Betti Numbers, Spectral Sequences and Algorithms for computing them

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- Subsets of $\mathbb{R}^k$ defined by a formula involving a finite number of polynomial equalities and inequalities.
- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.

Part I

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Bounds on the Complexity of Semi-algebraic Sets

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Uniform bounds on the number of connected components, Betti numbers etc.

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Dimension of the set itself : $\mathbf{k}'$

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- $\beta_i(S) =$ the number of i-cycles that do not bound.

The Torus in $\mathbb{R}^3$

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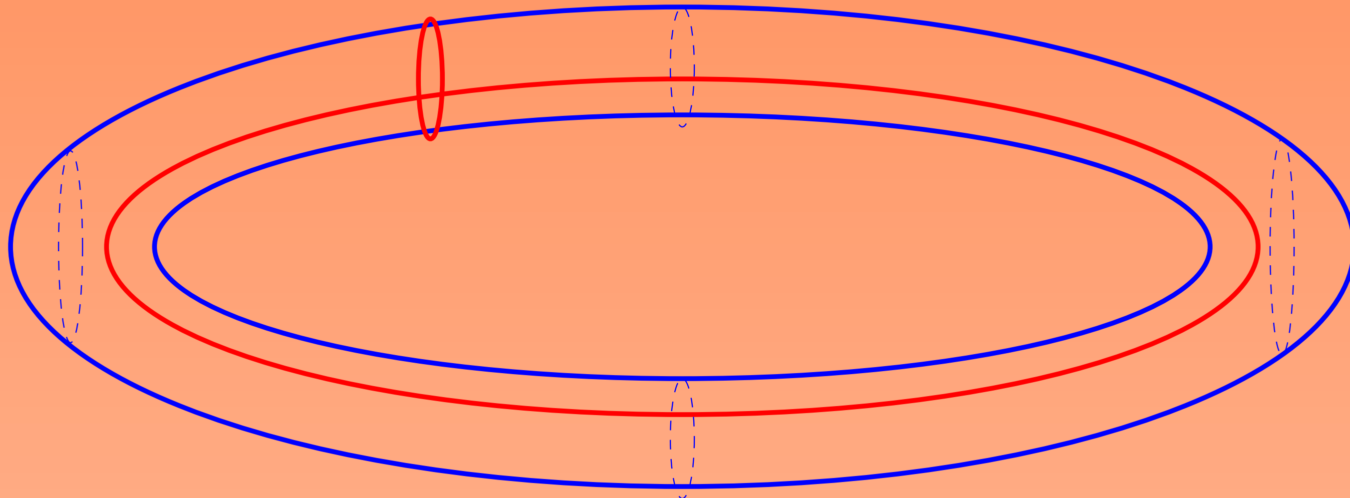

Betti Numbers of the Torus

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- $\beta_i(T) = 0, i > 2.$

Classical Result on the Topology of Semi-algebraic Sets

Theorem 1. *(Oleinik and Petrovsky, Thom, Milnor) Let* $S \subset \mathbb{R}^k$ *be the set defined by the conjunction of* n *inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

$deg(P_i) \leq d, 1 \leq i \leq n$. *Then,*

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$deg(P_i) \leq d, 1 \leq i \leq n$. *Then,*

