New bounds for Betti numbers of semi-algebraic sets and algorithms for computing them

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Semi-algebraic Sets

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- Subsets of $\mathbb{R}^k$ defined by a formula involving a finite number of polynomial equalities and inequalities.
- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.
Part I

Bounds on the Complexity of Semi-algebraic Sets
Complexity of Semi-algebraic Sets

Uniform bounds on the number of connected components, Betti numbers etc.

In terms of:
Complexity of Semi-algebraic Sets

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The number of polynomials : \( n \) (controls the combinatorial complexity)
Complexity of Semi-algebraic Sets

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In terms of:
The number of polynomials: \( n \) (controls the \textit{combinatorial complexity})
Degree bound: \( d \) (controls the \textit{algebraic complexity})
Complexity of Semi-algebraic Sets

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In terms of:

The number of polynomials : \( n \) (controls the \textit{combinatorial complexity})

Degree bound : \( d \) (controls the \textit{algebraic complexity})

Dimension of the ambient space : \( k \)
Complexity of Semi-algebraic Sets

Uniform bounds on the number of connected components, Betti numbers etc.

In terms of:
The number of polynomials : $n$ (controls the combinatorial complexity)
Degree bound : $d$ (controls the algebraic complexity)
Dimension of the ambient space : $k$
Dimension of the set itself : $k'$
Topological Complexity of Semi-algebraic Sets
An important measure of the topological complexity of a set $S$ are the Betti numbers: $\beta_i(S)$. 
An important measure of the topological complexity of a set $S$ are the Betti numbers. $\beta_i(S')$.

$\beta_i(S')$ is the rank of the $H^i(S')$ (the $i$-th co-homology group of $S$).
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$\beta_0(S) = \text{the number of connected components.}$
An important measure of the topological complexity of a set $S$ are the Betti numbers $\beta_i(S)$.

- $\beta_i(S)$ is the rank of the $H^i(S)$ (the $i$-th co-homology group of $S$).
- $\beta_0(S)$ = the number of connected components.
- $\beta_i(S)$ = the number of $i$-cycles that do not bound.
The Torus in $\mathbb{R}^3$

Let $T$ be the hollow torus.
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Betti Numbers of the Torus
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• $\beta_0(T) = 1$
Betti Numbers of the Torus

- $\beta_0(T) = 1$
- $\beta_1(T) = 2$
Betti Numbers of the Torus

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- $\beta_0(T) = 1$
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- $\beta_i(T) = 0, i > 2.$
Theorem 1. (Oleinik and Petrovsky, Thom, Milnor) Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of $n$ inequalities,

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$$\deg(P_i) \leq d, 1 \leq i \leq n.$$ Then,
Classical Result on the Topology of Semi-algebraic Sets

**Theorem 1.** *(Oleinik and Petrovsky, Thom, Milnor)* Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of $n$ inequalities,

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$\deg(P_i) \leq d, 1 \leq i \leq n$. Then,

$$\sum_i \beta_i(S) = nd(2nd - 1)^{k-1} = O(nd)^k.$$
Tightness

The above bound is actually quite tight. Example: Let

\[ P_i = L_{i,1}^2 \cdots L_{i,[d/2]}^2 - \epsilon, \]

where the \( L_{ij} \)'s are generic linear polynomials and \( \epsilon > 0 \) and sufficiently small. The set \( S \) defined by \( P_1 \geq 0, \ldots, P_n \geq 0 \) has \( \Omega(nd)^k \) connected components and hence \( \beta_0(S) = \Omega(nd)^k \).
What about the higher Betti Numbers?
What about the higher Betti Numbers?

- Cannot construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$. 
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- Cannot construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$.

- The technique used for proving the above result does not help:

  Replace the semi-algebraic set $S$ by another set bounded by a smooth algebraic hypersurface of degree $2nd$ having the same homotopy type as $S$. Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.
Connected component of $S$
**Theorem 2.** (B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of $n$ inequalities,

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$$\deg(P_i) \leq d, 1 \leq i \leq n.$$ contained in a variety $Z(Q)$ of real dimension $k'$, and $\deg(Q) \leq d$.

