Combinatorial Complexity in O-minimal Geometry

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Outline

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Semi-algebraic Sets and their Betti numbers

- Let $S \subset \mathbb{R}^k$ be defined by a Boolean formula whose atoms consists of $P > 0, P = 0, P < 0,\ P \in \mathcal{P}$, where $\mathcal{P}$ is a set of polynomials of degrees bounded by a parameter and $\#\mathcal{P} = n$.
- $$\sum_{i \geq 0} b_i(S) \leq n^{2k} O(d)^k. \quad \text{(Gabrielov-Vorobjov, 2005)}$$
- Bound for sign conditions: (B-Pollack-Roy, 2005)

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathcal{R}(\sigma)) \leq \sum_{j=0}^{k-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = n^{k-i} O(d)^k.$$

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Combinatorial Complexity

- Notice that the bounds in the previous page are products of two quantities – one that depends only on n (and k), and another part which is independent of n. We refer to the first part as the combinatorial part of the complexity, and the latter as the algebraic part.
- While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of interest (the algebraic part is assumed to be bounded by a constant).

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- While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of interest (the algebraic part is assumed to be bounded by a constant).

Definition of Arrangements

- Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, with each S_i belonging to some "simple" class of sets.
- For $I \subset \{1, \ldots, n\}$, let $\mathcal{A}(I)$denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{R}^k \setminus S_j,$$

and it is customary to call a connected component of $\mathcal{A}_I$ a cell of the arrangement $\mathcal{A}$ and we denote by $\mathcal{C}(\mathcal{A})$ the set of all non-empty cells of the arrangement $\mathcal{A}$.
- The cardinality of $\mathcal{C}(\mathcal{A})$ is called the combinatorial complexity of the arrangement $\mathcal{A}$.

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Objects of Bounded Description Complexity

- The class of sets usually considered in the study of arrangements are sets with "bounded description complexity". This means that each set in the arrangement is defined by a first order formula in the language of ordered fields involving at most a constant number polynomials whose degrees are also bounded by a constant.
- Additionally, there is often a requirement that the sets be in "general position". The precise definition of "general position" varies with context, but often involves restrictions such as: the sets in the arrangements are smooth manifolds, intersecting transversally.

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Definition of O-minimal Structures

An o-minimal structure on a real closed field R is a sequence $\mathcal{S}(\mathrm{R}) = (\mathcal{S}_n)_{n \in \mathbb{N}}$.

1. All algebraic subsets of R^n are in $\mathcal{S}_n$.
2. The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
3. If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
4. If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n co-ordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
5. The elements of $\mathcal{S}_1$ are precisely finite unions of points and intervals.

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Examples of O-minimal Structures I

- Our first example of an o-minimal structure $\mathcal{S}(\mathrm{R})$, is the o-minimal structure over a real closed field R where each $\mathcal{S}_n$ is exactly the class of semi-algebraic subsets of R^n.
- Let $\mathcal{S}_n$ be the images in $\mathbb{R}^n$ under the projection maps $\mathbb{R}^{n+k} \to \mathbb{R}^n$ of sets of the form $\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+k} \mid P(\mathbf{x}, \mathbf{y}, e^{\mathbf{x}}, e^{\mathbf{y}}) = 0\}$, where P is a real polynomial in $2(n+k)$ variables, and $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_n})$ and $e^{\mathbf{y}} = (e^{y_1}, \ldots, e^{y_k})$. We will denote this o-minimal structure over $\mathbb{R}$ by $\mathcal{S}_{\exp}(\mathbb{R})$.

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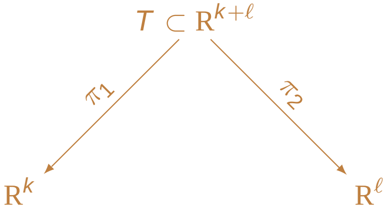
Examples of O-minimal Structures II

- Let $\mathcal{S}_n$ be the images in $\mathbb{R}^n$ under the projection maps $\mathbb{R}^{n+k} \to \mathbb{R}^n$ of sets of the form $\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+k} \mid P(\mathbf{x}, \mathbf{y}) = 0\}$, where P is a restricted analytic function in $2(n+k)$ variables.
  (A restricted analytic function in N variables is an analytic function defined on an open neighborhood of $[0,1]^N$ restricted to $[0,1]^N$ (and extended by 0 outside)).
  We will denote this o-minimal structure over $\mathbb{R}$ by $\mathcal{S}_{\mathrm{ana}}(\mathbb{R})$.

