Algorithms for Computing Betti Numbers of Semi-algebraic Sets – Recent Progress and Open Problems.

Saugata Basu

School of Mathematics and College of Computing
Georgia Tech

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Semi-algebraic Sets and their Betti numbers

- A semi-algebraic set, $S \subset \mathrm{R}^k$, is a subset of R^k defined by a Boolean formula whose atoms are polynomial equalities and inequalities. If all the polynomials involved belong to $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call S a $\mathcal{P}$-semi-algebraic set.
- $b_i(S)$ will denote the i-th Betti number of S.
- Classical result (though with some modern tweaks) (Oleinik, Petrovsky, Thom, Milnor, B., Gabrielov-Vorobjov)

$$\sum_{0 \le i \le k} b_i(S) \le (O(s^2 d))^k$$

  where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.
- *Even though the Betti numbers are bounded singly exponentially in k, there is no known algorithm with single exponential complexity for computing them.*

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Some Motivations

- Semi-algebraic sets occur as configuration spaces in applications.
- Studying certain questions in quantitative real algebraic geometry. For instance, existence of single exponential sized triangulations.
- Next step after deciding whether a formula is satisfiable. Analogous problem in the discrete case, are counting problems (perhaps generally computing zeta functions of varieties). Recent work on continuous versions of counting complexity classes (Burgisser, Cucker, Meer) develops this analogy more formally.
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Complexity of Algorithms

- Double exponential vs single exponential vs polynomial time.
- Problems that can be solved in single exponential time: Testing emptiness, deciding connectivity, computing descriptions of the connected components, computing the Euler-Poincaré characteristic, computing the dimension of a given semi-algebraic set.
- Problems for which no single exponential time algorithm is known: Computing the higher Betti numbers, computing semi-algebraic triangulations, computing semi-algebraic stratifications.

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Three Main Techniques

- It is possible to obtain a semi-algebraic triangulation of S in doubly exponential time using cylindrical algebraic decomposition (Collins,Schwartz-Sharir). Thus, algorithms with doubly exponential complexity ($(sd)^{2^{O(k)}}$) is known for computing all the Betti numbers.
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Computing the first Betti number. [B,Pollack,Roy, 2005]

There exists an algorithm that takes as input the description of a $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^k$, and outputs $b_1(S)$. The complexity of the algorithm is $(sd)^{k^{O(1)}}$.

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Basic Semi-algebraic Sets Defined By Quadratic Inequalities

- Let $S \subset \mathbb{R}^k$ be a semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0,$ with $\deg(P_i) \leq 2, 1 \leq i \leq s$.
- Such sets are in fact quite general, since every semi-algebraic set can be defined by (quantified) formulas involving only quadratic polynomials.
- Moreover, as in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large. For example, the set $S \subset \mathbb{R}^k$ defined by $X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$ has $b_0(S) = 2^k$.
- It is NP-hard to decide whether such a set is empty.

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Bounds on Betti Numbers of Sets Defined by Quadratic Inequalities

Using prior results of Barvinok and inequalities derived from Mayer-Vietoris exact sequence,

Theorem (B. 2003)

Let $S \subset R^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0, \ \textit{with } \deg(P_i) \leq 2, 1 \leq i \leq s.$$

Then,

$$b_{k-\ell}(S) \leq \binom{s}{\ell} k^{O(\ell)}.$$

Features of the bound

- For fixed $\ell \geq 0$ this gives a polynomial bound on the top ℓ Betti numbers of S (which could possibly be non-zero).
- Similar bounds do not hold for sets defined by polynomials of degree greater than two. For instance, the set defined by the single quartic equation,

$$\sum_{i=1}^{k} X_i^2(X_i - 1)^2 - \varepsilon = 0,$$

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Bounds on the Projection

Theorem (B.,Zell, 2005)

Let $S \subset \mathrm{R}^{k+m}$ *be a bounded semi-algebraic set defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0, \text{ with } \deg(P_i) \leq 2, 1 \leq i \leq \ell,$$

with $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m]$.
Let $\pi : \mathrm{R}^{k+m} \to \mathrm{R}^m$ *be the projection onto the last* m *coordinates. Then, for any* $q > 0,\ 0 \leq q \leq k$,

$$\sum_{i=0}^{q} b_i(\pi(S)) \leq (k+m)^{O(q\ell)}.$$

Results in the Quadratic Case I

Algorithm for deciding emptiness. [Barvinok,1993], [Grigoriev-Pasechnik, 2004].