Then,
**Theorem 2.** (B, 2001) Let \( S \subset \mathbb{R}^k \) be the set defined by the conjunction of \( n \) inequalities,

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P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],
\]

\[
\text{deg}(P_i) \leq d, 1 \leq i \leq n.
\]

contained in a variety \( Z(Q) \) of real dimension \( k' \), and \( \text{deg}(Q) \leq d \). Then,

\[
\beta_i(S) \leq \binom{n}{k' - i} (4d)^k.
\]
The case of the union

**Theorem 3.** (B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the disjunction of $n$ inequalities,

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$$\deg(P_i) \leq d, 1 \leq i \leq n.$$ 

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The case of the union

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$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$\deg(P_i) \leq d, 1 \leq i \leq n$. Then,

$$\beta_i(S) \leq \left( \binom{n}{i+1} \right) (4d)^k.$$
Sets defined by Quadratic Inequalities
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Theorem 4. (B, 2001) Let \( \ell \) be any fixed number and let \( S \subset \mathbb{R}^k \) be defined by \( P_1 \geq 0, \ldots, P_n \geq 0 \) with \( \deg(P_i) \leq 2 \). Then,
Sets defined by Quadratic Inequalities

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\[
\beta_{k-\ell}(S) \leq \binom{n}{\ell} k^{O(\ell)}.
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**Example:** \( X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0 \).
Betti Numbers of Sign Patterns I
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- Let $Q$ and $P$ be finite subsets of $\mathbb{R}[X_1, \ldots, X_k]$. A sign condition on $P$ is an element of $\{0, 1, -1\}^P$. 
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Let $b_i(\sigma)$ denote the $i$-th Betti number of the realization of $\sigma$, and let $b_i(Q, P) = \sum_\sigma b_i(\sigma)$.
Betti Numbers of Sign Patterns II
Let $b_i(d, k, k', n)$ be the maximum of $b_i(Q, P)$ over all $Q, P$ where $Q$ and $P$ are finite subsets of $\mathbb{R}[X_1, \ldots, X_k]$, whose elements have degree at most $d$, $\#(P) = n$ and the algebraic set $Z(Q)$ has dimension $k'$. 
Let $b_i(d, k, k', n)$ be the maximum of $b_i(Q, P)$ over all $Q, P$ where $Q$ and $P$ are finite subsets of $\mathbb{R}[X_1, \ldots, X_k]$, whose elements have degree at most $d$, $\#(P) = n$ and the algebraic set $Z(Q)$ has dimension $k'$.

Previously known (B, Pollack, Roy (1995))

$$b_0(d, k, k', n) = \binom{4n}{k'} d(2d - 1)^{k-1} = \binom{n}{k'} O(d)^k.$$
Theorem 5. (B, Pollack, Roy, 2002)

\[ b_i(d, k, k', n) \leq \sum_{0 \leq j \leq k' - i} \binom{n}{j} 4^j d (2d - 1)^{k-1} = \binom{n}{k' - i} O(d)^k. \]
Theorem 5. (B, Pollack, Roy, 2002)

\[ b_i(d, k, k', n) \leq \sum_{0 \leq j \leq k' - i} \binom{n}{j} 4^j d (2d - 1)^{k-1} = \binom{n}{k' - i} O(d)^k. \]

Applications?
Proofs

Uses spectral sequences associated to the Mayer-Vietoris double complex.
Generalized Mayer-Vietoris Exact Sequence
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- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex $A$ such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of $i$ co-chains of $A$, and $C^*(A) = \bigoplus_i C^i(A)$. 
Generalized Mayer-Vietoris Exact Sequence

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We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
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• We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

• The following sequence of homomorphisms is exact.