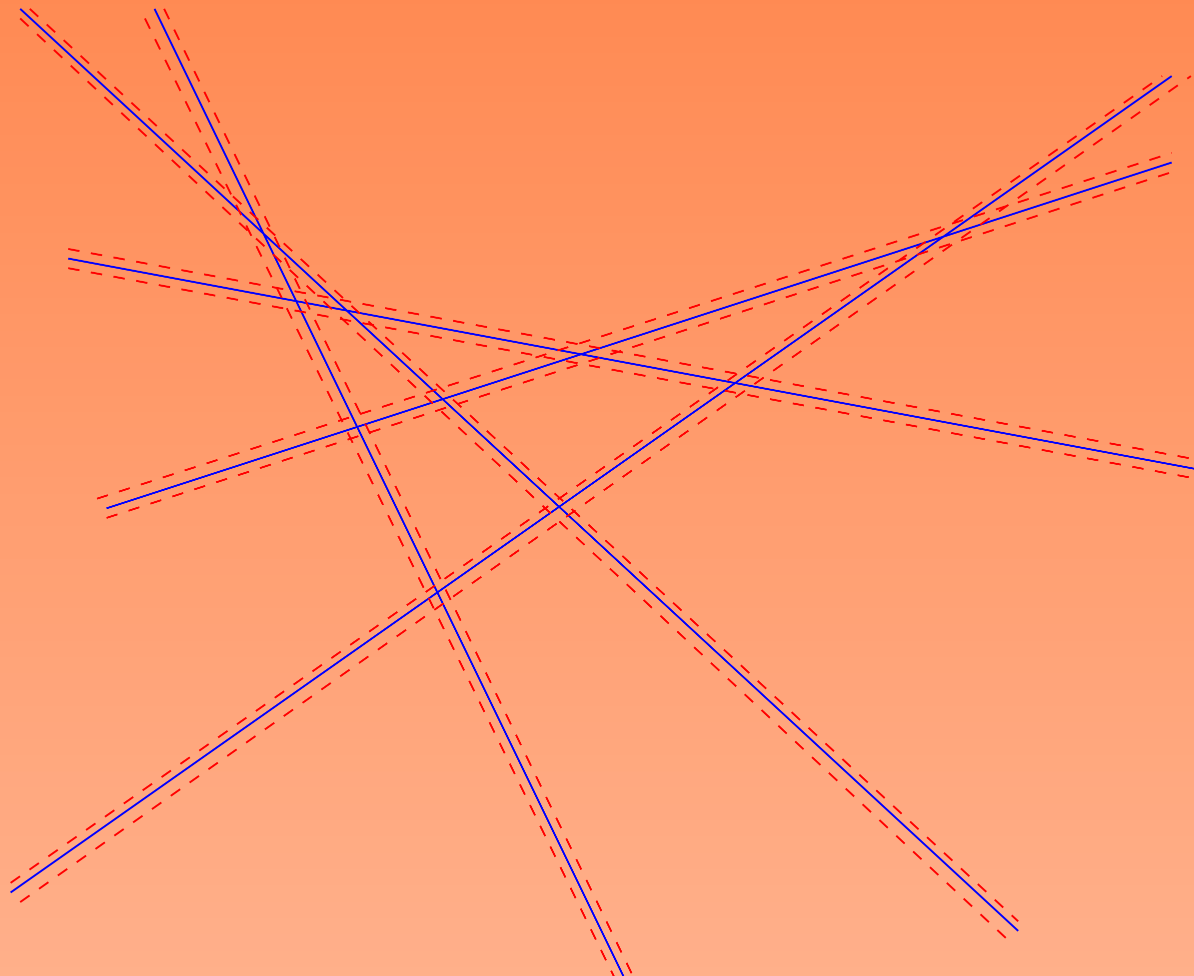
$$\sum_{\mathbf{i}} \beta_{\mathbf{i}}(\mathbf{S}) = \mathbf{nd(2nd-1)^{k-1} = O(nd)^k}.$$

Tightness

The above bound is actually quite tight. Example: Let

$$P_i = L_{i,1}^2 \cdots L_{i,\lfloor d/2 \rfloor}^2 - \epsilon,$$

where the L_{ij}'s are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_n \geq 0$ has $\Omega(nd)^k$ connected components and hence $\beta_0(S) = \Omega(nd)^k$.



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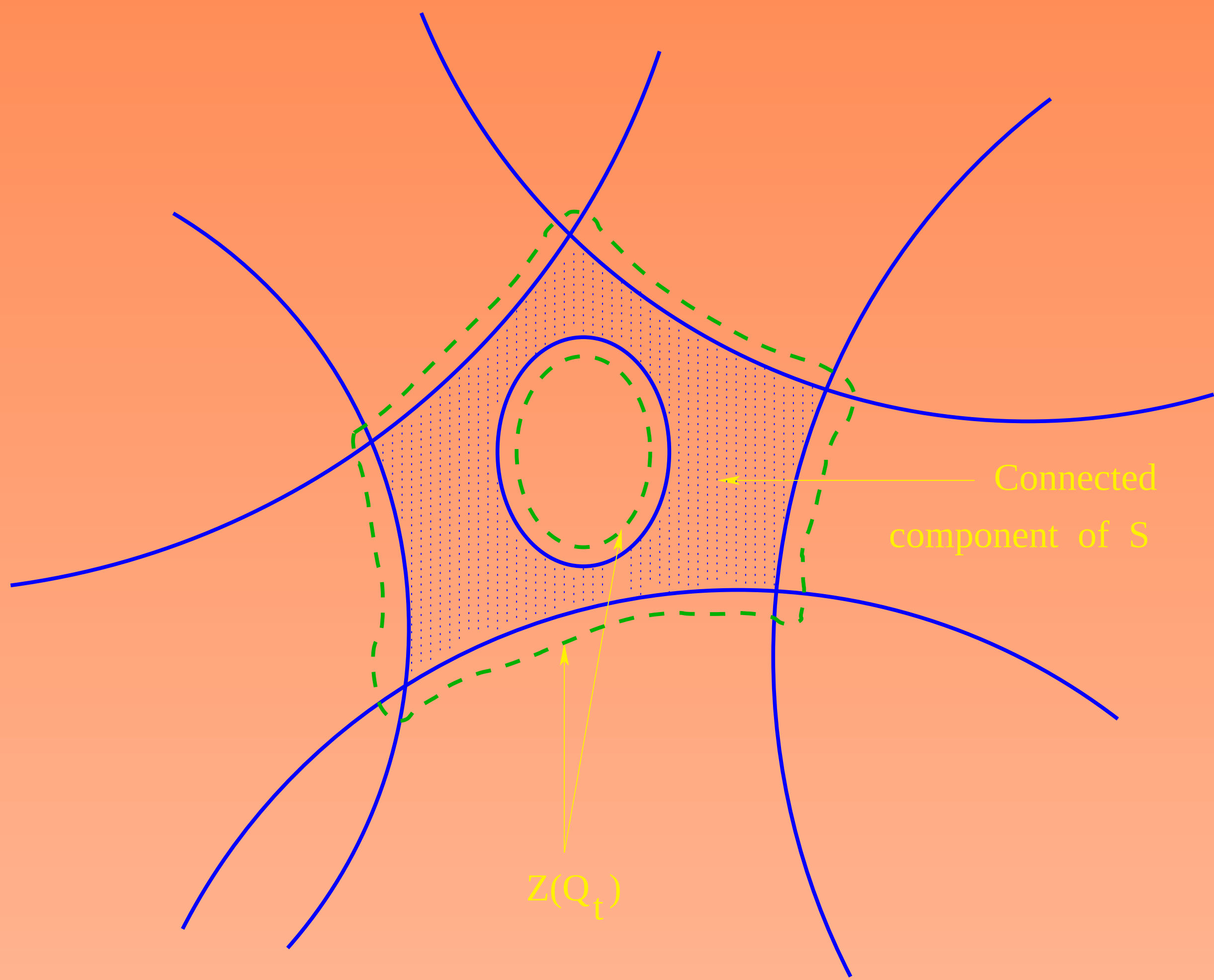
- Cannot construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$.
- The technique used for proving the above result does not help:

  Replace the semi-algebraic set S by another set bounded by a smooth algebraic hypersurface of degree $2nd$ having the same homotopy type as S. Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.


Connected component of S
Z(Q_t)

Graded Bounds

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$$\beta_{\mathbf{i}}(\mathbf{S}) \leq \binom{\mathbf{n}}{\mathbf{k}' - \mathbf{i}} (\mathbf{4d})^{\mathbf{k}}.$$

The case of the union

Theorem 3. *(B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the disjunction of n inequalities,*

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Example: $\mathbf{X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0}$.

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- Let $b_i(\sigma)$ denote the i-th Betti number of the realization of σ, and let $b_i(\mathcal{Q}, \mathcal{P}) = \sum_\sigma b_i(\sigma)$.

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- Let $b_i(n, d, k, k')$ be the maximum of $b_i(\mathcal{Q}, \mathcal{P})$ over all $\mathcal{Q}, \mathcal{P}$ where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of of $\mathbb{R}[X_1, \ldots, X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = n$ and the algebraic set $Z(\mathcal{Q})$ has dimension k'.

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- Previously known (B, Pollack, Roy (1995))

$$\mathbf{b_0(n, d, k, k') = \binom{4n}{k'} d(2d-1)^{k-1} = \binom{n}{k'} O(d)^k.}$$

Betti Numbers of Sign Patterns III

Theorem 5. *(B, Pollack,Roy, 2002)*

$$\mathbf{b_i(n,d,k,k') \le \sum_{0\le j\le k'-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = \binom{n}{k'-i} O(d)^k.}$$

Proofs

Uses the Mayer-Vietoris long exact sequence.

Case when $n = 2$:

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$$\cdots H_i(A_1 \cap A_2) \to H_i(A_1) \oplus H_i(A_2) \to H_i(A_1 \cup A_2) \to H_{i-1}(A_1 \cap A_2) \to \cdots$$

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gives

$$\beta_i(A_1 \cup A_2) \leq \beta_i(A_1) + \beta_i(A_2) + \beta_{i-1}(A_1 \cap A_2)$$

Case of many sets:

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Lemma 6. *Let* A *be a finite simplicial complex and* $A_1, \ldots, A_n$ *subcomplexes of* A *such that* $A = A_1 \cup \cdots \cup A_n$. *Suppose that for every* ℓ, $0 \leq \ell \leq i$, *and for every* $(\ell+1)$ *tuple* $A_{\alpha_0}, \ldots, A_{\alpha_\ell}$, $\beta_{i-\ell}(A_{\alpha_0,\ldots,\alpha_\ell}) \leq M$. *Then,* $\beta_i(A) \leq \sum_{0 \leq \ell \leq i} \binom{n}{\ell+1} M$.

(*Many in terms of few.*)

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Lemma 7. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the conjunction of the inequalities* $P_i \geq 0$. *Assume that* S *is bounded. Then,* $\sum_i \beta_i(S) = (4d)^k$.

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Theorem 3 follows.
Theorem 2 follows by a dual argument.
Theorem 4 follows using a result of Barvinok (1995).

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- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.

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- We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
- The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_{p+1}} C^*(A_{\alpha_0,\ldots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

Mayer-Vietoris Double Complex I

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

and ...

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
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 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Double Complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \cdots \\
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\end{array}$$

The Associated Total Complex

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\diagdown & & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \\
 & \xrightarrow{\delta} & C^{p-1,q+1} & \xrightarrow{\delta} & C^{p,q+1} & \xrightarrow{\delta} & C^{p+1,q+1} & \xrightarrow{\delta} & \cdots \\
\diagdown & & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \\
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\diagdown & & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \uparrow{\scriptstyle d} & \diagdown & \\
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$$D = d \pm \delta$$

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- $E_\infty = H^*(\text{Associated Total Complex})$.

Pictorially ...