Admissible Sets

- Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure on a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a fixed definable set.



- We will call S of R^k to be a (T, π_1, π_2)-set if

$$S = T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$$

for some $\mathbf{y} \in \mathrm{R}^\ell$.

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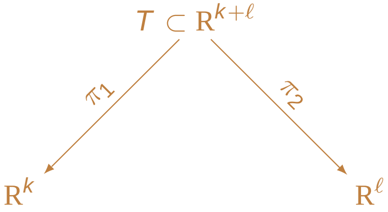

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Example I

Let $\mathcal{S}(\mathrm{R}) = \mathcal{S}_{\mathrm{sa}}(\mathrm{R})$ and Let $T \subset \mathrm{R}^{2k+1}$ be the semi-algebraic set defined by

$$T = \{(x_1, \ldots, x_k, a_1, \ldots, a_k, b) \mid \langle \mathbf{a}, \mathbf{x} \rangle - b = 0\}$$

(where we denote $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{x} = (x_1, \ldots, x_k)$), and π_1 and π_2 are the projections onto the first k and last $k + 1$ co-ordinates respectively. A (T, π_1, π_2)-set is clearly a hyperplane in R^k and vice versa.

Example II

Let $\mathcal{S}(\mathbb{R}) = \mathcal{S}_{\exp}(\mathbb{R})$ and

$$T = \{(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m, a_1, \ldots, a_m) \mid \mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^k,$$

$$a_1, \ldots, a_m \in \mathbb{R}, x_1, \ldots, x_k > 0, \sum_{i=0}^{m} a_i \mathbf{x}^{\mathbf{y}_i} = 0\},$$

with $\pi_1 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^{m(k+1)}$ be the projections onto the first k and the last $m(k+1)$ co-ordinates respectively. The (T, π_1, π_2)-sets in this example include (amongst others) all semi-algebraic sets consisting of intersections with the positive orthant of all real algebraic sets defined by a polynomial having at most m monomials (different sets of monomials are allowed to occur in different polynomials).

Outline

$\mathcal{A}$-sets I

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathrm{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \;\cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{R}^k \setminus S_j, \quad (1)$$

and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.

$\mathcal{A}$-sets II

We will call definable subsets $S \subset \mathbb{R}^k$ defined by a Boolean formula whose atoms are of the form, $x \in S_i, 1 \leq i \leq n$, a $\mathcal{A}$-set. A $\mathcal{A}$-set is thus a union of basic $\mathcal{A}$-sets. If T is closed, and the Boolean formula defining S has no negations, then S is closed by definition (since each S_i is closed) and we call such a set an $\mathcal{A}$-closed set.

Moreover, if V is any closed definable subset of $\mathbb{R}^k$, and S is an $\mathcal{A}$-set (resp. $\mathcal{A}$-closed set), then we will call $S \cap V$ to be an $(\mathcal{A}, V)$-set (resp. $(\mathcal{A}, V)$-closed set).

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Bounds on Betti Numbers I

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subsets of R^k the following holds. For every $i, 0 \leq i \leq k$,

$$\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \leq C \cdot n^{k-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$, is at most $C \cdot n^k$. The topological complexity of any m cells in the arrangement $\mathcal{A}$ is bounded by $m + C \cdot n^{k-1}$.

Lower dimensional

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$, $V \subset \mathrm{R}^k$ be closed definable sets with $\dim(V) = k'$. Then, there exists a constant $C = C(T, V) > 0$ depending only on T and V, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, of subsets of R^k, and for every $i, 0 \le i \le k'$,

$$\sum_{D \in \mathcal{C}(\mathcal{A}, V)} b_i(D) \le C \cdot n^{k'-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$ restricted to V, is bounded by $C \cdot n^{k'}$.

