $H^i(\text{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q})$, for $0 \leq i \leq \ell - \text{level}(X)$. (1)

2. For every admissible set $Y$, such that $X$ is an ancestor of $Y$, and $\text{level}(X) = \text{level}(Y)$, a restriction homomorphism: $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \rightarrow \mathcal{M}^{\bullet,\bullet}(Y)$, which induces the restriction homomorphisms between the cohomology groups:

$$r : H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q})$$

for $0 \leq i \leq \ell - \text{level}(X)$ via the isomorphisms in (1).
Double complex Associated to an Admissible Set

Given the different coverings described above, we now associate to each admissible set $X \subset S$ a double complex, $M^{\bullet,\bullet}(X)$, satisfying the following:

1. \[ H^i(\text{Tot}^{\bullet}(M^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q}), \text{ for } 0 \leq i \leq \ell - \text{level}(X). \] (1)

2. For every admissible set $Y$, such that $X$ is an ancestor of $Y$, and $\text{level}(X) = \text{level}(Y)$, a restriction homomorphism: $r_{X,Y} : M^{\bullet,\bullet}(X) \rightarrow M^{\bullet,\bullet}(Y)$, which induces the restriction homomorphisms between the cohomology groups:

\[ r : H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q}) \]

for $0 \leq i \leq \ell - \text{level}(X)$ via the isomorphisms in (1).
We now describe the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ and prove that it has the properties stated above. The double complex $\mathcal{M}^{\bullet,\bullet}(X)$ is constructed inductively using induction on $\text{level}(X)$:

The base case is when $\text{level}(X) = \ell$. In this case the double complex, $\mathcal{M}^{\bullet,\bullet}(X)$ is defined by:

$$
\mathcal{M}^{0,0}(X) = \bigoplus_{U_{\alpha_0} \in c(X)} C^0(U_{\alpha_0}),
$$

$$
\mathcal{M}^{1,0}(X) = \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in c(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0, \alpha_1}),
$$

$$
\mathcal{M}^{p,q}(X) = 0, \text{ if } q > 0 \text{ or } p > 1.
$$

Here $C^0(Y)$ is the $\mathbb{Q}$-vector space of $\mathbb{Q}$ valued locally constant functions on $Y$. 
Diagramatically

\[ \bigoplus_{\alpha_0 \in C(X)} C^0(U_{\alpha_0}) \xrightarrow{\delta} \bigoplus_{\alpha_0, \alpha_1 \in C(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0, \alpha_1}) \rightarrow 0 \]
Definition of the restriction homomorphism

For every admissible set $Y$, such that $X$ is an ancestor of $Y$, and $\text{level}(X) = \text{level}(Y) = \ell$, we define

$$r_{X,Y} : M^{0,0}(X) \to M^{0,0}(Y),$$

as follows. Recall that,

$$M^{0,0}(X) = \bigoplus_{U \in C(X)} C^0(U), \text{ and } M^{0,0}(Y) = \bigoplus_{V \in C(Y)} C^0(V).$$

Also, by definition of $C(Y)$, we have that for each $V \in C(Y)$ there is a unique $U \in C(X)$ (which we will denote by $a(V)$) such that $U$ is an ancestor of $V$.

For $x \in M^{0,0}(X)$ and $V \in C(Y)$ we define,

$$r_{X,Y}(x)_V = x_{a(V)}|_V.$$

We define $r_{X,Y} : M^{1,0}(X) \to M^{1,0}(Y)$, in a similar manner. More precisely, for $x \in M^{0,0}(X)$ and $V, V' \in C(Y)$, we define

$$r_{X,Y}(x)_V, V' = x_{a(V), a(V')}|_{V \cap V'}.$$
The inductive step

In general the $M^{p,q}(X)$ are defined as follows using induction on level$(X)$ and with $n = \ell - \text{level}(X) + 1$.

\[
\begin{align*}
M^{0,0}(X) &= \bigoplus_{\alpha_0 \in \mathcal{C}(X)} C^0(U_{\alpha_0}), \\
M^{0,q}(X) &= 0, \\
M^{p,q}(X) &= \bigoplus_{\alpha_0 < \ldots < \alpha_p, \; \alpha_i \in \mathcal{C}(X)} \text{Tot}^q(M^{\bullet, \bullet}(U_{\alpha_0}, \ldots, \alpha_p)), \quad 0 < p, \; 0 < p + q \leq n, \\
M^{p,q}(X) &= 0, \quad \text{else}.
\end{align*}
\]
Diagrammatically

\[ \begin{array}{c}
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{n-1}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{n-2}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
\vdots \\
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{2}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{1}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{0}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{-1}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{-2}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
\vdots \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{-n}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
d \\
0 \\
\oplus_{\alpha_0 < \alpha_1} \text{Tot}^{0}(M^\bullet, U_{\alpha_0, \alpha_1}) \xrightarrow{\delta} 0 \\
\end{array} \]
Key proposition