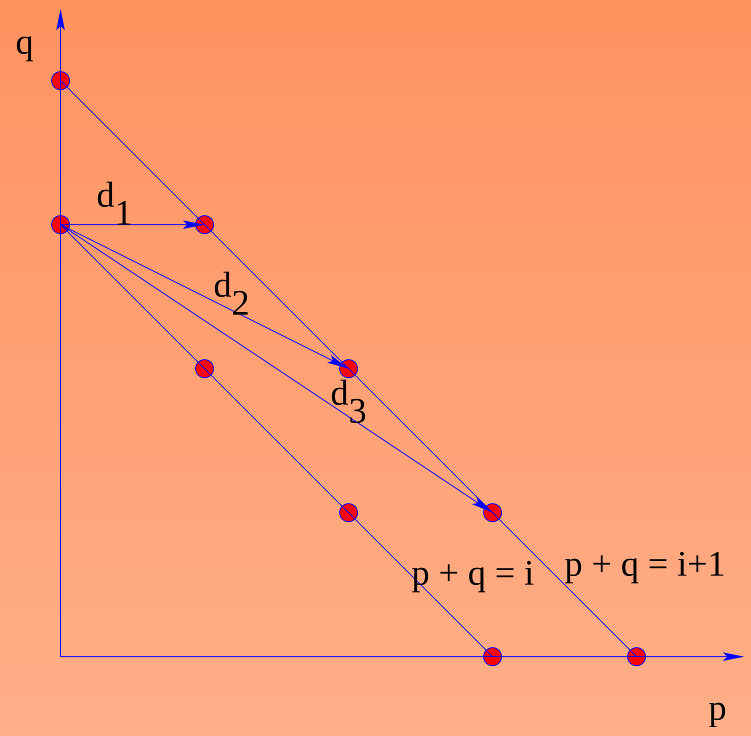

Figure 1: The differentials d_r in the spectral sequence (E_r, d_r)

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- $$\mathbf{E}'_\mathbf{1} = \mathbf{H_d}(\mathcal{M}), \;\; \mathbf{E}'_\mathbf{2} = \mathbf{H}_\delta\mathbf{H_d}(\mathcal{M})$$

$$E_1 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ C^3(A) & 0 & 0 \\ C^2(A) & 0 & 0 \\ C^1(A) & 0 & 0 \\ C^0(A) & 0 & 0 \end{array}$$

$$E_2 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ H^3(A) & 0 & 0 \\ H^2(A) & 0 & 0 \\ H^1(A) & 0 & 0 \\ H^0(A) & 0 & 0 \end{array}$$

The degeneration of this sequence at E_2 shows that

$$\mathbf{H}^*_{\mathbf{D}}(\mathcal{M}) \cong \mathbf{H}^*(\mathbf{A}).$$

$$E_1' = \begin{array}{ccc} \vdots & \vdots & \vdots \\ \prod_{\alpha_0} H^3(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0, \\ \prod_{\alpha_0} H^2(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0, \\ \prod_{\alpha_0} H^1(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0, \\ \prod_{\alpha_0} H^0(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0, \end{array}$$

Part II

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Algorithms in Computational Geometry.

Arrangements in Computational Geometry

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- Arrangements of balls or simplices in $\mathbb{R}^k$.