Topological Complexity of $\mathcal{A}$-sets

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$, $V \subset \mathrm{R}^k$ be closed definable sets with $\dim(V) = k'$. Then, there exists a constant $C = C(T, V) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$ with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S_1 \subset \mathrm{R}^k$, and an $\mathcal{A}$-set $S_2 \subset \mathrm{R}^k$,

$$\sum_{i=0}^{k'} b_i(S_1 \cap V) \leq C \cdot n^{k'} \quad \textit{and,}$$

$$\sum_{i=0}^{k'} b_i(S_2 \cap V) \leq C \cdot n^{2k'}.$$

Topological Complexity of Projections

Theorem (Topological Complexity of Projections)

Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure, and let $T \subset \mathbb{R}^{k+\ell}$ be a definable, closed and bounded set. Let $k = k_1 + k_2$ and let $\pi_3 : \mathbb{R}^k \to \mathbb{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates.
Then, there exists a constant $C = C(T) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$, with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S \subset \mathbb{R}^k$,

$$\sum_{i=0}^{k_2} b_i(\pi_3(S)) \leq C \cdot n^{(k_1+1)k_2}.$$

Outline

Definition of cdcd

A cdcd of R^k is a finite partition of R^k into definable sets $(C_i)_{i\in I}$ (called the cells of the cdcd) satisfying the following properties. If $k = 1$ then a cdcd of R is given by a finite set of points $a_1 < \cdots < a_N$ and the cells of the cdcd are the singletons $\{a_i\}$ as well as the open intervals, $(\infty, a_1), (a_1, a_2), \ldots, (a_N, \infty)$. If $k > 1$, then a cdcd of R^k is given by a cdcd, $(C_i')_{i\in I'}$, of R^{k-1} and for each $i \in I'$, a collection of cells, $\mathcal{C}_i$ defined by,

$$\mathcal{C}_i = \{\phi_i(C_i' \times D_j) \mid j \in J_i\},$$

Definition II

where

$$\phi_i : C_i' \times \mathrm{R} \to \mathrm{R}^k$$

is a definable homemorphism satisfying $\pi \circ \phi = \pi$, $(D_j)_{j \in J_i}$ is a cdcd of R, and $\pi : \mathrm{R}^k \to \mathrm{R}^{k-1}$ is the projection map onto the first $k-1$ coordinates. The cdcd of R^k is then given by

$$\bigcup_{i \in I'} \mathcal{C}_i.$$

Given a family of definable subsets $\mathcal{A} = \{S_1, \ldots, S_n\}$ of R^k, we say that a cdcd is adapted to $\mathcal{A}$, if each S_i is a union of cells of the given cdcd.

Quantitative cylindrical definable cell decomposition I

Theorem (Quantitative cylindrical definable cell decomposition)

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exist constants $C_1, C_2 > 0$ depending only on T, and definable sets,

$$\{T_i\}_{i \in I}, \quad T_i \subset \mathrm{R}^k \times \mathrm{R}^{2(2^k-1)\cdot\ell},$$

depending only on T, with $|I| \leq C_1$, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$ with $S_i = T_{\mathbf{y}_i}, \mathbf{y}_i \in \mathrm{R}^\ell, 1 \leq i \leq n$, some sub-collection of the sets

Quantitative cylindrical definable cell decomposition II

Theorem (Quantitative cylindrical definable cell decomposition)

$$\pi^{\leq k}_{k+2(2^k-1)\cdot\ell}\left({\pi^{>k}_{k+2(2^k-1)\cdot\ell}}^{-1}(\mathbf{y}_{i_1},\ldots,\mathbf{y}_{i_{2(2^k-1)}}) \cap T_i\right),$$
$$i \in I,\ 1 \leq i_1,\ldots,i_{2(2^k-1)} \leq n,$$

form a cdcd of R^k compatible with $\mathcal{A}$. Moreover, the cdcd has at most $C_2 \cdot n^{2(2^k-1)}$ cells.