There exists an algorithm which given a set of s polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq s$, decides if S is non-empty, where S is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is $k^{O(s)}$.

Algorithm for computing the Euler-Poincaré characteristic. [B., 2005].

There exists an algorithm for computing $\chi(S)$ whose complexity is $k^{O(s)}$.

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Results in the Quadratic Case II

Polynomial time algorithm for computing top Betti numbers. [B., 2005].

We have an algorithm which given a set of s polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq s$, computes $b_{k-1}(S), \ldots, b_{k-\ell}(S)$, where S is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is

$$\sum_{i=0}^{\ell+2} \binom{s}{i} k^{2^{O(\min(\ell,s))}} = s^{\ell+2} k^{2^{O(\ell)}}.$$

Results in the Quadratic Case III

Projections of sets defined by few quadratic inequalities. [B.,Zell, 2005].

For fixed ℓ and q, there exists an algorithm for computing the first q Betti numbers of $\pi(S)$ where $S \subset \mathrm{R}^{k+m}$ is a bounded basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_\ell \geq 0$, with $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m], \deg(P_i) \leq 2,\ 1 \leq i \leq \ell$. The complexity of the algorithm is

$$(k+m)^{2^{O(q\ell)}}.$$

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Main Steps in the General Case

- First reduce to the closed and bounded case using a recent construction of Gabrielov and Vorobjov.
- Instead of computing a triangulation (which we do not know how to do in single exponential time), we compute with single exponential complexity a family of closed, bounded and contractible semi-algebraic sets, $\{X\}_{i \in I}$ such that, $S = \cup_{i \in I} X_i$.
- Using the Roadmap Algorithm compute the connected components of the pairwise and triple-wise intersections of the elements of the covering and their inclusion relationships.

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Result Needed from Algebraic Topology

A simple spectral sequence argument yields:

Proposition

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$ and each A_i is acyclic. Then, $b_1(A) = \dim(\mathrm{Ker}(\delta_2)) - \dim(\mathrm{Im}(\delta_1))$, *with*

$$\prod_i H^0(A_i) \xrightarrow{\delta_1} \prod_{i<j} H^0(A_{i,j}) \xrightarrow{\delta_2} \prod_{i<j<\ell} H^0(A_{i,j,\ell})$$

The homomorphisms δ_i are induced by generalized restrictions. Corresponding result needed for computing the higher Betti numbers is much more complicated.

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Connecting paths

- Given a semi-algebraic set $S \subset \mathrm{R}^k$, $x, y \in S$, there exists an algorithm (Roadmap) with single exponential complexity which can decide whether x and y are in the same connected component of S and if so output a semi-algebraic path connecting x to y in S.
- Fix a finite set of distinguished points in every connected component of S and for $x \in S$, let $\gamma(x)$ denote the connecting path computed by the algorithm connecting x to a distinguished point.

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Important Property of Connecting Path

- The connecting path $\gamma(x)$ consists of two consecutive parts, $\gamma_0(x)$ and $\Gamma_1(x)$. The path $\gamma_0(x)$ is contained in $\mathrm{RM}(S)$ and the path $\Gamma_1(x)$ is contained in S_{x_1}.
- Moreover, $\Gamma_1(x)$ can again be decomposed into two parts, $\gamma_1(x)$ and $\Gamma_2(x)$ with $\Gamma_2(x)$ contained in S_{x_1,x_2} and so on.
- If $y = (y_1, \ldots, y_k) \in S$ is another point such that $x_1 \neq y_1$, then the images of $\Gamma_1(x)$ and $\Gamma_1(y)$ are disjoint. If the image of $\gamma_0(y)$ (which is contained in S) follows the same sequence of curve segments as $\gamma_0(x)$ starting at p, then the images of the paths $\gamma(x)$ and $\gamma(y)$ has the property that they are identical upto a point and they are disjoint after it.