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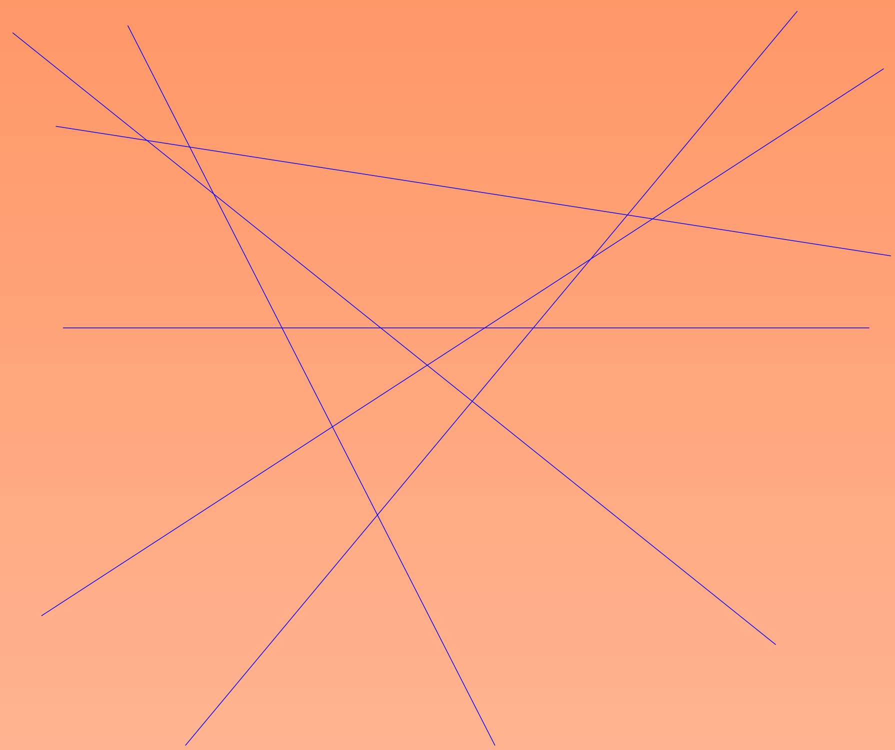
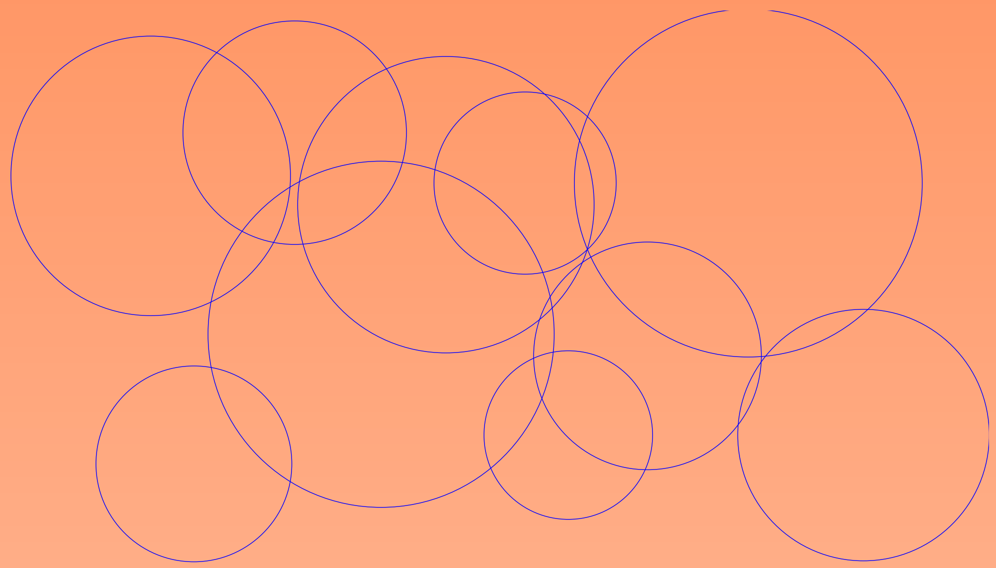
An arrangement in $\mathbb{R}^k$ is a collection of n objects in $\mathbb{R}^k$ each of *constant description complexity*.

- Arrangements of lines in the plane, or more generally hyperplanes in $\mathbb{R}^k$.
- Arrangements of balls or simplices in $\mathbb{R}^k$.
- Arrangements of semi-algebraic objects in $\mathbb{R}^k$, each defined by a fixed number of polynomials of constant degree.

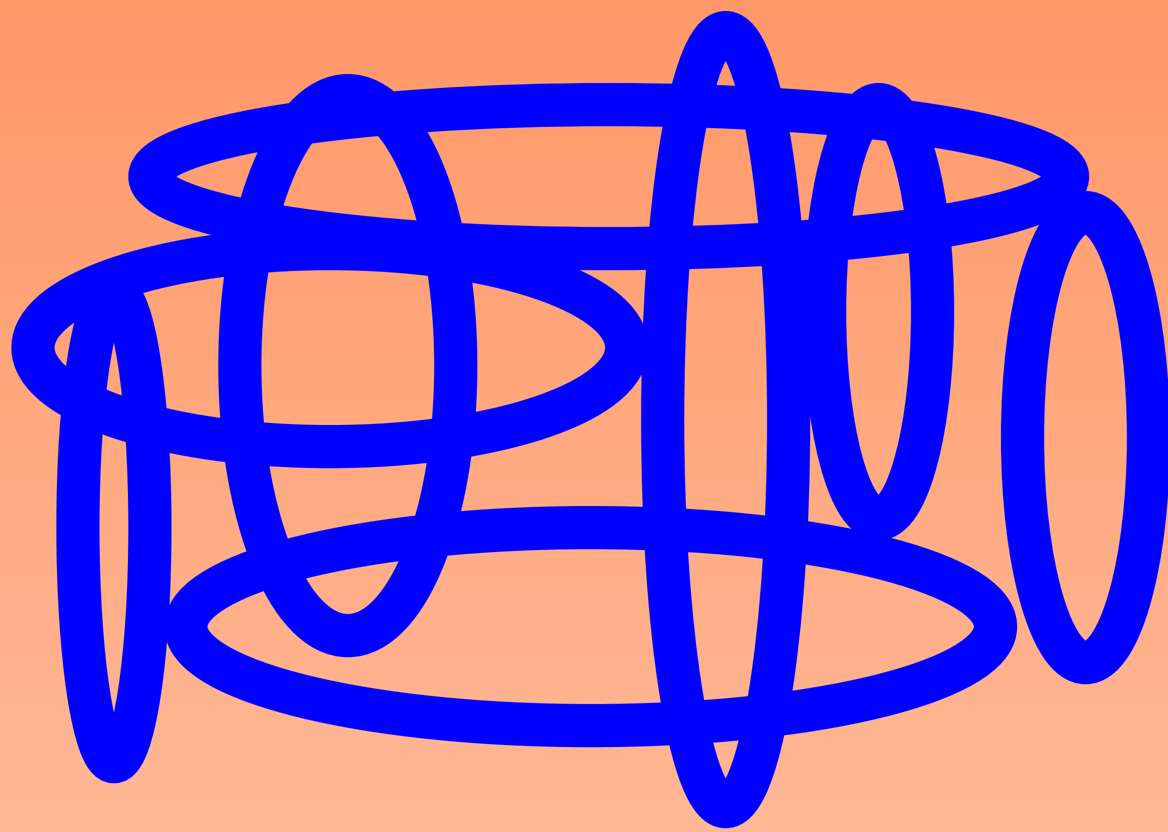
Arrangements of lines in the $\mathbb{R}^2$



