Combinatorial Complexity in O-minimal Geometry

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1. Introduction

Over the last twenty years there has been a lot of work on bounding the topological complexity (measured in terms of their Betti numbers) of several different classes of subsets of $\mathbb{R}^k$ – most notably semi-algebraic and semi-Pfaffian sets. The usual setting for proving these bounds is as follows. One considers a semi-algebraic (or semi-Pfaffian) set $S \subset \mathbb{R}^k$ defined by a Boolean formula whose atoms consists of $P > 0, P = 0, P < 0,\ P \in \mathcal{P}$, where $\mathcal{P}$ is a set of polynomials (resp. Pfaffian functions) of degrees bounded by a parameter (resp. whose Pfaffian complexity is bounded by certain parameters) and $\#\mathcal{P} = n$. It is possible to obtain bounds on the Betti numbers of S in terms of n, k and the parameters bounding the complexity of the functions in $\mathcal{P}$.

1.1. Known Bounds in the Semi-algebraic and Semi-Pfaffian cases.

In the semi-algebraic case, if we assume that the degrees of the polynomials in $\mathcal{P}$ are bounded by d, and denoting by $b_i(S)$ the i-th Betti number of S, then it is shown in [6] that, $\sum_{i\geq 0} b_i(S) \leq n^{2k} O(d)^k$.

A similar bound is also shown for semi-Pfaffian sets [6].

In another direction, we also have reasonably tight bounds on the sum of the Betti numbers of the realizations of all realizable sign conditions of the family $\mathcal{P}$. A sign condition on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$, and the realization of a sign condition σ is the set, $\mathcal{R}(\sigma) = \{\mathbf{x} \in \mathrm{R}^k \mid \mathrm{sign}(P(\mathbf{x})) = \sigma(P), \forall P \in \mathcal{P}\}$. It is shown in [3] that, $\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathcal{R}(\sigma)) \leq \sum_{j=0}^{k-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = n^{k-i} O(d)^k$.

2. New Results

Notice that the above bounds are products of two quantities – one that depends only on n (and k), and another part which is independent of n, but depends on the parameters controlling the complexity of individual elements of $\mathcal{P}$ (such as degrees of polynomials in the semi-algebraic case, or the degrees and the length of the Pfaffian chain defining the functions in the Pfaffian case). It is customary to refer to the first part as the *combinatorial part* of the complexity, and the latter as the algebraic (or Pfaffian) part.

While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of primary interest (the algebraic part is assumed to be bounded by a constant). The motivation behind this point of view is the following. In problems in discrete and computational geometry, one typically encounters arrangements of a large number of objects in $\mathbb{R}^k$ (for some fixed k), where each object is of "constant description complexity" (for example, defined by a polynomial inequality of degree bounded by a constant). Thus, it is the number of objects that constitutes the important parameter, and the algebraic complexity of the individual objects are thought of as small constants. It is this second setting that is our primary interest in this paper.

The main results of this paper generalize (combinatorial parts of) the above mentioned bounds to sets which are definable in an arbitrary o-minimal structure over a real closed field R. The proofs of the theorems stated below, as well as several other results, can be found in the full paper [2].

2.1. **Admissible Sets.** We now define the sets that will play the role of objects of "constant description complexity".

Definition 2.1. Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure on a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a definable set. Let $\pi_1 : \mathrm{R}^{k+\ell} \to \mathrm{R}^k$ (resp. $\pi_2 : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$), be the projections onto the first k (resp. last ℓ) co-ordinates.

We will call a subset S of R^k to be a (T, π_1, π_2)-set if $S = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$ for some $\mathbf{y} \in \mathrm{R}^\ell$, and when the context is clear we will denote $T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$. In this paper, we will consider finite families of (T, π_1, π_2)-sets, where T is some fixed definable set for each such family, and we will call a family of (T, π_1, π_2)-sets to be a (T, π_1, π_2)-family. We refer to a finite (T, π_1, π_2)-family as an *arrangement* of (T, π_1, π_2)-sets.