Outline

Ramsey type theorem

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let F be a closed definable subset of $\mathrm{R}^\ell \times \mathrm{R}^\ell$. Then, there exists a constant $1 > \varepsilon = \varepsilon(F) > 0$, depending only on F, such that for any set of n points,

$$\mathcal{F} = \{\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathrm{R}^\ell\}$$

there exists two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, with $|\mathcal{F}_1|, |\mathcal{F}_2| \geq \varepsilon n$ and either,

- *for all $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$, or*
- *for no $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$.*

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Interesting corollary

Corollary

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $1 > \varepsilon = \varepsilon(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, there exists two subfamilies $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, with $|\mathcal{A}_1|, |\mathcal{A}_2| \geq \varepsilon n$, and either,

- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j \neq \emptyset$, or*
- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j = \emptyset$.*

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Unions of definable families

Suppose that $T_1, \ldots, T_m \subset \mathrm{R}^{k+\ell}$ are closed, definable sets, $\pi_1 : \mathrm{R}^{k+\ell} \to \mathrm{R}^k$ and $\pi_2 : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$ the two projections.

Lemma

For any collection of (T_i, π_1, π_2) families $\mathcal{A}_i$, $1 \leq i \leq m$, the family $\cup_{1 \leq i \leq m} \mathcal{A}_i$ is a (T', π'_1, π'_2) family where,

$$T' = \bigcup_{i=1}^{m} T_i \times \{e_i\} \subset \mathrm{R}^{k+\ell+m},$$

with e_i the i-th standard basis vector in R^m, and $\pi'_1 : \mathrm{R}^{k+\ell+m} \to \mathrm{R}^k$ and $\pi'_2 : \mathrm{R}^{k+\ell+m} \to \mathrm{R}^{\ell+m}$, the projections onto the first k and the last $\ell + m$ coordinates respectively.

Notations

Given closed definable sets $X \subset V \subset \mathrm{R}^k$, and $\varepsilon > 0$, we denote

$$\mathrm{OT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) < \varepsilon\},$$

$$\mathrm{CT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) \leq \varepsilon\},$$

$$\mathrm{BT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) = \varepsilon\},$$

and finally for $\varepsilon_1 > \varepsilon_2 > 0$ we define

$$\mathrm{Ann}(X, V, \varepsilon_1, \varepsilon_2) = \{\mathbf{x} \in V \mid \varepsilon_2 < d_X(\mathbf{x}) < \varepsilon_1\},$$

$$\overline{\mathrm{Ann}}(X, V, \varepsilon_1, \varepsilon_2) = \{\mathbf{x} \in V \mid \varepsilon_2 \leq d_X(\mathbf{x}) \leq \varepsilon_1\}.$$

Key Proposition

Proposition

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ *be a collection of closed definable subsets of* R^k *and let* $V \subset \mathrm{R}^k$ *be a closed, and bounded definable set. Then, for all sufficiently small* $1 \gg \varepsilon_1 \gg \varepsilon_2 > 0$ *the following holds. For any connected component,* C*, of* $\mathcal{A}(I) \cap V$, $I \subset [1 \ldots n]$*, there exists a connected component, D, of the definable set,*

$$\bigcap_{1 \leq i \leq n} \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V$$

such that D *is definably homotopy equivalent to* C.

Proof of Proposition

- For all sufficiently small $\varepsilon_1 > 0$ and for each connected component C of $\mathcal{A}(I) \cap V$, there exists a connected component D' of

$$\bigcap_{i \in I} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{OT}(S_j, \varepsilon_1)^c \cap V,$$

homotopy equivalent to C.

- For $0 < \varepsilon_2 \ll \varepsilon_1$, and each connected component D' of $\bigcap_{i \in I} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{OT}(S_j, \varepsilon_1)^c \cap V$, there exists a connected component D of $\bigcap_{i \in I} \mathrm{CT}(S_i, \varepsilon_2) \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{OT}(S_j, \varepsilon_1)^c \cap V$, homotopy equivalent to D'.

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Proof of Proposition (cont).

- Now notice that D is connected and contained in the set

$$\bigcap_{1 \leq i \leq n} \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V.$$

Let D'' be the connected component of

$$\bigcap_{1 \leq i \leq n} \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V$$

containing D.

- We claim that $D = D''$, which will prove the proposition.
- Suppose $D'' \setminus D \neq \emptyset$. Let $\mathbf{x} \in D'' \setminus D$ and $\mathbf{y}$ any point in D. Since $\mathbf{x} \notin D$, either
  1. there exists $i \in I$ such that $\mathbf{x} \in OT(S_i, \varepsilon_1)^c$ or
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Proof of Proposition (cont).