Arrangement of circles in $\mathbb{R}^2$



Arrangement of tori in $\mathbb{R}^3$



Computing the Betti Numbers: Previous Work

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- Computing the Betti numbers of triangulated manifolds (Edelsbrunner, Dey, Guha et al).

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- We only count the number of algebraic operations and ignore the cost of doing linear algebra.

Two Approaches

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Global

vs

Local

First Approach (Global): Using Triangulations

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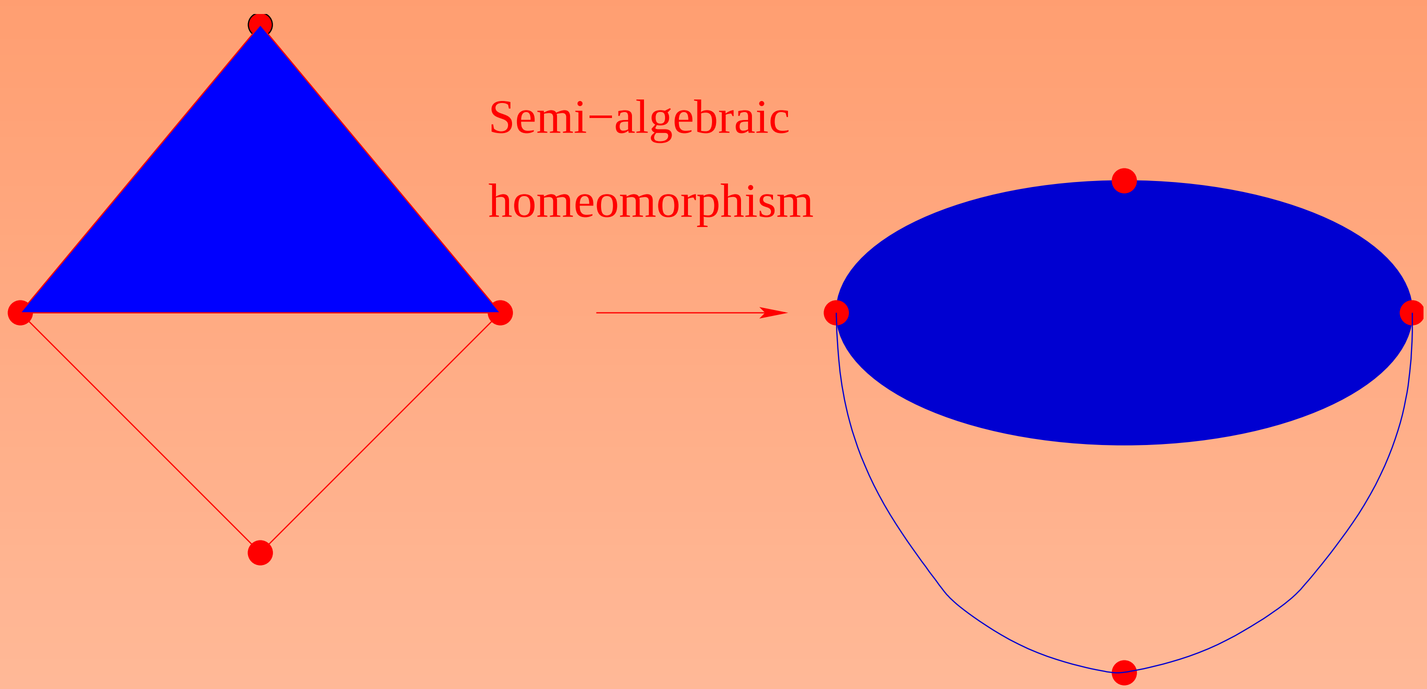
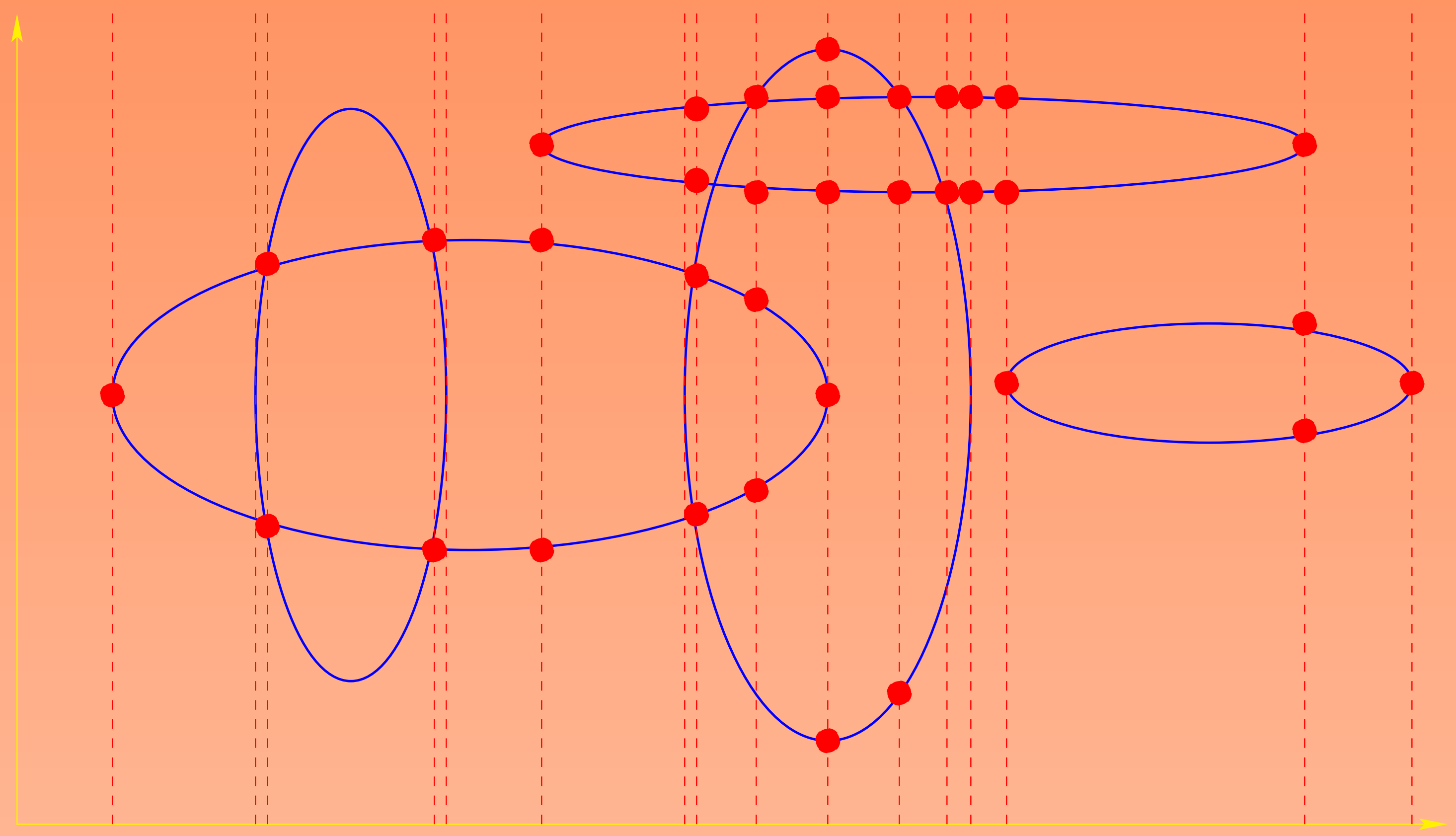
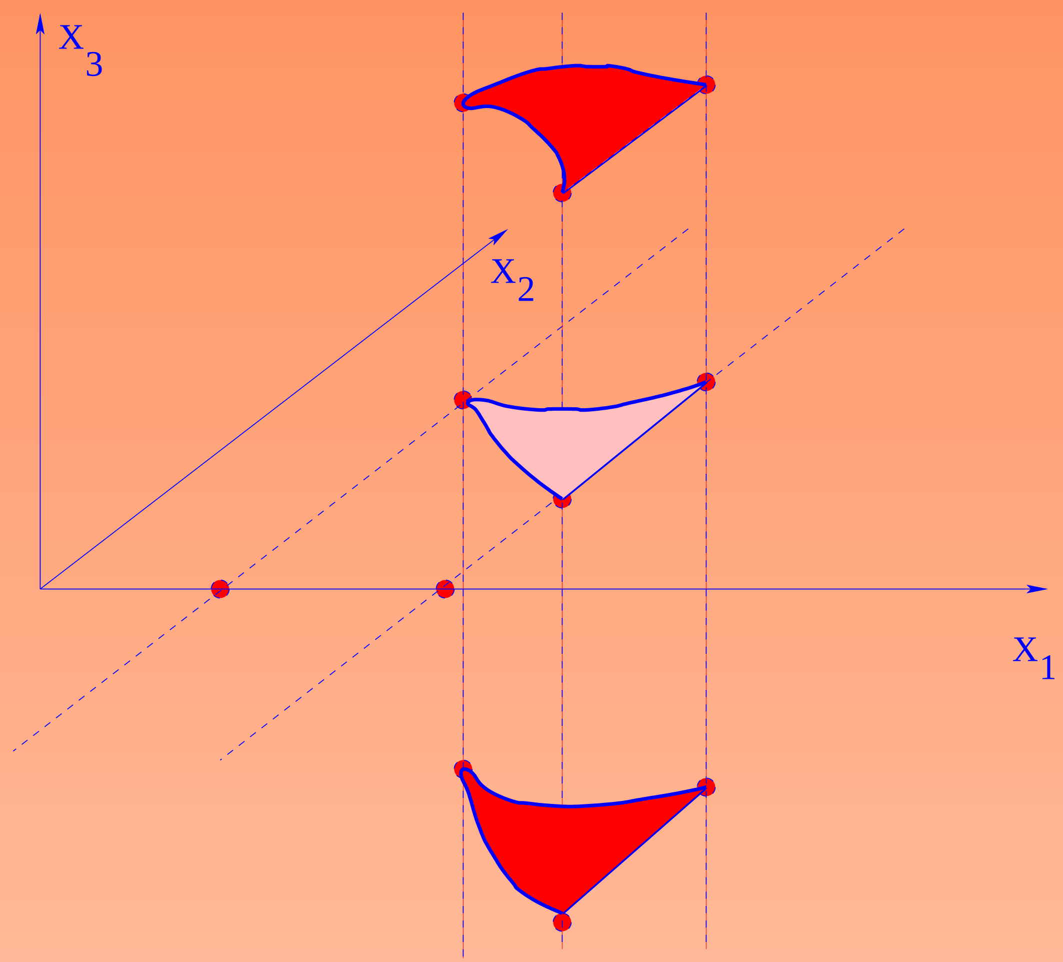