Definition 2.2. Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathrm{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set $\bigcap_{i \in I \subset [1\ldots n]} S_i \cap \bigcap_{j \in [1\ldots n] \setminus I} \mathrm{R}^k \setminus S_j$, and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.

For any definable set $X \subset \mathrm{R}^k$, we let $b_i(X)$ denote the i-th Betti number of X, and we let $b(X)$ denote $\sum_{i=0}^k b_i(X)$. We define the *topological complexity* of an arrangement $\mathcal{A}$ of (T, π_1, π_2)-sets to be the number $\sum_{D \in \mathcal{C}(\mathcal{A})} \sum_{i=0}^k b_i(D)$.

2.2. **Combinatorial and Topological Complexity of Arrangements.** We have the following theorems.

Theorem 2.3. *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field* R *and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subsets of R^k the following holds.*

(1) *For every $i, 0 \le i \le k$, $\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \le C \cdot n^{k-i}$. In particular, the combinatorial complexity of $\mathcal{A}$, which is equal to $\sum_{D \in \mathcal{C}(\mathcal{A})} b_0(D)$, is at most $C \cdot n^k$.*
(2) *The topological complexity of any m cells in the arrangement $\mathcal{A}$ is bounded by $m + C \cdot n^{k-1}$.*

Theorem 2.4 (Topological Complexity of Projections)**.** *Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure, and let $T \subset \mathbb{R}^{k+\ell}$ be a definable, closed and bounded set. Let $k = k_1 + k_2$ and let $\pi_3 : \mathbb{R}^k \to \mathbb{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates. Then,* *there exists a constant $C = C(T) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$, with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S \subset \mathbb{R}^k$, $\sum_{i=0}^{k_2} b_i(\pi_3(S)) \leq C \cdot n^{(k_1+1)k_2}$.*

2.3. **Cylindrical Definable Cell Decompositions.** The fact that given any finite family $\mathcal{A}$ of definable subsets of R^k, there exists a Cylindrical Definable Cell Decomposition (cdcd for short) of R^k adapted to $\mathcal{A}$ is classical (see [4, 5]). We prove a quantitative version of this result. Such quantitative versions are known in the semi-algebraic as well as semi-Pfaffian categories, but is missing in the general o-minimal setting.

Theorem 2.5 (Quantitative cylindrical definable cell decomposition)**.** *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field* R, *and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exist constants $C_1, C_2 > 0$ depending only on T, and definable sets, $\{T_i\}_{i\in I}$, $T_i \subset \mathrm{R}^k \times \mathrm{R}^{2(2^k-1)\cdot\ell}$, depending only on T, with $|I| \leq C_1$, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$ with $S_i = T_{\mathbf{y}_i}, \mathbf{y}_i \in \mathrm{R}^\ell, 1 \leq i \leq n$, some sub-collection of the sets*

$$\pi^{\leq k}_{k+2(2^k-1)\cdot\ell} \left({\pi^{>k}_{k+2(2^k-1)\cdot\ell}}^{-1}(\mathbf{y}_{i_1}, \ldots, \mathbf{y}_{i_{2(2^k-1)}}) \cap T_i \right), \quad i \in I,\ 1 \leq i_1, \ldots, i_{2(2^k-1)} \leq n,$$

form a cdcd of R^k *compatible with $\mathcal{A}$. Moreover, the cdcd has at most $C_2 \cdot n^{2(2^k-1)}$ cells.*

2.4. **Application.** We end with an application which generalizes a Ramsey-type result due to Alon et al. [1] from the class of semi-algebraic sets of constant description complexity to (T, π_1, π_2)-families.

Theorem 2.6. *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field* R, *and let F be a closed definable subset of $\mathrm{R}^\ell \times \mathrm{R}^\ell$. Then, there exists a constant $1 > \varepsilon = \varepsilon(F) > 0$, depending only on F, such that for any set of n points, $\mathcal{F} = \{\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathrm{R}^\ell\}$ there exists two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, with $|\mathcal{F}_1|, |\mathcal{F}_2| \geq \varepsilon n$ and either, for all $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$, or for no $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$.*

References

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