- Let $\gamma : [0,1] \to D''$ be a definable path with $\gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y}$. and let $d_i : D'' \to \mathrm{R}$ be the definable continuous function, $d_i(\mathbf{z}) = \mathrm{dist}(\mathbf{z}, S_i)$.
- Then, in the first case, $d_i(\mathbf{x}) = d_i(\gamma(0)) \geq \varepsilon_1$ and $d_i(\mathbf{y}) = d_i(\gamma(1)) < \varepsilon_2$, implying that there exists $t \in (0,1)$ with $\varepsilon_2 < d_i(\gamma(t)) < \varepsilon_1$ implying that $d_i(\gamma(t)) \notin \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c$ and hence not in D'' (a contradiction).
- In the second case, $d_i(\mathbf{x}) = d_i(\gamma(0)) < \varepsilon_2$ and $d_i(\mathbf{y}) = d_i(\gamma(1)) \geq \varepsilon_1$, implying that there exists $t \in (0,1)$ with $\varepsilon_2 < d_i(\gamma(t)) < \varepsilon_1$ again implying that $d_i(\gamma(t)) \notin \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c$ and hence not in D'' (a contradiction).

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Proof of Theorem on Topological Complexity

- For $1 \leq i \leq n$, let $\mathbf{y}_i \in \mathrm{R}^\ell$ such that

$$S_i = T_{\mathbf{y}_i},$$

and let

$$A_i(\varepsilon_1, \varepsilon_2) = \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V.$$

- Applying Mayer-Vietoris inequalities we have for $0 \leq i \leq k'$,

$$b_i(\bigcap_{j=1}^{n} A_j(\varepsilon_1, \varepsilon_2)) \leq b_{k'}(V) + \sum_{j=1}^{k'-i} \sum_{J \subset \{1,\ldots,n\}, \#(J)=j} \Big(b_{i+j-1}(A^J(\varepsilon_1, \varepsilon_2))$$

where $A^J(\varepsilon_1, \varepsilon_2) = \cup_{j \in J} A_j(\varepsilon_1, \varepsilon_2)$.

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where $A^J(\varepsilon_1, \varepsilon_2) = \cup_{j \in J} A_j(\varepsilon_1, \varepsilon_2)$.

Proof of Theorem on Topological Complexity (cont).

- Notice that each $\mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c, 1 \leq i \leq n$, is a $(\mathrm{Ann}(T, \varepsilon_1, \varepsilon_2)^c, \pi_1, \pi_2)$-set and moreover,

$$\mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c = T_{\mathbf{y}_i} \cap \mathrm{Ann}(T, \varepsilon_1, \varepsilon_2)^c;\ 1 \leq i \leq n.$$

- For $J \subset [1 \ldots n]$, we denote

$$S^J(\varepsilon_1, \varepsilon_2) = \bigcup_{j \in J} \mathrm{Ann}(S_j, \varepsilon_1, \varepsilon_2)^c.$$

There are only a finite number (depending on T) of topological types amongst $S^J(\varepsilon_1, \varepsilon_2)$. Restricting all the sets to V in the above argument, we obtain that there are only finitely many (depending on T and V) of topological types amongst the sets $A^J(\varepsilon_1, \varepsilon_2) = S^J(\varepsilon_1, \varepsilon_2) \cap V$.

Proof of Theorem on Topological Complexity (cont).

- Notice that each $\mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c, 1 \leq i \leq n$, is a $(\mathrm{Ann}(T, \varepsilon_1, \varepsilon_2)^c, \pi_1, \pi_2)$-set and moreover,

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Proof of Theorem on topological complexity(cont).

- Thus, there exists a constant $C(T,V)$ such that

$$C(T,V) = \max_{J \subset \{1,\ldots,n\}} \Big(b_{i+j-1}(A^J(\varepsilon_1,\varepsilon_2)) + b_{k'}(V) \Big) + b_{k'}(V).$$

- It now follows from inequality **??** and Proposition 10 that,

$$\sum_{D \in \mathcal{C}(\mathcal{A},V)} b_i(D) \leq C \cdot n^{k'-i}.$$

Proof of Theorem on topological complexity(cont).

