Different Bounds on the Different Betti Numbers of Semi-algebraic Sets

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Semi-algebraic Sets

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- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.

Topological Complexity

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- Intuitively $\beta_i(S)$ measures the number of "i-dimensional" holes in S.
- $\beta_0(S)$ is the number of connected components.
- For the hollow torus $T \subset R^3$, $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1$.

Uses of topological complexity

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Arrangements in Computational Geometry

Figure 1: An arrangement of circles in the plane.

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Topological Complexity of Semi-Algebraic Sets

Oleinik and Petrovsky (1949) Thom (1964) and Milnor (1965) proved that the sum of the Betti numbers of a semi-algebraic set $S \subset R^k$, defined by

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n,$$

is bounded by

$$(O(nd))^k.$$

The Nerve Complex

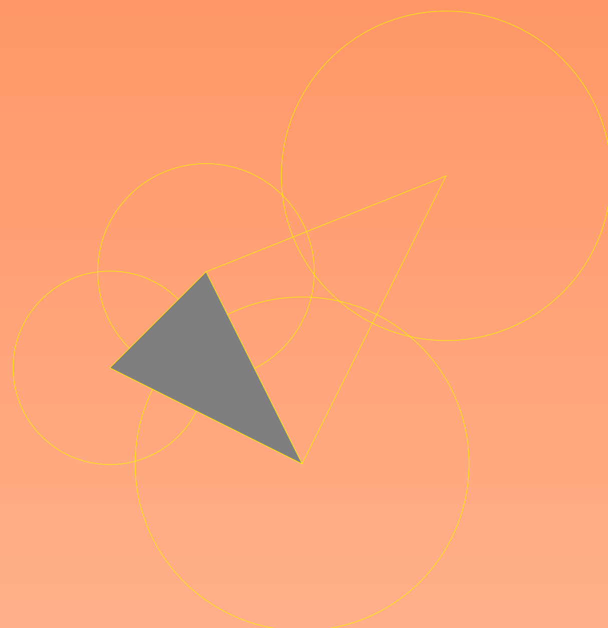

Figure 2: The nerve complex of a union of disks

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- What if the intersections are not acyclic but have bounded topology ?

Betti numbers for union

Theorem 1. *Let $S \subset R^k$ be the set defined by the disjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq n^{i+1} O(d)^k.$$

Betti numbers for intersections

Theorem 2. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq n^{k-i} O(d)^k.$$

Sets defined by Quadratic Inequalities

- Let $S \subset R^k$ be defined by

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- Can be topologically quite complicated. If S is defined by

$$X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$$

then clearly $\beta_0(S) = 2^k$ (exponential in the dimension).

But ...

Theorem 3. *Let ℓ be any fixed number and let $S \subset R^k$ be defined by*

$$P_1 \geq 0, \ldots, P_n \geq 0$$

with $\deg(P_i) \leq 2$. *Then,*

$$\beta_{k-\ell}(S) \leq n^\ell k^{O(\ell)}.$$

Note that this bound is polynomial in the dimension.

Ideas behind the proofs:

Let $A, B \subset R^k$ be compact semi-algebraic sets.
Mayer-Vietoris exact sequence:

$$0 \to H_{k-1}(A \cap B) \to H_{k-1}(A) \oplus H_{k-1}(B) \to H_{k-1}(A \cup B) \to$$

$$H_{k-2}(A \cap B) \to \cdots \to H_{i+1}(A \cup B) \to H_i(A \cap B) \to$$
$$H_i(A) \oplus H(B) \to H_i(A \cup B) \to \cdots$$

A preliminary lemma

Lemma 4. *Let* $S_1, \ldots, S_n \subset R^k$ *be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cup \cdots \cup S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq k - i$ *(that is the sum of the Betti numbers of the union of any* ℓ *of the sets for all* $\ell \leq k - i$ *is bounded by* M*). Let* $S = \cap_{1 \leq j \leq n} S_j$*. Then,*

$$\beta_i(S) \leq n^{k-i} M.$$

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$$0 \longrightarrow H_{k-1}(T_{n-1} \cap S_n) \longrightarrow H_{k-1}(T_{n-1}) \oplus H_{k-1}(S_n)$$

$$\longrightarrow H_{k-1}(T_{n-1} \cup S_n) \longrightarrow H_{k-2}(T_{n-1} \cap S_n) \longrightarrow \cdots$$
$$\longrightarrow H_{i+1}(T_{n-1} \cup S_n) \longrightarrow H_i(T_{n-1} \cap S_n) \longrightarrow$$
$$H_i(T_{n-1}) \oplus H_i(S_n) \longrightarrow H_i(T_{n-1} \cup S_n) \longrightarrow \cdots$$

Proof (cont):

- $\beta_{k-1}(T_n) = \beta_{k-1}(T_{n-1} \cap S_n) \leq \beta_{k-1}(T_{n-1}) + \beta_{k-1}(S_n)$. Unwinding the first term of right hand side we obtain that $\beta_{k-1}(S) \leq nM$.

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- Again from the Mayer-Vietoris sequence we get that,

$$\beta_i(S) \leq \beta_{i+1}(T_{n-1} \cup S_n) + \beta_i(T_{n-1}) + \beta_i(S_n).$$

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- $T_{n-1} \cup S_n = \cap_{1 \leq i \leq n-1}(S_i \cup S_n)$. The $n-1$ sets $S_i \cup S_n$ satisfies the assumption on at most $(k-i-1)$-ary unions and we can apply the induction hypothesis.

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- Thus, we have that $\beta_i(S) \le (n-1)^{k-i-1}M + (n-1)^{k-i}M + M \le n^{k-i}M$.

Dual lemma

Lemma 5. *Let* $S_1, \ldots, S_n \subset R^k$ *be compact semi-algebraic sets, such that,*

$$\sum_i \beta_i(S_{i_1} \cap \cdots \cap S_{i_\ell}) \leq M,$$

for all $1 \leq i_1 \leq \cdots \leq i_\ell \leq n, \ell \leq i+1$. *Let* $S = \cup_{1 \leq j \leq n} S_j$. *Then,*

$$\beta_i(S) \leq n^{i+1} M.$$

Sets defined by few inequalities:

Lemma 6. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the conjunction of the inequalities* $P_i \geq 0$. *Let* S *be bounded. Then,* $\sum_i \beta_i(S) = O(d)^k$.

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Lemma 7. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the disjunction of the inequalities* $P_i \geq 0$. *Then,* $\sum_i \beta_i(S) = O(d)^k$.