$$
0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \xrightarrow{\delta} \cdots
$$
We now consider the following bigraded double complex $M^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$M^{p,q} = \prod_{\alpha_0, \ldots, \alpha_p} C^q(A_{\alpha_0, \ldots, \alpha_p}).$$

and ...
\[
\begin{array}{cccccc}
0 & \rightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0},\alpha_1) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0},\alpha_1,\alpha_2) \\
& & d & & d & & d \\
0 & \rightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0},\alpha_1) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0},\alpha_1,\alpha_2) \\
& & d & & d & & d \\
0 & \rightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0},\alpha_1) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0},\alpha_1,\alpha_2) \\
& & d & & d & & d \\
0 & \rightarrow & \prod_{\alpha_0} C^0(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^0(A_{\alpha_0},\alpha_1) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^0(A_{\alpha_0},\alpha_1,\alpha_2) \\
& & d & & d & & d \\
0 & \rightarrow & 0 & \xrightarrow{} & 0 & \xrightarrow{} & 0
\end{array}
\]
Double Complex

\[ \cdots \]

\[ C^0,2 \xrightarrow{d} C^1,2 \xrightarrow{\delta} C^2,2 \xrightarrow{\delta} \cdots \]

\[ C^0,1 \xrightarrow{d} C^1,1 \xrightarrow{\delta} C^2,1 \xrightarrow{\delta} \cdots \]

\[ C^0,0 \xrightarrow{d} C^1,0 \xrightarrow{\delta} C^2,0 \xrightarrow{\delta} \cdots \]
The Associated Total Complex

\[
\begin{array}{ccc}
\delta \quad C^{p-1,q+1} & \delta \quad C^{p,q+1} & \delta \quad C^{p+1,q+1} \\
\vdots & \vdots & \vdots \\
\delta \quad C^{p-1,q} & \delta \quad C^{p,q} & \delta \quad C^{p+1,q} \\
\vdots & \vdots & \vdots \\
\delta \quad C^{p-1,q-1} & \delta \quad C^{p,q-1} & \delta \quad C^{p+1,q-1} \\
\vdots & \vdots & \vdots \\
\delta \quad C^{p-1,q-2} & \delta \quad C^{p,q-2} & \delta \quad C^{p+1,q-2} \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{array}
\]
\[ D = d \pm \delta \]
Spectral Sequence I

- A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
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- a sequence of bi-graded vector spaces and differentials \((E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})\),
• A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,

• a sequence of bi-graded vector spaces and differentials \((E_r, d_r : E^{p,q}_r \rightarrow E^{p+r,q-r+1}_r)\),

• \(E_{r+1} = H(E_r, d_r)\),
A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,

- a sequence of bi-graded vector spaces and differentials \((E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})\),
- \(E_{r+1} = H(E_r, d_r)\),
- \(E_\infty = H^*(\text{Associated Total Complex})\).
Figure 1: The differentials $d_r$ in the spectral sequence $(E_r, d_r)$
Two Spectral Sequences of interest

- There are two spectral sequences associated with $M^{p,q}$ both converging to $H^*_D(M)$. The first terms of these are:
Two Spectral Sequences of interest

- There are two spectral sequences associated with \( M^{p,q} \) both converging to \( H^*_D(M) \). The first terms of these are:

\[
E_1 = H_\delta(M), \quad E_2 = H_dH_\delta(M)
\]
Two Spectral Sequences of interest

• There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H_D^*(\mathcal{M})$. The first terms of these are:

• $E_1 = H_\delta(\mathcal{M})$, $E_2 = H_d H_\delta(\mathcal{M})$

• $E_1' = H_d(\mathcal{M})$, $E_2' = H_\delta H_d(\mathcal{M})$
\[ E_1 = \begin{array}{ccc}
\vdots & \vdots & \vdots \\
C^3(A) & 0 & 0 \\
C^2(A) & 0 & 0 \\
C^1(A) & 0 & 0 \\
C^0(A) & 0 & 0 \\
\end{array} \]
The degeneration of this sequence at \( E_2 \) shows that

\[
E_2 = \begin{array}{ccc}
\vdots & \vdots & \vdots \\
H^3(A) & 0 & 0 \\
H^2(A) & 0 & 0 \\
H^1(A) & 0 & 0 \\
H^0(A) & 0 & 0 \\
\end{array}
\]

\[H^*_D(\mathcal{M}) \cong H^*(A).\]
\[ E'_1 = \prod_{\alpha_0} H^3(A_{\alpha_0}) \quad \prod_{\alpha_0 < \alpha_1} H^3(A_{\alpha_0, \alpha_1}) \quad \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^3(A_{\alpha_0, \alpha_1, \alpha_2}) \]
\[ \prod_{\alpha_0} H^2(A_{\alpha_0}) \quad \prod_{\alpha_0 < \alpha_1} H^2(A_{\alpha_0, \alpha_1}) \quad \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^2(A_{\alpha_0, \alpha_1, \alpha_2}) \]
\[ \prod_{\alpha_0} H^1(A_{\alpha_0}) \quad \prod_{\alpha_0 < \alpha_1} H^1(A_{\alpha_0, \alpha_1}) \quad \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^1(A_{\alpha_0, \alpha_1, \alpha_2}) \]
\[ \prod_{\alpha_0} H^0(A_{\alpha_0}) \quad \prod_{\alpha_0 < \alpha_1} H^0(A_{\alpha_0, \alpha_1}) \quad \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^0(A_{\alpha_0, \alpha_1, \alpha_2}) \]
Lemma 6. Let $A$ be a finite simplicial complex and $A_1, \ldots, A_n$ subcomplexes of $A$ such that $A = A_1 \cup \cdots \cup A_n$. Suppose that for every $\ell$, $0 \leq \ell \leq i$, and for every $(\ell + 1)$ tuple $A_{\alpha_0}, \ldots, A_{\alpha_\ell}$, $\beta_{i-\ell}(A_{\alpha_0}, \ldots, A_{\alpha_\ell}) \leq M$. Then, $\beta_i(A) \leq \sum_{0 \leq \ell \leq i} \binom{n}{\ell+1} M$. 
Lemma 2

Lemma 7. Let $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $\deg(P_i) \leq d$, and $l \leq k$. Let $S$ be the set defined by the conjunction of the inequalities $P_i \geq 0$. Assume that $S$ is bounded. Then, $\sum_i \beta_i(S) = (4d)^k$.

Theorem 3 follows.
Theorem 2 follows by a dual argument.
Theorem 4 follows using a result of Barvinok (1995).
Part II
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Algorithms in Computational Geometry.
Arrangements in Computational Geometry

An arrangement in $\mathbb{R}^k$ is a collection of $n$ objects in $\mathbb{R}^k$ each of constant description complexity.
Arrangements in Computational Geometry

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- Arrangements of lines in the plane, or more generally hyperplanes in $\mathbb{R}^k$. 
Arrangements in Computational Geometry

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- Arrangements of balls or simplices in $\mathbb{R}^k$. 
Arrangements in Computational Geometry

An arrangement in $\mathbb{R}^k$ is a collection of $n$ objects in $\mathbb{R}^k$ each of constant description complexity.

- Arrangements of lines in the plane, or more generally hyperplanes in $\mathbb{R}^k$.

- Arrangements of balls or simplices in $\mathbb{R}^k$.

- Arrangements of semi-algebraic objects in $\mathbb{R}^k$, each defined by a fixed number of polynomials of constant degree.
Arrangements of lines in the $\mathbb{R}^2$
Arrangement of circles in $\mathbb{R}^2$
Arrangement of tori in $\mathbb{R}^3$
Computing the Betti Numbers: Previous Work
Schwartz and Sharir, in their seminal papers on the Piano Mover’s Problem (Motion Planning).
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- Computing the Betti numbers of arrangements of balls by Edelsbrunner et al (Molecular Biology).
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- Computing the Betti numbers of triangulated manifolds (Edelsbrunner, Dey, Guha et al).
Complexity of Algorithms
In computational geometry it is customary to study the *combinatorial complexity* of algorithms. The *algebraic complexity* (dependence on the degree) is considered to be a constant.
Complexity of Algorithms

- In computational geometry it is customary to study the \textit{combinatorial complexity} of algorithms. The \textit{algebraic complexity} (dependence on the degree) is considered to be a constant.