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Proof of Theorem for $\mathcal{A}$-sets

Key proposition:

Proposition

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ be a collection of closed definable subsets of R^k and let $V \subset \mathrm{R}^k$ be a closed, and bounded definable set and let S be an $(\mathcal{A}, V)$-closed set. Then, for all sufficiently small $1 \gg \varepsilon_1 \gg \varepsilon_2 \cdots \gg \varepsilon_n > 0$,

$$b(S) \leq \sum_{D \in \mathcal{C}(\mathcal{B}, V)} b(D),$$

where

$$\mathcal{B} = \bigcup_{i=1}^{n} \{S_i, \mathrm{BT}(S_i, \varepsilon_i), \mathrm{OT}(S_i, 2\varepsilon_i)^c\}.$$

Proof of Theorem on projections

Notice that for each $p, 0 \leq p \leq k_2$, and any $\mathcal{A}$-closed set $S \subset \mathbb{R}^{k_1+k_2}$, $W^p_{\pi_3}(S) \subset \mathrm{R}^{(p+1)k_1+k_2}$ is an $\mathcal{A}^p$-closed set where,

$$\mathcal{A}^p = \bigcup_{j=0}^{p} \mathcal{A}^{p,j},$$

$$\mathcal{A}^{p,j} = \bigcup_{i=1}^{n} \{S_i^{p,j}\},$$

where $S_i^{p,j} \subset \mathbb{R}^{(p+1)k_1+k_2}$ is defined by,

$$S_i^{p,j} = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p, \mathbf{y}) \mid \mathbf{x}_j \in \mathrm{R}^{k_1}, \mathbf{y} \in \mathbb{R}^{k_2}, (\mathbf{x}_j, \mathbf{y}) \in S_i\}.$$

and $W^i_f(X) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_i) \in X^{i+1} \mid f(\mathbf{x}_0) = \cdots = f(\mathbf{x}_i)\}$.

Proof of Theorem on Projections (cont).

- Also, note that $\mathcal{A}^{p,j}$ is a $(T^{p,j}, \pi_1^p, \pi_2^p)$ family, where

$$T^{p,j} = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p, \mathbf{y}, \mathbf{z}) \mid \mathbf{x}_j \in \mathbb{R}^{k_1}, \mathbf{y} \in \mathbb{R}^{k_2}, \mathbf{z} \in \mathbb{R}^{\ell}, (\mathbf{x}_j, \mathbf{y}, \mathbf{z}) \in T, \text{ for some } j, 0 \leq j \leq p\}.$$

  and $\pi_1^p : \mathbb{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathbb{R}^{(p+1)k_1+k_2}$, and $\pi_2^p : \mathbb{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathbb{R}^{\ell}$ are the appropriate projections.
- Since each $T^{p,j}$ is determined by T, we have using previous lemma that $\mathcal{A}^p$ is a (T', π_1', π_2')-family for some definable T' determined by T.

Proof of Theorem on Projections (cont).

- Also, note that $\mathcal{A}^{p,j}$ is a $(T^{p,j}, \pi_1^p, \pi_2^p)$ family, where

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 and $\pi_1^p : \mathbb{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathbb{R}^{(p+1)k_1+k_2}$, and $\pi_2^p : \mathbb{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathbb{R}^{\ell}$ are the appropriate projections.
- Since each $T^{p,j}$ is determined by T, we have using previous lemma that $\mathcal{A}^p$ is a (T', π_1', π_2')-family for some definable T' determined by T.

Proof of Theorem on projections (cont).