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Schematic Picture

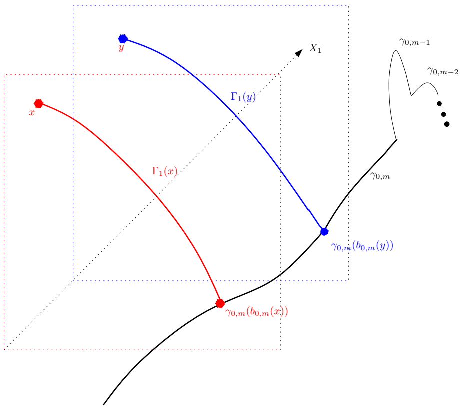

Figure: The connecting path $\Gamma(x)$

Parametrized Paths: Precise Definition

A parametrized path γ is a continuous semi-algebraic mapping, $V \subset \mathrm{R}^{k+1} \to \mathrm{R}^k$, a semi-algebraic continuous function $\ell : U \to [0, +\infty)$, with $U = \pi_{1\ldots k}(V) \subset \mathrm{R}^k$, and a in R^k, such that

1. $V = \{(x,t) \mid x \in U, 0 \le t \le \ell(x)\}$,
2. $\forall x \in U,\ \gamma(x,0) = a$,
3. $\forall x \in U,\ \gamma(x,\ell(x)) = x$,
4. $\forall x \in U, \forall y \in U, \forall s \in [0,\ell(x)], \forall t \in [0,\ell(y)]$ $(\gamma(x,s) = \gamma(y,t) \Rightarrow s = t)$,
5. $\forall x \in U, \forall y \in U, \forall s \in [0, \min(\ell(x),\ell(y))]$ $(\gamma(x,s) = \gamma(y,s) \Rightarrow \forall t \le s,\ \gamma(x,t) = \gamma(y,t))$.

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Parametrized Paths: Precise Definition

A parametrized path γ is a continuous semi-algebraic mapping, $V \subset \mathrm{R}^{k+1} \to \mathrm{R}^k$, a semi-algebraic continuous function $\ell : U \to [0, +\infty)$, with $U = \pi_{1\dots k}(V) \subset \mathrm{R}^k$, and a in R^k, such that

1. $V = \{(x, t) \mid x \in U, 0 \leq t \leq \ell(x)\}$,
2. $\forall x \in U,\ \gamma(x, 0) = a$,
3. $\forall x \in U,\ \gamma(x, \ell(x)) = x$,
4. $\forall x \in U, \forall y \in U, \forall s \in [0, \ell(x)], \forall t \in [0, \ell(y)]$ $(\gamma(x, s) = \gamma(y, t) \Rightarrow s = t)$,
5. $\forall x \in U, \forall y \in U, \forall s \in [0, \min(\ell(x), \ell(y))]$ $(\gamma(x, s) = \gamma(y, s) \Rightarrow \forall t \leq s,\ \gamma(x, t) = \gamma(y, t))$.

Schematic Picture

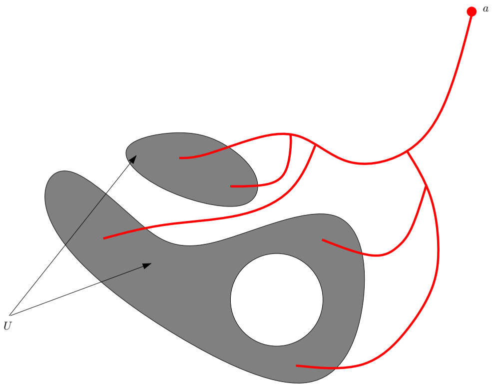

Figure: A Parametrized Path

Useful property of Parametrized Paths

Proposition

Let $\gamma : V \to R^k$ be a parametrized path such that $U = \pi_{1\ldots k}(V)$ is closed and bounded. Then, the image of γ is semi-algebraically contractible.