- We only count the number of algebraic operations and ignore the cost of doing linear algebra.
Two Approaches
Two Approaches

Global vs Local
First Approach (Global): Using Triangulations
First Approach (Global): Using Triangulations

Semi-algebraic homeomorphism
Using Collin’s Cylindrical Algebraic Decomposition
Picture of a cylinder
Computing Betti Numbers using Global Triangulations

- Compact semi-algebraic sets are finitely triangulable.
Computing Betti Numbers using Global Triangulations

• Compact semi-algebraic sets are finitely triangulable.

• First triangulate the arrangement using Cylindrical algebraic decomposition and then compute the Betti numbers of the corresponding simplicial complex.

• But ...
Computing Betti Numbers using Global Triangulations

• Compact semi-algebraic sets are finitely triangulable.

• First triangulate the arrangement using Cylindrical algebraic decomposition and then compute the Betti numbers of the corresponding simplicial complex.

• But ... CAD produces $O(n^{2k})$ simplices in the worst case.
Second Approach (Local): Using the Nerve Complex
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- If the sets have the special property that all their non-empty intersections are contractible we can use the nerve lemma (Leray, Folkman).
Second Approach (Local): Using the Nerve Complex

- If the sets have the special property that all their non-empty intersections are contractible we can use the nerve lemma (Leray, Folkman).

- The homology groups of the union are then isomorphic to the homology groups of a combinatorially defined complex called the nerve complex.
The Nerve Complex

Figure 2: The nerve complex of a union of disks
Computing the Betti Numbers via the Nerve Complex (local algorithm)

- The nerve complex has $n$ vertices, one vertex for each set in the union, and a simplex for each non-empty intersection among the sets.
Computing the Betti Numbers via the Nerve Complex (local algorithm)

- The nerve complex has \( n \) vertices, one vertex for each set in the union, and a simplex for each non-empty intersection among the sets.

- Thus, the \((\ell+1)\)-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible \( \sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2}) \) at most \((\ell + 2)\)-ary intersections among the \( n \) given sets.
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- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
What if the sets are not special?

- If the sets are such that the topology of the “small” intersections are controlled, then
- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
- The algorithmic version gives the first efficient algorithm for computing the Betti numbers, without the double-exponential complexity entailed in CAD.
Main Result

**Theorem 8.** Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity and let $S = \bigcup_{1 \leq i \leq n} S_i$, and $0 \leq \ell \leq k - 1$. Then, there is an algorithm to compute $\beta_0(S), \ldots, \beta_\ell(S)$, whose complexity is $O(n^{\ell+2})$. 
The Algorithm
The Algorithm

- Compute the spectral sequence \((E'_r, d_r)\) of the Mayer-Vietoris double complex.
The Algorithm

- Compute the spectral sequence \((E'_r, d_r)\) of the Mayer-Vietoris double complex.

- In order to compute \(\beta_\ell\), we only need to compute upto \(E'_{\ell+2}\).
The Algorithm

• Compute the spectral sequence \((E'_r, d_r)\) of the Mayer-Vietoris double complex.