Using Collin's Cylindrical Algebraic Decomposition



Picture of a cylinder



Computing Betti Numbers using Global Triangulations

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- Compact semi-algebraic sets are finitely triangulable.
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- But ... CAD produces $O(n^{2^k})$ simplices in the worst case.

Second Approach (Local): Using the Nerve Complex

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- If the sets have the special property that all their non-empty intersections are contractible we can use the *nerve lemma* (Leray, Folkman).
- The homology groups of the union are then isomorphic to the homology groups of a combinatorially defined complex called the *nerve complex*.

The Nerve Complex

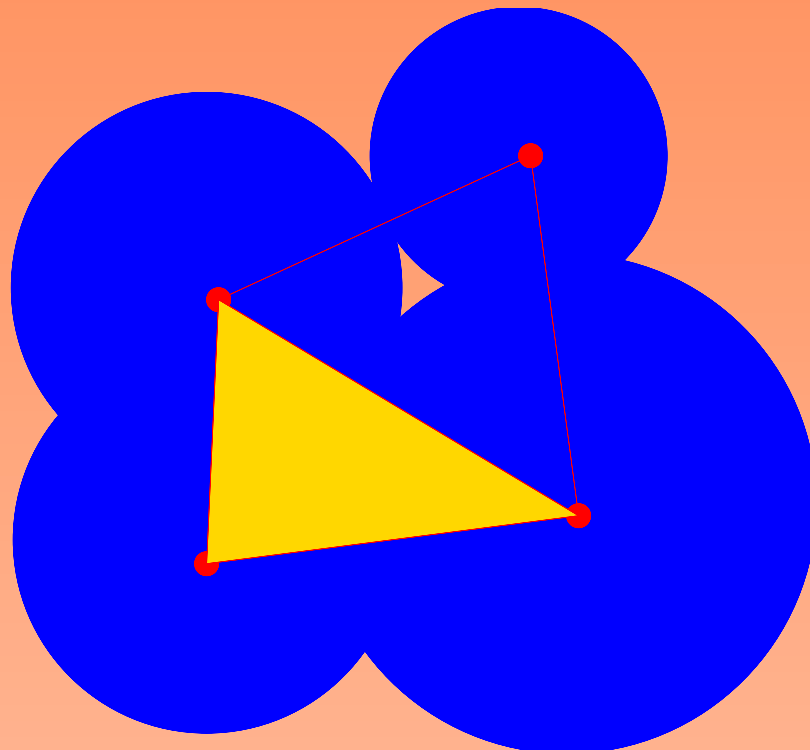

Figure 2: The nerve complex of a union of disks

Computing the Betti Numbers via the Nerve Complex (local algorithm)

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- Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1 \le j \le \ell+2} \binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets.

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- If the sets are such that the topology of the "small" intersections are controlled, then
- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
- The algorithmic version gives the first efficient algorithm for computing the Betti numbers, without the double-exponential complexity entailed in CAD.

Main Result

Theorem 8. *Let* $S_1, \ldots, S_n \subset \mathbb{R}^k$ *be compact semi-algebraic sets of constant description complexity and let* $S = \cup_{1 \leq i \leq n} S_i$, *and* $0 \leq \ell \leq k-1$. *Then, there is an algorithm to compute* $\beta_0(S), \ldots, \beta_\ell(S)$, *whose complexity is* $O(n^{\ell+2})$.

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The Algorithm

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- In order to compute β_ℓ, we only need to compute upto $E'_{\ell+2}$.*But the punchline is that:*

- In order to compute the differentials $d_r, 1 \leq r \leq \ell+1$, it suffices to have *independent triangulations of the different unions taken upto* $\ell+2$ *at a time.*

- For instance, it should be intuitively clear that in order to compute $\beta_0(\cup_i S_i)$ it suffices to triangulate pairs.

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- To what extent does topological simplicity aid algorithms in computational geometry ?