The images of the parametrized paths are the building blocks in the construction of the covering by contractible sets.

Outline

1. Introduction
   - Statement of the Problems
   - Motivation
   - Complexities of Different Problems
2. Recent Results
   - General Case
   - Quadratic Case
3. Outline of the Methods
   - General Case
   - Computing Covering by Contractible sets: Few Words
   - Quadratic Case
   - Mayer-Vietoris Exact Sequence
   - Projections
4. Open Problems

Main Ideas

- Consider S as the intersection of the individual sets, S_i defined by $P_i \geq 0$.
- The top dimensional homology groups of S are isomorphic to those of the total complex associated to a suitable truncation of the Mayer-Vietoris double complex.
- The terms appearing in the truncated complex depend on the unions of the S_i's taken at most $\ell + 2$ at a time. There are at most $\sum_{j=1}^{\ell+2} \binom{s}{j} = O(s^{\ell+2})$ such sets.
- Moreover, for such semi-algebraic sets we are able to compute in polynomial (in k) time a complex, whose homology groups are isomorphic to those of the given sets.

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Generalized Mayer-Vietoris Exact Sequence

Proposition

Let $A = A_1 \cap \cdots \cap A_n$ and $A^{\alpha_0,\ldots,\alpha_p}$ denote the union, $A_{\alpha_0} \cup \cdots \cup A_{\alpha_p}$. The following sequence is exact.

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} \bigoplus_{\alpha_0} C_\bullet(A^{\alpha_0}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\alpha_1} C_\bullet(A^{\alpha_0,\alpha_1}) \xrightarrow{\delta} \cdots$$

$$\xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_{p+1}} C_\bullet(A^{\alpha_0,\ldots,\alpha_{p+1}}) \xrightarrow{\delta} \cdots,$$

where i is induced by inclusion and the connecting homomorphisms δ are defined as follows:
for $c \in \oplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p})$,
$(\delta c)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum_{0\leq i\leq p+1} (-1)^i c_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{p+1}}$.

Mayer-Vietoris Double Complex

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_k(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_k(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_k(A^{\alpha_0,\alpha_1,\alpha_2}) \\
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0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-1}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-1}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-1}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-2}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-2}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-2}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-3}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-3}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-3}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

The Associated Total Complex

- The i-th homology group of A, $H_i(A)$ is isomorphic to the i-th homology group of the associated total complex of the double complex described above.
- For $0 \leq i \leq k$,

$$H_i(A) \cong H^i(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet})).$$

Moreover, if we denote by $\mathcal{N}_\ell^{\bullet,\bullet}$ the truncated complex defined by,

$$\begin{aligned} \mathcal{N}_\ell^{p,q} &= \mathcal{N}^{p,q}, & 0 \leq p \leq \ell, \; k-\ell \leq q \leq k, \\ &= 0, & \text{otherwise}, \end{aligned}$$

then it is clear that,

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Computing a quasi-isomorphic complex

- We cannot hope to compute even the truncated complex $\mathcal{N}_\ell^{\bullet,\bullet}$ since we do not know how to compute triangulations efficiently.
- We overcome this problem by computing another double complex $\mathcal{D}_\ell^{\bullet,\bullet}$, such that there exists a homomorphism of double complexes,

$$\psi : \mathcal{D}_\ell^{\bullet,\bullet} \to \mathcal{N}_\ell^{\bullet,\bullet},$$

  which induces an isomorphism between the E_1 terms of the spectral sequences associated to the double complexes $\mathcal{D}_\ell^{\bullet,\bullet}$ and $\mathcal{N}_\ell^{\bullet,\bullet}$.
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Topology of Unions