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- In order to compute the differentials \(d_r, 1 \leq r \leq \ell + 1\), it suffices to have independent triangulations of the different unions taken up to \(\ell + 2\) at a time.
For instance, it should be intuitively clear that in order to compute $\beta_0(\bigcup_i S_i)$ it suffices to triangulate pairs.
Spectral Sequences

Where do they come from?
Filtration I
Filtration II

\[ C^m_k = \bigoplus_{p+q=n, p \geq k} C^{p,q} \]
Filtration II

- $C^n_k = \bigoplus_{p+q=n, p \geq k} C^{p,q}$
- $Z^n_k = \{ z \in C^n_k | Dz = 0 \}$, 
Filtration II

• \( C_k^n = \bigoplus_{p+q=n, p \geq k} C^{p,q} \)

• \( Z_k^n = \{ z \in C_k^n | Dz = 0 \} \), \( B^n = D C_{n-1} \)
Filtration II

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- $H^n_k = Z^n_k / Z^n_k \cap B^n$
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- $H^n_k = Z^n_k / Z^n_k \cap B^n$
- We thus have a decreasing filtration, $\cdots \supset H^n_{k-1} \supset H^n_k \supset H^n_{k+1} \cdots$ of the cohomology group $H^n(C, D)$. We denote the successive quotients $H^n_k / H^n_{k+1}$ by $H^{k,n-k}$. 
Spectral Sequence II
• An element in $C^n = \bigoplus_{i+j=n} C^{i,j}$ will have a leading term at a position $(p, q)$, where $p$ denotes the smallest $i$ such that the component at position $(i, n - i)$ does not vanish.
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- Let $Z^{p,q}$ denote the set of the $(p, q)$ components of co-cycles whose leading term is at position $(p, q)$. 
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\[
\begin{align*}
\delta a &= 0 \\
\delta a &= -da^{(1)} \\
\delta a^{(1)} &= -da^{(2)} \\
\delta a^{(2)} &= -da^{(3)} \\
&\vdots
\end{align*}
\]

Here, \( a^{(i)} \in C^{p+i,q-i} \).
Pictorially ...
Pictorially ...
Figure 3: $\delta a^{(i)} + da^{(i+1)} = 0$
Spectral Sequence IV

Now, let $B^{p,q} \subset C^{p,q}$ consist of all $b$ with the property that the following system of equations admits a solution.

\[
d b^{(0)} + \delta b^{(-1)} = b
\]
\[
d b^{(-1)} + \delta b^{(-2)} = 0
\]
\[
d b^{(-2)} + \delta b^{(-3)} = 0
\]
\[
\vdots
\]

Here, $b^{(-i)} \in C^{p-i,q+i-1}$. 
Spectral Approximation

- Clearly, $H^{p,q} \cong Z^{p,q} / B^{p,q}$. 
• Clearly, $H^{p,q} \cong Z^{p,q}/B^{p,q}$.

•

$$Z_{r}^{p,q} = \{ a \in C^{p,q} | \exists (a^{(1)}, \ldots, a^{(r-1)}) | (a, a^{(1)}, \ldots, a^{(r-1)})$$

satisfying equations above\}. 

Spectral Approximation
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\[ B_{r}^{p,q} = \{ b \in C^{p,q} | \exists (b^{(0)}, b^{(-1)}, \ldots, b^{(-r+1)})(b, b^{(0)}, \ldots, b^{(-r+1)}) \text{ satisfying equations above} \} \]
Spectral Approximation

\[ B^{p,q}_r = \{ b \in C^{p,q} \mid \exists (b^{(0)}, b^{(-1)}, \ldots, b^{(-r+1)}) \parallel (b, b^{(0)}, \ldots, b^{(-r+1)}) \text{ satisfying equations above} \} \]

\[ E^{p,q}_r = Z^{p,q}_r/B^{p,q}_r. \]
Let $[a] \in E_{r}^{p,q}$ for some $a \in Z_{r}^{p,q}$. Then, there exists $a^{(1)}, \ldots, a^{(r-1)}$ satisfying equations above. We let

$$d_{r}[a] = [\delta a^{(r-1)}] \in E_{r}^{p+r,q-r+1}.$$
Figure 4: The differentials $d_r$ in the spectral sequence $(E_r, d_r)$
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To what extent does topological simplicity aid algorithms in computational geometry?