Now $W^p_{\pi_3}(S) \subset \mathbb{R}^{(p+1)k_1+k_2}$ is a $\mathcal{A}^p$-closed set and $\#\mathcal{A}^p = (p+1)n$. Applying previous theorem we get, for each p and j, $0 \leq p, j < k_2$,

$$b_j(W^p_{\pi_3}(S)) \leq C_1(T) \cdot n^{(p+1)k_1+k_2}$$

The theorem now follows, since for each $q, 0 \leq q < k_2$,

$$b_q(\pi_3(S)) \leq \sum_{i+j=q} b_j(W^i_{\pi_3}(S)) \leq C_2(T) \cdot n^{(q+1)k_1+k_2} \leq C(T) \cdot n^{(k_1+1)k_2}.$$

Proof of Ramsey type Theorem

- For each $i, 1 \leq i \leq n$, let

$$A_i = \pi_{2\ell}^{\leq \ell}(\pi_{2\ell}^{> \ell^{-1}}(\mathbf{y}_i) \cap F),$$

and $\mathcal{G} = \{A_i \mid 1 \leq i \leq n\}$. Note that $\mathcal{G}$ is a $(R, \pi_{2\ell}^{\leq \ell}, \pi_{2\ell}^{> \ell})$-family.

- We now use the Clarkson-Shor random sampling technique (using Theorem on cdcd instead of vertical decomposition). Applying Theorem on quantitative cdcd to some sub-family $\mathcal{G}_0 \subset \mathcal{G}$ of cardinality r, we get a decomposition of R^ℓ into at most $Cr^{2(2^\ell - 1)} = r^{O(1)}$ definable cells, each of them defined by at most $2(2^\ell - 1) = O(1)$ of the $\mathbf{y}_i$'s.

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Proof of Ramesey type Theorem (cont).

- Let τ be a cell of the cdcd of $\mathcal{G}_0$ and let $G \in \mathcal{G}$. We say that G crosses τ if $G \cap \tau \neq \emptyset$ and $\tau \not\subset G$. The standard theory of random sampling now ensures that we can choose $\mathcal{G}_0$ such that each cell of the cdcd of $\mathcal{G}_0$ is "crossed" by no more than $\frac{c_1 n \log r}{r}$ elements of $\mathcal{G}$, where c_1 is a constant depending only on F.
- For each cell τ of the cdcd of $\mathcal{G}_0$, let $\mathcal{G}_\tau$ denote the set of elements of $\mathcal{G}$ which cross τ and let $\mathcal{F}_\tau = \mathcal{F} \cap \tau$.
- Since the total number of cells in the cdcd of $\mathcal{G}_0$ is bounded by $r^{O(1)}$, there must exist a cell τ such that,

$$|\mathcal{F}_\tau| \geq \frac{n}{r^{O(1)}}.$$

Now, every element of $\mathcal{G} \setminus \mathcal{G}_\tau$ either fully contains τ or is disjoint from it.

Proof of Ramesey type Theorem (cont).

- Let τ be a cell of the cdcd of $\mathcal{G}_0$ and let $G \in \mathcal{G}$. We say that G crosses τ if $G \cap \tau \neq \emptyset$ and $\tau \not\subset G$. The standard theory of random sampling now ensures that we can choose $\mathcal{G}_0$ such that each cell of the cdcd of $\mathcal{G}_0$ is "crossed" by no more than $\frac{c_1 n \log r}{r}$ elements of $\mathcal{G}$, where c_1 is a constant depending only on F.
- For each cell τ of the cdcd of $\mathcal{G}_0$, let $\mathcal{G}_\tau$ denote the set of elements of $\mathcal{G}$ which cross τ and let $\mathcal{F}_\tau = \mathcal{F} \cap \tau$.
- Since the total number of cells in the cdcd of $\mathcal{G}_0$ is bounded by $r^{O(1)}$, there must exist a cell τ such that,

$$|\mathcal{F}_\tau| \geq \frac{n}{r^{O(1)}}.$$

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Proof of Ramsey type Theorem (cont).

- Setting $\alpha = \frac{1}{r^{O(1)}}$ and $\beta = \frac{1}{2}(1 - \frac{c_1 \log r}{r})$ we have that there exists a set $\mathcal{F}' = \mathcal{F}_\tau$ of cardinality at least αn, and a subset $\mathcal{G}'$ of cardinality at least βn such that either each element of $\mathcal{F}'$ is contained in every element of $\mathcal{G}'$, or no element of $\mathcal{F}'$ is contained in any element of $\mathcal{G}'$.
- The proof is complete by taking $\mathcal{F}_1 = \mathcal{F}'$, and $\mathcal{F}_2 = \{\mathbf{y}_i \mid A_i \in \mathcal{G}'\}$ and $\varepsilon = \min(\alpha, \beta)$.

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