- For quadratic forms $P_1, \ldots, P_s$, we denote by $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \rightarrow \mathrm{R}^s$, the map defined by the polynomials $P_1, \ldots, P_s$.
- Let $A = \cup_{P \in \mathcal{P}} \{x \in \mathbf{S}^k \mid P(x) \leq 0\}$, and $\Omega = \{\omega \in \mathbf{S}^s \mid \omega_i \leq 0, 1 \leq i \leq s\}$.
- For $\omega \in \Omega$ let $\omega P = \sum_{i=1}^{s} \omega_i P_i$, and let $B = \{(\omega, x) \mid \omega \in \Omega, x \in \mathbf{S}^k$ and $\omega P(x) \geq 0\}$.
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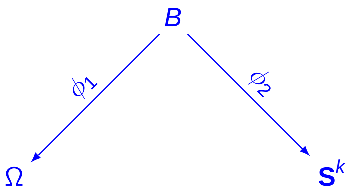
- 

Property of ϕ_2

Proposition (Agrachev)

The map ϕ_2 gives a homotopy equivalence between B and $\phi_2(B) = A$.

Thus, in order to compute a complex quasi-isomorphic to $C_\bullet(A)$ it suffices to construct one quasi-isomorphic to $C_\bullet(B)$.

Property of ϕ_1

Proposition

For $\omega \in \Omega$, $\phi_1^{-1}(\omega)$ is homotopy equivalent to the sphere $\mathbf{S}^{k-\mathrm{index}(\omega P)}$, where $\mathrm{index}(\omega P)$ is the number of negative eigenvalues of the quadratic form ωP.

Using this Proposition and an index invariant triangulation of Ω, it is possible to construct a complex quasi-isomorphic to $C_\bullet(B)$. Complexity is doubly exponential in ℓ.

Outline

Cohomological Descent

- Let, $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be semi-algebraic sets, and let $f : X \to Y$ be a semi-algebraic, continuous surjection, which is also an open mapping (it takes open sets to open sets).
- We denote by $W_f^i(X)$ the $(i+1)$-fold fibered product of X over f, that is, $W_f^i(X) = \{(x_0, \ldots, x_i) \in X^{i+1} \mid f(x_0) = \cdots = f(x_i)\}$.

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Descent Spectral Sequence

Exact sequence analogous to the Mayer-Vietoris exact sequence.

$$0 \longrightarrow C^\bullet(Y) \xrightarrow{f^*} C^\bullet(W_f^0(X)) \xrightarrow{\delta^0} C^\bullet(W_f^1(X)) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} C^\bullet(W_f^p(X)) \xrightarrow{\delta^p} C^\bullet(W_f^{p+1}(X)) \xrightarrow{\delta^{p+1}} \cdots$$

Now consider truncation of the corresponding double complex, and compute quasi-isomorphic complex using previous result.

Idea behind the algorithm

- Notice that the fibered product of q sets each defined by ℓ quadatic inequalities is defined by $q + \ell$ quadratic inequalities.
- Using the polynomial time algorithm described previously for computing a complex whose cohomology groups are isomorphic to those of a given semi-algebraic set defined by a constant number of quadratic inequalities,we are able to construct a certain double complex, whose associated total complex is quasi-isomorphic to a suitable truncation of the one obtained from the cohomological descent spectral sequence mentioned above. This complex is of much smaller size and can be computed in polynomial time.

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1. Single exponential time triangulation ?
2. Best complexity for computing Euler-Poincaré characteristic, or testing emptiness of an algebraic set V is $d^{O(k)}$. For computing $b_0(V)$ it is $d^{O(k^2)}$ and more generally for computing $b_\ell(V)$ is $d^{k^{O(\ell)}}$. Improve this or are the higher Betti numbers more difficult to compute ?
3. Real analogue of Toda's theorem ?
4. In view of recent results of D'Acunto and Kurdyka on bounding geodesic diameters of real algebraic varieties, can one improve the complexity of computing roadmaps to $d^{O(k)}$?
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