Proposition

For each admissible subset \( X \subset S \) the double complex \( M^{\bullet,\bullet}(X) \) satisfies the following properties:

1. \( H^i(\text{Tot}^\bullet(M^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q}) \) for \( 0 \leq i \leq \ell - \text{level}(X) \).

2. For every admissible set \( Y \), such that \( X \) is an ancestor of \( Y \), and \( \text{level}(X) = \text{level}(Y) \), the homomorphism, \( r_{X,Y} : M^{\bullet,\bullet}(X) \to M^{\bullet,\bullet}(Y) \) induces the restriction homomorphisms between the cohomology groups:

\[
r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})
\]

for \( 0 \leq i \leq \ell - \text{level}(X) \) via the isomorphisms in (1).
**Proposition**

For each admissible subset $X \subset S$ the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ satisfies the following properties:

1. $H^i(\text{Tot}^{\bullet}(\mathcal{M}^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q})$ for $0 \leq i \leq \ell - \text{level}(X)$.

2. For every admissible set $Y$, such that $X$ is an ancestor of $Y$, and $\text{level}(X) = \text{level}(Y)$, the homomorphism, $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$ induces the restriction homomorphisms between the cohomology groups:

$$r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$

for $0 \leq i \leq \ell - \text{level}(X)$ via the isomorphisms in (1).
Key proposition

**Proposition**

For each admissible subset $X \subset S$ the double complex $\mathcal{M}^{\bullet, \bullet}(X)$ satisfies the following properties:

1. $H^i(\text{Tot}^\bullet(\mathcal{M}^{\bullet, \bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q})$ for $0 \leq i \leq \ell - \text{level}(X)$.

2. For every admissible set $Y$, such that $X$ is an ancestor of $Y$, and $\text{level}(X) = \text{level}(Y)$, the homomorphism, $r_{X,Y} : \mathcal{M}^{\bullet, \bullet}(X) \to \mathcal{M}^{\bullet, \bullet}(Y)$ induces the restriction homomorphisms between the cohomology groups:

$$r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$

for $0 \leq i \leq \ell - \text{level}(X)$ via the isomorphisms in (1).
Proof.

The proof is by induction on $\text{level}(X)$. If $\text{level}(X) = \ell$ then the proposition is clear. Otherwise, by induction we can assume that the proposition is true for all admissible sets of the form, $U_{\alpha_0,\ldots,\alpha_p}$ with $U_{\alpha_i} \in C(X)$. Thus, the $p$-th column of the complex, $M^{\bullet,\bullet}(X)$, is the direct sum of the complexes,

\[
\begin{align*}
\text{Tot}^{n-p}(M^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \\
\vdots \\
\text{Tot}^1(M^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \\
\text{Tot}^0(M^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))
\end{align*}
\]
Proof (cont.)

Proof.

By induction hypothesis,
\[ H^i(\text{Tot}^\bullet(\mathcal{M}^\bullet,\bullet(U_{\alpha_0},...,\alpha_p))) \cong H^i(U_{\alpha_0},...,\alpha_p). \]
Moreover, the homomorphism,
\[ r_{U_{\alpha_0},...,\alpha_p} : \mathcal{M}^\bullet,\bullet(U_{\alpha_0},...,\alpha_p) \to \mathcal{M}^\bullet,\bullet(U_{\alpha_0},...,\alpha_{p+1}) \]
induces the restriction homomorphisms between the cohomology groups:
\[ r : H^i(U_{\alpha_0},...,\alpha_p, \mathbb{Q}) \to H^i(U_{\alpha_0},...,\alpha_{p+1}, \mathbb{Q}). \]

The proposition follows from comparing the spectral sequence of \( \mathcal{M}^\bullet,\bullet(X) \) with that of the truncated Mayer-Vietoris double complex associated to the covering, \( \mathcal{C}(X) \), of \( X \), which are isomorphic from the \( 'E_1 \) term onwards. \( \square \)